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Gaussian Fluctuations for the stochastic Burgers equation in dimension $d \geq 2$

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Abstract

The goal of the present paper is to establish a framework which allows to rigorously determine the large-scale Gaussian fluctuations for a class of singular SPDEs at and above criticality, and therefore beyond the range of applicability of pathwise techniques, such as the theory of Regularity Structures. To this purpose, we focus on a d-dimensional generalization of the Stochastic Burgers equation (SBE) introduced in [H. van Beijeren, R. Kutner and H. Spohn, Excess noise for driven diffusive systems, PRL, 1985]. In both the critical $d = 2$ and super-critical $d \geq 3$ cases, we show that the scaling limit of (the regularised) SBE is given by a stochastic heat equation with non-trivially renormalised coefficient, introducing a set of tools that we expect to be applicable more widely. For $d \geq 3$ the scaling adopted is the classical diffusive one, while in $d = 2$ it is the so-called *weak coupling* scaling which corresponds to tuning down the strength of the interaction in a scale-dependent way.

Key words and phrases. Stochastic Partial Differential Equations, critical and super-critical dimension, KPZ equation, weak coupling scaling, diffusion coefficient, asymmetric simple exclusion

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1 Introduction

The study of (singular) Stochastic Partial Differential Equations (SPDEs) has known tremendous advances in recent years. Thanks to the groundbreaking theory of regularity structures [Hai14], paracontrolled calculus [GIP15] or renormalization group techniques [Kup14], a local solution theory can be shown to hold for a large family of SPDEs in the so-called *sub-critical* dimensions (see [Hai14] for the notion of *sub-criticality* in this context). In contrast, SPDEs in the *critical* and *super-critical* dimensions are, from a mathematical viewpoint, still largely unexplored and poorly understood. The aim of the present paper is to establish a framework which allows to rigorously determine the large-scale (Gaussian) fluctuations for a class of SPDEs at and above criticality under (suitable) diffusive scaling.

To present the ideas and the techniques in our analysis, we will focus on a specific example, the Stochastic Burgers Equation (SBE), which, in any dimension, can (formally) be written as

$$\partial_t \eta = \tfrac{1}{2}\Delta\eta + \mathfrak{w} \cdot \nabla \eta^2 + \operatorname{div} \boldsymbol{\xi} \tag{1.1}$$

where the scalar field $\eta = \eta(x,t)$ depends on $t \geq 0$ and $x \in \mathbb{R}^d$ with $d \geq 1$, $\mathfrak{w} \in \mathbb{R}^d$ is a fixed vector and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$ is a d-dimensional space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$, i.e. a centred d-dimensional Gaussian process whose covariance is formally given by $\mathbf{E}[\xi_i(t,x)\xi_j(s,y)] = \delta(t-s)\delta(x-y)\mathbb{1}_{i=j}$.

The above equation was introduced by van Beijeren, Kutner and Spohn in [vBKS85] as an approximation of the fluctuations of general driven diffusion systems with one conserved quantity, e.g. the Asymmetric Simple Exclusion Process (ASEP) on $\mathbb{Z}^d$. The nonlinear term in (1.1) expresses the fact that the velocity at which the fluctuation density travels depends on its magnitude. In the context of ASEP, comes from the gradient of the current and is due to its asymmetry, so that in particular the vector $\mathfrak{w}$ models how the ASEP under consideration is asymmetric in the different spatial directions. In the *sub-critical* dimension $d = 1$, (1.1) reduces to the (derivative of the) celebrated Kardar–Parisi–Zhang (KPZ) equation [KPZ86] whose local solution theory is by now well-understood [Hai13, GP17, GJ14, GP18] and whose large-scale statistics have been recently determined [QS23, Vir20]. In higher dimensions the only available results are in the discrete setting. For $d \geq 3$, corresponding to the super-critical case, ASEP has been shown to be diffusive at large scales [EMY94, LY97a] – its fluctuations are Gaussian and given by the solution of a Stochastic Heat equation (SHE) with additive noise, see [CLO01, LOY04] and [KLO12] for a review. The critical dimension $d = 2$ is more subtle. Only qualitative information is available and no scaling limit has been obtained thus far.

The most important result is that of [Yau04], which states that two-dimensional ASEP is logarithmically superdiffusive, meaning that the diffusivity $D(t)$ satisfies $D(t) \sim (\log t)^{2/3}$ for t large.[1]

As mentioned above, our focus is on the critical and super-critical dimensions, i.e. we will be studying (1.1) (in the continuum) in $d \geq 2$. In both cases, we will prove that the large-scale behaviour is Gaussian and that the fluctuations evolve as a SHE with *renormalized coefficients* (see Theorem 1.3), but while for $d \geq 3$ the scaling adopted is the classical diffusive scaling, for $d = 2$ we will consider the so-called *weak coupling* scaling. This corresponds to tuning down the strength of the nonlinearity (morally, the "norm" of the vector $\mathfrak{w}$) at a suitable rate together with the scale parameter, see (1.6). The tuning is chosen so that we can "tame" the growth caused by the non-linear term without destroying its large-scale effects (see e.g. [CD20, CSZ20, Gu20, CES21, CETarb] for a similar choice of scaling in the critical dimension for different equations).

Let us mention that the study of large-scale Gaussian fluctuations of space-time random fields with local non-linear random dynamics is not restricted the context of SPDEs but is a classical topic in probability theory and mathematical physics. From the more general point of view of theoretical physics, large-scale Gaussian fluctuations are expected for equilibrium diffusive systems above their critical dimension and this belief is informally explained via (non-rigorous) Renormalization Group (RG) arguments and effective dynamical field theories.

A variety of tools have been used to mathematically justify these heuristics in the *static* setting, i.e. for the equilibrium measure of local non-linear systems. RG methods have been applied to derive the scaling limit of unbounded spin systems (and other statistical mechanical models) at and above their critical dimension $d \geq 4$ in [BBS15, BBS14, BBS19][2]. The related problem of triviality of Φ^4_d theories for $d > 4$ was treated in [Aiz82, Frö82, FFS92] using apriori bounds on correlation functions, an approach which ultimately lead to a proof in the more challenging critical dimension $d = 4$, see [AD21]. Another class of equilibrium models for which scaling limits have been successfully determined are equilibrium interface models, so called $\nabla\varphi$ models in all dimensions [BY90, NS97, FS97, GOS01], for which the Helffer-Sjöstrand representation of correlations together with homogeneization arguments was exploited. Analogous results have been proven, with very different methods, for *discrete* $\nabla\varphi$ models in dimension $d = 2$ (in particular dimer models, both integrable ones [Ken01] and their non-integrable perturbations [GMT20]).

On the other hand, establishing large-scale Gaussian fluctuations for *dynamical* problems has proven to be challenging and we are not aware of general results in this direction. The major issue seems to stem from the fact that there is currently no

[1] The diffusivity measures how the correlations of a process grow in space as a function of time so that, denoting by $\ell(t)$ the correlation length, it formally satisfies $\ell(t) \sim \sqrt{tD(t)}$.

[2] the authors further claim that "Our emphasis here is on the critical dimension $d = 4$, which is more difficult than dimensions $d > 4$."

framework which allows to naturally identify the law of the limit process. A lot of attention has been recently devoted to the above mentioned KPZ equation in high dimension, whose expression in its most general form is as (1.1) but with nonlinearity given by $\langle \nabla\eta, Q\nabla\eta\rangle$, for Q a $d \times d$ deterministic symmetric matrix, and a real valued space-time white noise in place of $\operatorname{div}\boldsymbol{\xi}$. Most of the works focus on the case in which Q is given by the identity matrix and the specific form of the resulting equation plays a crucial role. Indeed, one can use the Cole–Hopf transformation to linearise the SPDE and turn it into a *linear* multiplicative stochastic heat equation or the associated directed-polymer models. These latter models possess an "explicit" representation (via, e.g., Feynman-Kac formula) and one can then leverage Malliavin calculus tools to derive the scaling limit (see [DGRZ21, GRZ18b] for the multiplicative stochastic heat equation and [GRZ18a, CCM20, LZ22, CNN20] for KPZ). In this context, let us mention the recent striking results on the fluctuations of KPZ at the critical dimension $d = 2$ in the weak coupling scaling (or the intermediate disorder regime) starting with [CD20, CSZ20, Gu20] (for KPZ) and culminating with the characterisation of the stochastic heat flow [CSZ23]. An alternative approach is that used in [MU18], which is based on re-expressing the SPDE as a functional integral via the Martin–Siggia–Rose formalism (essentially a Girsanov transformation, which again is only possible in view of the specific form of the equation) and then leveraging constructive quantum field theory techniques and RG ideas to control the large scale fluctuations in the regime of small non-linearity.

In the critical dimension $d = 2$, for Q diagonal and with diagonal entries 1 and -1 (so-called Anisotropic KPZ or AKPZ), progress has been made in [CETara], where it is shown that the equation is logarithmically superdiffusive, and in [CETarb], where the large-scale behaviour in the weak coupling scaling is determined. In both articles, the tools adopted are closer to those in the present paper and we will comment more on them below.

Let us also mention recent work in extending the homogeneization methods (originated in the work of Naddaf and Spencer [NS97]) to dynamical problems [AD22, CDS22].

One of the main advantages of the approach taken in the present paper is that it moves away from the usual technique of Cole–Hopf transformation, which is *not applicable*, towards a set of tools which on the one hand makes a link with interacting particle systems, i.e. via the martingale problem and the associated infinite-dimensional generator, and on the other has better chances to apply more generally. A crucial tool for us is the fact that we have control over the invariant measure for (1.1), which significantly simplifies the already involved analysis of the generator. While this is another non-generic situation, there is a wide class of examples in which a similar control is available and to which our tools potentially directly apply, e.g. (super-)critical Stochastic Navier–Stokes [GT20, CK23, HZZ23], diffusions in divergence-free random vector fields [CHST22], lattice gas models [LRY05] and many others.

Our results open the way to gradually attack more challenging models and we

discuss some open problems below in Section 1.2.

Before turning to the core of the proof, in the next section we will precisely state our main result and outline the ideas behind our arguments.

1.1 The equation and main result

As written (1.1) is meaningless as the noise is too irregular for the non-linearity to be well-defined. Nonetheless, as we are interested in the large-scale behaviour, we regularise it, so to have a well-defined field at the microscopic level, and our goal then becomes to control its fluctuations while zooming out at the correct scale. We choose the regularisation in such a way to retain a fundamental property of the solution, namely its (formal) invariant measure. This amounts to smoothening the quadratic term via a Fourier cut-off as follows

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \mathfrak{w}\cdot \Pi_1 \nabla(\Pi_1\eta)^2 + (-\Delta)^{\frac{1}{2}}\xi, \tag{1.2}$$

where for $a > 0$, Π_a acts in Fourier space as

$$\widehat{\Pi_a \eta}(k) = \widehat{\eta}(k) 1_{|k|\le a}\,. \tag{1.3}$$

Without loss of generality, we also replaced $\operatorname{div}\boldsymbol{\xi}$ in (1.1) with $(-\Delta)^{\frac{1}{2}}\xi$ for ξ a space-time white noise (see (1.16) for the definition of the fractional Laplacian), since the two can be easily seen to have the same law.

Remark 1.1 In principle, the results below can be shown to hold using the same techniques of the present paper, even if instead of the Fourier cut-off we used a more general mollifier. That said, to simplify some (minor but annoying) technical points, we will stick to Π_1 and further assume that

- for $d \ge 3$, $1/\varepsilon \in \mathbb{N} + 1/2$ and that the norm in (1.3) is the sup-norm $|\cdot|_\infty$,
- for $d = 2$, the norm in (1.3) is the Euclidean norm $|\cdot|$ (no further assumption on ε is made).

We refer the reader to Remark 2.7 for some additional (technical) advantages of our choice.

In order to avoid technicalities related to infinite volume issues, we restrict (1.2) to the d-dimensional torus $\mathbb{T}^d_\varepsilon$ of side-length $2\pi/\varepsilon$. Here, ε is the "scale" parameter which will later be sent to 0. Note that via the diffusive rescaling

$$\eta^\varepsilon(x,t) \stackrel{\text{def}}{=} \varepsilon^{-\frac{d}{2}}\eta\big(x/\varepsilon, t/\varepsilon^2\big)\,, \tag{1.4}$$

equation (1.2) becomes

$$\partial_t \eta^\varepsilon = \frac{1}{2}\Delta\eta^\varepsilon + \lambda_\varepsilon \mathfrak{w}\cdot \Pi_{1/\varepsilon}\nabla(\Pi_{1/\varepsilon}\eta^\varepsilon)^2 + (-\Delta)^{\frac{1}{2}}\xi \tag{1.5}$$

where now the spatial variable takes values in the d-dimensional torus $\mathbb{T}^d \stackrel{\text{def}}{=} \mathbb{T}^d_1$ of side-length 2π, and ξ is a space-time white noise on $\mathbb{R}^+ \times \mathbb{T}^d$. The coefficient λ_ε depends on the dimension and is defined as

$$\lambda_\varepsilon \stackrel{\text{def}}{=} \begin{cases} \frac{1}{\sqrt{\log \varepsilon^{-1}}} & \text{if } d = 2 \\ \varepsilon^{\frac{d}{2}-1} & \text{if } d \geq 3. \end{cases} \tag{1.6}$$

While in the super-critical case $d \geq 3$, λ_ε is directly determined by (1.4), for $d = 2$, (1.1) is formally scale invariant under the diffusive scaling (another reason why such dimension is critical) so that one would have $\lambda_\varepsilon = 1$. The choice made in (1.6) corresponds to the so-called *weak coupling* scaling alluded to earlier and we will comment more on it after we state the main result. Before doing so, let us give the notion of solution for (1.5) we will be working with.

Definition 1.2 We say that η^ε is a stationary solution to (1.5) with coupling constant λ_ε if η^ε solves (1.5) and its initial condition is $\eta(0,\cdot) \stackrel{\text{def}}{=} \mu$, for μ a spatial white noise on $\mathbb{T}^d$, i.e. a mean zero Gaussian distribution on $\mathbb{T}^d$ such that

$$\mathbb{E}[\mu(\varphi)\mu(\psi)] = \langle \varphi, \psi \rangle_{L^2(\mathbb{T}^d)}, \qquad \text{for all } \varphi,\, \psi \in L^2_0(\mathbb{T}^d) \tag{1.7}$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T}^d)}$ is the usual scalar product in $L^2(\mathbb{T}^d)$ and $L^2_0(\mathbb{T}^d)$ is the space of zero-average square integrable functions on $\mathbb{T}^d$. We will denote by $\mathbb{P}$ the law of μ and by $\mathbb{E}$ the corresponding expectation.

The reason why the solution η^ε of (1.5) started from a spatial white noise is called *stationary* is that $\mathbb{P}$ is its invariant measure and, as we will prove in Lemma 2.1 below, this holds irrespective of the choice of $\mathfrak{w}, \varepsilon$. Instead, the law $\mathbf{P}$ (with corresponding expectation $\mathbf{E}$) of a stationary solution to (1.5) clearly depends on $\mathfrak{w}, \varepsilon$.

For $T > 0$, let us denote by $C([0,T], \mathcal{S}'(\mathbb{T}^d))$ the space of continuous functions with values in the space of tempered distributions $\mathcal{S}'(\mathbb{T}^d)$ and by $\mathcal{F}$ the Fourier transform on $\mathcal{S}'(\mathbb{T}^d)$ (see (1.13) for a definition). We are now ready to state the main result of the paper.

Theorem 1.3 *Let $\mathfrak{w} \in \mathbb{R}^d \setminus \{0\}$ and $T > 0$. For $\varepsilon > 0$, let λ_ε be defined according to (1.6) and η^ε be the stationary solution of (1.5) started from a spatial white noise μ. Then, there exists a constant $D_{\text{SHE}} > 0$, depending only on the dimension d and the norm $|\mathfrak{w}|$ of $\mathfrak{w}$, such that, as $\varepsilon \to 0$, η^ε converges in law in $C([0,T], \mathcal{S}'(\mathbb{T}^d))$ to the unique stationary solution η of*

$$\partial_t \eta = \tfrac{1}{2}(\Delta + D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)\eta + (-\Delta - D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)^{1/2}\xi\,, \qquad \eta(0,\cdot) = \mu\,, \tag{1.8}$$

where $(\mathfrak{w}\cdot\nabla)^2$ is the linear operator defined as $\mathcal{F}((\mathfrak{w}\cdot\nabla)^2\varphi)(k) \stackrel{\text{def}}{=} -(\mathfrak{w}\cdot k)^2\mathcal{F}(\varphi)(k)$, for $\varphi \in \mathcal{S}'(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$.

In the case $d = 2$, D_{SHE} is explicit and given by the formula

$$D_{\mathrm{SHE}} = \frac{1}{|\mathfrak{w}|^2}\left[\left(\frac{3|\mathfrak{w}|^2}{2\pi} + 1\right)^{\frac{2}{3}} - 1\right], \tag{1.9}$$

$|\mathfrak{w}|$ being the Euclidean norm of $\mathfrak{w}$.

Before proceeding to the proof of the above theorem, let us make a few comments on its statement and on the choice of λ_ε in (1.6). Notice that in any dimension $d \geq 2$, the constant D_{SHE} appearing in the limiting equation is *strictly* positive. Hence, despite the presence of the vanishing factor λ_ε in front of the nonlinearity, not only the quadratic term does not go to 0 but actually produces a new "Laplacian" and a new noise. Further, this new Laplacian (and noise), in a sense, "feels" the small-scale behaviour of the field η in (1.2) as it depends on the vector $\mathfrak{w}$, which in turn describes its microscopic dynamics. A similar phenomenon has already been observed for other critical and super critical SPDEs, even though we are not aware of examples in which a Laplacian of the form above was derived.

We have already pointed out that in $d = 2$ (1.1) is formally scale invariant. That said, the above mentioned result in [Yau04] for ASEP suggests that also (1.2) is logarithmically superdiffusive and the additional diffusivity can only come from the non-linear term. What the previous theorem shows is that the choice in (1.6) guarantees that the nonlinearity ultimately gives a non-vanishing order 1 contribution. What might appear puzzling is that, by translating the result of Yau to (1.5), the diffusivity $D^\varepsilon(t)$ of the scaled process grows as $|\log \varepsilon|^{2/3}$ so that one might be led to think that λ_ε should be chosen accordingly. While this is not the case, the expression in (1.9) formally implies the result of Yau, as can be seen by taking $\mathfrak{w} = \mathfrak{w}_\varepsilon$ such that $|\mathfrak{w}_\varepsilon| \stackrel{\text{def}}{=} \lambda_\varepsilon^{-1}$.

Furthermore, a similar scaling has been considered in the critical dimension $d = 2$ in the context of the KPZ equation [CD20, CSZ20, Gu20, CSZ23], of the Anisotropic KPZ equation [CES21, CETarb] and of the stochastic Navier–Stokes equation [CK23]. What is interesting is that, even though these latter examples in principle have different large scale diffusivity, i.e., $D(t) \sim t^\beta$ for some $\beta > 0$ [FT90] for KPZ, $D(t) \sim \sqrt{\log t}$ for AKPZ and Navier–Stokes [CETara, WAG71], they all display *non-trivial* (in that the non-linearity nontrivially contributes to the limit) *Gaussian fluctuations* under the same weak coupling scaling $\lambda_\varepsilon = |\log \varepsilon|^{-1/2}$, thus suggesting some sort of universality for it. This is to be compared with the one-dimensional case, in which, setting $\lambda_\varepsilon \stackrel{\text{def}}{=} \sqrt{\varepsilon}$, one recovers the so-called *weakly asymmetric* scaling under which convergence of ASEP to the one-dimensional KPZ equation was first shown in [BG97] and since then for a wide variety of models.

1.2 Open problems

Our results raise several interesting questions, as

- obtain Gaussian fluctuations for (1.1) in the case of more general non-linearity, i.e. $\mathfrak{w} \cdot \nabla \eta^2$ replaced by $\mathfrak{w} \cdot \nabla F(\eta)$, for instance for polynomial F. The

main difficulty is to obtain operator estimates similar to those of Lemma 2.4, which though seem highly non-trivial.
- Prove the analog of Theorem 1.3 (for $d = 2$) for ASEP on $\mathbb{Z}^2$ in the same limit of weak asymmetry.
- In dimension $d = 2$, define $\tilde{\eta}^\varepsilon(x,t) = \varepsilon^{-1}\eta(x/\varepsilon, t/(\varepsilon^2\tau(\varepsilon)))$ (to be compared with (1.4)) and find the right correction $\tau(\varepsilon)$ to the diffusive scaling so that a non-trivial scaling limit as $\varepsilon \to 0$ exists. This corresponds to a *strong coupling regime*. On the basis of [Yau04], the natural guess is $\tau(\varepsilon) = |\log \varepsilon|^{2/3}$ but the identification of the limit is hard as the regularising properties of the Laplacian vanish.

More challenging will be to move towards models for which the invariant measure is non-Gaussian or non-explicit. One guiding heuristic could be that the limiting Gaussian fluctuations and associated Gaussian stationary distribution could still provide a good setting for the analysis.

1.3 Idea of the proof

The idea of the proof finds its roots in the approach detailed in [KLO12] in the context of interacting particle systems. To summarise it and see how it translates to the present context, let us consider the weak formulation of (1.5) started from a spatial white noise μ, which, for a given test function φ, reads

$$\eta_t^\varepsilon(\varphi) - \mu(\varphi) - \tfrac{1}{2}\int_0^t \eta_s^\varepsilon(\Delta\varphi)\mathrm{d}s - \int_0^t \mathcal{N}_\varphi^\varepsilon(\eta_s^\varepsilon)\mathrm{d}s = M_t(\varphi) \tag{1.10}$$

where $\mathcal{N}_\varphi^\varepsilon$ stands for the nonlinearity tested against φ and M is the martingale associated to the space-time white noise ξ. Now, upon assuming the sequence $\{\eta^\varepsilon\}_\varepsilon$ to be tight in the space $C([0,T]\mathcal{S}'(\mathbb{T}^d))$, we see that all the terms in the above expression converge (at least along subsequences) but we have no information concerning the nonlinearity. For Theorem 1.3 to be true, we need this latter term to produce both a *dissipation* term of the form of $D_{\mathrm{SHE}}\mathcal{L}_0^{\mathfrak{w}}\eta$, where $\mathcal{L}_0^{\mathfrak{w}} \stackrel{\text{def}}{=} \frac{1}{2}(\mathfrak{w}\cdot\nabla)^2$, and a *fluctuation* part which should encode the additional noise in (1.8). In other words, the problem is to identify a *fluctuation-dissipation* relation [LY97a], i.e. to determine $D_{\mathrm{SHE}} > 0$ and $\mathscr{W}^\varepsilon$ such that

$$\mathcal{N}_\varphi^\varepsilon(\eta^\varepsilon) = -\mathcal{L}^\varepsilon\mathscr{W}^\varepsilon(\eta^\varepsilon) + D_{\mathrm{SHE}}\mathcal{L}_0^{\mathfrak{w}}\eta^\varepsilon(\varphi) \tag{1.11}$$

where $\mathcal{L}^\varepsilon$ is the generator of η^ε. Indeed, since by Dynkin's formula, there exists a martingale $\mathscr{M}^\varepsilon(\mathscr{W}^\varepsilon)$ such that

$$\mathscr{W}^\varepsilon(\eta_t^\varepsilon) - \mathscr{W}^\varepsilon(\mu) - \int_0^t \mathcal{L}^\varepsilon\mathscr{W}^\varepsilon(\eta_s^\varepsilon)\mathrm{d}s = \mathscr{M}_t^\varepsilon(\mathscr{W}^\varepsilon),$$

given (1.11), we could rewrite the nonlinear term in (1.10) as

$$\int_0^t \mathcal{N}_\varphi^\varepsilon(\eta_s^\varepsilon)\mathrm{d}s = \int_0^t D_{\mathrm{SHE}}\mathcal{L}_0^{\mathfrak{w}}\eta_s^\varepsilon(\varphi)\mathrm{d}s - \mathscr{M}_t^\varepsilon(\mathscr{W}^\varepsilon) + o(1)\,, \tag{1.12}$$

where $o(1)$ is a vanishing summand which contains the boundary terms. The advantage of the above is that we have expressed the nonlinearity in terms of a drift which captures the additional diffusivity and a martingale part which instead econdes the extra noise. At this point, tightness would ensure convergence of the former and we would be left to prove that the sequence $\mathscr{M}^\varepsilon(\mathscr{W}^\varepsilon)$ converges to a martingale with the correct quadratic variation.

The (hardest) problem is clearly the derivation of the fluctuation-dissipation relation and this is the point from which our analysis departs from that in [KLO12]. Even though, as we will see, it is enough to determine (1.11) approximately (i.e. that the difference of left and right hand side is small in a suitable sense), in the equation there are two unknowns, $D_{\mathrm{SHE}}\mathscr{L}_0^{\mathfrak{w}}$ which is *not* apriori given to us, and $\mathscr{W}^\varepsilon$ which, even if we were given the previous, is the solution of an infinite dimensional equation. To separate the two issues, we introduce a suitable truncation which removes the second summand from the right hand side of (1.11), so that we can first solve for $\mathscr{W}^\varepsilon$ and then project back and determine $D_{\mathrm{SHE}}\mathscr{L}_0^{\mathfrak{w}}$. This is done in $d \geq 3$ and $d = 2$ in very different ways. In the former case, we will control $\mathscr{W}^\varepsilon$ similarly to [LY97a], as we can show it satisfies a graded sector condition in the spirit of [KLO12, Section 2.7.4]. Nonetheless, their functional analytic approach does not apply in our setting and in particular, for the identification of D_{SHE} and $\mathscr{L}_0^{\mathfrak{w}}$, we devise a new method which is detailed in Section 2.4. For $d = 2$ instead, we introduce a novel ansatz which is based on the idea that at large scales the generator of η^ε should approximate a *modulated version* of the generator of (1.8). This ansatz simplifies dramatically the (iterative) analysis performed in [CETarb] and is heavily based on the Replacement Lemma in Section 2.1.2.

Organisation of the article

The rest of this work is organised as follows. Below we introduce notations, conventions, function spaces and elements of Wiener space analysis which will be used throughout. Section 2 is the bulk of the paper. In Section 2.1, after recalling basic properties of (1.5) and its generator, we discuss the two main technical tools - integrability estimates for $d \geq 3$ and the Replacement Lemma for $d = 2$. In Section 2.2 we determine apriori estimates on the solution to the generator equation and in the next two we identify D_{SHE} and $\mathscr{L}_0^{\mathfrak{w}}$, so to obtain the fluctuation-dissipation relation (1.11). Section 3 is devoted to the proof of Theorem 1.3 which, given the analysis in Section 2, essentially reduces to the control of the martingales $\mathscr{M}^\varepsilon(\mathscr{W}^\varepsilon)$ in (1.12) which is carried out in Section 3.3. At last, Appendix A contains some technical steps necessary in the proof of the Replacement Lemma.

Notations, function spaces and Wiener space analysis

We let $\mathbb{T}^d$ be the d-dimensional torus of side length 2π. We denote by $\{e_k\}_{k\in\mathbb{Z}^d}$ the Fourier basis defined via $e_k(x) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{d/2}} e^{ik\cdot x}$ which, for all $j,\ k \in \mathbb{Z}^d$, satisfies $\langle e_k, e_{-j}\rangle_{L^2(\mathbb{T}^d)} = \mathbb{1}_{k=j}$.

The Fourier transform of a function $\varphi \in L^2(\mathbb{T}^d)$ will be denoted by $\mathcal{F}(\varphi)$ or $\hat{\varphi}$ and, for $k \in \mathbb{Z}^d$ is given by the formula

$$\mathcal{F}(\varphi)(k) = \hat{\varphi}(k) \stackrel{\text{def}}{=} \int_{\mathbb{T}^d} \varphi(x) e_{-k}(x) \mathrm{d}x \,, \tag{1.13}$$

so that in particular

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} \hat{\varphi}(k) e_k(x) \,, \qquad \text{for all } x \in \mathbb{T}^d. \tag{1.14}$$

Let $\mathcal{S}(\mathbb{T}^d)$ be the space of smooth functions on $\mathbb{T}^d$ and $\mathcal{S}'(\mathbb{T}^d)$ the space of real-valued distributions given by the dual of $\mathcal{S}(\mathbb{T}^d)$. For any $\eta \in \mathcal{S}'(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$, we will denote its Fourier transform by

$$\hat{\eta}(k) \stackrel{\text{def}}{=} \eta(e_{-k}) \,. \tag{1.15}$$

Note that $\overline{\hat{\eta}(k)} = \hat{\eta}(-k)$. Since the zero mode $\hat{\eta}(0)$ of the solution is automatically zero, we will only care about $\hat{\eta}(k)$ for $k \in \mathbb{Z}^d_0 \stackrel{\text{def}}{=} \mathbb{Z}^d \setminus \{0\}$. Moreover, we recall that the Laplacian Δ on $\mathbb{T}^d$ has eigenfunctions $\{e_k\}_{k \in \mathbb{Z}^d}$ with eigenvalues $\{-|k|^2 \,:\, k \in \mathbb{Z}^d\}$, so that the operator $(-\Delta)^{\frac{1}{2}}$ in (1.2) is defined by its action on the basis elements

$$(-\Delta)^{\frac{1}{2}} e_k \stackrel{\text{def}}{=} |k| e_k \,, \qquad k \in \mathbb{Z}^d \,. \tag{1.16}$$

In particular, $(-\Delta)^{\frac{1}{2}}$ is an invertible linear bijection on distributions with null 0-th Fourier mode. We denote by $H^1(\mathbb{T}^d)$ the space of mean-zero functions φ such that the norm

$$\|\varphi\|^2_{H^1(\mathbb{T}^d)} \stackrel{\text{def}}{=} \|(-\Delta)^{\frac{1}{2}} \varphi\|^2_{L^2(\mathbb{T}^d)} \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}^d_0} |k|^2 |\hat{\varphi}(k)|^2$$

is finite.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and η be a mean-zero spatial white noise on the d-dimensional torus $\mathbb{T}^d$, i.e. η is a Gaussian field with covariance

$$\mathbb{E}[\eta(\varphi)\eta(\psi)] = \langle \varphi, \psi \rangle_{L^2(\mathbb{T}^d)} \tag{1.17}$$

where $\varphi, \psi \in H \stackrel{\text{def}}{=} L^2_0(\mathbb{T}^d)$, the space of square-integrable functions with 0 total mass, and $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T}^d)}$ is the usual scalar product in $L^2(\mathbb{T}^d)$. For $n \in \mathbb{N}$, let $\mathscr{H}_n$ be the *n-th homogeneous Wiener chaos*, i.e. the closed linear subspace of $L^2(\Omega)$ generated by the random variables $H_n(\eta(h))$, where H_n is the n-th Hermite polynomial, and $h \in H$ has norm 1. By [Nua06, Theorem 1.1.1], $\mathscr{H}_n$ and $\mathscr{H}_m$ are orthogonal whenever $m \neq n$ and $L^2(\Omega) = \bigoplus_n \mathscr{H}_n$. Moreover, there exists a canonical contraction $I : \bigoplus_{n \geq 0} L^2(\mathbb{T}^{2n}) \to L^2(\Omega)$, which restricts to an isomorphism $I : \Gamma L^2 \to L^2(\Omega)$ on the Fock space $\Gamma L^2 := \bigoplus_{n \geq 0} \Gamma L^2_n$, where ΓL^2_n denotes the space $L^2_{\text{sym}}(\mathbb{T}^{dn})$ of functions in $L^2_0(\mathbb{T}^{dn})$ which are symmetric with respect to permutation of variables. The restriction I_n of I to ΓL^2_n, is itself an isomorphism

from ΓL^2_n to $\mathscr{H}_n$ and, by [Nuao6, Theorem 1.1.2], for every $F \in L^2(\Omega)$ there exists a family of kernels $(f_n)_{n\in\mathbb{N}} \in \Gamma L^2$ such that $F = \sum_{n\geq 0} I_n(f_n)$ and

$$\mathbb{E}[F^2] \stackrel{\text{def}}{=} \|F\|^2 = \sum_{n\geq 0} n!\|f_n\|^2_{L^2(\mathbb{T}^{dn})} \tag{1.18}$$

and we take the right hand side as the definition of the scalar product on ΓL^2, i.e.

$$\langle f, g\rangle = \sum_{n\geq 0}\langle f_n, g_n\rangle \stackrel{\text{def}}{=} \sum_{n\geq 0} n!\langle f_n, g_n\rangle_{L^2(\mathbb{T}^{dn})}. \tag{1.19}$$

Remark 1.4 In view of the isometry between ΓL^2 and $L^2(\Omega)$, throughout the paper, we will abuse notation and denote with the same symbol operators acting on ΓL^2 and their composition with the isometry I, which is an operator acting on $L^2(\Omega)$.

For $F(\eta) = f(\eta(\varphi_1), \dots, \eta(\varphi_n))$, $f\colon \mathbb{R}^n \to \mathbb{R}$ smooth and growing at most polynomially at infinity and $\varphi_1, \dots, \varphi_n \in L^2_0(\mathbb{T}^d)$, we define the Malliavin derivative DF according to

$$DF(\eta)(\cdot) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \partial_i f(\eta(\varphi_1), \dots, \eta(\varphi_n))\varphi_i(\cdot) \tag{1.20}$$

and, for $k \in \mathbb{Z}^d_0$, denote its Fourier transform by

$$D_k F(v) \stackrel{\text{def}}{=} \mathscr{F}(DF(v))(k) = \langle DF, e_{-k}\rangle\,. \tag{1.21}$$

Notice that, for $p \in \mathbb{Z}$, the action of D_p on $f \in \Gamma L^2_n$ is

$$\mathscr{F}(D_p f)(k_{1:n-1}) = n\hat{f}_n(k_{1:n-1}, p)\,, \tag{1.22}$$

so that in particular $D_p f \in \Gamma L^2_{n-1}$. At last, we also recall the following integration by parts formula on Wiener space, i.e.

$$\mathbb{E}[GD_k F] = \mathbb{E}[G\langle DF, e_{-k}\rangle] = \mathbb{E}[-FD_k G + FG\hat{\eta}(k)]\,. \tag{1.23}$$

Throughout the paper, we will write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$. We will adopt the previous notations only in case in which the hidden constants do not depend on any quantity which is relevant for the result. When we write $\lesssim_T$ for some quantity T, it means that the constant C implicit in the bound depends on T.

2 The equation and its generator

In this section, we first collect a number of preliminary properties of the solution η^ε to the regularised Burgers equation (1.5) and its generator (Section 2.1). Then, we carry out a detailed analysis of the generator equation (a.k.a. resolvent equation) and obtain the main estimates we will need in order to determine the large–scale behaviour of η^ε.

2.1 Multidimensional Burgers generator

As in Section 1.3, let us first write (1.5) in its weak formulation. For $\varphi \in \mathcal{S}(\mathbb{T}^d)$ and $t \geq 0$, it reads

$$\eta_t^\varepsilon(\varphi) - \eta_0^\varepsilon(\varphi) = \int_0^t \eta_s^\varepsilon(\Delta\varphi)\mathrm{d}s + \int_0^t \mathcal{N}_\varphi^\varepsilon(\eta^\varepsilon)\mathrm{d}s + \int_0^t \xi(\mathrm{d}s, (-\Delta)^{1/2}\varphi) \tag{2.1}$$

where η_0^ε is the initial condition, ξ is a space-time white noise so that

$$\xi(\mathrm{d}t, (-\Delta)^{1/2}\varphi) = \sum_{k \in \mathbb{Z}_0^d} |k|\hat{\varphi}(k)\mathrm{d}B_t(k)$$

for $B(k)$'s complex valued Brownian motions satisfying $\overline{B(k)} = B(-k)$ and $\mathrm{d}\langle B(k), B(\ell)\rangle_t = \mathbb{1}_{\{k+\ell=0\}}\,\mathrm{d}t$, and $\mathcal{N}_\varphi^\varepsilon(\eta)$ is the nonlinearity tested against φ, i.e.

$$\begin{aligned} \mathcal{N}_\varphi^\varepsilon(\eta) &\stackrel{\text{def}}{=} \lambda_\varepsilon \mathfrak{w} \cdot \Pi_{1/\varepsilon}\nabla(\Pi_{1/\varepsilon}\eta)^2(\varphi) \\ &= \frac{\iota}{(2\pi)^{\frac{d}{2}}}\lambda_\varepsilon \sum_{\ell,m} \mathbb{J}_{\ell,m}^\varepsilon \mathfrak{w} \cdot (\ell+m)\hat{\varphi}(-\ell-m)\hat{\eta}(\ell)\hat{\eta}(m) \end{aligned} \tag{2.2}$$

with $\iota = \sqrt{-1}$ and

$$\mathbb{J}_{\ell,m}^\varepsilon = \mathbb{1}\{0 < \varepsilon|\ell|_\infty \leq 1, 0 < \varepsilon|m|_\infty \leq 1, 0 < \varepsilon|\ell+m|_\infty \leq 1\}\,. \tag{2.3}$$

In accordance with Remark 1.1, for $d = 2$ the sup-norm $|\cdot|_\infty$ in (2.3) is replaced by the Euclidean norm $|\cdot|$ instead.

From (2.1) follows that the Fourier modes $\hat{\eta}^\varepsilon(k)$, $|k| \geq \varepsilon^{-1}$ evolve like independent (and ε-independent) Ornstein-Uhlenbeck processes.

Lemma 2.1 *For every deterministic initial condition η_0, the solution $t \mapsto \eta_t^\varepsilon$ of (2.1) exists globally in time and is a strong Markov process. The generator $\mathscr{L}^\varepsilon$ of η^ε can be written as $\mathscr{L}^\varepsilon = \mathscr{L}_0 + \mathscr{A}^\varepsilon$ and the action of $\mathscr{L}_0$ and $\mathscr{A}^\varepsilon$ on smooth cylinder functions F is given by*

$$(\mathscr{L}_0 F)(\eta) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{k \in \mathbb{Z}_0^d} |k|^2(-\hat{\eta}(-k)D_k + D_{-k}D_k)F(\mu) \tag{2.4}$$

$$(\mathscr{A}^\varepsilon F)(\eta) \stackrel{\text{def}}{=} \frac{\iota}{(2\pi)^{d/2}}\lambda_\varepsilon \sum_{m,\ell \in \mathbb{Z}_0^d} \mathbb{J}_{\ell,m}^\varepsilon \mathfrak{w} \cdot (\ell+m)\hat{\eta}(m)\hat{\eta}(\ell)D_{-m-\ell}F(\eta). \tag{2.5}$$

Moreover, $\mathscr{L}_0$ is symmetric with respect to $\mathbb{P}$ while $\mathscr{A}^\varepsilon$ is skew-symmetric. Finally, the law $\mathbb{P}$ of the average-zero space white noise is stationary.

Proof. Very similar arguments were provided in a number of references for equations which share features similar to those of (2.1), e.g. [CES21, GJ13] and, more comprehensively, [Gub19], so we will limit ourselves to sketch some of the proofs.

For the global in time existence of (2.1) for fixed ε, we refer the reader to [CES21, Prop. 3.4] or [GJ13, Section 3], while the expressions (2.4) and (2.5) can be easily derived by Itô's formula and the definition of Malliavin derivative in (1.20).

To see that $\mathbb{P}$ is stationary, we start by noting that stationarity for the linear equation with $w = 0$ is well-known (and easy to check), see e.g. [Gub19, Section 2.3], where it is further shown that $\mathscr{L}_0$ is symmetric with respect to $\mathbb{P}$. Hence, by [Ech82], it suffices to check that $\mathbb{E}[\mathscr{A}^\varepsilon F] = 0$ for every cylinder function F. In fact, this follows immediately if we prove that $\mathscr{A}^\varepsilon$ is skew-symmetric, since, if so, $\mathbb{E}[\mathscr{A}^\varepsilon F] = -\mathbb{E}[F\mathscr{A}^\varepsilon 1] = 0$ as the Malliavin derivative of the constant random variable $F = 1$ is 0. Now, to prove skew-symmetry for $\mathscr{A}^\varepsilon$, we apply Gaussian integration by parts (1.23) and get

$$\begin{aligned}\mathbb{E}[G\mathscr{A}^\varepsilon F] =& -\mathbb{E}[F\mathscr{A}^\varepsilon G]\\ &+\frac{\iota\lambda_\varepsilon}{(2\pi)^{d/2}}\sum_{m,\ell\in\mathbb{Z}^d_0}\mathbb{J}^\varepsilon_{\ell,m}\mathfrak{w}\cdot(\ell+m)\mathbb{E}\Big[\hat\eta(-\ell-m)\hat\eta(m)\hat\eta(\ell)F(\eta)G(\eta)\Big]\\ =& -\mathbb{E}F\mathscr{A}^\varepsilon G+\mathbb{E}\Big[\langle\mathcal{N}^\varepsilon_\varphi(\eta),\eta\rangle_{L^2(\mathbb{T}^d)}F(\eta)G(\eta)\Big]\end{aligned}$$

so that, since

$$\langle\mathcal{N}^\varepsilon_\varphi(\eta),\eta\rangle_{L^2(\mathbb{T}^d)}=\lambda_\varepsilon\langle\mathfrak{w}\cdot\nabla(\Pi_{1/\varepsilon}\eta)^2,(\Pi_{1/\varepsilon}\eta)\rangle=\frac{\lambda_\varepsilon}{3}\langle\mathfrak{w}\cdot\nabla(\Pi_{1/\varepsilon}\eta)^3,1\rangle=0 \tag{2.6}$$

the proof is completed. □

Next, we want to determine the action of $\mathscr{L}_0$ and $\mathscr{A}^\varepsilon$ on the Fock space ΓL^2. To lighten notations, for a set of integers I, we will denote by k_I the vector $(k_i)_{i\in I}$ and

$$|k_I|^2\stackrel{\text{def}}{=}\sum_{i\in I}|k_i|^2\qquad\text{and}\qquad k_{1:n}\stackrel{\text{def}}{=}k_{\{1:n\}}\,, \tag{2.7}$$

where $\{1:n\}\stackrel{\text{def}}{=}\{1,\dots,n\}$.

Lemma 2.2 *Let $\varepsilon>0$, $\mathscr{L}_0$ and $\mathscr{A}^\varepsilon$ be the operators defined according to* (2.4) *and* (2.5) *respectively. Then, $\mathscr{A}^\varepsilon$ can be written as*

$$\mathscr{A}^\varepsilon=\mathscr{A}^\varepsilon_+ +\mathscr{A}^\varepsilon_-,\qquad\textit{with}\qquad\mathscr{A}^\varepsilon_-=-(\mathscr{A}^\varepsilon_+)^*\,. \tag{2.8}$$

Moreover, for any $n\in\mathbb{N}$, the action of the operators $\mathscr{L}_0,\mathscr{A}^\varepsilon_-,\mathscr{A}^\varepsilon_+$ on $f\in\Gamma L^2_n$ is given by

$$\begin{aligned}\mathscr{F}(\mathscr{L}_0 f)(k_{1:n})=&-\tfrac12|k_{1:n}|^2\hat f(k_{1:n})\\ \mathscr{F}(\mathscr{A}^\varepsilon_+ f)(k_{1:n+1})=&-\frac{2\iota}{(2\pi)^{d/2}}\frac{\lambda_\varepsilon}{n+1}\times\\ &\times\sum_{1\le i<j\le n+1}[\mathfrak{w}\cdot(k_i+k_j)]\mathbb{J}^\varepsilon_{k_i,k_j}\hat f(k_i+k_j,k_{\{1:n+1\}\setminus\{i,j\}})\end{aligned} \tag{2.9}$$

$$\mathscr{F}(\mathscr{A}^\varepsilon_- f)(k_{1:n-1}) = \frac{2\iota}{(2\pi)^{d/2}} n\lambda_\varepsilon \sum_{j=1}^{n-1} (\mathfrak{w}\cdot k_j) \sum_{\ell+m=k_j} \mathbb{J}^\varepsilon_{\ell,m} \hat{f}(\ell, m, k_{\{1:n-1\}\setminus\{j\}}).$$

and, if $n = 1$, then $\mathscr{A}^\varepsilon_- f$ is identically 0.

Proof. The proof is identical to that of [CES21, Lemma 3.5], to which we refer the interested reader. □

In order to state the following lemma, we introduce the so-called *number operator*.

Definition 2.3 We define the *number operator* $\mathcal{N} : \Gamma L^2 \to \Gamma L^2$ to be the linear diagonal operator acting on $\psi = (\psi_n)_n \in \Gamma L^2$ as $(\mathcal{N}\psi)_n \stackrel{\text{def}}{=} n\psi_n$.

Our analysis will need some preliminary estimates on the asymmetric part of the generator, which corresponds to proving the so-called *graded sector condition* in [KLO12, Section 2.7.4].

Lemma 2.4 *For $d \geq 2$ here exist a constant $C = C(d) > 0$ such that for every $\psi \in \Gamma L^2$ the following estimate holds*

$$\|(-\mathscr{L}_0)^{-\frac{1}{2}} \mathscr{A}^\varepsilon_\sigma \psi\| \leq C \|\sqrt{\mathcal{N}}(-\mathscr{L}_0)^{\frac{1}{2}}\psi\|\,, \tag{2.10}$$

for $\sigma \in \{+,-\}$. In particular, for $\sigma \in \{+,-\}$, it implies

$$\|(-\mathscr{L}_0)^{-\frac{1}{2}} \mathscr{A}^\varepsilon_\sigma (-\mathscr{L}_0)^{-\frac{1}{2}}\psi\| \leq C \|\sqrt{\mathcal{N}}\psi\|\,. \tag{2.11}$$

Proof. Clearly, once we establish (2.10), (2.11) follows immediately by choosing ψ therein to be $(-\mathscr{L}_0)^{-1/2}\varrho$ for $\varrho \in \Gamma L^2$.

For (2.10), we claim that we only need to prove that for any $\psi \in \Gamma L^2_n$ and $\varrho \in \Gamma L^2_{n+1}$, we have

$$|\langle \varrho, \mathscr{A}^\varepsilon_+ \psi\rangle| \lesssim \sqrt{n}\Big(\gamma \|(-\mathscr{L}_0)^{1/2}\psi\|^2 + \frac{1}{\gamma}\|(-\mathscr{L}_0)^{1/2}\varrho\|^2\Big)\,. \tag{2.12}$$

We will show (2.12) at the end. Assuming it holds, we first derive (2.10) for $\mathscr{A}^\varepsilon_+$. The variational characterisation of the $(-\mathscr{L}_0)^{-1/2}\Gamma L^2$-norm gives

$$\begin{aligned}
\|(-\mathscr{L}_0)^{-1/2}\mathscr{A}^\varepsilon_+\psi\|^2 &= \sup_{\varrho\in\Gamma L^2_{n+1}} \Big(2\langle \varrho, \mathscr{A}^\varepsilon_+\psi\rangle - \|(-\mathscr{L}_0)^{\frac{1}{2}}\varrho\|^2\Big) \\
&\leq \sup_{\varrho\in\Gamma L^2_{n+1}} \left(2C_0\sqrt{n}\Big(\gamma\|(-\mathscr{L}_0)^{1/2}\psi\|^2 + \frac{1}{\gamma}\|(-\mathscr{L}_0)^{1/2}\varrho\|^2\Big) - \|(-\mathscr{L}_0)^{\frac{1}{2}}\varrho\|^2\right) \\
&\lesssim n\|(-\mathscr{L}_0)^{1/2}\psi\|^2 = \|\sqrt{\mathcal{N}}(-\mathscr{L}_0)^{1/2}\psi\|^2
\end{aligned}$$

where in the first step we used (2.12) (and C_0 is the universal constant implicit in that inequality) while in the last we chose $\gamma \stackrel{\text{def}}{=} 2C_0\sqrt{n}$.

For $\mathscr{A}_-^\varepsilon$ instead, we use that, by (2.8), $\mathscr{A}_-^\varepsilon = -(\mathscr{A}_+^\varepsilon)^*$. Invoking again the variational formula above, we deduce

$$\begin{aligned}
\|(-\mathscr{L}_0)^{-1/2}\mathscr{A}_-^\varepsilon\psi\|^2 &= \sup_{\varrho\in\Gamma L^2_{n-1}}\Big(2\langle\varrho,\mathscr{A}_-^\varepsilon\psi\rangle - \|(-\mathscr{L}_0)^{\frac12}\varrho\|^2\Big)\\
&= \sup_{\varrho\in\Gamma L^2_{n-1}}\Big(-2\langle\mathscr{A}_+^\varepsilon\varrho,\psi\rangle - \|(-\mathscr{L}_0)^{\frac12}\varrho\|^2\Big)\\
&\le \sup_{\varrho\in\Gamma L^2_{n-1}}\bigg(2C_0\sqrt{n}\Big(\gamma\|(-\mathscr{L}_0)^{1/2}\varrho\|^2+\frac{1}{\gamma}\|(-\mathscr{L}_0)^{1/2}\psi\|^2\Big) - \|(-\mathscr{L}_0)^{\frac12}\varrho\|^2\bigg)\\
&\lesssim n\|(-\mathscr{L}_0)^{1/2}\psi\|^2 = \|\sqrt{\mathcal{N}}(-\mathscr{L}_0)^{1/2}\psi\|^2
\end{aligned}$$

where this time we chose $\gamma \stackrel{\text{def}}{=} 1/(2C_0\sqrt{n})$.

It remains to prove (2.12). By definition of $\mathscr{A}_+^\varepsilon$, we have

$$\begin{aligned}
|\langle\varrho,\mathscr{A}_+^\varepsilon\psi\rangle| =& n!\frac{2}{(2\pi)^{\frac d2}}\lambda_\varepsilon\Big|\sum_{k_{1:n+1}}\hat\varrho(-k_{1:n+1})\times\\
&\times\sum_{1\le i<j\le n+1}[\mathfrak{w}\cdot(k_i+k_j)]\mathbb{J}^\varepsilon_{k_i,k_j}\hat\psi(k_i+k_j,k_{\{1:n+1\}\setminus\{i,j\}})\Big|\\
=&\frac{(n+1)!n}{(2\pi)^{\frac d2}}\lambda_\varepsilon\Big|\sum_{k_{1:n+1}}\hat\varrho(-k_{1:n+1})[\mathfrak{w}\cdot(k_1+k_2)]\mathbb{J}^\varepsilon_{k_1,k_2}\hat\psi(k_1+k_2,k_{3:n+1})\Big|.
\end{aligned}$$

Let us look at the sum over k_1,k_2. This equals

$$\begin{aligned}
&\Big|\sum_{k_{1:2}}\hat\varrho(-k_{1:n+1})[\mathfrak{w}\cdot(k_1+k_2)]\mathbb{J}^\varepsilon_{k_1,k_2}\hat\psi(k_1+k_2,k_{3:n+1})\Big|\\
&=\Big|\sum_q(\mathfrak{w}\cdot q)\hat\psi(q,k_{3:n+1})\sum_{k_1+k_2=q}\mathbb{J}^\varepsilon_{k_1,k_2}\hat\varrho(-k_{1:n+1})\Big|\\
&\lesssim\Big(\sum_q(\mathfrak{w}\cdot q)^2|\hat\psi(q,k_{3:n+1})|^2\Big)^{1/2}\Big(\sum_q\Big|\sum_{k_1+k_2=q}\mathbb{J}^\varepsilon_{k_1,k_2}\hat\varrho(-k_{1:n+1})\Big|^2\Big)^{1/2}\\
&=: A\times(\mathrm{I})
\end{aligned}\tag{2.13}$$

We leave A as it stands and focus on (I). Note that this can be bounded by

$$\begin{aligned}
(\mathrm{I}) &= \Big(\sum_q\Big|\sum_{k_1+k_2=q}\mathbb{J}^\varepsilon_{k_1,k_2}\hat\varrho(-k_{1:n+1})\Big|^2\Big)^{1/2}\\
&\le\Big(\sum_q\sum_{k_1+k_2=q}|k_{1:2}|^2|\hat\varrho(-k_{1:n+1})|^2\sum_{k_1+k_2=q}\frac{\mathbb{J}^\varepsilon_{k_1,k_2}}{|k_{1:2}|^2}\Big)^{1/2}\\
&\lesssim\lambda_\varepsilon^{-1}\Big(\sum_q\sum_{k_1+k_2=q}|k_{1:2}|^2|\hat\varrho(k_{1:n+1})|^2\Big)^{1/2} =: \lambda_\varepsilon^{-1}B
\end{aligned}\tag{2.14}$$

where we used that for $d = 2$, $\lambda_\varepsilon^2 = (\log \varepsilon^{-2})^{-1}$ and

$$\lambda_\varepsilon^2 \sum_{k_1+k_2=q} \frac{\mathbb{J}^\varepsilon_{k_1,k_2}}{|k_{1:2}|^2} \le \frac{\varepsilon^2}{\log \varepsilon^{-2}} \sum_{\varepsilon<|\varepsilon k_1|\le 1} \frac{1}{|\varepsilon k_1|^2} \lesssim \frac{1}{\log \varepsilon^{-2}} \int_{|x|\in[\varepsilon,1]} \frac{\mathrm{d}x}{|x|^2} \lesssim 1 \tag{2.15}$$

while for $d \ge 3$, $\lambda_\varepsilon^2 = \varepsilon^{d-2}$ and

$$\varepsilon^{d-2} \sum_{k_1+k_2=q} \frac{\mathbb{J}^\varepsilon_{k_1,k_2}}{|k_{1:2}|^2} \le \varepsilon^d \sum_{|\varepsilon k_1|\le 1} \frac{1}{|\varepsilon k_1|^2} \lesssim \int_{|x|\le 1} \frac{\mathrm{d}x}{|x|^2} \lesssim 1\,. \tag{2.16}$$

As a consequence, for any $\gamma > 0$, we have

$$\begin{aligned} |\langle \varrho, \mathscr{A}^\varepsilon_+ \psi\rangle| &\lesssim (n+1)! n \sum_{k_{3:n+1}} AB \le (n+1)! n \Big(\frac{\gamma}{\sqrt{n}} \sum_{k_{3:n+1}} A^2 + \frac{\sqrt{n}}{\gamma} \sum_{k_{3:n+1}} B^2 \Big) \\ &\le (n+1)! n \Big(\frac{1}{n!} \frac{\gamma}{n^{3/2}} \|(-\mathscr{L}_0)^{1/2}\psi\|^2 + \frac{1}{(n+1)!} \frac{n^{-1/2}}{\gamma} \|(-\mathscr{L}_0)^{1/2}\varrho\|^2 \Big) \\ &\le \sqrt{n}\gamma \|(-\mathscr{L}_0)^{1/2}\psi\|^2 + \frac{\sqrt{n}}{\gamma} \|(-\mathscr{L}_0)^{1/2}\varrho\|^2 \end{aligned} \tag{2.17}$$

where we used that, by the definition of A and B in (2.13) and (2.14) respectively, we have

$$\sum_{k_{3:n+1}} A^2 \lesssim \sum_{k_{1:n}} |k_1|^2 |\hat\psi(k_{1:n})|^2 \le \frac{n^{-1}}{n!} \|(-\mathscr{L}_0)^{1/2}\psi\|^2$$

and similarly for B. Then, (2.12) follows. □

For $d = 2$ and $d \ge 3$, we need some further but distinct estimates, which we separately give in the two sections below.

2.1.1 Integrability estimates: $d \ge 3$

In the supercritical regime $d \ge 3$, in order to identify the limit process we will resort to use a sort of *uniform integrability condition*. This is better expressed in terms of modified $\mathscr{A}^\varepsilon_+$ and $\mathscr{A}^\varepsilon_-$ operators, which we now introduce. For $\delta \ge 0$, let $\mathscr{A}^{\varepsilon,\delta}_\pm$, be the operators whose action on $f \in \Gamma L^2_n$ is given by

$$\begin{aligned} \mathscr{F}(\mathscr{A}^{\varepsilon,\delta}_+ f)(k_{1:n+1}) &= \varepsilon^{d/2} \Big(\frac{2\varepsilon^{-1}}{(2\pi)^{d/2}(n+1)} \Big)^{1+\delta} \\ &\quad \times \sum_{1\le i<j\le n+1} |\mathfrak{w}\cdot(k_i+k_j)|^{1+\delta} \mathbb{J}^\varepsilon_{k_i,k_j} \hat f(k_i+k_j, k_{\{1:n+1\}\setminus\{i,j\}}) \\ \mathscr{F}(\mathscr{A}^{\varepsilon,\delta}_- f)(k_{1:n-1}) &= \varepsilon^{d/2} \Big(\frac{n\varepsilon^{-1}}{(2\pi)^{d/2}} \Big)^{1+\delta} \\ &\quad \times \sum_{j=1}^{n-1} |\mathfrak{w}\cdot k_j|^{1+\delta} \sum_{\ell+m=k_j} \mathbb{J}^\varepsilon_{\ell,m} \hat f(\ell, m, k_{\{1:n-1\}\setminus\{j\}})\,. \end{aligned} \tag{2.18}$$

In the next lemma, we establish suitable bounds on the previous operators, which are similar in flavour to those obtained in Lemma 2.4 for $\mathscr{A}^\varepsilon$.

Lemma 2.5 *Let $d \geq 3$. Then, there exists a $C = C(d) > 0$ such that for any function $\psi \in \Gamma L^2$ and $\delta \leq (d-2)/2$, the following estimate holds*

$$\|(-\mathscr{L}_0)^{-\frac{1+\delta}{2}} \mathscr{A}_\sigma^{\varepsilon,\delta} \psi\| \leq C \|\mathcal{N}^{1+\delta} (-\mathscr{L}_0)^{\frac{1+\delta}{2}} \psi\| \,, \tag{2.19}$$

where $\sigma \in \{+,-\}$ and $\mathcal{N}$ is the number operator given in Definition 2.3. In particular, for $\sigma \in \{+,-\}$, the previous implies

$$\|(-\mathscr{L}_0)^{-\frac{1+\delta}{2}} \mathscr{A}_\sigma^{\varepsilon,\delta} (-\mathscr{L}_0)^{-\frac{1+\delta}{2}} \psi\| \leq C \|\mathcal{N}^{1+\delta} \psi\| \,. \tag{2.20}$$

With respect to Lemma 2.4, here we do not aim at the best possible estimate in terms of growth with respect to $\mathcal{N}$, and the exponent $1+\delta$ is certainly not optimal. In fact, for the purposes of Section 2.4 (which is where the Lemma is used), the precise $\mathcal{N}$-dependence is irrelevant.

Proof. As in the proof of Lemma 2.4, it suffices to show (2.19). We begin with $\mathscr{A}_+^{\varepsilon,\delta}$. We first apply Cauchy-Schwarz to the sums in i, j and relabel the variables so that, for $\psi \in \Gamma L^2_n$, we obtain

$$\begin{aligned} &\|(-\mathscr{L}_0)^{-\frac{1+\delta}{2}} \mathscr{A}_+^{\varepsilon,\delta} \hat{\psi}\|^2 \\ &\lesssim n! n^3 \sum_{k_{1:n}} |k_1|^{2+2\delta} |\hat{\psi}(k_{1:n})|^2 \, \varepsilon^{d-2(1+\delta)} \sum_{\ell+m=k_1} \frac{\mathbb{J}^\varepsilon_{\ell,m}}{(|\ell|^2 + |m|^2 + |k_{2:n}|^2)^{1+\delta}} \\ &\lesssim n! n^2 \sum_{k_{1:n}} (|k_{1:n}|^2)^{1+\delta} |\hat{\psi}(k_{1:n})|^2 \, \varepsilon^{d-2(1+\delta)} \sum_{\ell+m=k_1} \frac{\mathbb{J}^\varepsilon_{\ell,m}}{(|\ell|^2 + |m|^2)^{1+\delta}} \,. \end{aligned} \tag{2.21}$$

On the other hand, arguing as in (2.16),

$$\varepsilon^{d-2(1+\delta)} \sum_{\ell+m=k_1} \frac{\mathbb{J}^\varepsilon_{\ell,m}}{(|\ell|^2 + |m|^2)^{1+\delta}} \leq \varepsilon^d \sum_{|\varepsilon\ell| \leq 1} \frac{}{|\varepsilon \ell|^{2+2\delta}} \lesssim_d \int_{|x| \leq 1} \frac{\mathrm{d}x}{|x|^{2+2\delta}} \tag{2.22}$$

with the right hand side independent of k_1, so that (2.19) follows provided that $\delta < (d-2)/2$.

Next, we turn to $\mathscr{A}_-^{\varepsilon,\delta}$. From its expression in (2.18), we have for $\psi \in \Gamma L^2_n$

$$\begin{aligned} &\|(-\mathscr{L}_0)^{-\frac{1+\delta}{2}} \mathscr{A}_-^{\varepsilon,\delta} \psi\|^2 \\ &\lesssim \varepsilon^{d-2(1+\delta)} (n-1)! n^{4+2\delta} \sum_{k_{1:n-1}} \frac{|k_1|^{2(1+\delta)}}{|k_{1:n-1}|^{2(1+\delta)}} \Big(\sum_{\ell+m=k_1} \mathbb{J}^\varepsilon_{\ell,m} \hat{\psi}(\ell, m, k_{2:n-1}) \Big)^2 \\ &= \varepsilon^{d-2(1+\delta)} (n-1)! n^{4+2\delta} \sum_{k_{1:n-1}} \frac{|k_1|^{2(1+\delta)}}{|k_{1:n-1}|^{2(1+\delta)}} \times \end{aligned}$$

$$\times \Big(\sum_{\ell+m=k_1} \mathbb{J}^{\varepsilon}_{\ell,m} \frac{(|\ell|^2+|m|^2)^{\frac{1+\delta}{2}} \hat{\psi}(\ell,m,k_{2:n-1})}{(|\ell|^2+|m|^2)^{\frac{1+\delta}{2}}} \Big)^2$$

$$\leq n! n^{3+2\delta} \sum_{k_{1:n}} (|k_1|^2+|k_2|^2)^{1+\delta} |\hat{\psi}(k_{1:n+1})|^2 \Big(\sum_{k_1+k_2=q} \frac{\varepsilon^{d-2(1+\delta)} \mathbb{J}^{\varepsilon}_{\ell,m}}{(|\ell|^2+|m|^2)^{1+\delta}} \Big)$$

$$\lesssim_d n! n^{2+2\delta} \sum_{k_{1:n}} (|k_{1:n}|^2)^{1+\delta} |\hat{\psi}(k_{1:n})|^2 = n^{2+2\delta} \|(-\mathscr{L}_0)^{\frac{1+\delta}{2}} \psi\|^2$$

where again we used (2.22). The proof of (2.19) is complete. □

2.1.2 The replacement lemma: $d = 2$

The reason why the estimates of Lemma 2.5 were possible is that, when suitably scaled, the inverse Laplacian in dimension $d \geq 3$ is integrable close to the origin, which in our context means that the estimate (2.22) holds for δ sufficiently small but positive. In $d = 2$, this is not the case. To overcome the lack of integrability, we devise an alternative route, which (morally) approximates the inverse of the generator with a linear and *diagonal* (both in Fourier and in the chaos) operator, given by a non-trivial order 1 perturbation of $\mathscr{L}_0$.

To be more precise, we need to introduce some notation. For $\varepsilon > 0$, let L^{ε} be the function defined on $[1/2, \infty)$ as

$$\mathrm{L}^{\varepsilon}(x) \stackrel{\text{def}}{=} \lambda_{\varepsilon}^2 \log\left(1 + \frac{1}{\varepsilon^2 x}\right). \tag{2.23}$$

Further, let G be the function

$$G(x) \stackrel{\text{def}}{=} \frac{1}{|\mathfrak{w}|^2} \left[\left(\frac{3|\mathfrak{w}|^2}{2\pi} x + 1 \right)^{\frac{2}{3}} - 1 \right], \tag{2.24}$$

and $\mathscr{G}^{\varepsilon}$ be the operator on ΓL^2 given by

$$\mathscr{G}^{\varepsilon} \stackrel{\text{def}}{=} G(\mathrm{L}^{\varepsilon}(-\mathscr{L}_0)) \tag{2.25}$$

which means that for $\psi \in \Gamma L^2_n$, $n \in \mathbb{N}$, the action of $\mathscr{G}^{\varepsilon}$ on ψ is

$$\mathscr{F}(\mathscr{G}^{\varepsilon}\psi)(k_{1:n}) \stackrel{\text{def}}{=} G(\mathrm{L}^{\varepsilon}(\tfrac{1}{2}|k_{1:n}|^2)) \hat{\psi}(k_{1:n}), \qquad k_{1:n} \in \mathbb{Z}_0^{2n}.$$

We are now ready to state the main result of this section, namely, the replacement lemma. Recall the definition of the scalar product $\langle \cdot, \cdot \rangle$ on ΓL^2 given in (1.19).

Lemma 2.6 (Replacement Lemma) *There exists a constant $C > 0$ such that for every $\psi_1, \psi_2 \in \Gamma L^2$ we have*

$$|\langle [- \mathscr{A}^{\varepsilon}_{-} (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^{\varepsilon})^{-1} \mathscr{A}^{\varepsilon}_{+} + \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^{\varepsilon}] \psi_1, \psi_2 \rangle| \tag{2.26}$$

$$\leq C\lambda_\varepsilon^2 \|\mathcal{N}(-\mathscr{L}_0)^{1/2}\psi_1\| \|\mathcal{N}(-\mathscr{L}_0)^{1/2}\psi_2\|$$

where $\mathscr{G}^\varepsilon$ is the operator defined according to (2.25) *and* (2.24) *and, for $n \in \mathbb{N}$, $\mathscr{L}_0^{\mathfrak{w}}$ is the operator acting on $\psi \in \Gamma L_n^2$ as*

$$\mathscr{F}(\mathscr{L}_0^{\mathfrak{w}}\psi)(k_{1:n}) = -\tfrac{1}{2}(\mathfrak{w}\cdot k)_{1:n}^2 \hat{\psi}(k_{1:n})\,, \qquad \textit{for all } k_{1:n} \in \mathbb{Z}_0^{2n} \tag{2.27}$$

and $(\mathfrak{w}\cdot k)_{1:n}^2 \stackrel{def}{=} \sum_{i=1}^n (\mathfrak{w}\cdot k_i)^2$.

Remark 2.7 As it appears from the proof of Proposition A.1 in the appendix, the form of the function G in (2.23) is dictated by the following fixed point equation

$$G(x) = \frac{1}{\pi}\int_0^x \frac{\mathrm{d}y}{\sqrt{1+|\mathfrak{w}|^2 G(y)}}\,. \tag{2.28}$$

Now, if we had chosen a different regularisation for the nonlinearity, e.g. the $|\cdot|_\infty$-distance in (2.3) instead of the Euclidean one or a smooth regularisation instead of the Fourier cut-off in (2.2), then we would have not obtained an explicit G but statement above would have remained true.

Apart from working with an explicit function, the advantage of our choice lies in the fact that it allows us to make an explicit connection with the work of Yau [Yau04] on the two-dimensional ASEP. Indeed, the exponent $2/3$ in (2.24) precisely reflects that obtained therein and strongly suggests that, if λ_ε were taken to be a constant independent of ε, then the diffusivity would diverge as $(\log t)^{2/3}$ for $t \to \infty$.

Proof. Notice first that, for any $n \in \mathbb{N}$, the operator $\mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon$ maps ΓL_n^2 into itself, so that, to establish (2.26), it suffices to consider $\psi_1, \psi_2 \in \Gamma L_n^2$. Moreover, we can write

$$\langle [-\mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon]\psi_1, \psi_2\rangle = \langle (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon\psi_1, \mathscr{A}_+^\varepsilon\psi_2\rangle. \tag{2.29}$$

Recalling the form of $\mathscr{A}_+^\varepsilon$ in (2.9), we see that the scalar product at the right hand side can be split into a *diagonal part*, corresponding to making the same choice for the indices i, j in the two occurrences of $\mathscr{A}_+^\varepsilon$, an *off-diagonal part of type 1*, corresponding to choosing i, j and i', j' with one element in common, and *off-diagonal part of type 2*, corresponding to all remaining choices. (See for instance [CETara, Lemma 3.6], where a similar distinction was made.)

To write this in formulas, let $\mathcal{S}^\varepsilon \stackrel{\text{def}}{=} (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}$ and denote with $\sigma^\varepsilon = (\sigma_n^\varepsilon)_{n\geq 1}$ its Fourier multiplier, i.e. for $\psi \in \Gamma L_n^2$ $\mathscr{F}(\mathcal{S}^\varepsilon\psi)(k_{1:n}) = \sigma_n^\varepsilon(k_{1:n})\hat{\psi}(k_{1:n})$ where

$$\sigma_n^\varepsilon(k_{1:n}) = \frac{2}{|k_{1:n}|^2 + (\mathfrak{w}\cdot k)_{1:n}^2 G(\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2))}\,. \tag{2.30}$$

Then, the scalar product in (2.29) satisfies

$$\langle \mathcal{S}^\varepsilon\mathscr{A}_+^\varepsilon\psi_1, \mathscr{A}_+^\varepsilon\psi_2\rangle = \langle \mathcal{S}^\varepsilon\mathscr{A}_+^\varepsilon\psi_1, \mathscr{A}_+^\varepsilon\psi_2\rangle_{\text{diag}} + \sum_{i=1}^2 \langle \mathcal{S}^\varepsilon\mathscr{A}_+^\varepsilon\psi_1, \mathscr{A}_+^\varepsilon\psi_2\rangle_{\text{off}_i}$$

where diagonal part, given by the first summand, is defined as

$$\langle \mathcal{S}^\varepsilon \mathcal{A}_+^\varepsilon \psi_1, \mathcal{A}_+^\varepsilon \psi_2 \rangle_{\text{diag}} \stackrel{\text{def}}{=} n!\, n\, \frac{4\lambda_\varepsilon^2}{(2\pi)^2} \times \tag{2.31}$$
$$\times \sum_{k_{1:n}} \tfrac{1}{2}(\mathfrak{w}\cdot k_1)^2 \overline{\hat\psi_1(k_{1:n})}\hat\psi_2(k_{1:n}) \sum_{\ell+m=k_1} \sigma_{n+1}^\varepsilon(\ell, m, k_{2:n}) \mathbb{J}_{\ell,m}^\varepsilon$$

while off-diagonal parts of type 1 and 2 are respectively given by

$$\langle \mathcal{S}^\varepsilon \mathcal{A}_+^\varepsilon \psi_1, \mathcal{A}_+^\varepsilon \psi_2 \rangle_{\text{off}_1} \stackrel{\text{def}}{=} n!\, n\, c_{\text{off}_1}(n) \sum_{k_{1:n+1}} \sigma_{n+1}^\varepsilon(k_{1:n+1}) \mathbb{J}_{k_1,k_2}^\varepsilon \mathbb{J}_{k_1,k_3}^\varepsilon \times \tag{2.32}$$
$$\times (\mathfrak{w}\cdot(k_1+k_2))(\mathfrak{w}\cdot(k_1+k_3)) \overline{\hat\psi_1(k_1+k_2, k_3, k_{4:n+1})}\hat\psi_2(k_1+k_3, k_2, k_{4:n+1})\,,$$

and

$$\langle \mathcal{S}^\varepsilon \mathcal{A}_+^\varepsilon \psi_1, \mathcal{A}_+^\varepsilon \psi_2 \rangle_{\text{off}_2} \stackrel{\text{def}}{=} n!\, n\, c_{\text{off}_2}(n) \sum_{k_{1:n+1}} \sigma_{n+1}^\varepsilon(k_{1:n+1}) \mathbb{J}_{k_1,k_2}^\varepsilon \mathbb{J}_{k_1,k_3}^\varepsilon \times \tag{2.33}$$
$$\times (\mathfrak{w}\cdot(k_1+k_2))(\mathfrak{w}\cdot(k_3+k_4)) \overline{\hat\psi_1(k_1+k_2, k_{3:4}, k_{5:n+1})}\hat\psi_2(k_3+k_4, k_{1:2}, k_{5:n+1})$$

where, for $i = 1, 2$, $c_{\text{off}_i}(n)$ is an explicit positive constant only depending on n and such that $c_{\text{off}_i}(n) = O(n^{i+1})$.

We are ready to control the left hand side of (2.26). At first, we split all the diagonal parts from the off-diagonal

$$|\langle [-\mathcal{A}_-^\varepsilon(-\mathcal{L}_0 - \mathcal{L}_0^{\mathfrak{w}}\mathcal{G}^\varepsilon)^{-1}\mathcal{A}_+^\varepsilon + \mathcal{L}_0^{\mathfrak{w}}\mathcal{G}^\varepsilon]\psi_1, \psi_2\rangle| \tag{2.34}$$
$$\le \Big|\langle \mathcal{S}^\varepsilon \mathcal{A}_+^\varepsilon \psi_1, \mathcal{A}_+^\varepsilon \psi_2\rangle_{\text{diag}} + \langle \mathcal{L}_0^{\mathfrak{w}}\mathcal{G}^\varepsilon \psi_1, \psi_2\rangle\Big| + \sum_{i=1,2} |\langle \mathcal{S}^\varepsilon \mathcal{A}_+^\varepsilon \psi_1, \mathcal{A}_+^\varepsilon \psi_2\rangle_{\text{off}_i}|\,.$$

Let us begin by estimating the first summand. By (2.25), (2.27) and (2.31), we see that

$$\Big|\langle(-\mathcal{L}_0 - \mathcal{L}_0^{\mathfrak{w}}\mathcal{G}^\varepsilon)^{-1}\mathcal{A}_+^\varepsilon\psi_1, \mathcal{A}_+^\varepsilon\psi_2\rangle_{\text{diag}} + \langle \mathcal{L}_0^{\mathfrak{w}}\mathcal{G}^\varepsilon\psi_1, \psi_2\rangle\Big|$$
$$= n!n\Big|\sum_{k_{1:n}} \tfrac{1}{2}(\mathfrak{w}\cdot k_1)^2 \hat\psi_1(-k_{1:n})\hat\psi_2(k_{1:n})\Big(P^\varepsilon(k_{1:n}) - G(\mathrm{L}^\varepsilon(\tfrac{1}{2}|k_{1:n}|^2))\Big)\Big|$$

where P^ε is defined as

$$P^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2} \sum_{\ell+m=k_1} \frac{\mathbb{J}_{\ell,m}^\varepsilon}{\tilde\Gamma + \tilde\Gamma^w G(\mathrm{L}^\varepsilon(\tilde\Gamma))} \tag{2.35}$$

and, to shorten the notations, we wrote

$$\tilde\Gamma \stackrel{\text{def}}{=} \tfrac{1}{2}(|\ell|^2 + |m|^2 + |k_{2:n}|^2)\,, \qquad \tilde\Gamma^w \stackrel{\text{def}}{=} \tfrac{1}{2}((\mathfrak{w}\cdot\ell)^2 + (\mathfrak{w}\cdot m)^2 + (\mathfrak{w}\cdot k)^2_{2:n})\,.$$

By Proposition A.1, we can upper bound the above by

$$\begin{aligned} & n!n \sum_{k_{1:n}} \tfrac{1}{2}(\mathfrak{w}\cdot k_1)^2 |\hat{\psi}_1(k_{1:n})||\hat{\psi}_2(k_{1:n})| \Big| P^\varepsilon(k_{1:n}) - G(\mathrm{L}^\varepsilon(\tfrac{1}{2}|k_{1:n}|^2)) \Big| \\ & \lesssim \lambda_\varepsilon^2 n!n \sum_{k_{1:n}} |k_1|^2 |\hat{\psi}_1(k_{1:n})||\hat{\psi}_2(k_{1:n})| = \lambda_\varepsilon^2 n! \sum_{k_{1:n}} |k_{1:n}|^2 |\hat{\psi}_1(k_{1:n})||\hat{\psi}_2(k_{1:n})| \\ & \lesssim \lambda_\varepsilon^2 \|(-\mathscr{L}_0)^{1/2}\psi_1\| \|(-\mathscr{L}_0)^{1/2}\psi_2\| \end{aligned} \tag{2.36}$$

so that this term satisfies (2.26).

We are now left to estimate the off diagonal terms. This is done by following *mutatis mutandis* the same steps as in the proof of [CETarb, Lemma 3.4], so we will only point out the necessary changes. For the off-diagonal terms of type 1, we need to control

$$\begin{aligned} |\langle \mathcal{S}^\varepsilon \mathscr{A}_+^\varepsilon \psi_1, \mathscr{A}_+^\varepsilon \psi_2 \rangle_{\mathrm{off}_1}| \lesssim n! c_{\mathrm{off}_1}(n) \lambda_\varepsilon^2 \sum_{j_{1:3}, k_{3:n}} |j_1 + j_2|\, |j_1 + j_3| \\ \times \overline{\hat{\psi}_1(j_1 + j_2, j_3, k_{3:n})} \hat{\psi}_2(j_1 + j_3, j_2, k_{3:n}) \sigma_{n+1}^\varepsilon(j_{1:3}, k_{3:n}) \end{aligned} \tag{2.37}$$

where σ_{n+1}^ε, given in (2.30), is the Fourier multiplier of $\mathcal{S}^\varepsilon$ in ΓL_{n+1}^2 and, as usual, we have bounded $|\mathfrak{w}\cdot k| \lesssim |k|$. As in [CETarb, Eq. (3.21)], an application of Cauchy-Schwarz shows that (2.37) is upper bounded by

$$\lambda_\varepsilon^2 c_{\mathrm{off}_1}(n) \prod_{i=1}^2 \Big(n! \sum_{k_{1:n}} |\hat{\psi}_i(k_{1:n})|^2 |k_1|^2 |k_2| \sum_{j_1 + j_2 = k_1} \frac{1}{|j_2|} \sigma_{n+1}^\varepsilon(j_{1:2}, k_{2:n}) \Big)^{\frac{1}{2}} . \tag{2.38}$$

This is the same expression as in [CETarb, Eq. (3.21)], with σ_{n+1}^ε replacing J^N there. Since σ_{n+1}^ε satisfies

$$\sigma_{n+1}^\varepsilon(j_{1:2}, k_{2:n}) \lesssim \frac{1}{|j_{1:2}|^2 + |k_{2:n}|^2},$$

which is the same estimate that J^N satisfies, one can argue as in [CETarb] and deduce

$$\sum_{j_1 + j_2 = k_1} \frac{1}{|j_2|} \sigma_{n+1}^\varepsilon(j_{1:2}, k_{2:n}) \lesssim \frac{1}{|k_{1:n}|} \leq \frac{1}{|k_2|}. \tag{2.39}$$

Now, since $c_{\mathrm{off}_1(n)} \lesssim n^2$, (2.38) is upper bounded by a constant times $\lambda_\varepsilon^2 \|\sqrt{\mathcal{N}}(-\mathscr{L}_0)^{\frac{1}{2}}\psi_1\|$ $\|\sqrt{\mathcal{N}}(-\mathscr{L}_0)^{\frac{1}{2}}\psi_2\|$.

As for the off-diagonal terms of type 2, we proceed similarly, recalling that, this time, $c_{\mathrm{off}_2}(n) \lesssim n^3$. Namely, applying Cauchy-Schwarz and exploiting once more (2.39), we bound them as

$$\begin{aligned} & |\langle \mathcal{S}^\varepsilon \mathscr{A}_+^\varepsilon \psi_1, \mathscr{A}_+^\varepsilon \psi_2 \rangle_{\mathrm{off}_2}| \\ & \lesssim \lambda_\varepsilon^2 c_{\mathrm{off}_2}(n) \prod_{i=1}^2 \Big(n! \sum_{k_{1:n}} |\hat{\psi}_i(k_{1:n})|^2 |k_1|^2 |k_2||k_3| \sum_{j_1 + j_2 = k_1} \frac{\sigma_{n+1}^\varepsilon(j_{1:2}, k_{2:n})}{|j_1||j_2|} \Big)^{\frac{1}{2}} \end{aligned}$$

$$\lesssim \lambda_\varepsilon^2 n! c_{\mathrm{off}_2}(n) \prod_{i=1}^{2} \Big(\sum_{k_{1:n}} |\hat{\psi}_i(k_{1:n})|^2 \frac{|k_1||k_2||k_3|}{|k_{1:n}|} \Big)^{1/2}$$

$$\lesssim \lambda_\varepsilon^2 n^3 \prod_{i=1}^{2} \Big(n! \sum_{k_{1:n}} |\hat{\psi}_i(k_{1:n})|^2 |k_1|^2 \Big)^{1/2} \lesssim \lambda_\varepsilon^2 \prod_{i=1}^{2} \|\mathcal{N}(-\mathscr{L}_0)^{1/2} \psi_i\|$$

as desired. □

2.2 A priori estimates on the truncated generator equation

The goal of this section is to derive a priori estimates on suitable elements of the n-th inhomogeneous Fock space, which will be crucial in characterising the limit $\varepsilon \to 0$ of the solution to (1.5). The specific functions we need will depend on the dimension, as for $d = 2$ we want to take full advantage of the Replacement Lemma 2.6. Let us begin with some definitions.

Let $n \in \mathbb{N}$ and P_2^n be the projection onto n-th inhomogeneous Fock space with the first chaos removed, i.e. onto $\bigoplus_{j=2}^n \Gamma L_j^2$. The truncated generator $\mathscr{L}_{2,n}^\varepsilon$, and the corresponding truncated operators $\mathscr{A}_{2,n}^\varepsilon$, $\mathscr{A}_{2,n}^{\varepsilon,+}$ and $\mathscr{A}_{2,n}^{\varepsilon,-}$ are respectively defined as

$$\mathscr{L}_{2,n}^\varepsilon \stackrel{\text{def}}{=} \mathscr{L}_0 + \mathscr{A}_{2,n}^\varepsilon\,, \quad \mathscr{A}_{2,n}^\varepsilon \stackrel{\text{def}}{=} P_2^n \mathscr{A}^\varepsilon P_2^n \quad \text{and} \quad \mathscr{A}_{2,n}^{\varepsilon,\sigma} \stackrel{\text{def}}{=} P_2^n \mathscr{A}_\sigma^\varepsilon P_2^n\,, \tag{2.40}$$

for $\sigma \in \{+,-\}$.

Let $\psi \in \bigoplus_{j=2}^m \Gamma L_j^2$, $2 \le m \le n$. For $d \ge 2$, let $u^{\varepsilon,n}$ be the unique solution of the *truncated generator equation* which is defined as

$$-\mathscr{L}_{2,n}^\varepsilon u^{\varepsilon,n} = \psi\,. \tag{2.41}$$

In the specific case of $d = 2$, we further consider $\tilde{u}^\varepsilon$ which satisfies

$$\tilde{u}^\varepsilon \stackrel{\text{def}}{=} (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathscr{A}_+^\varepsilon \tilde{u}^\varepsilon + (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \psi \tag{2.42}$$

where $\mathscr{L}_0^{\mathfrak{w}}$ and $\mathscr{G}^\varepsilon$ are defined according to (2.27) and (2.25) respectively. Note that, even if in (2.42) there is no truncation in the chaos, the equation admits a unique solution. Indeed, it is a triangular equation which can be explicitly solved starting from $\tilde{u}_1^\varepsilon = 0$, and then inductively setting

$$\tilde{u}_j^\varepsilon \stackrel{\text{def}}{=} (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathscr{A}_+^\varepsilon \tilde{u}_{j-1}^\varepsilon + (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \psi_j\,, \qquad j \ge 2\,.$$

That said, we will be concerned with the projection of $\tilde{u}^\varepsilon$ onto $\oplus_{j=2}^n \Gamma L_j^2$, that we denote by $\tilde{u}^{\varepsilon,n}$ and can be easily seen to solve

$$\tilde{u}^{\varepsilon,n} \stackrel{\text{def}}{=} (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathscr{A}_+^\varepsilon P_2^{n-1} \tilde{u}^{\varepsilon,n} + (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \psi\,. \tag{2.43}$$

Since the difficult term in (1.5) is the quadratic one, we need to find a way to replace it with objects we are able to control. To do so, we will choose the input function ψ at the right hand side of (2.41) and (2.43) to be the (kernel in ΓL^2 of the) nonlinearity of (1.5).

For a smooth function $\varphi \in \mathcal{S}(\mathbb{T}^d)$, recall that, as in (2.2), $\mathcal{N}^\varepsilon_\varphi(\eta)$ is the nonlinearity tested against φ. Now, notice that this equals

$$\mathcal{N}^\varepsilon_\varphi(\eta) = \sum_{\ell,m} \frac{\iota}{(2\pi)^{\frac{d}{2}}} \lambda_\varepsilon \mathbb{J}^\varepsilon_{\ell,m} \mathfrak{w} \cdot (\ell+m)\hat{\varphi}(-\ell-m) : \hat{\eta}(\ell)\hat{\eta}(m) := I_2(\mathfrak{n}^\varepsilon_\varphi) \quad (2.44)$$

where $: \hat{\eta}(\ell)\hat{\eta}(m) :\stackrel{\text{def}}{=} \hat{\eta}(\ell)\hat{\eta}(m) - \mathbb{E}[\hat{\eta}(\ell)\hat{\eta}(m)]$ denotes the Wick's product and we could replace the usual product with it since for $\ell = -m$ the summand vanishes. In the above, we defined $\mathfrak{n}^\varepsilon_\varphi$ to be the kernel of $\mathcal{N}^\varepsilon_\varphi(\eta)$ in ΓL^2_2.

From now on, we consider a given test function $\varphi \in \mathcal{S}(\mathbb{T}^d)$ and we will study $v^{\varepsilon,n}$ (that we distinguish from $u^{\varepsilon,n}$ as the input function is fixed), which is the solution to the equation

$$-\mathscr{L}^\varepsilon_{2,n} v^{\varepsilon,n} = \mathfrak{n}^\varepsilon_\varphi \,. \quad (2.45)$$

and for $d = 2$, we consider also $\tilde{v}^{\varepsilon,n}$ which solves

$$\tilde{v}^{\varepsilon,n} = (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathscr{A}^\varepsilon_+ P_2^{n-1} \tilde{v}^{\varepsilon,n} + (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathfrak{n}^\varepsilon_\varphi \,. \quad (2.46)$$

We are ready to state the main result of this section.

Theorem 2.8 *Let $\varphi \in \mathcal{S}(\mathbb{T}^d)$ and $\mathfrak{n}^\varepsilon_\varphi$ be defined according to* (2.44)*. For $n \in \mathbb{N}$, let $w^{\varepsilon,n}$ be the solution of the truncated generator equation $v^{\varepsilon,n}$ in* (2.45) *if $d \geq 3$ or be $\tilde{v}^{\varepsilon,n}$ given by* (2.46) *if $d = 2$. Then, for any $k \in \mathbb{N}$, there exists a constant $C > 0$, depending only on k, m and the dimension d, and $\varepsilon_d(n) > 0$ such that for any $\varepsilon < \varepsilon_d(n)$ the following estimate holds*

$$\|(-\mathscr{L}_0)^{-\frac{1}{2}}(-\mathscr{L}^\varepsilon w^{\varepsilon,n} - \mathfrak{n}^\varepsilon_\varphi + \mathscr{A}^\varepsilon_- w^{\varepsilon,n}_2)\| \leq \frac{C}{n^k} \|(-\mathscr{L}_0)^{-\frac{1}{2}} \mathfrak{n}^\varepsilon_\varphi\| \,, \quad (2.47)$$

and the latter norm at the right hand side satisfies

$$\|(-\mathscr{L}_0)^{-\frac{1}{2}} \mathfrak{n}^\varepsilon_\varphi\| \lesssim \|\varphi\|_{H^1(\mathbb{T}^d)} \quad (2.48)$$

uniformly over ε. If $d \geq 3$, $\varepsilon_d(n)$ can be taken to be 1.

As a first step in the proof of the previous theorem, we derive weighted estimates on the solution to (2.45) which hold in *any* dimension $d \geq 2$. The argument we exploit follows closely that of [LY97b, Lemma 2.5] and since the proof works for any choice of ψ, we formulate and prove the result in the more general case of (2.41).

Proposition 2.9 *Let $m \in \mathbb{N}$ and $\psi \in \bigoplus_{j=2}^{m} \Gamma L_j^2$. For $n \in \mathbb{N}$, $n \geq m$, let $u^{\varepsilon,n}$ be the solution of the truncated generator equation in* (2.41) *in any dimension $d \geq 2$. Then, there exists a positive constant $C = C(m,k)$ independent of n, ε and ψ, such that*

$$\|\mathcal{N}^k(-\mathscr{L}_0)^{\frac{1}{2}} u^{\varepsilon,n}\| \leq C\|(-\mathscr{L}_0)^{-\frac{1}{2}}\psi\|\,, \tag{2.49}$$

where $\mathcal{N}$ is the number operator in Definition 2.3.

Proof. As n and ε are fixed throughout the proof, we will write u and u_j, $j = 2,\dots,n$ in place of $u^{\varepsilon,n}$ and $u_j^{\varepsilon,n}$, respectively. Also, by convention, we let $u_{n+1} = 0 = u_1$. To denote constants which do not depend on ε, n or ψ we will use C, and such C might change from line to line.

Let us test the j-th component, $j \geq 2$, with $\mathscr{L}^\varepsilon u$, which, since by Lemma 2.2 $(\mathscr{A}_+^\varepsilon)^* = -\mathscr{A}_-^\varepsilon$, gives

$$\begin{aligned}\langle u_j, \mathscr{L}^\varepsilon u\rangle &= \langle u_j, \mathscr{L}_0 u_j\rangle + \langle u_j, \mathscr{A}_+^\varepsilon u_{j-1}\rangle + \langle u_j, \mathscr{A}_-^\varepsilon u_{j+1}\rangle \\ &= \langle u_j, \mathscr{L}_0 u_j\rangle - \Big[\langle u_{j+1}, \mathscr{A}_+^\varepsilon u_j\rangle - \langle u_j, \mathscr{A}_+^\varepsilon u_{j-1}\rangle\Big].\end{aligned}$$

Via a summation by parts, for any $a > 0$, we have

$$\begin{aligned}\sum_{j=2}^{n}(a + j^{2k})\langle u_j, (-\mathscr{L}_0)u_j\rangle =& \sum_{j=2}^{n}(a + j^{2k})\langle u_j, (-\mathscr{L}^\varepsilon)u\rangle \\ &+ \sum_{j=2}^{n}(j^{2k} - (j-1)^{2k})\langle u_j, \mathscr{A}_+^\varepsilon u_{j-1}\rangle.\end{aligned}$$

Next, note that $-\mathscr{L}^\varepsilon u = \psi - \mathscr{A}_+^\varepsilon u_n - \mathscr{A}_-^\varepsilon u_2$ so that, by orthogonality of Fock spaces with different indices,

$$|\langle u_j, (-\mathscr{L}^\varepsilon)u\rangle| = |\langle u_j, \psi_j\rangle| \leq \frac{1}{2}\langle u_j, (-\mathscr{L}_0)u_j\rangle + \frac{1}{2}\langle \psi_j, (-\mathscr{L}_0)^{-1}\psi_j\rangle.$$

As a consequence, there exists some finite strictly positive constant $C = C(a,m,k)$, which might change from line to line, such that

$$\begin{aligned}&\sum_{j=2}^{n}(a + j^{2k})\langle u_j, (-\mathscr{L}_0)u_j\rangle \\ &\qquad \leq C\Big(\|(-\mathscr{L}_0)^{-\frac{1}{2}}\psi\|^2 + \sum_{j=2}^{n}(j^{2k} - (j-1)^{2k})\langle u_j, \mathscr{A}_+^\varepsilon u_{j-1}\rangle\Big) \qquad (2.50)\\ &\qquad \leq C\Big(\|(-\mathscr{L}_0)^{-\frac{1}{2}}\psi\|^2 + \sum_{j=2}^{n} j^{2k-1}\langle u_j, \mathscr{A}_+^\varepsilon u_{j-1}\rangle\Big)\end{aligned}$$

where $C = C(k)$ is a constant chosen in such a way that $(j^{2k} - (j-1)^{2k})\sqrt{j} \leq Cj^{2k-\frac{1}{2}}$. To handle the last sum, we bound

$$\begin{aligned} |\langle u_j, \mathscr{A}_+^\varepsilon u_{j-1}\rangle| &\leq \|(-\mathscr{L}_0)^{\frac{1}{2}} u_j\| \|(-\mathscr{L}_0)^{-\frac{1}{2}} \mathscr{A}_+^\varepsilon u_{j-1}\| \\ &\leq C\sqrt{j}\|(-\mathscr{L}_0)^{\frac{1}{2}} u_j\| \|(-\mathscr{L}_0)^{\frac{1}{2}} u_{j-1}\| \\ &\leq C\sqrt{j}\Big(\frac{1}{2}\|(-\mathscr{L}_0)^{\frac{1}{2}} u_j\|^2 + \frac{1}{2}\|(-\mathscr{L}_0)^{\frac{1}{2}} u_{j-1}\|^2\Big) \end{aligned} \tag{2.51}$$

where in the second bound we used (2.10) in Lemma 2.4. We now plug the result into (2.50) and rearrange the terms, so that we conclude

$$\sum_{j=1}^{n} j^{2k}\Big(1 + \frac{a}{j^{2k}} - \frac{C}{\sqrt{j}}\Big)\langle u_j, (-\mathscr{L}_0) u_j\rangle \leq C(1 + \|(-\mathscr{L}_0)^{-\frac{1}{2}}\psi\|^2)\,. \tag{2.52}$$

Choosing a sufficiently large, in such a way that $(1 + \frac{a}{j^{2k}} - \frac{C}{\sqrt{j}}) \geq 1/2$ for every $j \geq 2$, (2.49) follows. □

In the next two lemmas, we consider the specific case of $d = 2$. In the first one, we show that, for any ψ, $\tilde{u}^{\varepsilon,n}$ defined according to (2.43) can be estimated in terms of ψ.

Lemma 2.10 *For $d = 2$, let $\psi \in \bigoplus_{j=2}^m \Gamma L_j^2$, $n \in \mathbb{N}$ and $\tilde{u}^{\varepsilon,n}$ be given by* (2.43)*. Then, there exists a constant $C > 0$ independent of n, ε and $\varepsilon_2(n) > 0$ such that for every $\varepsilon < \varepsilon_2(n)$*

$$\|(-\mathscr{L}_0)^{1/2}\tilde{u}^{\varepsilon,n}\|^2 \leq C\|(-\mathscr{L}_0)^{-1/2}\psi\|^2\,. \tag{2.53}$$

Proof. At first, we study how the generator $\mathscr{L}^\varepsilon$ acts on $\tilde{u}^{\varepsilon,n}$. We have

$$\begin{aligned} -\mathscr{L}^\varepsilon\tilde{u}^{\varepsilon,n} =& (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)\tilde{u}^{\varepsilon,n} - \mathscr{A}_+^\varepsilon\tilde{u}^{\varepsilon,n} - \mathscr{A}_-^\varepsilon\tilde{u}^{\varepsilon,n} + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon\tilde{u}^{\varepsilon,n} \\ =& \psi - \mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\psi - \mathscr{A}_+^\varepsilon\tilde{u}_n^{\varepsilon,n} + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon\tilde{u}_n^{\varepsilon,n} \\ &+ [\,-\mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]P_2^{n-1}\tilde{u}^{\varepsilon,n} \end{aligned} \tag{2.54}$$

where in the last step we used (2.43). Now, $\mathscr{A}_+^\varepsilon\tilde{u}_n^{\varepsilon,n} \in \Gamma L_{n+1}^2$, so that, since $\tilde{u}^{\varepsilon,n} \in \oplus_{j=2}^n \Gamma L_j^2$, the two are orthogonal. Therefore, by testing both sides by $\tilde{u}^{\varepsilon,n}$, we obtain

$$\begin{aligned} &\|(-\mathscr{L}_0)^{1/2}\tilde{u}^{\varepsilon,n}\|^2 \\ &\quad= \langle \tilde{u}^{\varepsilon,n}, \psi - \mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\psi\rangle + \langle \tilde{u}_n^{\varepsilon,n}, \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon\tilde{u}_n^{\varepsilon,n}\rangle \\ &\qquad+ \langle \tilde{u}^{\varepsilon,n}, [\,-\mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]P_2^{n-1}\tilde{u}^{\varepsilon,n}\rangle \\ &\quad\leq \tfrac{1}{2}\|(-\mathscr{L}_0)^{1/2}\tilde{u}^{\varepsilon,n}\|^2 + \tfrac{1}{2}\|(-\mathscr{L}_0)^{-1/2}[\psi - \mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\psi]\|^2 \\ &\qquad- \|(-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{1/2}\tilde{u}_n^{\varepsilon,n}\|^2 + C\lambda_\varepsilon^2 n^2\|(-\mathscr{L}_0)^{1/2}\tilde{u}^{\varepsilon,n}\|^2 \\ &\quad\leq C'\|(-\mathscr{L}_0)^{-1/2}\psi\|^2 + (\tfrac{1}{2} + Cn^2\lambda_\varepsilon^2)\|(-\mathscr{L}_0)^{1/2}\tilde{u}^{\varepsilon,n}\|^2 \end{aligned} \tag{2.55}$$

where in the last step we neglected the negative term at the right hand side, we used (2.11) together with the positivity of $-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon$ and the fact that the operators $\mathscr{L}_0, \mathscr{L}_0^{\mathfrak{w}}$ and $\mathscr{G}^\varepsilon$ commute to control the second term containing ψ, and the Replacement Lemma 2.6 to bound the third summand. Now, we choose $\varepsilon_2(n)$ in such a way that $Cn^2\lambda^2_{\varepsilon_2(n)} < 1/2$. Therefore, (2.53) follows upon rearranging the terms in (2.55). □

In the next lemma, we turn our attention to $\tilde{v}^{\varepsilon,n}$ given in (2.46) and prove that it satisfies the same weighted estimates as those in (2.49). To do so, in view of Proposition 2.9 and the previous Lemma, it suffices to control the difference between $\tilde{v}^{\varepsilon,n}$ and the solution of the truncated generator equation $v^{\varepsilon,n}$.

Lemma 2.11 *Let $d = 2$. For $n \in \mathbb{N}$, let $\tilde{v}^{\varepsilon,n}$ be given by* (2.46)*. Then, for any $k \in \mathbb{N}$ there exists a constant $C > 0$ and $\varepsilon_2(n,k) > 0$ such that for every $\varepsilon < \varepsilon_2(n,k)$*

$$\|\mathscr{N}^k(-\mathscr{L}_0)^{1/2}\tilde{v}^{\varepsilon,n}\| \le C\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\|\,. \tag{2.56}$$

Proof. Let $v^{\varepsilon,n}$ be the solution of (2.45) in $d = 2$. Notice that, trivially,

$$\|\mathscr{N}^k(-\mathscr{L}_0)^{1/2}\tilde{v}^{\varepsilon,n}\| \le n^k\|(-\mathscr{L}_0)^{1/2}(\tilde{v}^{\varepsilon,n} - v^{\varepsilon,n})\| + \|\mathscr{N}^k(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}\|\,. \tag{2.57}$$

Now, in view of Proposition 2.9, the second summand is bounded by (a constant times) $\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\|$, so we only need to control the first. To do so, we claim that for any j there exists $C > 0$ such that for ε small enough, we have

$$\|(-\mathscr{L}_0)^{1/2}(\tilde{v}^{\varepsilon,n} - v^{\varepsilon,n})\|^2 \le C(n^{-j} + n^2\lambda_\varepsilon^2)\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\|^2\,. \tag{2.58}$$

Assuming the claim, the proof is concluded as it suffices to choose in (2.58) $j = k$ and $\varepsilon_2(n,k) > 0$ such that $n^{k+2}\lambda_{\varepsilon_2(n,k)} < 1$, to ensure that also the first summand in (2.57) is bounded.

We now prove (2.58). To shorten the notation, set $\psi^\sharp_\varepsilon \stackrel{\text{def}}{=} (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathfrak{n}^\varepsilon_\varphi$. We begin by evaluating $-\mathscr{L}^\varepsilon$ on $\tilde{v}^{\varepsilon,n} - v^{\varepsilon,n}$. To do so, we exploit (2.54) and the fact that, as noted in the proof of Proposition 2.9, since $v^{\varepsilon,n}$ solves (2.45), we have

$$-\mathscr{L}^\varepsilon v^{\varepsilon,n} - \mathfrak{n}^\varepsilon_\varphi + \mathscr{A}^\varepsilon_- v^{\varepsilon,n}_2 = -\mathscr{A}^\varepsilon_+ v^{\varepsilon,n}_n\,.$$

This means that

$$\begin{aligned}-\mathscr{L}^\varepsilon(\tilde{v}^{\varepsilon,n} - v^{\varepsilon,n}) = &-\mathscr{A}^\varepsilon_-(\psi^\sharp_\varepsilon - v^{\varepsilon,n}_2) - \mathscr{A}^\varepsilon_+(\tilde{v}^{\varepsilon,n}_n - v^{\varepsilon,n}_n) + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon\tilde{v}^{\varepsilon,n}_n \\ &+ [\,-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]P_2^{n-1}\tilde{v}^{\varepsilon,n}.\end{aligned} \tag{2.59}$$

Now, since $\psi^\sharp_\varepsilon \in \Gamma L^2_2$, $\mathscr{A}^\varepsilon_-(\psi^\sharp_\varepsilon - v^{\varepsilon,n}_2) \in \Gamma L^2_1$ and $\mathscr{A}^\varepsilon_+(v^{\varepsilon,n}_n - \tilde{v}^{\varepsilon,n}_n) \in \Gamma L^2_{n+1}$, which implies that they are orthogonal to both $\tilde{v}^{\varepsilon,n}$ and $v^{\varepsilon,n}$ as these belong to $\oplus_{j=2}^n \Gamma L^2_j$. Hence, by testing both sides of (2.59) by $\tilde{v}^{\varepsilon,n} - v^{\varepsilon,n}$, we obtain

$$\|(-\mathscr{L}_0)^{1/2}(\tilde{v}^{\varepsilon,n} - v^{\varepsilon,n})\|^2 = \langle \tilde{v}^{\varepsilon,n} - v^{\varepsilon,n}, \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon\tilde{v}^{\varepsilon,n}_n\rangle \tag{2.60}$$

$$+\langle \tilde v^{\varepsilon,n}-v^{\varepsilon,n}, [-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]P_2^{n-1}\tilde v^{\varepsilon,n}\rangle\,.$$

Let us analyse the two terms at the right hand side separately. For the first, note that the operator $\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon$ is negative, so that

$$\begin{aligned}\langle \tilde v^{\varepsilon,n}-v^{\varepsilon,n}, \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon \tilde v_n^{\varepsilon,n}\rangle &= \langle v_n^{\varepsilon,n}, \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon \tilde v_n^{\varepsilon,n}\rangle - \langle \tilde v_n^{\varepsilon,n}, (-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)\tilde v_n^{\varepsilon,n}\rangle\\ &\le \langle -v_n^{\varepsilon,n}, (-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)\tilde v_n^{\varepsilon,n}\rangle\\ &\lesssim \|(-\mathscr{L}_0)^{1/2}v_n^{\varepsilon,n}\|\|(-\mathscr{L}_0)^{1/2}\tilde v_n^{\varepsilon,n}\|\end{aligned}$$

and in the last bound we used that $-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon \lesssim -\mathscr{L}_0$ and that $\mathscr{L}_0, \mathscr{L}_0^{\mathfrak{w}}$ and $\mathscr{G}^\varepsilon$ commute. Now, the a priori estimates in Proposition 2.9 and Lemma 2.10 allow to upper bound the previous by

$$n^{-k}\|\mathscr{N}^k(-\mathscr{L}_0)^{1/2}\tilde v^{\varepsilon,n}\|\|(-\mathscr{L}_0)^{1/2}v_n^{\varepsilon,n}\| \lesssim n^{-k}\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\|^2\,.$$

For the second term in (2.60), we apply the Replacement Lemma 2.6 first and the same a priori estimates as above, thus getting

$$\begin{aligned}\langle \tilde v^{\varepsilon,n}-v^{\varepsilon,n}, &[\mathscr{A}^\varepsilon_-(-\mathscr{L}_0-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]P_2^{n-1}\tilde v^{\varepsilon,n}\rangle\\ &\le C\lambda_\varepsilon^2 n^2\|(-\mathscr{L}_0)^{1/2}(\tilde v^{\varepsilon,n}-v^{\varepsilon,n})\|\|(-\mathscr{L}_0)^{1/2}\tilde v^{\varepsilon,n}\|\\ &\le \tfrac12 C\lambda_\varepsilon^2 n^2\Big(\|(-\mathscr{L}_0)^{1/2}(\tilde v^{\varepsilon,n}-v^{\varepsilon,n})\|^2 + \|(-\mathscr{L}_0)^{1/2}\tilde v^{\varepsilon,n}\|^2\Big)\\ &\le \tfrac12 C\lambda_\varepsilon^2 n^2\Big(\|(-\mathscr{L}_0)^{1/2}(\tilde v^{\varepsilon,n}-v^{\varepsilon,n})\|^2 + \|(-\mathscr{L}_0)^{1/2}\mathfrak{n}^\varepsilon_\varphi\|^2\Big)\,,\end{aligned}$$

where the constant C changed in the last two lines. Now, by using that for $\varepsilon < \varepsilon_2(n,k)$, $Cn^2\lambda_\varepsilon^2 < 1$, collecting the previous bounds and rearranging the terms, (2.58) follows. □

We are now ready for the proof of Theorem 2.8, which is an easy consequence of Lemma 2.4 and, for $d \ge 3$, Proposition 2.9, while, for $d = 2$ Lemmas 2.10 and 2.11.

Proof of Theorem 2.8. Let us first treat the case of $d \ge 3$. As noted in the proof of Proposition 2.9, since $v^{\varepsilon,n}$ solves (2.45), we have

$$-\mathscr{L}^\varepsilon v^{\varepsilon,n} - \mathfrak{n}^\varepsilon_\varphi + \mathscr{A}^\varepsilon_- v_2^{\varepsilon,n} = -\mathscr{A}^\varepsilon_+ v_n^{\varepsilon,n}\,. \tag{2.61}$$

By (2.10), for any $k \ge 1$, we have

$$\|(-\mathscr{L}_0)^{-\frac12}\mathscr{A}^\varepsilon_+ v_n^{\varepsilon,n}\| \lesssim \sqrt{n}\|(-\mathscr{L}_0)^{\frac12}v_n^{\varepsilon,n}\| \le \frac{1}{n^k}\|\mathscr{N}^{k+1}(-\mathscr{L}_0)^{\frac12}v^{\varepsilon,n}\|\,. \tag{2.62}$$

Hence, (2.47) follows directly by (2.49).

We now turn to $d = 2$, in which case $\tilde v^{\varepsilon,n}$ satisfies (2.46). Notice that, by the recursive definition of $\tilde v^{\varepsilon,n}$, $\tilde v_1^{\varepsilon,n} = 0$ and therefore

$$\tilde v_2^{\varepsilon,n} = (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathfrak{n}^\varepsilon_\varphi\,.$$

Hence, upon replacing ψ with $\mathfrak{n}^\varepsilon_\varphi$ in (2.54), we see that

$$\begin{aligned}-\mathscr{L}^\varepsilon \tilde{v}^{\varepsilon,n} - \mathfrak{n}^\varepsilon_\varphi + \mathscr{A}^\varepsilon_- \tilde{v}^{\varepsilon,n}_2 &= -\mathscr{L}^\varepsilon \tilde{v}^{\varepsilon,n} - \mathfrak{n}^\varepsilon_\varphi + \mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathfrak{n}^\varepsilon_\varphi \\ &= -\mathscr{A}^\varepsilon_+ \tilde{v}^{\varepsilon,n}_n + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon \tilde{v}^{\varepsilon,n}_n \\ &\quad + [-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon] P_2^{n-1}\tilde{v}^{\varepsilon,n}\,.\end{aligned}$$

Now, for the first two terms we use Lemma 2.4 and $-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon \lesssim -\mathscr{L}_0$, so that we can bound them by

$$\begin{aligned}\|(-\mathscr{L}_0)^{-1/2}[-\mathscr{A}^\varepsilon_+ \tilde{v}^{\varepsilon,n}_n + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon \tilde{v}^{\varepsilon,n}_n]\| &\lesssim \sqrt{n}\|(-\mathscr{L}_0)^{1/2}\tilde{v}^{\varepsilon,n}_n\| \\ &\lesssim n^{-k}\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\|\end{aligned}$$

where the last step follows by Lemma 2.11. For the third term, we note that the ΓL^2-norm satisfies a variational formulation, i.e.

$$\begin{aligned}&\|(-\mathscr{L}_0)^{-1/2}[-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon] P_2^{n-1}\tilde{v}^{\varepsilon,n}\| \\ &\quad= \sup_{\|\psi\|=1} \langle \psi, (-\mathscr{L}_0)^{-1/2}[-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon] P_2^{n-1}\tilde{v}^{\varepsilon,n}\rangle \\ &\quad= \sup_{\|\psi\|=1} \langle (-\mathscr{L}_0)^{-1/2}\psi, [-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon] P_2^{n-1}\tilde{v}^{\varepsilon,n}\rangle \\ &\quad\lesssim C\lambda_\varepsilon^2 \sup_{\|\psi\|=1} \|\psi\|\|\mathcal{N}^2(-\mathscr{L}_0)^{1/2}\tilde{v}^{\varepsilon,n}\| = C\lambda_\varepsilon^2\|\mathcal{N}^2(-\mathscr{L}_0)^{1/2}\tilde{v}^{\varepsilon,n}\| \\ &\quad\le C\lambda_\varepsilon^2 n^2 \|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\| \qquad (2.63)\end{aligned}$$

where in the third step we used the Replacement Lemma 2.6 and in the last, Lemma 2.10. By choosing $\varepsilon_2(n,k) > 0$ such that $\lambda^2_{\varepsilon_2(n,k)} n^{2+k} < 1$, (2.47) follows at once.

It remains to argue (2.48). For this, notice that for any φ, (2.44) and (2.9) imply that the following equality holds

$$\mathscr{A}^\varepsilon_+ \varphi = \mathfrak{n}^\varepsilon_\varphi\,. \qquad (2.64)$$

Hence, by Lemma 2.4 we have

$$\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\| = \|(-\mathscr{L}_0)^{-1/2}\mathscr{A}^\varepsilon_+\varphi\| \lesssim \|(-\mathscr{L}_0)^{1/2}\varphi\| = \|\varphi\|_{H^1(\mathbb{T}^d)}\,,$$

so that the proof is concluded. □

2.3 The diffusivity

The main advantage of Theorem 2.8 is that it allows to replace the nonlinearity with the sum of two terms: one involves the generator of our process, and therefore encodes its fluctuations, while the other consists of $\mathscr{A}^\varepsilon_-$ applied to an element in the Fock space of degree 2, so that overall it is of degree 1. This suggests that this latter term should give us the additional diffusivity appearing in the limit (1.8). The goal of this section is to analyse this object and determine its behaviour.

We begin our analysis with a lemma that provides a representation for $\mathscr{A}^\varepsilon_- w^{\varepsilon,n}_2$, which will help us identify its limit.

Lemma 2.12 *Let $\mathfrak{n}^\varepsilon_\varphi$ be defined according to* (2.44)*. For $n \in \mathbb{N}$, let $v^{\varepsilon,n}$ be the solution of the truncated generator equation in* (2.45) *and, for $d = 2$, $\tilde v^{\varepsilon,n}$ be given by* (2.46) *if $d = 2$. Then, for $d \geq 3$, we have*

$$\mathscr{A}^\varepsilon_- v_2^{\varepsilon,n} = -\sum_k \hat\varphi(k) \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}[-k]\|^2 e_k \tag{2.65}$$

where, for $k \in \mathbb{Z}^d$, $v^{\varepsilon,n}[k]$ is the solution to (2.45) *with $\varphi = e_{-k}$.*

For $d = 2$ instead, we have

$$\mathscr{A}^\varepsilon_- \tilde v_2^{\varepsilon,n} = \mathscr{A}^\varepsilon_- (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathscr{A}^\varepsilon_+ \varphi \,. \tag{2.66}$$

Proof. Recall (2.64). Then, in $d = 2$, by the recursive definition of $\tilde v^{\varepsilon,n}$, we have that $\tilde v_2^{\varepsilon,n} = (-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}} \mathscr{G}^\varepsilon)^{-1} \mathfrak{n}^\varepsilon_\varphi$ which immediately gives (2.66).

We now show (2.65). It is convenient to observe that $\mathfrak{n}^\varepsilon_\varphi$ can be written as

$$\mathfrak{n}^\varepsilon_\varphi = \sum_k \hat\varphi(k) \mathfrak{n}^\varepsilon_{-k} \tag{2.67}$$

where, to shorten the notation, we set $\mathfrak{n}^\varepsilon_{-k} \stackrel{\text{def}}{=} \mathfrak{n}^\varepsilon_{e_k}$. By linearity of $\mathscr{L}^\varepsilon_{2,n}$ and uniqueness for (2.45), we have $v^{\varepsilon,n} = \sum_k \hat\varphi(k) v^{\varepsilon,n}[-k]$, where, as in the statement, $v^{\varepsilon,n}[k]$ is the solution of the truncated generator equation whose input is $\mathfrak{n}^\varepsilon_k$. Hence, applying $\mathscr{A}^\varepsilon_-$ to $v_2^{\varepsilon,n}$ we obtain

$$\mathscr{A}^\varepsilon_- v_2^{\varepsilon,n} = \sum_k \hat\varphi(k) \mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}[-k] \,,$$

so that the proof is concluded once we verify that

$$\mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}[k] = -\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}[k]\|^2 e_{-k} \,, \qquad k \in \mathbb{Z}^d_0 \,. \tag{2.68}$$

Since $\mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}[k] \in \Gamma L^2_1$, we can expand it in the Fourier basis, i.e.

$$\begin{aligned} \mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}[k] &= \sum_{j \in \mathbb{Z}^d_0} \langle \mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}[k], e_{-j} \rangle e_j = -\sum_{j \in \mathbb{Z}^d_0} \langle v_2^{\varepsilon,n}[k], \mathscr{A}^\varepsilon_+ e_{-j} \rangle e_j \\ &= -\langle v_2^{\varepsilon,n}[k], \mathscr{A}^\varepsilon_+ e_k \rangle e_{-k} = -\langle v_2^{\varepsilon,n}[k], \mathfrak{n}^\varepsilon_{-k} \rangle e_{-k} \,, \end{aligned} \tag{2.69}$$

where the third equality is a consequence of the translation invariance of the equation (which can be easily seen to hold by verifying that, for any $i = 1, \dots, d$, the momentum operator M_i defined as $\mathcal{F}(M_i f)(k_{1:n}) = (\sum_j k^i_j) \hat f(k_{1:n})$, $f \in \Gamma L^2_n$, commutes with $\mathscr{L}_0, \mathscr{A}^\varepsilon_+$ and $\mathscr{A}^\varepsilon_-$), while the last, of (2.64). Therefore, we can now exploit orthogonality of Fock spaces with different indices and the antisymmetry of $\mathscr{A}^\varepsilon$, to see that

$$\begin{aligned} \langle v_2^{\varepsilon,n}[k], \mathscr{A}^\varepsilon_+ e_k \rangle &= \langle v_2^{\varepsilon,n}[k], \mathfrak{n}^\varepsilon_{-k} \rangle = \langle v^{\varepsilon,n}[k], \mathfrak{n}^\varepsilon_{-k} \rangle = \langle v^{\varepsilon,n}[k], -\mathscr{L}^\varepsilon v^{\varepsilon,n}[-k] \rangle \\ &= \langle v^{\varepsilon,n}[k], -\mathscr{L}_0 v^{\varepsilon,n}[-k] \rangle = \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}[k]\|^2 \,, \end{aligned}$$

from which, by (2.69), (2.68) and therefore the statement, follow at once. □

From the previous statement and the Replacement Lemma 2.6, we can easily obtain the $\varepsilon \to 0$ (and $n \to \infty$) limit of $\mathscr{A}_-^\varepsilon v_2^{\varepsilon,n}$ for $d = 2$. This is summarised in the following theorem, which also shows that the ΓL^2-norm of $v^{\varepsilon,n}$ goes to 0.

Theorem 2.13 *Let $d = 2$. For $n \in \mathbb{N}$, $n \geq 2$, $\varepsilon > 0$ and $\varphi \in \mathcal{S}(\mathbb{T}^2)$, let $\tilde{v}^{\varepsilon,n}$ be given by* (2.46)*. Then*

$$\lim_{\varepsilon \to 0} \|\mathscr{A}_-^\varepsilon \tilde{v}_2^{\varepsilon,n} - \mathscr{D}\varphi\| = 0\,, \tag{2.70}$$

where $\mathscr{D}$ is the operator introduced in Theorem 1.3, given by $\mathscr{D} \stackrel{def}{=} D_{\text{SHE}}\mathscr{L}_0^{\mathfrak{w}}$ for $\mathscr{L}_0^{\mathfrak{w}}$ defined according to (2.27) *and $D_{\text{SHE}} = G(1)$, G being the function given in* (2.24)*. Further, we have*

$$\lim_{\varepsilon \to 0} \|\tilde{v}^{\varepsilon,n}\| = 0\,. \tag{2.71}$$

Proof. We begin by showing (2.70). By (2.66) in Lemma 2.12, we have

$$\|\mathscr{A}_-^\varepsilon \tilde{v}_2^{\varepsilon,n} - \mathscr{D}\varphi\| \leq \|[\mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]\varphi\| + \|[\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon - \mathscr{D}]\varphi\|\,.$$

For the first term at the right hand side, we use the same variational principle as in (2.63) so that

$$\begin{aligned}
\|[- \mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]\varphi\| \\
&= \sup_{\|\psi\|=1} \langle \psi, [- \mathscr{A}_-^\varepsilon(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]\varphi\rangle \\
&\lesssim \lambda_\varepsilon^2 \sup_{\|\psi\|=1} \|\psi\|\|(-\mathscr{L}_0)\varphi\| = \lambda_\varepsilon^2 \|\varphi\|_{H^2(\mathbb{T}^2)}
\end{aligned}$$

where in the last step we used the Replacement Lemma 2.6 but in which we distributed the $-\mathscr{L}_0$ in (2.36) differently (note that, as $\varphi \in \Gamma L_1^2$ there are no off-diagonal terms). For the second, we use the explicit expression of $\mathscr{L}_0^{\mathfrak{w}}$, $\mathscr{G}^\varepsilon$ and $\mathscr{D}$, which give

$$\|[\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon - \mathscr{D}]\varphi\|^2 = \tfrac{1}{4}\sum_k (\mathfrak{w}\cdot k)^2 \Big[G(\mathrm{L}^\varepsilon(\tfrac{1}{2}|k|^2)) - D_{\text{SHE}}\Big]^2 |\hat{\varphi}(k)|^2\,.$$

Since φ is smooth we can take the limit in ε inside the sum and, provided we take $D_{\text{SHE}} = G(1)$, we conclude that the right hand side goes to 0. Hence, the proof of (2.70) is completed.

Concerning (2.71), we note that for any $j \in \mathbb{N}$ and $\psi \in \Gamma L_j^2$,

$$\begin{aligned}
\|(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon\psi\|^2 \\
&\lesssim j^2\lambda_\varepsilon^2 \sum_{k_{1:j+1}} \frac{\mathbb{J}^\varepsilon_{k_1,k_2}(\mathfrak{w}\cdot(k_1+k_2))^2|\hat{\psi}(k_1+k_2,k_{3:j+1})|^2}{[\frac{1}{2}|k_{1:j+1}|^2 + \frac{1}{2}(\mathfrak{w}\cdot k)_{1:j+1}^2 G(\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:j+1}|^2))]^2} \\
&\lesssim j^2\lambda_\varepsilon^2 \sum_{k_{1:j}} (\mathfrak{w}\cdot k_1)^2 |\hat{\psi}(k_{1:j})|^2 \sum_{\ell+m=k_1} \frac{\mathbb{J}^\varepsilon_{\ell,m}}{(|\ell|^2+|m|^2)^2}
\end{aligned}$$

Now, the inner sum is clearly finite, and therefore we deduce

$$\|(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon\psi\|^2 \lesssim \lambda_\varepsilon^2\|\mathscr{N}(-\mathscr{L}_0)^{1/2}\psi\|^2\,. \tag{2.72}$$

We apply the previous estimate to the definition of $\tilde{v}^{\varepsilon,n}$ together with (2.64), so that we obtain

$$\|\tilde{v}^{\varepsilon,n}\|^2 \le \|(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}_+^\varepsilon[\tilde{v}^{\varepsilon,n} + \varphi]\|^2 \lesssim \lambda_\varepsilon^2\|\mathscr{N}(-\mathscr{L}_0)^{1/2}[\tilde{v}^{\varepsilon,n} + \varphi]\|^2 \tag{2.73}$$

where in the first step, we replaced the truncated operator $\mathscr{A}_{2,n}^{\varepsilon,+}$ with $\mathscr{A}_+^\varepsilon$. By the weighted a priori estimate on $\tilde{v}^{\varepsilon,n}$ in Lemma 2.11 and the bound (2.48) on $\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}_\varphi^\varepsilon\|$, we see that the norm at the right hand side of (2.73) is bounded uniformly in ε. Now, since λ_ε goes to 0 as $\varepsilon \to 0$, (2.71) follows. □

We now turn to the case $d \ge 3$, whose analysis is more delicate. Our goal is to prove the following theorem.

Theorem 2.14 *Let $d \ge 3$. For $n \in \mathbb{N}$, $n \ge 2$, $\varepsilon > 0$ and $\varphi \in \mathcal{S}(\mathbb{T}^d)$, let $v^{\varepsilon,n}$ be the solution to* (2.45)*. Then, there exists a unique constant $D_{\mathrm{SHE}} > 0$ independent of φ such that*

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\|(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}\|^2 = \langle\varphi, (-\mathscr{D})\varphi\rangle\,. \tag{2.74}$$

where $\mathscr{D}$ is the operator introduced in Theorem 1.3, given by $\mathscr{D} \overset{def}{=} D_{\mathrm{SHE}}\mathscr{L}_0^{\mathfrak{w}}$ for $\mathscr{L}_0^{\mathfrak{w}}$ defined according to (2.27)*. As a consequence,*

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\|\mathscr{A}_-^\varepsilon v_2^{\varepsilon,n} - \mathscr{D}\varphi\| = 0\,. \tag{2.75}$$

The proof of (2.74) requires to be able to pass to the limit for $\|(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}\|$, first in ε and then in n. Since the first limit is the most delicate, we single it out in the statement below, which further provides essential information concerning the convergence of $v^{\varepsilon,n}$ to 0 in ΓL^2, and postpone its proof to the next subsection.

Proposition 2.15 *In the setting of Theorem 2.14, let n be fixed. Then, there exists a unique constant $D^n > 0$ such that*

$$\lim_{\varepsilon\to 0}\|(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}\|^2 = \langle\varphi, (-\mathscr{D}^n)\varphi\rangle\,, \tag{2.76}$$

where $\mathscr{D}^n$ is the operator given by $\mathscr{D}^n \overset{def}{=} D^n\mathscr{L}_0^{\mathfrak{w}}$ for $\mathscr{L}_0^{\mathfrak{w}}$ defined according to (2.27)*.*

Further, as $\varepsilon \to 0$, $v^{\varepsilon,n}$ converges to 0 in ΓL^2.

We now turn to the proof of (2.74) and consequently of Theorem 2.14, for which, thanks to Proposition 2.15 we only need to ensure that the limit in n of the constants D^n is unique and strictly positive.

Proof of Theorem 2.14. Notice that, once (2.74) is established, by Lemma 2.12 and smoothness of φ, (2.75) immediately follows.

For (2.74), thanks to Proposition 2.15 it remains to argue uniqueness and that D_{SHE} is strictly positive. We start with uniqueness.

Let $v^{\varepsilon,m}$ and $v^{\varepsilon,m+1}$ be the solutions to the truncated generator equation (2.45) with $n = m$ and $n = m+1$ respectively, and consider the difference $v^{\varepsilon,m+1} - v^{\varepsilon,m}$. Notice that

$$\begin{aligned}-\mathscr{L}^{\varepsilon}(v^{\varepsilon,m+1} - v^{\varepsilon,m}) = &- \mathscr{L}^{\varepsilon}_{2,m+1} v^{\varepsilon,m+1} - \mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m+1}_{m+1} - \mathscr{A}^{\varepsilon}_{-} v^{\varepsilon,m+1}_{2} \\ &+ \mathscr{L}^{\varepsilon}_{2,m} v^{\varepsilon,m} + \mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m}_{m} + \mathscr{A}^{\varepsilon}_{-} v^{\varepsilon,m}_{2} \\ = &- \mathscr{A}^{\varepsilon}_{+}(v^{\varepsilon,m+1}_{m+1} - v^{\varepsilon,m}_{m}) - \mathscr{A}^{\varepsilon}_{-}(v^{\varepsilon,m+1}_{2} - v^{\varepsilon,m}_{2}) \,.\end{aligned}$$

We now test both sides by $v^{\varepsilon,m+1} - v^{\varepsilon,m}$ and, using the antisymmetry of $\mathscr{A}^{\varepsilon}$, obtain

$$\begin{aligned}\|(-\mathscr{L}_0)^{1/2}(v^{\varepsilon,m+1} - v^{\varepsilon,m})\|^2 &= \langle v^{\varepsilon,m+1} - v^{\varepsilon,m}, -\mathscr{A}^{\varepsilon}_{+}(v^{\varepsilon,m+1}_{m+1} - v^{\varepsilon,m}_{m})\rangle \\ &= \langle v^{\varepsilon,m+1} - v^{\varepsilon,m}, \mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m}_{m}\rangle \end{aligned} \tag{2.77}$$

where the terms containing $\mathscr{A}^{\varepsilon}_{-}(v^{\varepsilon,m+1}_{2} - v^{\varepsilon,m}_{2}) \in \Gamma L^2_1$ and $\mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m+1}_{m+1} \in \Gamma L^2_{m+2}$ dropped out because of orthogonality of Fock spaces with different indices (recall that $v^{\varepsilon,m+1}_1 = v^{\varepsilon,m}_1 = 0 = v^{\varepsilon,m+1}_{m+2} = v^{\varepsilon,m}_{m+1}$). Now, the right hand side can be bounded above by

$$\begin{aligned}&\langle v^{\varepsilon,m+1} - v^{\varepsilon,m}, \mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m}_{m}\rangle \\ &\qquad \le \tfrac{1}{2}\|(-\mathscr{L}_0)^{1/2}(v^{\varepsilon,m+1} - v^{\varepsilon,m})\|^2 + 2\|(-\mathscr{L}_0)^{-1/2}\mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m}_{m}\|^2 \,.\end{aligned}$$

We can go back to (2.77) and get

$$\|(-\mathscr{L}_0)^{1/2}(v^{\varepsilon,m+1} - v^{\varepsilon,m})\|^2 \lesssim \|(-\mathscr{L}_0)^{-1/2}\mathscr{A}^{\varepsilon}_{+} v^{\varepsilon,m}_{m}\|^2 \lesssim \|\mathcal{N}(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m}_{m}\|^2 \tag{2.78}$$

where the last step follows by Lemma 2.4. To control the last term, we apply Proposition 2.9, which, for any $j > 1$, gives

$$\|\mathcal{N}(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m}_{m}\|^2 \le m^{-2j}\|\mathcal{N}^j(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m}_{m}\|^2 \lesssim m^{-2j}\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^{\varepsilon}_{\varphi}\|^2 \,.$$

Since $\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^{\varepsilon}_{\varphi}\|^2$ is bounded uniformly in ε by (2.48), by applying the triangular inequality at the left hand side of (2.78) we get

$$\Big|\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m+1}\| - \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m}\|\Big|^2 \lesssim m^{-2j} \,,$$

which, multiplying both sides by $(\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m+1}\| + \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m}\|)^2$ and invoking (2.49) once more, implies

$$\begin{aligned}&\Big|\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m+1}\|^2 - \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m})\|^2\Big|^2 \\ &\qquad \lesssim m^{-2j}\Big(\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m+1}\| + \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,m})\|\Big)^2 \lesssim m^{-2j} \,.\end{aligned}$$

By Proposition 2.15, for any n, the limit as $\varepsilon \to 0$ of $\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}\|^2$ exists, is unique and given by $\langle \varphi, (-\mathscr{D}^n)\varphi\rangle = \frac{1}{2} D^n \sum_k (\mathfrak{w}\cdot k)^2 |\hat\varphi(k)|^2$, hence we can take the ε-limit in the above inequality and obtain

$$\big| D^{m+1} - D^m \big| \sum_k (\mathfrak{w}\cdot k)^2 |\hat\varphi(k)|^2 \lesssim m^{-j}\,,$$

which holds for any φ (in particular for at least one which is not concentrated on k's orthogonal to w). Therefore, the sequence $\{D^n\}_n$ is Cauchy, thus it converges to a unique limit.

We now show that D_{SHE} is strictly positive. To do so, we consider once more the solution $v^{\varepsilon,n}$ of (2.45). Notice that thanks to the antisymmetry of $\mathscr{A}^\varepsilon_{2,n}$ in (2.40)

$$\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}\|^2 = \langle v^{\varepsilon,n}, (-\mathscr{L}^\varepsilon_{2,n}) v^{\varepsilon,n}\rangle = \langle \mathfrak{n}^\varepsilon_\varphi, (-\mathscr{L}^\varepsilon_{2,n})^{-1} \mathfrak{n}^\varepsilon_\varphi\rangle\,.$$

Thanks to [KLO12, Theorem 4.1], the term at the right hand side can be written as

$$\begin{aligned}
&\langle \mathfrak{n}^\varepsilon_\varphi, (-\mathscr{L}^\varepsilon_{2,n})^{-1} \mathfrak{n}^\varepsilon_\varphi\rangle \qquad (2.79)\\
&\quad = \sup_{\psi\in\Gamma L^2} \big\{ 2\langle \psi, \mathfrak{n}^\varepsilon_\varphi\rangle - \langle \psi, (-\mathscr{L}_0)\psi\rangle - \langle \psi, (-\mathscr{A}^\varepsilon_{2,n})^* (-\mathscr{L}_0)^{-1} (-\mathscr{A}^\varepsilon_{2,n})\psi\rangle \big\}\\
&\quad \geq \sup_{\psi\in\Gamma L^2_2} \big\{ 2\langle \psi, \mathfrak{n}^\varepsilon_\varphi\rangle - \langle \psi, (-\mathscr{L}_0)\psi\rangle - \langle \mathscr{A}^\varepsilon_+ \psi, (-\mathscr{L}_0)^{-1} \mathscr{A}^\varepsilon_+ \psi\rangle \big\}\,,
\end{aligned}$$

where in the last step we restricted the supremum to ΓL^2_2, thus making it smaller, and exploited Lemma 2.2 and the truncation P^n_2 in the definition of $\mathscr{A}^\varepsilon_{2,n}$ in (2.40) to simplify the last term. According to Lemma 2.10, for $\psi \in \Gamma L^2_2$, there exists a constant $C > 0$ such that

$$\langle \mathscr{A}^\varepsilon_+ \psi, (-\mathscr{L}_0)^{-1} \mathscr{A}^\varepsilon_+ \psi\rangle = \|(-\mathscr{L}_0)^{-1/2} \mathscr{A}^\varepsilon_+ \psi\|^2 \leq C \|(-\mathscr{L}_0)^{1/2}\psi\|^2$$

which implies a further lower bound of the form

$$\begin{aligned}
\langle \mathfrak{n}^\varepsilon_\varphi, (-\mathscr{L}^\varepsilon_{2,n})^{-1} \mathfrak{n}^\varepsilon_\varphi\rangle &\geq \sup_{\psi\in\Gamma L^2_2} \big\{ 2\langle \psi, \mathfrak{n}^\varepsilon_\varphi\rangle - \langle \psi, (-\mathscr{L}_0)\psi\rangle - C\langle \psi, (-\mathscr{L}_0)\psi\rangle \big\}\\
&= \frac{1}{1+C} \|(-\mathscr{L}_0)^{-1/2} \mathfrak{n}^\varepsilon_\varphi\|^2\,.
\end{aligned}$$

We are left with estimating $\|(-\mathscr{L}_0)^{-1/2} \mathfrak{n}^\varepsilon_\varphi\|^2$ from below. Thanks to (2.44) and (2.9), we have an explicit expression for the latter, which reads

$$\begin{aligned}
\|(-\mathscr{L}_0)^{-1/2} \mathfrak{n}^\varepsilon_\varphi\|^2 &= \frac{2}{(2\pi)^d} \lambda_\varepsilon^2 \sum_{\ell,m} \mathbb{J}^\varepsilon_{\ell,m} \frac{(\mathfrak{w}\cdot(\ell+m))^2}{|\ell|^2+|m|^2} |\hat\varphi(\ell+m)|^2 \qquad (2.80)\\
&= \frac{2}{(2\pi)^d} \sum_k (\mathfrak{w}\cdot k)^2 |\hat\varphi(k)|^2 \left(\lambda_\varepsilon^2 \sum_{\ell+m=k} \frac{\mathbb{J}^\varepsilon_{\ell,m}}{|\ell|^2+|m|^2} \right).
\end{aligned}$$

We can now take the limit for $\varepsilon \to 0$. Since φ is smooth, by dominated convergence, we can pass the limit in ε inside the sum in k and observe that the quantity in parenthesis converges to a strictly positive constant $\bar{C} \stackrel{\text{def}}{=} \int_{|x|_\infty \le 1} \mathrm{d}x/(2|x|^2)$.

By the first part of the statement, we know that the limit as $\varepsilon \to 0$ first and $n \to \infty$ after of $\|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}\|$ exists, is unique and given by $\langle \varphi, -\mathscr{D}\varphi \rangle$. In conclusion, we have shown that

$$D_{\text{SHE}} \sum_k (\mathfrak{w}\cdot k)^2 |\hat{\varphi}(k)|^2 = \langle \varphi, -\mathscr{D}\varphi\rangle = \lim_n \lim_\varepsilon \|(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}\|^2$$
$$\ge \frac{1}{1+C} \lim_\varepsilon \|(-\mathscr{L}_0)^{-1/2} \mathfrak{n}_\varphi^\varepsilon\|^2 \ge \frac{4\bar{C}}{1+C} \sum_k (\mathfrak{w}\cdot k)^2 |\hat{\varphi}(k)|^2 .$$

Since the previous holds for any φ, D_{SHE} is strictly positive. □

2.4 Proof of Proposition 2.15

The proof of Proposition 2.15 is the most delicate of the present paper. Since in $d \ge 3$ we have no replacement lemma (and we do not even expect it to hold), we need to resort to a different strategy. Our proof is centred around a suitable expansion of the norms of $v^{\varepsilon,n}$ (see e.g. (2.89) below) which involves several subsequent applications of the operators $\mathscr{A}_+^\varepsilon$, $\mathscr{A}_-^\varepsilon$ interposed by inverses of $-\mathscr{L}_0$. In order to see why such an expansion is meaningful, it is convenient to write it terms of operators which are skew-Hermitian and bounded in ΓL^2 (uniformly in ε), and this is the first thing we will do.

Let $n \in \mathbb{N}$, $n \ge 2$, be fixed throughout the section. Let $\varepsilon > 0$ and $\mathscr{A}_{2,n}^\varepsilon$ be given as in (2.40). We introduce the operators $T_{2,n}^\varepsilon$ and for $\sigma \in \{+,-\}$, $T^{\varepsilon,\sigma}, T_{2,n}^{\varepsilon,\sigma}$, according to

$$\begin{aligned} T_{2,n}^\varepsilon &\stackrel{\text{def}}{=} (-\mathscr{L}_0)^{-1/2} (\mathscr{A}_{2,n}^\varepsilon)(-\mathscr{L}_0)^{-1/2} , \\ T^{\varepsilon,\sigma} &\stackrel{\text{def}}{=} (-\mathscr{L}_0)^{-1/2} (\mathscr{A}_\sigma^\varepsilon)(-\mathscr{L}_0)^{-1/2} , \\ T_{2,n}^{\varepsilon,\sigma} &\stackrel{\text{def}}{=} (-\mathscr{L}_0)^{-1/2} (\mathscr{A}_{2,n}^{\varepsilon,\sigma})(-\mathscr{L}_0)^{-1/2} . \end{aligned} \tag{2.81}$$

Note that, by Lemma 2.4 and in particular (2.11), each of the operators above is uniformly bounded in ΓL^2 in ε (the bound depends on n, but this is irrelevant as n is fixed) and, by Lemma 2.2, it is easy to see that $T_{2,n}^\varepsilon$ is skew-Hermitian.

To have a notation for multiple applications of the operators $T^{\varepsilon,\sigma}$, $\sigma \in \{+,-\}$, we set, for $m \ge 1$ and $a \ge 0$, $\Pi_{a,m}^{(n)}$ to be the set of simple random walk paths $p = (p_0, \dots, p_a)$ consisting of a steps, starting at height 1, i.e. $p_0 = 1$, ending at height m, i.e. $p_a = m$, that do not reach height $n+1$ and that can have height 1 only at the endpoints. Given $p \in \Pi_{a,m}^{(n)}$ we let $\mathscr{T}_p^\varepsilon$ be defined according to

$$\mathscr{T}_p^\varepsilon \stackrel{\text{def}}{=} T^{\varepsilon,\sigma_a} T^{\varepsilon,\sigma_{a-1}} \dots T^{\varepsilon,\sigma_1} \tag{2.82}$$

where $\sigma_i = +$, if $p_i - p_{i-1} = 1$, and $\sigma_i = -$, if $p_i - p_{i-1} = -1$. In other words, there is a factor $T^{\varepsilon,+}$ (resp. $T^{\varepsilon,-}$) whenever the path takes a step up (resp. down)[3].

[3]For instance, for $a = 4$ the path $p = (1,2,3,2,3) \in \Pi_{4,3}^{(n)}, n \ge 4$ corresponds to $\mathscr{T}_p^\varepsilon = T^{\varepsilon,+} T^{\varepsilon,-} T^{\varepsilon,+} T^{\varepsilon,+}$.

We enclose the most technical aspect of the proof of Proposition 2.15 in Proposition 2.16 below. Before giving the proof of the latter, we will show how it implies the former.

Proposition 2.16 *In the notations introduced above, for every $a > 0, m \geq 1$ and $p \in \Pi^{(n)}_{a,m}$, the limit*

$$\lim_{\varepsilon\to 0}\langle e_{-k}, \mathcal{T}^\varepsilon_p e_k\rangle = c(p)\frac{(\mathfrak{w}\cdot k)^2}{|k|^2} \tag{2.83}$$

exists, with $c(p)$ depending only on the path p. Moreover, $c(p) = 0$ unless $a \in 2\mathbb{N}, m = 1$. Furthermore, for every $a > 0, m \geq 2$, $q_1, \dots, q_m \in \mathbb{Z}^d_0$, $m' \geq 1$ and $p \in \Pi^{(n)}_{a,m'}$, one has

$$\lim_{\varepsilon\to 0}\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{T}^\varepsilon_p e_k\rangle = 0\,. \tag{2.84}$$

Proof of Proposition 2.15 given Proposition 2.16. We start with (2.76). Let $v^{\varepsilon,n}$ be the solution of (2.45). As shown in the proof of Lemma 2.12, we have that $v^{\varepsilon,n} = \sum_k \hat\varphi(k) v^{\varepsilon,n}[-k]$, where, for $j \in \mathbb{Z}^d_0$, $v^{\varepsilon,n}[j]$ solves the truncated generator equation (2.45) with $\varphi = e_{-j}$. Hence, recalling (2.64) and (2.67), the object whose limit we want to determine equals

$$\begin{aligned}\langle v^{\varepsilon,n}, (-\mathscr{L}_0)v^{\varepsilon,n}\rangle &= \langle v^{\varepsilon,n}, (-\mathscr{L}^\varepsilon_{2,n})v^{\varepsilon,n}\rangle = \langle v^{\varepsilon,n}, \mathfrak{n}^\varepsilon_\varphi\rangle\\ &= \sum_{k,j}\hat\varphi(k)\hat\varphi(-j)\langle v^{\varepsilon,n}[-k], \mathfrak{n}^\varepsilon_j\rangle\\ &= \sum_k |\hat\varphi(k)|^2\langle v^{\varepsilon,n}[-k], \mathfrak{n}^\varepsilon_k\rangle = \sum_k |\hat\varphi(k)|^2\|(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}[-k]\|^2\end{aligned}$$

where, as in (2.69), we used translation invariance to remove summands in which $j \neq -k$. Each of the squared norms at the right hand side are uniformly (in ε) bounded by a quantity that grows at most quadratically in k. In fact, by Proposition 2.9, each squared norm is bounded by a constant times $\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_k\|^2$. In turn, by (2.80) we have

$$\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_k\|^2 = 2(\mathfrak{w}\cdot k)^2\left(\lambda_\varepsilon^2\sum_{\ell+m=k}\frac{\mathbb{J}^\varepsilon_{\ell,m}}{|\ell|^2+|m|^2}\right)$$

and the sum is uniformly bounded by a constant independent of ε and k. Since φ is smooth, to prove (2.76) it is enough (by dominated convergence) to show that there exists a unique D^n such that for all k

$$\lim_{\varepsilon\to 0}\|(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}[-k]\|^2 = \lim_{\varepsilon\to 0}\langle \mathfrak{n}^\varepsilon_k, v^{\varepsilon,n}[-k]\rangle = \tfrac12 D^n(\mathfrak{w}\cdot k)^2\,. \tag{2.85}$$

Recall the definition of the operators $T^\varepsilon_{2,n}$ and $T^{\varepsilon,\sigma}$, $\sigma \in \{+,-\}$ in (2.81). Note that, thanks to (2.64) and (2.45),

$$v^{\varepsilon,n}[-k] = (-\mathscr{L}^\varepsilon_{2,n})^{-1}\mathscr{A}^\varepsilon_+ e_k = \frac{|k|}{\sqrt{2}}(-\mathscr{L}_0)^{-1/2}(I - T^\varepsilon_{2,n})^{-1}T^{\varepsilon,+}e_k\,, \tag{2.86}$$

where in the last equality we further used (2.9). Therefore,

$$\langle \mathfrak{n}_k^\varepsilon, v^{\varepsilon,n}[-k]\rangle = \frac{|k|^2}{2}\langle T^{\varepsilon,+}e_{-k}, (I - T_{2,n}^\varepsilon)^{-1}T^{\varepsilon,+}e_k\rangle\,. \tag{2.87}$$

Note that $T_{2,n}^\varepsilon$ is skew-Hermitian and, by Lemma 2.4 and in particular (2.11), it is bounded uniformly in ε (for n fixed). Then, we write the inverse operator above as

$$\begin{aligned}\frac{2}{|k|^2}\langle \mathfrak{n}_k^\varepsilon, v^{\varepsilon,n}[-k]\rangle &= \langle T^{\varepsilon,+}e_{-k}, \Big(\int_0^\infty e^{-s}e^{sT_{2,n}^\varepsilon}\mathrm{d}s\Big)\,T^{\varepsilon,+}e_k\rangle \\ &= \int_0^\infty e^{-s}\langle T^{\varepsilon,+}e_{-k}, e^{sT_{2,n}^\varepsilon}T^{\varepsilon,+}e_k\rangle \mathrm{d}s\end{aligned} \tag{2.88}$$

where in the second step we applied Fubini, which is allowed as the operator $e^{sT_{2,n}^\varepsilon}$ has norm at most 1, since the spectrum of $T_{2,n}^\varepsilon$ is purely imaginary. For the same reason, once we take the ε-limit at both sides we can apply dominated convergence and pass the limit inside the integral in s, i.e.

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{2}{|k|^2}\langle \mathfrak{n}_k^\varepsilon, v^{\varepsilon,n}[-k]\rangle &= \int_0^\infty e^{-s}\lim_{\varepsilon\to 0}\langle T^{\varepsilon,+}e_{-k}, e^{sT_{2,n}^\varepsilon}T^{\varepsilon,+}e_k\rangle \mathrm{d}s \\ &= \int_0^\infty e^{-s}\lim_{\varepsilon\to 0}\langle T^{\varepsilon,+}e_{-k}, \sum_{b\geq 0}\frac{s^b}{b!}(T_{2,n}^\varepsilon)^b T^{\varepsilon,+}e_k\rangle \mathrm{d}s \\ &= \int_0^\infty e^{-s}\sum_{b\geq 0}\frac{s^b}{b!}\lim_{\varepsilon\to 0}\langle T^{\varepsilon,+}e_{-k}, (T_{2,n}^\varepsilon)^b T^{\varepsilon,+}e_k\rangle \mathrm{d}s\end{aligned} \tag{2.89}$$

where in the last step we could exchange the limit with both the summation over b and the scalar product because, for fixed n, s, the norm of sT_n^ε is uniformly bounded (recall that after the first, in the subsequent steps s can be treated as fixed). Therefore, we are left to determine, for every $b \geq 0$, the limit of

$$\langle T^{\varepsilon,+}e_{-k}, (T_{2,n}^\varepsilon)^b T^{\varepsilon,+}e_k\rangle = -\langle e_{-k}, T^{\varepsilon,-}(T_{2,n}^\varepsilon)^b T^{\varepsilon,+}e_k\rangle\,.$$

Let us begin with a few remarks. First of all, since $T_{2,n}^\varepsilon$ is skew-Hermitian, the scalar product is zero unless b is even, hence we set $a \in \mathbb{N}$ to be such that $b = 2a$. Next, we expand

$$\begin{aligned}\langle e_{-k}, T^{\varepsilon,-}(T_{2,n}^\varepsilon)^{2a}T^{\varepsilon,+}e_k\rangle &= \langle e_{-k}, T^{\varepsilon,-}(T_{2,n}^{\varepsilon,-} + T_{2,n}^{\varepsilon,+})^{2a}T^{\varepsilon,+}e_k\rangle \\ &= \sum_{p\in\widetilde{\Pi}_{2a+2,1}} \langle e_{-k}, T^{\varepsilon,\sigma_{2a+2}}T_{2,n}^{\varepsilon,\sigma_{2a+1}}\dots T_{2,n}^{\varepsilon,\sigma_2}T^{\varepsilon,\sigma_1}e_k\rangle\end{aligned} \tag{2.90}$$

where $\widetilde{\Pi}_{2a+2,1}$ is the set of all walks p which start at time 1 from 1, i.e. $p_0 = 1$, ends at time $2a+2$ at 1, i.e. $p_{2a+2} = 1$, have increments of size 1 and are such that $p_1 - p_0 = 1 = p_{2a+1} - p_{2a+2}$. If either of the previous conditions were violated we would be taking the scalar product of quantities in Fock spaces with different indices, which would be 0. In the above, for $p \in \widetilde{\Pi}_{2a+2,1}$ and $i = 1, \dots, 2a+2$,

$\sigma_i = +$ if $p_i - p_{i-1} = 1$ and $\sigma_i = -$ if $p_i - p_{i-1} = -1$. Note that the last condition on the p's ensures that $\sigma_1 = 1 = -\sigma_{2a+2}$ (so that the $T^{\varepsilon,-}$ and $T^{\varepsilon,+}$ in the line above are present).

Because of the projector P_2^n entering the definition of $T_{2,n}^{\varepsilon,\sigma}$, if a path p reaches height $n+1$ or height 1 (except at the endpoints), then the corresponding scalar product is annihilated. Hence, we can replace the index set of the sum at the right hand side of (2.90) to $\Pi_{2a+2,1}^{(n)}$, and get

$$\begin{aligned}\langle e_{-k}, T^{\varepsilon,-}(T_{2,n}^{\varepsilon})^{2a}T^{\varepsilon,+}e_k\rangle &= \sum_{p\in\Pi_{2a+2,1}^{(n)}} \langle e_{-k}, T^{\varepsilon,\sigma_{2a+1}}T_{2,n}^{\varepsilon,\sigma_{2a}}\dots T_{2,n}^{\varepsilon,\sigma_2}T^{\varepsilon,\sigma_1}e_k\rangle \\ &= \sum_{p\in\Pi_{2a+2,1}^{(n)}} \langle e_{-k}, T^{\varepsilon,\sigma_{2a+1}}T^{\varepsilon,\sigma_{2a}}\dots T^{\varepsilon,\sigma_2}T^{\varepsilon,\sigma_1}e_k\rangle \\ &= \sum_{p\in\Pi_{2a+2,1}^{(n)}} \langle e_{-k}, \mathscr{T}_p^{\varepsilon}e_k\rangle\,,\end{aligned}$$

where in the second step replaced all instances of $T_{2,n}^{\varepsilon,\sigma}$ with $T^{\varepsilon,\sigma}$, since for $p \in \Pi_{2a+2,1}^{(n)}$ the projection P_2^n acts as the identity operator.

At last, getting back to (2.89) and applying (2.83) in Proposition 2.16, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle \mathfrak{n}_k^{\varepsilon}, v^{\varepsilon,n}[-k]\rangle &= \frac{|k|^2}{2}\int_0^\infty e^{-s}\sum_{a\geq 0}\frac{s^{2a}}{(2a)!}\lim_{\varepsilon\to 0}\sum_{p\in\Pi_{2a+2,1}^{(n)}}\langle e_{-k}, \mathscr{T}_p^{\varepsilon}e_k\rangle \mathrm{d}s \\ &= \frac{|k|^2}{2}\int_0^\infty e^{-s}\sum_{a\geq 0}\frac{s^{2a}}{(2a)!}\sum_{p\in\Pi_{2a+2,1}^{(n)}}\lim_{\varepsilon\to 0}\langle e_{-k}, \mathscr{T}_p^{\varepsilon}e_k\rangle \mathrm{d}s \\ &= \frac{(\mathfrak{w}\cdot k)^2}{2}\int_0^\infty e^{-s}\sum_{a\geq 0}\frac{s^{2a}}{(2a)!}\sum_{p\in\Pi_{2a+2,1}^{(n)}} c(p)\mathrm{d}s \\ &=: \frac{1}{2}D^n(\mathfrak{w}\cdot k)^2\end{aligned}$$

where the last constant D^n is uniquely defined by the expression before, so that (2.76) follows.

We are left to show that $v^{\varepsilon,n}$ converges to zero in ΓL^2. Let us begin by writing

$$\|v^{\varepsilon,n}\|^2 = \sum_{m=2}^n \sum_{k_{1:m}} |v_m^{\varepsilon,n}(k_{1:m})|^2,$$

with $v_m^{\varepsilon,n}$ the component of $v^{\varepsilon,n}$ in the m^{th} Wiener chaos. Let $N > 0$. We split the norm as

$$\|v^{\varepsilon,n}\|^2 = \sum_{m=2}^n \sum_{|k_{1:m}|>N} |v_m^{\varepsilon,n}(k_{1:m})|^2 + \sum_{m=2}^n \sum_{|k_{1:m}|\leq N} |v_m^{\varepsilon,n}(k_{1:m})|^2$$

$$\leq \frac{2}{N^2}\sum_{m=2}^{n}\sum_{|k_{1:m}|>N}\frac{|k_{1:m}|^2}{2}|v_m^{\varepsilon,n}(k_{1:m})|^2+\sum_{m=2}^{n}\sum_{|k_{1:m}|\leq N}|v_m^{\varepsilon,n}(k_{1:m})|^2$$
$$\leq \frac{2}{N^2}\|(-\mathscr{L}_0)^{1/2}v^{\varepsilon,n}\|^2+\sum_{m=2}^{n}\sum_{|k_{1:m}|\leq N}|v_m^{\varepsilon,n}(k_{1:m})|^2\,.$$

By the just proved (2.76), if we first take the limit $\varepsilon\to 0$ and then $N\to\infty$, we deduce that the convergence of $v^{\varepsilon,n}$ to zero in ΓL^2 follows provided that

$$\lim_{\varepsilon\to 0}|v_m^{\varepsilon,n}(k_{1:m})|^2=\lim_{\varepsilon\to 0}|\langle e_{k_1}\otimes\cdots\otimes e_{k_m},v^{\varepsilon,n}\rangle|^2=0 \tag{2.91}$$

for every fixed $2\leq m\leq n$ and any given $k_{1:m}$. Since $v^{\varepsilon,n}=\sum_k\hat\varphi(k)v^{\varepsilon,n}[-k]$ and φ is smooth, it is enough to show that

$$\lim_{\varepsilon\to 0}\langle e_{k_1}\otimes\cdots\otimes e_{k_m},v^{\varepsilon,n}[-k]\rangle=0$$

for every fixed k. To this purpose, by (2.86) and, following steps similar to those after (2.87), we come to

$$\lim_{\varepsilon\to 0}\langle e_{k_1}\otimes\cdots\otimes e_{k_m},v^{\varepsilon,n}[-k]\rangle$$
$$=\frac{|k|}{|k_{1:m}|}\int_0^\infty e^{-s}\sum_{b\geq 0}\frac{s^b}{b!}\lim_{\varepsilon\to 0}\langle e_{k_1}\otimes\cdots\otimes e_{k_m},(T_{2,n}^{\varepsilon,+}+T_{2,n}^{\varepsilon,-})^bT^{\varepsilon,+}e_k\rangle$$
$$=\frac{|k|}{|k_{1:m}|}\int_0^\infty e^{-s}\sum_{b\geq 0}\frac{s^b}{b!}\sum_{p\in\Pi_{b,m}^{(n)}}\lim_{\varepsilon\to 0}\langle e_{k_1}\otimes\cdots\otimes e_{k_m},\mathscr{T}_p^\varepsilon e_k\rangle\,.$$

Finally, (2.84) implies that the right-hand side of the previous expression is zero, thereby proving (2.91). □

We now turn to the proof of Proposition 2.16, for which we will need a preliminary lemma. Given $p\in\Pi_{a,m}^{(n)}$, $\mathscr{T}_p^\varepsilon e_k$ belongs to the m^{th} Fock space and we write the Fourier transform of its kernel in the form

$$f^\varepsilon(k_{1:m},k)=\varepsilon^{\frac{d}{2}(m-1)}F_p^\varepsilon(\varepsilon k_{1:m};\varepsilon k)\,, \tag{2.92}$$

in which we singled out the scaling factor $\varepsilon^{\frac{d}{2}(m-1)}$. We derive an expression for F_p^ε in terms of polynomials, whose form and main properties are summarised in the following lemma.

Lemma 2.17 *Given $p\in\Pi_{a,m}^{(n)}$ and $a\geq 1$, let $M=(1+a-m)/2\,(\geq 0)$ be the number of $T^{\varepsilon,-}$ in the product $\mathscr{T}_p^\varepsilon$. The function $F_p^\varepsilon(x_{1:m};x)$ in (2.92) can be written as*

$$F_p^\varepsilon(x_{1:m};x)=\mathbb{1}_{x_1+\cdots+x_m=x}\frac{(\mathfrak{w}\cdot x)}{|x|}\frac{\iota^a}{|x_{1:m}|}\times$$
$$\times\sum_{g\in\mathscr{G}[p]}\varepsilon^{dM}\sum_{\substack{y_{1:M}\in(\varepsilon\mathbb{Z}_0^d)^M:\\ |y_i|_\infty\leq 1,i\leq M}}\frac{P_g(y_{1:M};x_{1:m})}{Q_g(y_{1:M};x_{1:m})}I_g(y_{1:M};x_{1:m})\,, \tag{2.93}$$

where

(i) $\mathcal{G}[p]$ *is a finite set whose cardinality depends only on the path* p*,*
(ii) I_g *is a product of indicator functions, each imposing that certain linear combinations of its arguments have* $|\cdot|_\infty$ *norm in* $(0,1]$*,*
(iii) P_g *(resp.* Q_p*) is a homogenous real-valued polynomial of degree* $a-1$ *(resp.* $2(a-1)$*) in* $y_{1:M}, x_{1:m}$*. Further,* $Q_g \geq 0$ *and it does not vanish on the support of* I_g*,*
(iv) if $m=1$*, then* $Q_g(y_{1:M};0) > 0$ *for almost every* $y_{1:M}$*.*

In particular, if $m=1$ *then there exists a polynomial* $\tilde{P}_g$ *of degree* $a-2$ *such that*

$$\langle e_{-k_1}, \mathcal{T}_p^\varepsilon e_k\rangle = \iota^a \mathbb{1}_{k_1=k} \frac{(\mathfrak{w}\cdot k)^2}{|k|^2} \times \sum_{g\in\mathcal{G}[p]} \varepsilon^{dM} \sum_{\substack{y_{1:M}\in(\varepsilon\mathbb{Z}_0^d)^M:\\ |y_i|_\infty\leq 1, i\leq M}} \frac{\tilde{P}_g(y_{1:M};\varepsilon k)}{Q_g(y_{1:M};\varepsilon k)} I_g(y_{1:M};\varepsilon k)\,. \tag{2.94}$$

We emphasise that the functions P_g, Q_g, I_g and $\tilde{P}_g$ are ε*-independent*. The only ε-dependence in F_p^ε comes from the fact that the Riemann sums runs over $\varepsilon\mathbb{Z}_0^d$.

Proof. The statement is proven via induction on a. In order to verify condition *(iv)*, we will simultaneously show that the Q_g's further satisfy

(iii') $Q_g(y_{1:M};x_{1:m})$ is the product of homogeneous quadratic polynomials q_r, $1\leq r\leq a-1$, each being the sum of squared norms of linear combinations of its arguments, and, for every $1\leq r\leq a-1$, $q_r \neq |x_1+\cdots+x_m|^2$.

From now on, since we are in dimension $d\geq 3$, we will not write λ_ε but its value $\varepsilon^{d/2-1}$ so that the scaling exponents become more transparent.

We begin our induction with $a=1$. In this case, if $p\in\Pi_{a,m}^{(n)}$, then necessarily $m=2, M=0$ and $\mathcal{T}_p^\varepsilon = T^{\varepsilon,+}$. Then, from (2.81) and (2.9) with $n=1$ (see also (2.96) below), one sees that

$$F_p^\varepsilon(x_{1:2};x) = \mathbb{1}_{x_1+x_2=x} \frac{(\mathfrak{w}\cdot x)}{|x|} \frac{\iota}{|x_{1:2}|} \frac{-4}{(2\pi)^{d/2}} \mathbb{J}^1_{x_1,x_2} \tag{2.95}$$

because the kernel of e_k equals $\mathbb{1}_{\cdot=k}$. This expression is of the correct form (2.93), with $\mathcal{G}[p]=\emptyset$, $P_g \equiv -4/(2\pi)^{d/2}$, $Q_g\equiv 1$, $I_g \equiv \mathbb{J}^1_{x_1,x_2}$, so that $(i)-(iii)$ and (iii') are clearly satisfied.

As for the induction step, we assume the validity of the statement for a given $a\geq 2$ and take $p\in\Pi_{a+1,m}^{(n)}$. Write $\mathcal{T}_p^\varepsilon = T^{\varepsilon,\sigma}\mathcal{T}_{p'}^\varepsilon$ for some $p'\in\Pi_{a,m-\sigma_{a-1}}^{(n)}$. Consider first the case $\sigma=+$, so that $\sigma_{a-1}=1$. Recalling the definition of $T^{\varepsilon,+}$ and (2.92), we have

$$(T^{\varepsilon,+} f_{p'}^\varepsilon)(k_{1:m};k) = -\frac{1}{(2\pi)^{d/2}} \frac{4\varepsilon^{d/2-1+d/2(m-2)}}{m} \frac{\iota}{|k_{1:m}|} \times$$

$$\times \sum_{1\le i<j\le m} \frac{\mathfrak{w}\cdot(k_i+k_j)\mathbb{J}^\varepsilon_{k_i,k_j}}{\sqrt{|k_i+k_j|^2+|k_{1:m\setminus\{i,j\}}|^2}} F^\varepsilon_{p'}(\varepsilon k_i+\varepsilon k_j, \varepsilon k_{1:m\setminus\{i,j\}};\varepsilon k)$$
$$= \varepsilon^{d/2(m-1)} F^\varepsilon_p(\varepsilon k_{1:m};\varepsilon k), \tag{2.96}$$

with

$$F^\varepsilon_p(x_{1:m};x) = \frac{\iota}{|x_{1:m}|} \sum_{1\le i<j\le m} \frac{-4}{m(2\pi)^{d/2}} \times \tag{2.97}$$
$$\times \frac{\mathfrak{w}\cdot(x_i+x_j)}{\sqrt{|x_i+x_j|^2+|x_{1:m\setminus\{i,j\}}|^2}} \mathbb{J}^1_{x_i,x_j} F^\varepsilon_{p'}(x_i+x_j, x_{1:m\setminus\{i,j\}};x).$$

By induction, $F_{p'}$ has the form in (2.93), and therefore so does F_p once we set $\mathcal{G}[p]=\{(g',(i,j)), g'\in\mathcal{G}[p'], 1\le i<j\le m\}$ and, for $g=(g',(i,j))\in\mathcal{G}[p]$,

$$\begin{aligned} P_g(y_{1:M};x_{1:m}) &= \frac{-4}{m(2\pi)^{d/2}}(\mathfrak{w}\cdot(x_i+x_j))P_{g'}(y_{1:M};x_i+x_j,x_{1:m\setminus\{i,j\}}),\\ Q_g(y_{1:M};x_{1:m}) &= (|x_i+x_j|^2+|x_{1:m\setminus\{i,j\}}|^2)Q_{g'}(y_{1:M};x_i+x_j,x_{1:m\setminus\{i,j\}})\\ I_g(y_{1:M};x_{1:m}) &= \mathbb{J}^1_{x_i,x_j} I_{g'}(y_{1:M};x_i+x_j,x_{1:m\setminus\{i,j\}})\,. \end{aligned} \tag{2.98}$$

Similarly, if $\sigma=-$ (so that $\sigma_{a-1}=-1$), by (2.9) one has

$$\begin{aligned} (T^\varepsilon_- f^\varepsilon_{p'})(k_{1:m};k) =& \frac{\varepsilon^{d/2-1}}{(2\pi)^{d/2}} \frac{4\iota(m+1)}{|k_{1:m}|} \sum_{j=1}^m (\mathfrak{w}\cdot k_j)\\ &\times \sum_{y,z\in\mathbb{Z}^d: y+z=k_j} \mathbb{J}^\varepsilon_{y,z} \frac{\varepsilon^{dm/2} F^\varepsilon_{p'}(\varepsilon y,\varepsilon z,\varepsilon k_{\{1:m\}\setminus\{j\}};\varepsilon k)}{\sqrt{|y|^2+|z|^2+|k_{\{1:m\}\setminus\{j\}}|^2}}\\ =& \varepsilon^{d/2(m-1)} F^\varepsilon_p(\varepsilon k_{1:m};\varepsilon k), \end{aligned}$$

where in this case

$$F^\varepsilon_p(x_{1:m};x) = \frac{4\iota(m+1)}{(2\pi)^{d/2}} \frac{1}{|x_{1:m}|} \sum_{j=1}^m (\mathfrak{w}\cdot x_j)\mathbb{1}_{0<|x_j|_\infty\le 1} \tag{2.99}$$
$$\times \sum_{y\in\varepsilon\mathbb{Z}^d_0:|y|_\infty\le 1} \varepsilon^d \mathbb{1}_{0<|y-x_j|_\infty\le 1} \frac{F^\varepsilon_{p'}(y,x_j-y,x_{\{1:m\}\setminus\{j\}};x)}{\sqrt{|y|^2+|x_j-y|^2+|x_{\{1:m\}\setminus\{j\}}|^2}}.$$

Again, as $F^\varepsilon_{p'}$ is of the form (2.93) by the induction hypothesis, the same holds for F^ε_p, where this time we need to set $\mathcal{G}[p]=\{(g',j), g'\in\mathcal{G}[p'], 1\le j\le m\}$ and, for $g=(g',j)\in\mathcal{G}[p]$,

$$\begin{aligned} P_g(y_{1:M};x_{1:m}) &= \frac{4(m+1)}{(2\pi)^{d/2}}(\mathfrak{w}\cdot x_j)P_{g'},\\ Q_g(y_{1:M};x_{1:m}) &= (|y_M|^2+|x_j-y_M|^2+|x_{1:m\setminus\{j\}}|^2)Q_{g'} \end{aligned}$$

$$I_g(y_{1:M};x_{1:m}) = \mathbb{1}_{0<|x_j|_\infty\le 1, 0<|y_M-x_j|\le 1} I_{g'}\,, \tag{2.100}$$

where the omitted variables of the functions $P_{g'}$, $Q_{g'}$ and $I_{g'}$ are $y_{1:M-1}; y_M, x_j - y_M, x_{1:m\setminus\{j\}}$.

Now, for both $\sigma = +,-$, (2.98) and (2.100) clearly ensure that if $\mathcal{G}[p']$, $P_{g'}$, $Q_{g'}$ and $I_{g'}$ satisfy conditions $(i)-(iii)$ and (iii') so do $\mathcal{G}[p]$, P_g, Q_g and I_g. As for property (iv), note that by (iii'), Q_g can be written as the product of quadratic polynomials q_r, $1 \le r \le a-1$, and if q_r depends on at least one of the y_i, $i \le M$, then necessarily $q_r(y_{1:M};0) \neq 0$ for almost all $y_{1:M}$. The only potentially problematic case arises when one of the q_r is such that $q_r(y_{1:M};x) = |x|^2$ but this is ruled out by (iii') for $m = 1$.

At last, to verify (2.94), we note that (2.92) and (2.93) give

$$\begin{aligned}\langle e_{-k_1}\mathcal{T}^\varepsilon_p e_k\rangle =& \iota^a \mathbb{1}_{k_1=k}\frac{(\mathfrak{w}\cdot k)}{|k|}\frac{1}{\varepsilon|k|} \\ &\times \sum_{g\in\mathcal{G}[p]}\varepsilon^{dM}\sum_{\substack{y_{1:M}\in(\varepsilon\mathbb{Z}^d_0)^M:\\ |y_i|\le 1, i\le M}}\frac{P_g(y_{1:M};\varepsilon k)}{Q_g(y_{1:M};\varepsilon k)}I_g(y_{1:M};\varepsilon k)\,.\end{aligned} \tag{2.101}$$

But, since $m = 1$, the left-most operator in the product $\mathcal{T}^\varepsilon_p$ is $T^{\varepsilon,-}$, so that $\mathcal{T}^\varepsilon_p e_k = T^{\varepsilon,-}\mathcal{T}^\varepsilon_{p'}e_k$ and $\mathcal{T}^\varepsilon_{p'}e_k \in \Gamma L^2_2$. Then, from (2.100) we see that

$$P_g(y_{1:M};\varepsilon k) = \frac{4}{(2\pi)^{d/2}}(\mathfrak{w}\cdot(\varepsilon k))P_{g'}(y_{1:M-1};y_M,\varepsilon k - y_M)$$

which, once plugged into (2.101), reduces to (2.94), upon defining

$$\tilde P_g(y_{1:M};\varepsilon k) \stackrel{\text{def}}{=} \frac{8}{(2\pi)^{d/2}}P_{g'}(y_{1:M-1};y_M,\varepsilon k - y_M)\,.$$

□

Proof of Proposition 2.16. We start with the proof of (2.83). If a is odd or $m > 1$, $\mathcal{T}^\varepsilon_p e_k$ belongs to a homogeneous Wiener chaos of index different from 1 and the scalar product is identically zero. Therefore, we assume that $p \in \Pi^{(n)}_{2a,1}, a > 0$, so that $M = a$. In this case, we want to show that the sum in (2.94) has a limit as $\varepsilon \to 0$ and that the limit depends only on p (in particular, not on k). Formally, the $\varepsilon \to 0$ limit of (2.94) is

$$\iota^a\mathbb{1}_{k_1=k}\frac{(\mathfrak{w}\cdot k)^2}{|k|^2}2^{ad}\int\mu(\mathrm{d}y_{1:a})\sum_{g\in\mathcal{G}[p]}\frac{P_g(y_{1:a};0)}{Q_g(y_{1:a};0)}I_g(y_{1:a};0)$$

where[4] μ is the uniform probability measure on $[-1,1]^{ad}$ and the integrand is well-defined and finite (except possibly on a zero-measure set) thanks to the properties

[4]note that we added the volume factor 2^{ad} to pass from the Lebesgue measure to the uniform one. The scope of this change of measure will soon become apparent.

of Q_g stated in Lemma 2.17. It remains to prove that the sum indeed converges to the integral, and that the integral is finite.

At first, we want to write the Riemann-sum in (2.94) as an integral with respect to the uniform probability measure on $[-1,1]^{ad}$. This is quite standard as it suffices to take a piece-wise constant extension of the polynomials P_g and Q_g, but since the quotient displays singularities, we provide the details below.

Recall that we assume $1/\varepsilon \in \mathbb{N} + 1/2$. Then, the box $[-1,1]^d$ contains exactly $(2/\varepsilon)^d$ points of $\varepsilon\mathbb{Z}^d$ and it can be written as the union of cubes $C(y^*)$ of side ε, centered at each of the points $y^* \in \varepsilon\mathbb{Z}^d \cap [-1,1]^d$. Given $y \in [-1,1]^d$, we let $y^*(y)$ denote the (unique, up to Lebesgue-measure zero sets) y^* such that $y \in C(y^*)$. Note that we can rewrite the sum in (2.94) as

$$2^{ad} \int \mu(\mathrm{d}y_{1:a}) \sum_{g \in \mathcal{G}[p]} \frac{P_g^{(\varepsilon)}(y_{1:a};\varepsilon k)}{Q_g^{(\varepsilon)}(y_{1:a};\varepsilon k)} I_g^{(\varepsilon)}(y_{1:a};\varepsilon k), \tag{2.102}$$

where μ is the uniform probability measure on $[-1,1]^{ad}$, $P_g^{(\varepsilon)}$ is given by

$$P_g^{(\varepsilon)}(y_{1:a};\varepsilon k) \stackrel{\text{def}}{=} P_g(y^*(y_1),\dots,y^*(y_a))\mathbb{1}_{y_i \notin [-\varepsilon/2,\varepsilon/2]^d, i \le a}\,,$$

and similarly $Q_g^{(\varepsilon)}, I_g^{(\varepsilon)}$, and the indicator function has been added because in (2.94) the sums are restricted to $y_i \neq 0$. Note that, by construction, the integrand is piecewise constant.

At this point, existence, finiteness of the $\varepsilon \to 0$ limit of (2.102) and its independence w.r.t. k follow if we can prove uniform integrability of the integrand. For this, we will introduce operators $U^{\varepsilon,\pm}$ analogous to the $T^{\varepsilon,\pm}$ but with $\mathscr{A}^{\varepsilon,\delta}$ in (2.18) replacing $\mathscr{A}^{\varepsilon}$, and their composition $\mathcal{U}_p^{\varepsilon}$ which is going to give us the same expression as in (2.102) but in which the integrand is raised to the power $1+\delta$ for δ small but positive. Upon proving that the latter is bounded, the claimed uniform integrability will follow. Let us now provide the details.

Let $\delta \in (0,(d-2)/2)$ be independent of ε and define the operators $U^{\varepsilon,\pm}$ in Fock space as $U^{\varepsilon,\pm} \stackrel{\text{def}}{=} (-\mathscr{L}_0)^{-(1+\delta)/2}\, \mathscr{A}_{\pm}^{\varepsilon,\delta} (-\mathscr{L}_0)^{-(1+\delta)/2}$. More explicitly, $U^{\varepsilon,+}$ acts on $f \in \Gamma L_n^2$, $n \ge 2$, as

$$\begin{aligned} U^{\varepsilon,+} f(k_{1:n+1}) &= \varepsilon^{d/2} \left(\frac{4}{(2\pi)^{d/2}(n+1)} \frac{1}{|\varepsilon k_{1:n+1}|} \right)^{1+\delta} \\ &\times \sum_{1 \le i < j \le n+1} \frac{|\mathfrak{w} \cdot (\varepsilon k_i + \varepsilon k_j)|^{1+\delta}\, \mathbb{J}^{\varepsilon}_{k_i,k_j}}{\Big(|\varepsilon(k_i+k_j)|^2 + |\varepsilon k_{1:n+1\setminus\{i,j\}}|^2 \Big)^{(1+\delta)/2}} f(k_i+k_j, k_{1:n+1\setminus\{i,j\}}) \end{aligned} \tag{2.103}$$

while $U^{\varepsilon,-}$ as

$$U^{\varepsilon,-} f(k_{1:n-1}) = \varepsilon^{d/2} \left(\frac{4n}{(2\pi)^{d/2}|\varepsilon k_{1:n-1}|} \right)^{1+\delta}$$

$$\times \sum_{j=1}^{n-1} |\mathfrak{w}\cdot(\varepsilon k_j)|^{1+\delta} \sum_{\ell+m=k_j} \mathbb{J}^{\varepsilon}_{\ell,m} \frac{f(\ell,m,k_{\{1:n-1\}\setminus\{j\}})}{\Big(|\varepsilon\ell|^2+|\varepsilon m|^2+|\varepsilon k_{\{1:n-1\}\setminus\{j\}}|^2\Big)^{(1+\delta)/2}}\,.$$

Also, define the product of $U^{\varepsilon,\pm}$ operators $\mathcal{U}^{\varepsilon}_p$ corresponding to the path p, in analogy with $\mathcal{T}^{\varepsilon}_p$, just with $T^{\varepsilon,\pm}$ replaced by $U^{\varepsilon,\pm}$. Then, the same argument as in Lemma 2.17 together with the derivation of (2.102), shows that

$$\begin{aligned}\langle e_{-k}, \mathcal{U}^{\varepsilon}_p e_k\rangle =& 2^{da}\frac{|\mathfrak{w}\cdot k|^{2(1+\delta)}}{|k|^{2(1+\delta)}} \\ &\times \int \mu(\mathrm{d}y_{1:a}) \sum_{g\in\mathcal{G}[p]} \left|\frac{P^{(\varepsilon)}_g(y_{1:a},\varepsilon k)}{Q^{(\varepsilon)}_g(y_{1:a},\varepsilon k)}\right|^{1+\delta} I^{(\varepsilon)}_g(y_{1:a},\varepsilon k)\,.\end{aligned} \tag{2.104}$$

Note that, as anticipated, the integral at the right hand side of (2.104) is the same as that in (2.102), but in which the modulus of the integrand is raised to the power $1+\delta$. Now, for any $\delta\in(0,(d-2)/2)$, the quantity at the left hand side is bounded uniformly in ε (but the bound might depend on m and a, which is though irrelevant as they are fixed) since $\mathcal{U}^{\varepsilon}_p$ is just a composition of $U^{\varepsilon,\pm} = (-\mathscr{L}_0)^{-(1+\delta)/2}\,\mathscr{A}^{\varepsilon,\delta}_{\pm}(-\mathscr{L}_0)^{-(1+\delta)/2}$. In turn, these operators are uniformly bounded on ΓL^2 by (2.20), which means that so is the integral at the right hand side of (2.104). Hence, the uniform integrability of the integrand in (2.102) is proven and (2.83) follows at once.

Next, we move to the proof of (2.84). First of all, the scalar product is zero unless $q_1+\cdots+q_m=k$ (by translation invariance) and $p\in\Pi^{(n)}_{a,m'}$ with $m'=m$ (by orthogonality of Wiener chaoses). We assume therefore that these conditions are satisfied. If $p\in\Pi^{(n)}_{a,m}$, the kernel of $\mathcal{T}^{\varepsilon}_p e_k$ is of the form (2.92), with F^{ε}_p as in Lemma 2.17. The scalar product in (2.84) is then

$$\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{T}^{\varepsilon}_p e_k\rangle = \varepsilon^{d/2(m-1)} F^{\varepsilon}_p(\varepsilon q_{1:m};\varepsilon k).$$

with $q_1+\cdots+q_m=k$. By Lemma 2.17 and arguing as in the derivation of (2.102), we get

$$\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{T}^{\varepsilon}_p e_k\rangle = 2^{dM}\frac{(\mathfrak{w}\cdot k)}{|k|}\int \mu(\mathrm{d}y_{1:M}) W^{\varepsilon}_p(y_{1:M},\varepsilon q_{1:m})\,, \tag{2.105}$$

where

$$W^{\varepsilon}_p(\ell_{1:M},\varepsilon q_{1:m}) \stackrel{\text{def}}{=} \varepsilon^{d/2(m-1)} \sum_{g\in\mathcal{G}[p]} \frac{\iota^a}{|q_{1:m}|}\frac{P^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})}{Q^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})} I^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m}).$$

Let us stress that the main difference w.r.t. (2.102) is the presence of the factor $\varepsilon^{d/2(m-1)}$, which tends to zero as $\varepsilon\to 0$, because $m\geq 2$, and this is the fact we

want to leverage. For this, we will consider once again the operators $U^{\varepsilon,\pm}$ and their composition $\mathcal{U}^\varepsilon$. Let $\delta \in (0,(d-2)/2)$. Proceeding as in (2.104), we see that

$$\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{U}^\varepsilon_p e_k\rangle = 2^{dM}\frac{|\mathfrak{w}\cdot k|^{1+\delta}}{|k|^{1+\delta}}\times \tag{2.106}$$
$$\times\int \mu(\mathrm{d}y_{1:M})\Big[\varepsilon^{d/2(m-1)}\sum_{g\in\mathcal{G}[p]}\Big|\frac{1}{|q_{1:m}|}\frac{P^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})}{Q^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})}\Big|^{1+\delta} I^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})\Big].$$

Now, notice that, Jensen's inequality (which holds as μ is a probability measure) applied at the right hand side of (2.105) gives

$$\begin{aligned}
&|\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{T}^\varepsilon_p e_k\rangle|^{1+\delta}\\
&\lesssim \frac{|\mathfrak{w}\cdot k|^{1+\delta}}{|k|^{1+\delta}}\int \mu(\mathrm{d}y_{1:M})|W^\varepsilon_p(y_{1:M},\varepsilon q_{1:m})|^{1+\delta}\\
&\lesssim \varepsilon^{d/2(m-1)\delta}\frac{|\mathfrak{w}\cdot k|^{1+\delta}}{|k|^{1+\delta}}\\
&\times\int \mu(\mathrm{d}y_{1:M})\Big[\varepsilon^{d/2(m-1)}\sum_{g\in\mathcal{G}[p]}\Big|\frac{1}{|q_{1:m}|}\frac{P^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})}{Q^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})}\Big|^{1+\delta} I^{(\varepsilon)}_g(y_{1:M},\varepsilon q_{1:m})\Big]\\
&\lesssim \varepsilon^{d/2(m-1)\delta}\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{U}^\varepsilon_p e_k\rangle \lesssim \varepsilon^{d/2(m-1)\delta}
\end{aligned}$$

where we used (2.106) and, in the last step, that, since by Lemma 2.4, the operators $U^{\varepsilon,\pm}$ are uniformly bounded (in ε) in ΓL^2, so is the scalar product $\langle e_{-q_1}\otimes\cdots\otimes e_{-q_m}, \mathcal{U}^\varepsilon_p e_k\rangle$. Hence, (2.84) follows at once and the proof of Proposition 2.16 is completed. □

3 Convergence of Burgers to renormalised SHE

Thanks to the results in the previous section, we have all the ingredients we need in order to prove Theorem 1.3. To do so, we will first show that the sequence η^ε is tight and then verify that every limit point solves the stationary stochastic heat equation (SHE)

$$\partial_t\eta = \tfrac{1}{2}(\Delta + D_{\mathrm{SHE}}(\mathfrak{w}\cdot\nabla)^2)\eta + (-\Delta - D_{\mathrm{SHE}}(\mathfrak{w}\cdot\nabla)^2)^{1/2}\xi\,, \qquad \eta(0,\cdot)=\mu \tag{3.1}$$

where μ is a zero-average space white noise and $D_{\mathrm{SHE}}>0$ is the constant identified in Theorem 2.13 for $d=2$ and Theorem 2.14 for $d\geq 3$.

3.1 Tightness

The proof of tightness follows a by now standard route (see e.g. [GJ13, GT20, CK23, CES21]) which exploits Mitoma's criterion [Mit83], Kolmogorov's continuity theorem and, more importantly, the so-called Itô trick, introduced in [GJ13] and since then exploited in a variety of contexts, see [Gub19] for a pedagogical introduction. For the reader's convenience, we recall below the statement of the latter and refer the reader to [CES21, Lemma 4.1] for a proof.

Lemma 3.1 (Itô trick) *Let $d \geq 2$, η^ε be the stationary solution to (1.5) with λ_ε given as in (1.6). For any $p \geq 2$, $T > 0$ and $F \in L^2(\Omega)$ with finite chaos expansion, i.e. $F \in \oplus_{j=1}^n \mathscr{H}_j$ for some $n \in \mathbb{N}$, there exists a constant $C = C(p,n) > 0$ such that*

$$\mathbf{E}\left[\sup_{t\in[0,T]}\left|\int_0^t F(\eta_s^\varepsilon)\mathrm{d}s\right|^p\right]^{1/p} \leq CT^{1/2}\|(-\mathscr{L}_0)^{-1/2}F\|\,. \tag{3.2}$$

We now move to the proof of tightness.

Proposition 3.2 *The sequence $\{\eta^\varepsilon\}_\varepsilon$ is tight in $C([0,T],\mathcal{S}'(\mathbb{T}^d))$.*

Proof. By Mitoma's criterion [Mit83], the sequence $\{\eta^\varepsilon\}_\varepsilon$ is tight in $C([0,T],\mathcal{S}'(\mathbb{T}^d))$ if and only if $\{\eta^\varepsilon(\varphi)\}_\varepsilon$ is tight in $C([0,T],\mathbb{R})$ for all $\varphi \in \mathcal{S}(\mathbb{T}^d)$. Therefore, we fix $\varphi \in \mathcal{S}(\mathbb{T}^d)$ and consider the process $\{\eta_t^\varepsilon(\varphi)\}_{t\in[0,T]}$. In view of (2.1), it suffices to show that each of the terms at the right hand side is tight. This is clear for the last, as it is independent of ε, while for the first and second, Lemma 3.1 implies that, for every $0 \leq t \leq T$ and $p > 2$, we have

$$\mathbf{E}\left[\left(\int_0^t \eta_s^\varepsilon(\Delta\varphi)\mathrm{d}s\right)^p\right]^{\frac{1}{p}} \lesssim t^{1/2}\|(-\mathscr{L}_0)^{-1/2}(\Delta\varphi)\| = t^{1/2}\|\varphi\|_{H^1(\mathbb{T}^d)} \tag{3.3}$$

and

$$\mathbf{E}\left[\left(\int_0^t \mathscr{N}_\varphi^\varepsilon(\eta^\varepsilon)\mathrm{d}s\right)^p\right]^{\frac{1}{p}} \lesssim t^{1/2}\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}_\varphi^\varepsilon\| \lesssim t^{1/2}\|\varphi\|_{H^1(\mathbb{T}^d)} \tag{3.4}$$

where in the last step we used (2.48). Since the process $\{\eta_t^\varepsilon(\varphi)\}_{t\in[0,T]}$ is Markov, (3.3) and (3.4) together with Kolmogorov's criterion ensure that the sequence $\{\eta^\varepsilon(\varphi)\}_\varepsilon$ is tight in $C([0,T],\mathbb{R})$, so that the proof of the statement is concluded. □

3.2 Convergence

In order to identify the limit equation, we will prove that any limit point satisfies the martingale problem associated to (3.1) which we now recall.

Definition 3.3 Let $T > 0$, $\Omega = C([0,T],\mathcal{S}'(\mathbb{T}^d))$ and $\mathscr{B}$ the canonical Borel σ-algebra on $C([0,T],\mathcal{S}'(\mathbb{T}^d))$. Let μ be a zero-average space white noise on $\mathbb{T}^d$. We say that a probability measure $\mathbf{P}$ on $(\Omega,\mathcal{G})$ *solves the martingale problem for* $\mathscr{L}^{\text{eff}} \overset{def}{=} \mathscr{L}_0 + \mathscr{D} = \frac{1}{2}(\Delta + D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)$ *with initial distribution* μ, if for all $\varphi \in \mathcal{S}(\mathbb{T}^d)$, the canonical process η under $\mathbf{P}$ is such that

$$\begin{aligned}\mathbf{M}_t(\varphi) &\overset{\text{def}}{=} \eta_t(\varphi) - \mu(\varphi) - \int_0^t \mathscr{L}^{\text{eff}}\eta_s(\varphi)\mathrm{d}s\\ \Gamma_t(\varphi) &\overset{\text{def}}{=} (\mathbf{M}_t(\varphi))^2 - t\,\|(-\Delta - D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)^{1/2}\varphi\|^2_{L^2(\mathbb{T}^2)}\end{aligned} \tag{3.5}$$

are local martingales.

In the next theorem, we state well-posedness for the previous martingale problem. The proof is classical (see for example [MW17, Theorem D.1]) therefore we omit it.

Theorem 3.4 *The martingale problem for $\mathscr{L}^{\text{eff}}$ with initial distribution μ in Definition 3.3 has a unique solution and uniquely characterises the law of the solution to* (3.1) *on* $C([0,T],\mathcal{S}'(\mathbb{T}^d))$.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Proposition 3.2, we know that the sequence $\{\eta^\varepsilon\}_\varepsilon$ is tight in $C([0,T],\mathcal{S}'(\mathbb{T}^d))$, hence it converges along subsequences. If we prove that any limit point is a solution of the martingale problem in Definition 3.3 then the statement follows by Theorem 3.4.

Let $\eta \in C([0,T],\mathcal{S}'(\mathbb{T}^d))$ be a limit point. To verify that η satisfies the martingale problem, we need to show that for every given $\varphi \in \mathcal{S}(\mathbb{T}^d)$ the processes $\mathbf{M}(\varphi)$ and $\Gamma(\varphi)$, defined according to (3.5), are local martingales. Notice that, by definition of quadratic variation, $\Gamma(\varphi)$ is a local martingale if and only if $\mathbf{M}(\varphi)$ has quadratic variation given by $t \mapsto t\|(-\Delta - D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)^{1/2}\varphi\|^2_{L^2(\mathbb{T}^2)}$, and we will prove this latter statement. Before delving into the details, we proceed as in Section 1.3 and write the equation for η^ε in such a way that we can identify the elements which are relevant to the limit.

Let us fix a smooth test function $\varphi \in \mathcal{S}(\mathbb{T}^d)$. Recall the weak formulation of (1.5) given in (2.1), i.e.

$$\eta^\varepsilon_t(\varphi) - \mu(\varphi) - \tfrac{1}{2}\int_0^t \eta^\varepsilon_s(\Delta\varphi)\mathrm{d}s - \int_0^t \mathcal{N}^\varepsilon_\varphi(\eta^\varepsilon_s)\mathrm{d}s = M_t(\varphi) \tag{3.6}$$

where $M_t(\varphi)$ is the martingale

$$M_t(\varphi) = \int_0^t \xi(\mathrm{d}s,(-\Delta)^{\frac{1}{2}}\varphi) = \sum_k |k|\hat\varphi(k)B_t(k) = \int_0^t \sum_k |k| D_k\eta^\varepsilon_s(\varphi)\mathrm{d}B_s(k) \tag{3.7}$$

and in the last step, we used the notion of Malliavin derivative in (1.21).

In order to describe the term containing the nonlinearity, we consider the random variable $\mathscr{W}^{\varepsilon,n} \in L^2(\mathbb{P})$, for $n \in \mathbb{N}$, which has kernel $w^{\varepsilon,n}$ given by the solution $v^{\varepsilon,n}$ of the truncated generator equation (2.45) if $d \geq 3$ and by $\tilde v^{\varepsilon,n}$ as in (2.46) if $d = 2$. By Itô's formula applied to $\mathscr{W}^{\varepsilon,n}$, we have

$$\mathscr{W}^{\varepsilon,n}(\eta^\varepsilon_t) - \mathscr{W}^{\varepsilon,n}(\mu) - \int_0^t \mathscr{L}^\varepsilon\mathscr{W}^{\varepsilon,n}(\eta^\varepsilon_s)\mathrm{d}s = \mathscr{M}^\varepsilon_t(\mathscr{W}^{\varepsilon,n}) \tag{3.8}$$

where $\mathscr{M}^\varepsilon(\mathscr{W}^{\varepsilon,n})$ is the martingale given by

$$\mathscr{M}^\varepsilon_t(\mathscr{W}^{\varepsilon,n}) = \int_0^t \sum_k |k|[D_k\mathscr{W}^{\varepsilon,n}](\eta^\varepsilon_s)dB_s(k)\,. \tag{3.9}$$

We now go back to (3.6). In view of Theorem 2.8, we rewrite it as

$$\eta^\varepsilon_t(\varphi) - \mu(\varphi) - \tfrac{1}{2}\int_0^t \eta^\varepsilon_s\Big((\Delta + \mathscr{D})\varphi\Big)\mathrm{d}s + R^{\varepsilon,n}(t) = M_t(\varphi) + \mathscr{M}^\varepsilon_t(\mathscr{W}^{\varepsilon,n}) \quad (3.10)$$

where $R^{\varepsilon,n}(t) \stackrel{\text{def}}{=} \text{(I)} + \text{(II)} + \text{(III)}$ and

$$\begin{aligned}
\text{(I)} &\stackrel{\text{def}}{=} \mathscr{W}^{\varepsilon,n}(\eta^\varepsilon_t) - \mathscr{W}^{\varepsilon,n}(\mu)\\
\text{(II)} &\stackrel{\text{def}}{=} -\int_0^t \Big(\mathscr{L}^\varepsilon \mathscr{W}^{\varepsilon,n} + \mathscr{N}^\varepsilon_\varphi - \mathscr{A}^\varepsilon_- \mathscr{W}^{\varepsilon,n}_2\Big)(\eta^\varepsilon_s)\mathrm{d}s\\
\text{(III)} &\stackrel{\text{def}}{=} -\int_0^t \Big(\mathscr{A}^\varepsilon_- \mathscr{W}^{\varepsilon,n}_2(\eta^\varepsilon_s) - \eta^\varepsilon_s(\mathscr{D}\varphi)\Big)\mathrm{d}s\,.
\end{aligned}$$

Notice that

$$\mathbf{E}[(\text{I})^2] \lesssim \mathbf{E}[\mathscr{W}^{\varepsilon,n}(\eta_t)^2] + \mathbf{E}[\mathscr{W}^{\varepsilon,n}(\mu)^2] = 2\mathbb{E}[\mathscr{W}^{\varepsilon,n}(\eta)^2] = 2\|w^{\varepsilon,n}\|^2 \quad (3.11)$$

$$\begin{aligned}
\mathbf{E}[(\text{III})^2] &\lesssim t^2\mathbf{E}\Big[\frac{1}{t}\int_0^t \Big(\mathscr{A}^\varepsilon_- \mathscr{W}^{\varepsilon,n}_2(\eta^\varepsilon_s) - \eta^\varepsilon_s(\mathscr{D}\varphi)\Big)^2\mathrm{d}s\Big]\\
&= t^2\|\mathscr{A}^\varepsilon_- w^{\varepsilon,n}_2 - \mathscr{D}\varphi\|^2 \qquad (3.12)
\end{aligned}$$

where the second follows by Jensen's inequality, and, by the Itô trick, Lemma 3.1, we have

$$\mathbf{E}[(\text{II})^2] \lesssim t\|(-\mathscr{L}_0)^{-\frac{1}{2}}(\mathscr{L}^\varepsilon w^{\varepsilon,n} + \mathfrak{n}^\varepsilon_\varphi - \mathscr{A}^\varepsilon_- w^{\varepsilon,n}_2)\|^2\,. \quad (3.13)$$

The right hand sides of (3.11), (3.12) and (3.13) all converge to 0 in the double limit as $\varepsilon \to 0$ first and $n \to \infty$ then, (3.13) in view of Theorem 2.8, while (3.11) and (3.12) by Proposition 2.15 and (2.75) in Theorem 2.14 for $d \geq 3$, and by Theorem 2.13 for $d = 2$. Hence, $R^{\varepsilon,n}$ converges to 0 in $L^2(\Omega)$ and consequently, almost surely (at least along subsequences).

It follows that the left hand side of (3.10) converges to

$$\eta_t(\varphi) - \mu(\varphi) - \tfrac{1}{2}\int_0^t \eta_s\Big((\Delta + D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)\varphi\Big)\mathrm{d}s$$

which by definition, equals $\mathbf{M}_t(\varphi)$. Hence, we can determine a common subsequence such that the convergence holds for any t in a countable dense set of $[0,T]$. Since tightness holds in $C([0,T],\mathbb{R})$, $\mathbf{M}(\varphi)$ is the almost sure pointwise (in time) limit of the sequence of martingales $\{M_t(\varphi) + \mathscr{M}^\varepsilon_t(\mathscr{W}^{\varepsilon,n})\}_\varepsilon$. Hence, it suffices to show that the sequence $\{M_t(\varphi) + \mathscr{M}^\varepsilon_t(\mathscr{W}^{\varepsilon,n})\colon t \in [0,T]\}_\varepsilon$ converges to a martingale whose quadratic variation is $t \mapsto t\|(-\Delta - D_{\text{SHE}}(\mathfrak{w}\cdot\nabla)^2)^{1/2}\varphi\|^2_{L^2(\mathbb{T}^2)}$. This in turn is the content of Theorem 3.5 in the next section, hence the proof of Theorem 1.3 is concluded. □

3.3 The Martingales

In this section, we want to prove that for any $\varphi \in \mathcal{S}(\mathbb{T}^d)$, the martingale part at the right hand side of (3.10) converges to a Brownian motion with a deterministic quadratic variation which can be suitably expressed in terms of the operators $\mathscr{L}_0$ and $\mathscr{D}$. To do so, recall first that by the definition of $M_t(\varphi)$ and $\mathscr{M}_t^\varepsilon(\mathscr{W}^{\varepsilon,n})$ in (3.7) and (3.9) respectively, we have

$$\mathbf{M}_t^{n,\varepsilon}(\varphi) \stackrel{\text{def}}{=} M_t(\varphi) + \mathscr{M}_t^\varepsilon(w^{\varepsilon,n}) = \int_0^t \sum_k |k| D_k \Big(\eta_s^\varepsilon(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta_s^\varepsilon) \Big) \mathrm{d}B_s(k)\,, \tag{3.14}$$

where $\mathscr{W}^{\varepsilon,n} \in L^2(\mathbb{P})$ is the random variable whose kernel $w^{\varepsilon,n} \in \Gamma L^2$ is either $v^{\varepsilon,n}$, the solution of the truncated generator equation in (2.45), for $d \geq 3$ or $\tilde{v}^{\varepsilon,n}$ given by (2.46) if $d = 2$. The quadratic variation of $\mathbf{M}_\cdot^{n,\varepsilon}(\varphi)$ is

$$\langle \mathbf{M}_t^{n,\varepsilon}(\varphi) \rangle_t = \int_0^t F^{\varepsilon,n}(\eta_s) \mathrm{d}s\,. \tag{3.15}$$

for $F^{\varepsilon,n}(\eta)$ the random variable defined as

$$F^{\varepsilon,n}(\eta) = \sum_k |k|^2 |D_k(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta))|^2\,. \tag{3.16}$$

Our goal is to prove the following theorem.

Theorem 3.5 *For $\varphi \in \mathcal{S}(\mathbb{T}^d)$, $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\mathscr{W}^{\varepsilon,n}$ be the random variable whose kernel $w^{\varepsilon,n} \in \Gamma L^2$ is either $v^{\varepsilon,n}$, the solution of the truncated generator equation in* (2.45)*, for $d \geq 3$, or $\tilde{v}^{\varepsilon,n}$ given by* (2.46) *if $d = 2$. Then, in the limit as $\varepsilon \to$ first and $n \to \infty$ after, the sequence of martingales $\{\mathbf{M}^{n,\varepsilon}(\varphi)\}_{n,\varepsilon}$ defined in* (3.14) *converges in law to a Brownian motion whose quadratic variation is given by*

$$t \quad \mapsto \quad 2t\langle \varphi, (-\mathscr{L}_0 - \mathscr{D})\varphi \rangle = t \|(-\Delta - D_{\mathrm{SHE}}(\mathfrak{w} \cdot \nabla)^2)^{1/2} \varphi\|^2 \tag{3.17}$$

where the operator $\mathscr{D}$ was defined in Theorem 2.14 for $d \geq 3$ and in Theorem 2.13 if $d = 2$.

The proof of the previous statement is a direct consequence of the following propositions. In the first, we show that the expectation of (3.15) converges to (3.17).

Proposition 3.6 *In the same setting as in Theorem 3.5, let $\mathbf{M}_t^{n,\varepsilon}(\varphi)$ be the martingale in* (3.14)*. Then, for any $t > 0$*

$$\lim_{n\to\infty} \lim_{\varepsilon \to 0} \mathbf{E} \langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi) \rangle_t = 2t\langle \varphi, (-\mathscr{L}_0 - \mathscr{D})\varphi \rangle\,. \tag{3.18}$$

Proof. By (3.15) and stationarity, we have

$$\mathbf{E} \langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi) \rangle_t = \int_0^t \mathbf{E}[F^{\varepsilon,n}(\eta_s)] \mathrm{d}t = t\mathbb{E}[F^{\varepsilon,n}(\eta)]\,.$$

so that we only need to focus on the latter expectation. To compute it, we notice first that for any random variable $G \in L^2(\mathbb{P})$, Gaussian integration by parts (1.23) gives

$$\mathbb{E}[G(\eta)D_{-k}D_kG(\eta)] = \mathbb{E}[G(\eta)\hat{\eta}(-k)D_kG(\eta)] - \mathbb{E}\big[|D_kG(\eta)|^2\big]$$

so that, by (2.4), we get

$$\begin{aligned}\mathbb{E}[G(\eta)(-\mathscr{L}_0)G(\eta)] &= \frac{1}{2}\sum_k |k|^2\mathbb{E}[G(\eta)(\eta_{-k}D_kG(\eta) - D_{-k}D_kG(\eta))] \\ &= \frac{1}{2}\sum_k |k|^2\mathbb{E}\big[|D_kG(\eta)|^2\big]\,.\end{aligned}$$

As a consequence, we see that

$$\tfrac{1}{2}\mathbb{E}[F^{\varepsilon,n}(\eta)] = \|(-\mathscr{L}_0)^{\frac{1}{2}}(\varphi+w^{\varepsilon,n})\|^2 = \|(-\mathscr{L}_0)^{\frac{1}{2}}\varphi\|^2+\|(-\mathscr{L}_0)^{\frac{1}{2}}w^{\varepsilon,n}\|^2\,, \quad (3.19)$$

where the last step holds as $w^{\varepsilon,n} \in \oplus_{j=2}^n \Gamma L_j^2$, while $\varphi \in \Gamma L_1^2$, so that in particular the two are orthogonal. For $d \geq 3$, the result then follows immediately by Theorem 2.14.

For $d = 2$ (recall that $w^{\varepsilon,n} = \tilde{v}^{\varepsilon,n}$ in (2.46)), by (2.54), we can write

$$\begin{aligned}\|(-\mathscr{L}_0)^{\frac{1}{2}}\tilde{v}^{\varepsilon,n}\|^2 =& \langle \tilde{v}^{\varepsilon,n}, (-\mathscr{L}^\varepsilon)\tilde{v}^{\varepsilon,n}\rangle \\ =& \langle \tilde{v}^{\varepsilon,n}, \mathfrak{n}^\varepsilon_\varphi\rangle + \langle \tilde{v}^{\varepsilon,n}_n, \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon\tilde{v}^{\varepsilon,n}_n\rangle \\ &+ \langle \tilde{v}^{\varepsilon,n}, [\,-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+ + \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon]P_2^{n-1}\tilde{v}^{\varepsilon,n}\rangle\end{aligned}$$

where we removed $-\mathscr{A}^\varepsilon_-(-\mathscr{L}_0-\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathfrak{n}^\varepsilon_\varphi$ and $\mathscr{A}^\varepsilon_+\tilde{v}^{\varepsilon,n}_n$ as they are orthogonal to $\tilde{v}^{\varepsilon,n}$. Now, the second term at the right hand side only involves the n-th coordinate of $\tilde{v}^{\varepsilon,n}$ so that it goes to 0 as $n \to \infty$ by the weighted a priori estimates in Lemma 2.11. The last term can be bounded via the replacement Lemma 2.6, and after controlling $\|(-\mathscr{L}_0)^{1/2}\tilde{v}^{\varepsilon,n}\|$ with Lemma 2.10 first and (2.48) then, we can conclude that, for any n fixed, it converges to 0 as $\varepsilon \to 0$. Hence, it remains to consider the first term for which, by (2.64), we have

$$\langle \tilde{v}^{\varepsilon,n}, \mathfrak{n}^\varepsilon_\varphi\rangle = \langle \tilde{v}^{\varepsilon,n}_2, \mathscr{A}^\varepsilon_+\varphi\rangle = \langle -\mathscr{A}^\varepsilon_-\tilde{v}^{\varepsilon,n}_2, \varphi\rangle,$$

and the right hand side converges to $\langle \varphi, (-\mathscr{D})\varphi\rangle$ by (2.70) in Theorem 2.13. □

It remains to show that the quadratic variation of $\mathbf{M}^{n,\varepsilon}_\cdot(\varphi)$ concentrates around its mean. Since we always take the limit in ε before that in n, from now on we will fix n and prove that, as $\varepsilon \to 0$, the variance of $\langle \mathbf{M}^{n,\varepsilon}_\cdot(\varphi)\rangle$ converges to 0. To do so, we will actually prove a stronger statement, namely that the sequence of random variables $F^{\varepsilon,n}$ in (3.16) is asymptotically deterministic.

Proposition 3.7 *Let $n \in \mathbb{N}$ be fixed and for $\varepsilon > 0$, let $F^{\varepsilon,n}(\eta)$ be the random variable defined according to* (3.16)*. Then,*

$$\lim_{\varepsilon\to 0} \mathrm{Var}(F^{\varepsilon,n}(\eta)) = 0\,. \quad (3.20)$$

Proof. Notice that, by Poincaré inequality for i.i.d. Gaussian measures, we have

$$\mathrm{Var}(F^{\varepsilon,n}(\eta)) \le \mathbb{E}[\|DF^{\varepsilon,n}(\eta)\|^2_{L^2(\mathbb{T}^d)}] = \mathbb{E}\left[\sum_{p\in\mathbb{Z}^d_0} |D_p F^{\varepsilon,n}(\eta)|^2\right] = \sum_{p\in\mathbb{Z}^d_0} \|D_p F^{\varepsilon,n}\|^2 ,$$

where the last step follows by monotone convergence theorem. Therefore, if we were able to show that the expectation of the norm of the Malliavin derivative of $F^{\varepsilon,n}(\eta)$ vanished, we would be done. In what follows, we will indeed prove that, for any $p \in \mathbb{Z}^d_0$ fixed, $\|D_p F^{\varepsilon,n}\|$ goes to 0, but exchanging the limit in ε and the sum in $p \in \mathbb{Z}^d_0$ would require a much stronger control. To bypass the issue, we introduce a martingale approximation $\{M^{\varepsilon,n}_\ell\}_\ell$ of $F^{\varepsilon,n}$, and to prove (3.20), we show that for ℓ fixed and $\varepsilon \to 0$, $M^{\varepsilon,n}_\ell$ converges to a constant, and that the limit in L^2 of the difference between $M^{\varepsilon,n}_\ell$ and $F^{\varepsilon,n}$ vanishes in the double limit, first in ε and then in $\ell \to \infty$. We will now detail the idea presented above.

For $\ell \in \mathbb{N}$, denote by $\mathscr{F}_\ell$ the σ-algebra generated by $\{\hat\eta(k)\colon |k| \le \ell\}$ and define

$$M^{\varepsilon,n}_\ell \stackrel{\text{def}}{=} \mathbb{E}[F^{\varepsilon,n}|\mathscr{F}_\ell] .$$

Clearly, $\{M^{\varepsilon,n}_\ell\}_\ell$ is a martingale in ℓ and for all ℓ

$$\mathbb{E}[M^{\varepsilon,n}_\ell] = \mathbb{E}[F^{\varepsilon,n}] . \tag{3.21}$$

We now bound the variance of $F^{\varepsilon,n}$ by

$$\mathrm{Var}(F^{\varepsilon,n}) = \mathbb{E}[(F^{\varepsilon,n} - \mathbb{E}[F^{\varepsilon,n}])^2] \lesssim \mathbb{E}[(F^{\varepsilon,n} - M^{\varepsilon,n}_\ell)^2] + \mathbb{E}[(M^{\varepsilon,n}_\ell - \mathbb{E}[M^{\varepsilon,n}_\ell])^2]$$

where we used (3.21). Notice that upon proving

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} \mathbb{E}[(F^{\varepsilon,n} - M^{\varepsilon,n}_\ell)^2] = 0 \tag{3.22}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}[(M^{\varepsilon,n}_\ell - \mathbb{E}[M^{\varepsilon,n}_\ell])^2] = 0 \tag{3.23}$$

the conclusion follows, and this is what we will show next.

Proof of (3.22) At first we want to ensure that, as $\varepsilon \to 0$, at least along subsequences, $F^{\varepsilon,n}$ and $M^{\varepsilon,n}_\ell$ converge to square integrable random variables.

Now, $F^{\varepsilon,n}$ lives in the n-th inhomogeneous chaos, i.e. in $\oplus_{j\le n}\mathscr{H}_j$, hence Gaussian hypercontractivity [Jan97, Theorem 5.10] implies that, for any $p \ge 1$ there exists a constant $C = C(n,p) > 0$ such that

$$\mathbb{E}[(F^{\varepsilon,n})^p]^{\frac{1}{p}} \le C\,\mathbb{E}[F^{\varepsilon,n}] .$$

By (3.19) and Proposition 2.9 for $d \ge 3$ and Lemma 2.10 for $d = 2$, the expectation is bounded by

$$\mathbb{E}[F^{\varepsilon,n}] = 2\|(-\mathscr{L}_0)^{\frac{1}{2}}(\varphi + w^{\varepsilon,n})\|^2 = 2\|\varphi\|^2_{H^1(\mathbb{T}^d)} + 2\|(-\mathscr{L}_0)^{\frac{1}{2}} w^{\varepsilon,n}\|^2$$

$$\lesssim \|\varphi\|^2_{H^1(\mathbb{T}^d)} + \|(-\mathscr{L}_0)^{-\frac{1}{2}}\mathfrak{n}^\varepsilon_\varphi\|^2 \lesssim \|\varphi\|^2_{H^1(\mathbb{T}^d)}$$

where in the last step we used (2.48).

Hence, uniformly in ε, $F^{\varepsilon,n}$ has bounded moments of all orders, which ensures that the sequence $\{F^{\varepsilon,n}\}_\varepsilon$ is tight. As a consequence, along subsequences (that, abusing the notation, will still be denoted by $\{F^{\varepsilon,n}\}_\varepsilon$) $\{F^{\varepsilon,n}\}_\varepsilon$ can be embedded in a common probability space and there exists F^n such that $\{F^{\varepsilon,n}\}_\varepsilon$ converges almost surely and in $L^2(\mathbb{P})$ to F^n, so that in particular

$$\lim_{\varepsilon\to 0} \mathbb{E}[(F^{\varepsilon,n} - F^n)^2] = 0\,. \tag{3.24}$$

Taking the same subsequence (in ε) for $M^{\varepsilon,n}_\ell$, by Jensen, we immediately have that $M^{\varepsilon,n}_\ell$ has uniformly bounded moments in ℓ and ε, and that, along this subsequence,

$$\lim_{\varepsilon\to 0} \mathbb{E}[(M^{\varepsilon,n}_\ell - M^n_\ell)^2] = 0$$

where M^n_ℓ is given by $M^n_\ell = \mathbb{E}[F^n|\mathscr{F}_\ell]$. Notice that, by definition $\{M^n_\ell\}_\ell$ is again a square-integrable martingale. Hence, by Martingale convergence theorem, we conclude that, as $\ell \to \infty$ M^n_ℓ converges in $L^2(\mathbb{P})$ to $\mathbb{E}[F^n|\mathscr{F}_\infty] = F^n$. Summarising, we showed that

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} \mathbb{E}[(F^{\varepsilon,n} - M^{\varepsilon,n}_\ell)^2] \lesssim \lim_{\ell\to\infty} \mathbb{E}[(F^n - M^n_\ell)^2] = 0$$

along the above-mentioned subsequence in ε. However, we also get (3.22) along *any* sequence $\{\varepsilon_k\}$ converging to zero, simply because (by exactly the same argument as above) for any such sequence one can extract a subsequence for which the convergence holds.

Proof of (3.23) Poincaré's inequality implies

$$\mathbb{E}[(M^{\varepsilon,n}_\ell - \mathbb{E}[M^{\varepsilon,n}_\ell])^2] \le \mathbb{E}[\|DM^{\varepsilon,n}_\ell\|^2_{L^2(\mathbb{T}^d)}] = \sum_{|p|\le\ell} \|D_p M^{\varepsilon,n}_\ell\|^2\,,$$

where the last equality follows by the fact that $M^{\varepsilon,n}_\ell$ is a random variable that only depends on $\{\hat\eta(k)\colon |k| \le \ell\}$ and therefore, for $|p| > \ell$, $D_p M^{\varepsilon,n}_\ell = 0$. Since ℓ is fixed, in order to prove that the right hand side above converges to 0, it suffices to show that, for every p fixed, $\lim_{\varepsilon\to 0} \|D_p M^{\varepsilon,n}_\ell\|$ vanishes. Note that

$$\begin{aligned}\|D_p M^{\varepsilon,n}_\ell\| &= \mathbb{E}[|D_p\mathbb{E}[F^{\varepsilon,n}|\mathscr{F}_\ell]|^2]^{1/2}\\ &= \mathbb{E}[|\mathbb{E}[D_p F^{\varepsilon,n}|\mathscr{F}_\ell]|^2]^{1/2} \le \mathbb{E}[|D_p F^{\varepsilon,n}|^2]^{1/2} = \|D_p F^{\varepsilon,n}\|\end{aligned}$$

so that we can focus on the latter. By (3.16), the p-th Fourier mode of the Malliavin derivative of $F^{\varepsilon,n}$ is given by

$$D_p F^{\varepsilon,n}(\eta) = \sum_k |k|^2 D_p |D_k(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta))|^2$$

$$= \sum_k |k|^2 \Big[D_k(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta)) D_p D_{-k}(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta)) \\ + D_p D_k(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta)) D_{-k}(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta)) \Big]$$

so that applying Cauchy-Schwarz in both $\mathbb{E}$ and the sum in k, we have

$$\|D_p F^{\varepsilon,n}\|^2 \lesssim \mathbb{E}\left[\sum_k |k|^2 |D_k(\eta(\varphi) + \mathscr{W}^{\varepsilon,n}(\eta))|^2\right] \mathbb{E}\left[\sum_k |k|^2 |D_k D_p \mathscr{W}^{\varepsilon,n}(\eta)|^2\right] \\ = 4\|(-\mathscr{L}_0)^{1/2}(\varphi + w^{\varepsilon,n})\|^2 \|(-\mathscr{L}_0)^{1/2} D_p w^{\varepsilon,n}\|^2 \,,$$

where we used the fact that $D_p D_k \eta(\varphi) = 0$ for any $p, k \in \mathbb{Z}_0^d$. Now, the first factor at the right hand side is uniformly bounded in ε (see the first part of the proof of (3.22)). For the second, we need some more refined estimates, which will be given in the next subsection. More specifically, in Proposition 3.10, we will show that

$$\lim_{\varepsilon \to 0} \|(-\mathscr{L}_0)^{1/2} D_p w^{\varepsilon,n}\|^2 = 0$$

and therefore the proof of (3.23) is complete. □

Thanks to the previous propositions, the proof of Theorem 3.5 is straightforward.

Proof of Theorem 3.5. $\{\mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\}_\varepsilon$ is a sequence of continuous martingales which has uniformly bounded (in ε) moments of all orders. Indeed, by Burkholder-Davis Gundy inequality, we have

$$\mathbf{E}[\sup_{s \le t} |\mathbf{M}_s^{n,\varepsilon}(\varphi)|^p] \lesssim_p \mathbf{E}\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t^{p/2} \lesssim_t \mathbb{E}[|F^{\varepsilon,n}(\eta)|^{p/2}]$$

where we used the definition of the quadratic variation of $\mathbf{M}_\cdot^{n,\varepsilon}(\varphi)$ in (3.15) and Jensen's inequality. Now, in the proof of Proposition 3.7, we showed that the right hand side is uniformly bounded in ε, from which the claim follows. According to [EK86, Theorem 7.1.4], to complete the proof it suffices to show that, for any $t > 0$ the sequence of quadratic variations $\{\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t\}_{\varepsilon,n}$ converges to the deterministic linear function of t given by (3.17). Let us abbreviate $Q(\varphi) \stackrel{\text{def}}{=} \langle \varphi, (-\mathscr{L}_0 - \mathscr{D})\varphi \rangle$. Since

$$\mathbf{E}\left[\Big(\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t - 2tQ(\varphi)\Big)^2\right] \\ \lesssim \mathrm{Var}(\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t) + \Big(\mathbb{E}[\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t] - 2tQ(\varphi)\Big)^2 \\ = \mathbb{E}\left[\Big(\int_0^t (F^{\varepsilon,n}(\eta_s) - \mathbb{E}[F^{\varepsilon,n}(\eta_s)])\mathrm{d}s\Big)^2\right] + \Big(\mathbb{E}\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t - 2tQ(\varphi)\Big)^2 \\ \lesssim_t \mathrm{Var}(F^{\varepsilon,n}(\eta)) + \Big(\mathbb{E}\langle \mathbf{M}_\cdot^{n,\varepsilon}(\varphi)\rangle_t - 2tQ(\varphi)\Big)^2 ,$$

by taking the limit in ε first, Proposition 3.7 implies that the first summand goes to 0, and taking the one in n after also the second summand vanishes, this time by Proposition 3.6. □

3.3.1 Bounds on the Malliavin derivative

The aim of this section is to show that, for any given p, $\|(-\mathscr{L}_0)^{\frac{1}{2}} D_p w^{\varepsilon,n}\|$ converges to 0 as $\varepsilon \to 0$, where $w^{\varepsilon,n} \in \Gamma L^2$ is either the solution $v^{\varepsilon,n}$ of the truncated generator equation in (2.45) for $d \geq 3$ or $\tilde{v}^{\varepsilon,n}$ given by (2.46) for $d = 2$. The proof boils down to obtain a priori estimates on the Malliavin derivative of $w^{\varepsilon,n}$ for which we need suitable bounds on the commutators between $\mathscr{L}_0$, $\mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon$, $\mathscr{A}_+^\varepsilon$ and $\mathscr{A}_-^\varepsilon$, and D_p, where we recall that, for two operators A, B the commutator between A and B is defined as $[A, B] \stackrel{\text{def}}{=} AB - BA$. In the next lemma, we identify an explicit expression for the action of the commutators above on functions in the Fock space.

Lemma 3.8 *Let $f \in \Gamma L_n^2$ and $\mathscr{A}_+^\varepsilon$, $\mathscr{A}_-^\varepsilon$ be the operators defined in* (2.9) *and $\mathcal{S}$ an operator which, for every n, maps ΓL_n^2 to itself and whose action on $\varphi \in \Gamma L_n^2$ is given in terms of the symmetric Fourier multiplier $\sigma_n \colon \mathbb{Z}_0^{dn} \to \mathbb{R}$ as*

$$\mathscr{F}(\mathcal{S}\varphi)(k_{1:n}) = \sigma_n(k_{1:n})\hat{\varphi}(k_{1:n})\,.$$

For any $p \in \mathbb{Z}_0^d$, the commutators $[\mathcal{S}, D_p]$, $[\mathscr{A}_+^\varepsilon, D_p]$, $[\mathscr{A}_-^\varepsilon, D_p]$ act on f as

$$\mathscr{F}([\mathcal{S}, D_p]f)(k_{1:n-1}) = (\sigma_{n-1}(k_{1:n-1}) - \sigma_n(k_{1:n-1}, p))\mathscr{F}(D_p f)(k_{1:n-1}) \quad (3.25)$$

$$\mathscr{F}([\mathscr{A}_+^\varepsilon, D_p]f)(k_{1:n}) = -2\iota\lambda_\varepsilon \sum_{i=1}^n [\mathfrak{w}\cdot(k_i+p)]\mathbb{J}_{k_i,p}^\varepsilon \hat{f}(k_i+p, k_{\{1:n\}\setminus\{i\}}) \quad (3.26)$$

$$\mathscr{F}([\mathscr{A}_-^\varepsilon, D_p]f)(k_{1:n-2}) = -\iota n(n-1)\lambda_\varepsilon\,(\mathfrak{w}\cdot p) \sum_{\ell+m=p} \mathbb{J}_{\ell,m}^\varepsilon \hat{f}(\ell, m, k_{1:n-2})\,. \quad (3.27)$$

Proof. Recall how the Malliavin derivative D_p acts on ΓL^2 in (1.22). Then,

$$\begin{aligned}\mathscr{F}([\mathcal{S}, D_p]f)(k_{1:n-1}) &= n(\sigma_{n-1}(k_{1:n-1}) - \sigma_n(k_{1:n-1}, p))f(k_{1:n-1}, p)\\ &= (\sigma_{n-1}(k_{1:n-1}) - \sigma_n(k_{1:n-1}, p))D_p f(k_{1:n-1})\,.\end{aligned}$$

We now turn to the commutator of $\mathscr{A}_+^\varepsilon$ and D_p. We have

$$\begin{aligned}\mathscr{F}(D_p\mathscr{A}_+^\varepsilon f)(k_{1:n}, p) =& 2\iota\lambda_\varepsilon \sum_{1\leq i<j\leq n} [\mathfrak{w}\cdot(k_i+k_j)]\mathbb{J}_{k_i,k_j}^\varepsilon \hat{f}(k_i+k_j, k_{\{1:n\}\setminus\{i,j\}}, p)\\ &+ 2\iota\lambda_\varepsilon \sum_{i=1}^n [\mathfrak{w}\cdot(k_i+p)]\mathbb{J}_{k_i,p}^\varepsilon \hat{f}(k_i+p, k_{\{1:n\}\setminus\{i\}})\\ =& \mathscr{F}(\mathscr{A}_+^\varepsilon D_p f)(k_{1:n}) + 2\iota\lambda_\varepsilon \sum_{i=1}^n [\mathfrak{w}\cdot(k_i+p)]\mathbb{J}_{k_i,p}^\varepsilon \hat{f}(k_i+p, k_{\{1:n\}\setminus\{i\}})\end{aligned}$$

from which (3.26) follows. At last, for the commutator of $\mathscr{A}_-^\varepsilon$ and D_p, note that

$$\mathscr{F}(D_p\mathscr{A}_-^\varepsilon f)(k_{1:n-2}) = \iota n(n-1)\lambda_\varepsilon \sum_{j=1}^{n-2}(\mathfrak{w}\cdot k_j) \sum_{\ell+m=k_j} \mathbb{J}_{\ell,m}^\varepsilon \hat{f}(\ell, m, k_{\{1:n-2\}\setminus\{j\}}, p)$$

$$+ \iota n(n-1)\lambda_\varepsilon\,(\mathfrak{w}\cdot p)\sum_{\ell+m=p}\mathbb{J}^\varepsilon_{\ell,m}\hat f(\ell,m,k_{1:n-2})$$
$$= \mathscr{F}(\mathscr{A}^\varepsilon_- D_p f)(k_{1:n-2}) + \iota n(n-1)\lambda_\varepsilon\,(\mathfrak{w}\cdot p)\sum_{\ell+m=p}\mathbb{J}^\varepsilon_{\ell,m}\hat f(\ell,m,k_{1:n-2})$$

which proves (3.27). □

We now determine bounds on the commutators in the previous lemma.

Lemma 3.9 *Let $n \geq 1$, $f \in \Gamma L^2_n$ and $p \in \mathbb{Z}^d_0$. Then, there exists a constant $C > 0$ independent of n, ε and p for which following estimates hold*

$$\|\mathcal{N}[\mathscr{A}^\varepsilon_+, D_p]f\| \leq C\lambda_\varepsilon \|\mathcal{N}^{3/2}(-\mathscr{L}_0)^{1/2} f\|\,, \tag{3.28}$$
$$\|\mathcal{N}[\mathscr{A}^\varepsilon_-, D_p]f\| \leq C|\mathfrak{w}\cdot p|\|\mathcal{N}^{5/2}(-\mathscr{L}_0)^{1/2} f\|\,,\,, \tag{3.29}$$

and

$$\|(-\mathscr{L}_0)^{-1/2}[-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon, D_p]f\|^2 \\ \leq C|p|^4\Big(\|\mathcal{N} f\|^2 + \lambda_\varepsilon^4\|\mathcal{N}(-\mathscr{L}_0)^{1/2} f\|^2\Big)\,. \tag{3.30}$$

Proof. We begin with (3.28). By (3.26), we have

$$\|\mathcal{N}[\mathscr{A}^\varepsilon_+, D_p]f\|^2 = 4n^2\lambda_\varepsilon^2\sum_{k_{1:n}}\Big|\sum_{i=1}^n[\mathfrak{w}\cdot(k_i+p)]\mathbb{J}^\varepsilon_{k_i,p}\hat f(k_i+p, k_{\{1:n\}\setminus\{i\}})\Big|^2$$
$$\leq 4n^3\lambda_\varepsilon^2\sum_{k_{1:n}}\sum_{i=1}^n[\mathfrak{w}\cdot(k_i+p)]^2|\hat f(k_i+p, k_{\{1:n\}\setminus\{i\}})|^2$$
$$\lesssim n^3\lambda_\varepsilon^2\sum_{k_{1:n}}\frac{1}{2}|k_{1:n}|^2|f(k_{1:n})|^2 = \lambda_\varepsilon^2\|\mathcal{N}^{3/2}(-\mathscr{L}_0)^{1/2} f\|^2\,.$$

For (3.29) instead, we have

$$\|\mathcal{N}[\mathscr{A}^\varepsilon_-, D_p]f\|^2 = n^4(n-1)^2(\mathfrak{w}\cdot p)^2\sum_{k_{1:n-2}}\Big|\lambda_\varepsilon\sum_{\ell+m=p}\mathbb{J}^\varepsilon_{\ell,m}\hat f(\ell,m,k_{1:n-2})\Big|^2$$
$$\leq n^4(n-1)^2(\mathfrak{w}\cdot p)^2\Big(\lambda_\varepsilon^2\sum_\ell\frac{\mathbb{J}^\varepsilon_{\ell,p-\ell}}{|\ell|^2+|p-\ell|^2}\Big)\times$$
$$\times\sum_{k_{1:n-1}}(|k_1|^2+|p-k_1|^2)|\hat f(k_1,p-k_1,k_{2:n-1})|^2$$

where we simply applied Cauchy-Schwarz. Now, the second sum at the right hand side is bounded uniformly in p and ε, hence the previous is bounded above by

$$n^4(n-1)^2(\mathfrak{w}\cdot p)^2\sum_{k_{1:n}}|k_1|^2|f(k_{1:n})|^2 \lesssim (\mathfrak{w}\cdot p)^2\|\mathcal{N}^{5/2}(-\mathscr{L}_0)^{1/2} f\|^2\,.$$

At last, we consider (3.30). By (3.25), we have

$$\begin{aligned}
&\|(-\mathscr{L}_0)^{-1/2}[-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon, D_p]f\|^2 \\
&\quad= \tfrac{n^2}{4}\sum_{k_{1:n-1}} \frac{1}{|k_{1:n-1}|^2}|f(k_{1:n-1},p)|^2\Big[|p|^2 + (\mathfrak{w}\cdot p)^2 G(\mathrm{L}^\varepsilon(\tfrac12|k_{1:n-1}|^2 + \tfrac12|p|^2)) \\
&\qquad + (\mathfrak{w}\cdot k)_{1:n-1}^2\Big(G(\mathrm{L}^\varepsilon(\tfrac12|k_{1:n-1}|^2 + \tfrac12|p|^2)) - G(\mathrm{L}^\varepsilon(\tfrac12(|k_{1:n-1}|^2)))\Big)\Big]^2 \\
&\quad\lesssim n^2 \sum_{k_{1:n-1}} |f(k_{1:n-1},p)|^2 \times \\
&\qquad\times\Big[|p|^4 + |k_{1:n}|^2\Big(G(\mathrm{L}^\varepsilon(\tfrac12|k_{1:n-1}|^2 + \tfrac12|p|^2)) - G(\mathrm{L}^\varepsilon(\tfrac12(|k_{1:n-1}|^2)))\Big)^2\Big]
\end{aligned}$$

where we used that G in (2.24) is locally bounded and L^ε in (2.23) is bounded uniformly in ε for $x \geq 1/2$. Now, to control the last term, we use that the function $x \mapsto G(\mathrm{L}^\varepsilon(x))$ has first derivative uniformly bounded by a constant times λ_ε^2 so that (3.30) follows at once. □

The following is the main result of this section, which completes the proof of Proposition 3.7.

Proposition 3.10 *For $n \in \mathbb{N}$ and $\varepsilon > 0$, let $w^{\varepsilon,n} \in \Gamma L^2$ be either the solution $v^{\varepsilon,n}$ of the truncated generator equation in* (2.45) *if $d \geq 3$ or $\tilde v^{\varepsilon,n}$ given by* (2.46) *if $d = 2$. Then, for any $p \in \mathbb{Z}_0^d$, there exists a constant $C = C(p) > 0$ such that*

$$\lim_{\varepsilon\to 0} \|(-\mathscr{L}_0)^{\frac12} D_p w^{\varepsilon,n}\|^2 = 0\,. \tag{3.31}$$

Proof. Let us separately consider the cases $d = 2$ and $d \geq 3$, starting with the former.

For $d = 2$, note that by (2.46) we have

$$\begin{aligned}
(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)D_p\tilde v^{\varepsilon,n} &= [-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon, D_p]\tilde v^{\varepsilon,n} + D_p(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)\tilde v^{\varepsilon,n} \\
&= [-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon, D_p]\tilde v^{\varepsilon,n} + D_p\mathscr{A}_+^\varepsilon P_2^{n-1}\tilde v^{\varepsilon,n} + D_p\mathfrak{n}_\varphi^\varepsilon \\
&= \mathscr{A}_+^\varepsilon P_2^{n-1} D_p\tilde v^{\varepsilon,n} + \psi^\sharp
\end{aligned} \tag{3.32}$$

where $\psi^\sharp$ is given by

$$\psi^\sharp \stackrel{\text{def}}{=} [-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon, D_p]\tilde v^{\varepsilon,n} + [\mathscr{A}_+^\varepsilon, D_p]\tilde v^{\varepsilon,n} + D_p\tilde v_2^{\varepsilon,n} + D_p\mathfrak{n}_\varphi^\varepsilon\,. \tag{3.33}$$

Equation (3.32) implies that $D_p\tilde v^{\varepsilon,n}$ solves (2.43) in which ψ is replaced by $\psi^\sharp$. Hence, by Lemma 2.10, it satisfies the bound

$$\|(-\mathscr{L}_0)^{1/2} D_p\tilde v^{\varepsilon,n}\| \lesssim \|(-\mathscr{L}_0)^{-1/2}\psi^\sharp\|\,.$$

This means that (3.31) follows if we prove that the $(-\mathscr{L}_0)^{-1/2}\Gamma L^2$-norm of each of the summands in (3.33) vanishes. For last three terms we upperbound $(-\mathscr{L}_0)^{-1/2}$ by 1. Then, for the fourth we use that by (1.22) and (2.44), we have

$$\|D_p \mathfrak{n}^\varepsilon_\varphi\|^2 \le \lambda_\varepsilon^2 \sum_\ell \mathbb{J}^\varepsilon_{\ell,p}(\mathfrak{w}\cdot(\ell+p))^2 |\hat\varphi(\ell+p)|^2 \lesssim \lambda_\varepsilon^2 \|\varphi\|^2_{H^1(\mathbb{T})}\,. \tag{3.34}$$

For the third, we proceed as

$$\|D_p \tilde v_2^{\varepsilon,n}\| \lesssim \|(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathfrak{n}^\varepsilon_\varphi\| = \|(-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon)^{-1}\mathscr{A}^\varepsilon_+\varphi\| \le \lambda_\varepsilon \|\varphi\|^2_{H^1(\mathbb{T})} \tag{3.35}$$

where we used (2.64) and (2.72). For the second term, we exploit (3.28), which gives

$$\|[\mathscr{A}^\varepsilon_+, D_p]\tilde v^{\varepsilon,n}\| \lesssim \lambda_\varepsilon \|\mathcal{N}^{5/2}(-\mathscr{L}_0)^{1/2}\tilde v^{\varepsilon,n}\| \lesssim \lambda_\varepsilon \|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\|$$

where in the last step we used (2.56). Hence, invoking also (2.48), the last three terms in (3.33) converge to 0, since so does λ_ε. For the first, by (3.30), we have

$$\begin{aligned}\|(-\mathscr{L}_0)^{-1/2}[-\mathscr{L}_0 - \mathscr{L}_0^{\mathfrak{w}}\mathscr{G}^\varepsilon, D_p]\tilde v^{\varepsilon,n}\|^2 \\ \lesssim n^2\|\tilde v^{\varepsilon,n}\|^2 + \lambda_\varepsilon^4 \|\mathcal{N}(-\mathscr{L}_0)^{1/2}\tilde v^{\varepsilon,n}\|^2\end{aligned}$$

Now, the ΓL^2-norm of $\tilde v^{\varepsilon,n}$ converges to 0 as $\varepsilon \to 0$ by Theorem 2.13, while as argued above, $\|\mathcal{N}(-\mathscr{L}_0)^{1/2}\tilde v^{\varepsilon,n}\|$ is uniformly bounded in both n and ε, so that the presence of the λ_ε at the second summand ensures it converges to 0 as well.

We now turn to the case $d \ge 3$. Since $v^{\varepsilon,n}$ solves (2.45), it satisfies

$$\begin{aligned}-\mathscr{L}^\varepsilon D_p v^{\varepsilon,n} &= [-\mathscr{L}^\varepsilon, D_p]v^{\varepsilon,n} - D_p\mathscr{L}^\varepsilon v^{\varepsilon,n} \\ &= [-\mathscr{L}^\varepsilon, D_p]v^{\varepsilon,n} + D_p\mathfrak{n}^\varepsilon_\varphi - D_p\mathscr{A}^\varepsilon_+ v_n^{\varepsilon,n} - D_p\mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}\,.\end{aligned} \tag{3.36}$$

We test both sides by $D_p v^{\varepsilon,n}$, so that, using the antisymmetry of $\mathscr{A}^\varepsilon$, we get

$$\begin{aligned}&\|(-\mathscr{L}_0)^{\frac12} D_p v^{\varepsilon,n}\|^2 \\ &= \langle [-\mathscr{L}^\varepsilon, D_p]v^{\varepsilon,n} + D_p\mathfrak{n}^\varepsilon_\varphi, D_p v^{\varepsilon,n}\rangle - \langle D_p\mathscr{A}^\varepsilon_+ v_n^{\varepsilon,n} + D_p\mathscr{A}^\varepsilon_- v_2^{\varepsilon,n}, D_p v^{\varepsilon,n}\rangle \\ &= \langle [-\mathscr{L}^\varepsilon, D_p]v^{\varepsilon,n} + D_p\mathfrak{n}^\varepsilon_\varphi, D_p v^{\varepsilon,n}\rangle \\ &\le \langle [-\mathscr{A}^\varepsilon_+, D_p]v^{\varepsilon,n} + [-\mathscr{A}^\varepsilon_-, D_p]v^{\varepsilon,n} + D_p\mathfrak{n}^\varepsilon_\varphi, D_p v^{\varepsilon,n}\rangle\end{aligned} \tag{3.37}$$

where, in the second step, we used the orthogonality of Fock spaces with different indices, while in the last that, by (3.25), we have

$$\langle [-\mathscr{L}_0, D_p]v^{\varepsilon,n}, D_p v^{\varepsilon,n}\rangle = -\frac{1}{2}|p|^2\|D_p v^{\varepsilon,n}\|^2 \le 0\,.$$

At this point we apply Cauchy-Schwarz to the right hand side of (3.37), so that

$$\begin{aligned}
&\|(-\mathscr{L}_0)^{\frac{1}{2}} D_p v^{\varepsilon,n}\|^2 \\
&\quad \le (\|\mathcal{N}[-\mathscr{A}^\varepsilon_+, D_p]v^{\varepsilon,n}\| + \|\mathcal{N}[-\mathscr{A}^\varepsilon_-, D_p]v^{\varepsilon,n}\| + \|\mathcal{N} D_p \mathfrak{n}^\varepsilon_\varphi\|)\|\mathcal{N}^{-1} D_p v^{\varepsilon,n}\| \\
&\quad \lesssim \Big(\lambda_\varepsilon \|\mathcal{N}^{3/2}(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}\| \\
&\qquad\qquad + |\mathfrak{w}\cdot p| \|\mathcal{N}^{5/2}(-\mathscr{L}_0)^{1/2} v^{\varepsilon,n}\| + \|\mathcal{N} D_p \mathfrak{n}^\varepsilon_\varphi\|\Big)\|\mathcal{N}^{-1} D_p v^{\varepsilon,n}\| \\
&\quad \lesssim \Big(\|(-\mathscr{L}_0)^{-1/2}\mathfrak{n}^\varepsilon_\varphi\| + \|D_p \mathfrak{n}^\varepsilon_\varphi\|\Big)\|v^{\varepsilon,n}\|
\end{aligned} \tag{3.38}$$

where in the second step we applied Lemma 3.9, while in the last the a priori estimate in Proposition 2.9 and the fact that by (1.22), we have $\|\mathcal{N}^{-1} D_p v^{\varepsilon,n}\| \le \|v^{\varepsilon,n}\|$. Now, since the term containing the norm of $\mathfrak{n}^\varepsilon_\varphi$ is bounded uniformly in ε and, by Proposition 2.15, $v^{\varepsilon,n}$ converges to 0 in ΓL^2, (3.31) follows. □

Appendix A Some technical results

In this appendix, we show the technical part of the Replacement Lemma 2.6.

Proposition A.1 *For $\varepsilon > 0$, $n \in \mathbb{N}$ and $k_{1:n} \in \mathbb{Z}^{2n}_0$, define P^ε as in (2.35), i.e.*

$$P^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2} \sum_{\ell+m=k_1} \frac{\mathbb{J}^\varepsilon_{\ell,m}}{\tilde\Gamma + \tilde\Gamma^w G(\mathrm{L}^\varepsilon(\tilde\Gamma))} \tag{A.1}$$

for

$$\tilde\Gamma \stackrel{\text{def}}{=} \tfrac{1}{2}(|\ell|^2 + |m|^2 + |k_{2:n}|^2)\,, \qquad \tilde\Gamma^w \stackrel{\text{def}}{=} \tfrac{1}{2}((\mathfrak{w}\cdot\ell)^2 + (\mathfrak{w}\cdot m)^2 + (\mathfrak{w}\cdot k)^2_{2:n})$$

and G is as in (2.24). Then, there exists a constant $C > 0$ independent of ε, n such that

$$\sup_{n\in\mathbb{N}, k_{1:n}\in\mathbb{Z}^{2n}_0} \Big| P^\varepsilon(k_{1:n}) - G(\mathrm{L}^\varepsilon(\tfrac{1}{2}|k_{1:n}|^2))\Big| \le C\lambda_\varepsilon^2\,. \tag{A.2}$$

Proof. Note first of all that we can assume that $|k_1| \le 1/(2\varepsilon)$. Indeed, in the complementary case both $P^\varepsilon(k_{1:n})$ and $G(\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2))$ would be $o(\lambda_\varepsilon^2)$ (uniformly in n and $k_{1:n}$). For $G(\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2))$, this easily follows from the definition of G and L^ε. As for $P^\varepsilon(k_{1:n})$, if $|k_1| \ge 1/(2\varepsilon)$, then either $|\ell|$ or $|m|$ are larger than $1/(4\varepsilon)$. Assume, for instance, that the former is the case. Then,

$$P^\varepsilon(k_{1:n}) \lesssim \lambda_\varepsilon^2 \sum_{\frac{1}{4\varepsilon}\le|\ell|\le\frac{1}{\varepsilon}} \frac{1}{|\ell|^2} \lesssim \lambda_\varepsilon^2. \tag{A.3}$$

For $|k_1| \le 1/(2\varepsilon)$, we rewrite (A.1) as

$$P^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2} \sum_{\substack{\ell+m=k_1 \\ |\ell|,|m|\le\varepsilon^{-1}}} \frac{1}{\tilde\Gamma + \tilde\Gamma^w G(\mathrm{L}^\varepsilon(\tilde\Gamma))} \tag{A.4}$$

Note that, when $\ell + m = k_1$,

$$\tilde{\Gamma} = \Gamma - \ell \cdot k_1, \qquad \Gamma \stackrel{\text{def}}{=} |\ell|^2 + \frac{1}{2}|k_{1:n}|^2,$$
$$\tilde{\Gamma}^w = \Gamma^w - (\mathfrak{w} \cdot k_1)(\mathfrak{w} \cdot \ell), \qquad \Gamma^w \stackrel{\text{def}}{=} (\mathfrak{w} \cdot \ell)^2 + \frac{1}{2}(\mathfrak{w} \cdot k)^2_{1:n}.$$

For future reference, we point out that Γ and $\tilde{\Gamma}$ are comparable in that, by the triangular inequality, we have

$$\Gamma \lesssim \tilde{\Gamma} \lesssim \Gamma \tag{A.5}$$

uniformly in $\ell, m = k_1 - \ell$ and $k_{1:n} \in \mathbb{Z}^{2n}_0$.

The proof of Proposition A.1 follows from a sequence of technical steps in which the summand in the definition of P^ε is progressively simplified and the Riemann sum is replaced by an integral. More precisely, we will first replace $P^\varepsilon_0 \stackrel{\text{def}}{=} P^\varepsilon$ with P^ε_1 and then P^ε_i with P^ε_{i+1}, $i = 1, \dots, 4$, in such a way that for all $i = 0, \dots, 5$,

$$\sup_{n \in \mathbb{N}, k_{1:n} \in \mathbb{Z}^{2n}_0} \left| P^\varepsilon_i(k_{1:n}) - P^\varepsilon_{i+1}(k_{1:n}) \right| \lesssim \lambda^2_\varepsilon \,. \tag{A.6}$$

At last, we will compute P^ε_5 and show that it equals $G(\mathrm{L}^\varepsilon)$.

Step 1 First of all, we define $P^\varepsilon_1(k_{1:n})$ as $P^\varepsilon(k_{1:n})$ in (A.4), except that $\tilde{\Gamma}$ and $\tilde{\Gamma}^w$ are replaced by Γ and Γ^w, respectively. The absolute value of the difference is upper bounded by a constant times

$$\lambda^2_\varepsilon \sum_{\substack{\ell+m=k_1 \\ |\ell|,|m| \le 1/\varepsilon}} \frac{A + B}{(|\ell|^2 + |k_1|^2)^2}$$

where in the denominator we used $\Gamma \wedge \tilde{\Gamma} \gtrsim |\ell|^2 + |k_1|^2$ and $\Gamma^w \wedge \tilde{\Gamma}^w \ge 0$, while in the numerator A and B are defined as

$$A \stackrel{\text{def}}{=} |\ell \cdot k_1 + (\mathfrak{w} \cdot \ell)(\mathfrak{w} \cdot k_1) G(\mathrm{L}^\varepsilon(\tilde{\Gamma}))|,$$
$$B \stackrel{\text{def}}{=} |G(\mathrm{L}^\varepsilon(\tilde{\Gamma})) - G(\mathrm{L}^\varepsilon(\Gamma))| \Big(|\ell|^2 + \frac{1}{2}|k_{1:n}|^2 \Big) .$$

Now, $A \lesssim |\ell||k_1|$ so that

$$\lambda^2_\varepsilon \sum_{\substack{\ell+m=k_1 \\ |\ell|,|m| \le 1/\varepsilon}} \frac{A}{(|\ell|^2 + |k_1|^2)^2} \lesssim \lambda^2_\varepsilon,$$

as can be immediately seen by splitting the sum according to whether $|\ell| \le |k_1|$ or the converse. The term containing B is dealt with similarly. Indeed, since $|\Gamma - \tilde{\Gamma}| \le |\ell||k_1|$, one has

$$|G(\mathrm{L}^\varepsilon(\tilde{\Gamma})) - G(\mathrm{L}^\varepsilon(\Gamma))| \lesssim |\ell||k_1| \sup_{y \in [\Gamma, \tilde{\Gamma}]} \left| \frac{1}{|\log \varepsilon|} \frac{G'(\mathrm{L}^\varepsilon(y))}{y(\varepsilon y + 1)} \right| \lesssim \frac{|\ell||k_1|}{|\ell|^2 + \frac{1}{2}|k_{1:n}|^2}$$

where we further used that the derivative of G is bounded and (A.5) holds. Hence, $B \lesssim |\ell||k_1|$ and one concludes as above. Inequality (A.6) for $i = 0$ is then proven.

Step 2 Secondly, we define $P_2^\varepsilon(k_{1:n})$ as $P_1^\varepsilon(k_{1:n})$, except that Γ^w is replaced by $(\mathfrak{w} \cdot \ell)^2$, that is,

$$P_2^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2} \sum_{\substack{\ell+m=k_1 \\ |\ell|,|m|\le\varepsilon^{-1}}} \frac{1}{\Gamma + (\mathfrak{w}\cdot\ell)^2 G(\mathrm{L}^\varepsilon(\Gamma))}\,.$$

The difference between P_1^ε and P_2^ε is bounded by a constant times

$$\lambda_\varepsilon^2 \sum_{\substack{\ell+m=k_1 \\ |\ell|,|m|\le\varepsilon^{-1}}} \frac{(\mathfrak{w}\cdot k)_{1:n}^2}{(|\ell|^2 + |k_{1:n}|^2)^2} \lesssim \lambda_\varepsilon^2 \sum_{\substack{\ell+m=k_1 \\ |\ell|,|m|\le\varepsilon^{-1}}} \frac{|k_{1:n}|^2}{(|\ell|^2 + |k_{1:n}|^2)^2} \lesssim \lambda_\varepsilon^2\,,$$

where we used that $G(\mathrm{L}^\varepsilon(\Gamma)) \lesssim 1$.

Step 3 As a third step, we replace $P_2^\varepsilon(k_{1:n})$ with

$$P_3^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2} \sum_{|\ell|\le\varepsilon^{-1}} \frac{1}{\Gamma + (\mathfrak{w}\cdot\ell)^2 G(\mathrm{L}^\varepsilon(\Gamma))}.$$

With respect to P_2^ε, we have only removed the indicator function of $|m| = |k_1 - \ell| \le \varepsilon^{-1}$. Hence, the difference between P_2^ε and P_3^ε only involves the sum over those values of ℓ such that $|k_1 - \ell| \ge 1/\varepsilon$ while $|\ell| \le 1/\varepsilon$. But, since $|k_1| \le 1/(2\varepsilon)$, this corresponds to $1/(2\varepsilon) \le |\ell| \le 1/\varepsilon$ and the resulting sum can be bounded as in (A.3).

Step 4 Next, we turn the sum into an integral. Setting $\alpha_\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}|\varepsilon k_{1:n}|^2$, we define P_4^ε as

$$P_4^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2} \int_{|x|\le 1} \frac{\mathrm{d}x}{|x|^2 + \alpha_\varepsilon + (x\cdot\mathfrak{w})^2 G(\mathrm{L}^\varepsilon(\varepsilon^{-2}(|x|^2+\alpha_\varepsilon)))}\,. \tag{A.7}$$

To obtain (A.6) for $i = 3$, we argue as in [CETara, Lemma C.7]. Denoting by I the integrand in (A.7) and by Q_ℓ^ε the square of side-length ε centred at $\ell \in \mathbb{Z}_0^2$, the difference between P_3^ε and P_4^ε is bounded above by

$$\begin{aligned}\lambda_\varepsilon^2 \sum_{1\le|\ell|\le\varepsilon^{-1}} \int_{Q_\ell^\varepsilon} |I(\varepsilon\ell) - I(x)|\mathrm{d}x &\le \lambda_\varepsilon^2 \varepsilon \sum_{1\le|\ell|\le\varepsilon^{-1}} \int_{Q_\ell^\varepsilon} \sup_{y\in Q_\ell^\varepsilon} |\nabla I(y)|\mathrm{d}x \\ &\le \lambda_\varepsilon^2 \varepsilon^3 \sum_{1\le|\ell|\le\varepsilon^{-1}} \sup_{y\in Q_\ell^\varepsilon} |\nabla I(y)|\,.\end{aligned}$$

Now, a simple computation yields

$$\sup_{y\in Q_\ell^\varepsilon} |\nabla I(y)| \lesssim \lambda_\varepsilon^2 \frac{1}{(|\varepsilon\ell|^2 + \alpha_\varepsilon)^{3/2}}$$

so that (A.6) follows at once.

Before moving to the next step, note that, by passing to polar coordinates, i.e. setting $x = rv_\theta$ for $v_\theta = (\cos\theta, \sin\theta)$, we have

$$\begin{aligned} P_4^\varepsilon(k_{1:n}) &= \frac{\lambda_\varepsilon^2}{\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^1 \frac{r\mathrm{d}r}{r^2 + \alpha_\varepsilon + r^2(\mathfrak{w}\cdot v_\theta)^2 G(\mathrm{L}^\varepsilon(\varepsilon^{-2}(r^2+\alpha_\varepsilon)))} \\ &= \frac{\lambda_\varepsilon^2}{2\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^1 \frac{\mathrm{d}r}{r + \alpha_\varepsilon + r(\mathfrak{w}\cdot v_\theta)^2 G(\mathrm{L}^\varepsilon(\varepsilon^{-2}(r+\alpha_\varepsilon)))} \\ &= \frac{\lambda_\varepsilon^2}{2\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^1 \frac{\mathrm{d}r}{r + \alpha_\varepsilon + r|\mathfrak{w}|^2\cos^2(\theta - \theta_w) G(\mathrm{L}^\varepsilon(\varepsilon^{-2}(r+\alpha_\varepsilon)))} \\ &= \frac{\lambda_\varepsilon^2}{\pi^2}\int_0^{1} \mathrm{d}r \int_0^{\pi} \frac{\mathrm{d}\theta}{r + \alpha_\varepsilon + r|\mathfrak{w}|^2\cos^2(\theta) G(\mathrm{L}^\varepsilon(\varepsilon^{-2}(r+\alpha_\varepsilon)))} \end{aligned}$$

where we set $w = |w|v_{\theta_w}$ and used periodicity of the integrand in θ.

Step 5 In this step, we insert two elements at the denominator of the integrand of P_4^ε, that will allow us to make a simple change of variables. More precisely, we replace P_4^ε with P_5^ε, the latter being defined as

$$P_5^\varepsilon(k_{1:n}) \stackrel{\text{def}}{=} \frac{\lambda_\varepsilon^2}{\pi^2}\int_0^{\pi}\int_0^1 \frac{\mathrm{d}\theta\mathrm{d}r}{(r+\alpha_\varepsilon)(r+\alpha_\varepsilon+1)[1+|\mathfrak{w}|^2\cos^2(\theta)G(\mathrm{L}^\varepsilon(\varepsilon^{-2}(r+\alpha_\varepsilon)))]}\,.$$

Now, (A.6) for $i = 4$ can be easily seen to hold by arguing as in the proof of [CETarb, Eq. (A.9)].

At this point, it remains to analyse the last integral. With the change of variables

$$\begin{aligned} y &= \mathrm{L}^\varepsilon(\varepsilon^{-2}(r+\alpha_\varepsilon)) = \lambda_\varepsilon^2 \log\Big(1 + \frac{1}{r+\alpha_\varepsilon}\Big), \\ \frac{\mathrm{d}r}{(r+\alpha_\varepsilon)(1+r+\alpha_\varepsilon)} &= -\lambda_\varepsilon^{-2}\mathrm{d}y, \end{aligned}$$

one finds

$$P_5^\varepsilon(k_{1:n}) = \frac{1}{\pi^2}\int_0^\pi \mathrm{d}\theta \int_{\mathrm{L}^\varepsilon(\varepsilon^{-2}+\frac{1}{2}|k_{1:n}|^2)}^{\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2)} \frac{\mathrm{d}y}{1+|\mathfrak{w}|^2\cos^2(\theta)G(y)}$$

It is then immediate to verify that, again at the price of an error of order λ_ε^2, we can replace the lower integration index by 0, and obtain

$$\begin{aligned} &\frac{1}{\pi^2}\int_0^{\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2)} \mathrm{d}y \int_0^\pi \frac{\mathrm{d}\theta}{1+|\mathfrak{w}|^2\cos^2(\theta)G(y)} \\ &\qquad = \frac{1}{\pi}\int_0^{\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2)} \frac{\mathrm{d}y}{\sqrt{1+|\mathfrak{w}|^2 G(y)}} = G(\mathrm{L}^\varepsilon(\frac{1}{2}|k_{1:n}|^2)) \end{aligned} \tag{A.8}$$

where the last step is a consequence of our choice of G (actually, *we have chosen G in such a way that it holds*). Hence, (A.2) follows at once and the proof of the proposition is concluded. □

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