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The elliptic stochastic quantization of some two dimensional Euclidean QFTs

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Abstract

We study a class of elliptic SPDEs with additive Gaussian noise on $\mathbb{R}^2 \times M$, with M a d-dimensional manifold equipped with a positive Radon measure, and a real-valued non linearity given by the derivative of a smooth potential V, convex at infinity and growing at most exponentially. For quite general coefficients and a suitable regularity of the noise we obtain, via the dimensional reduction principle discussed in [10], the identity between the law of the solution to the SPDE evaluated at the origin with a Gibbs type measure on the abstract Wiener space $L^2(M)$. The results are then applied to the elliptic stochastic quantization equation for the scalar field with polynomial interaction over $\mathbb{T}^2$, and with exponential interaction over $\mathbb{R}^2$ (known also as Høegh-Krohn or Liouville model in the literature). In particular for the exponential interaction case, the existence and uniqueness properties of solutions to the elliptic equation over $\mathbb{R}^{2+2}$ is derived as well as the dimensional reduction for all values of the "charge parameter" $\sigma = \frac{\alpha}{2\sqrt{\pi}} < \sqrt{8\pi}$ for which the model has an Euclidean invariant probability measure (hence also permitting to get the corresponding relativistic invariant model on the two dimensional Minkowski space).

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1 Introduction

The Euclidean approach to quantization of relativistic non-linear wave equations requires the construction of certain probability measures supported on distributions satisfying a set of quite strong requirements among which invariance under the full group of rigid motions of the Euclidean space and reflection positivity. Such constructions have succeeded in the case where the dimension d of the Euclidean space is equal to $1, 2, 3$ (that is also the dimension of the space-time of the original relativistic equation). The non-linearity is quite general if $d = 1$, whereas it is restricted to the derivatives of polynomials of even degree respectively suitable superpositions of trigonometric or exponential functions for $d = 2$, or a cubic monomial function for $d = 3$ (the φ^4_3 model). In these cases a number of interesting physical and mathematical properties of the quantized relativistic fields models have been put in evidence.

In recent years new methods coming from the study of singular partial stochastic differential equations (SSPDEs) have been developed and have opened the possibility of new constructions of such Euclidean measures (and other similar measures associated with problems coming from other areas of applications, like statistical mechanics, hydrodynamics, wave propagation in random media ...). In these approaches the relevant Euclidean invariant (probability) measures are obtained as invariant measures for certain parabolic semilinear SPDEs called stochastic quantization equations (SQEs).

Typically the measures to be constructed have the heuristic form

$$\mu(\mathrm{d}\varphi) = Z^{-1} e^{-S(\varphi)} \mathcal{D}\varphi,$$

where $\varphi = \varphi(x) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\mathcal{D}\varphi = \Pi_{x\in\mathbb{R}^d} \mathrm{d}\varphi(x)$ is a "flat measure", Z is a "normalization constant", $S(\varphi) = S_0(\varphi) + \lambda S_{\mathrm{int}}(\varphi)$ with $S_0(\varphi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 \mathrm{d}x + m^2 \int_{\mathbb{R}^d} |\varphi(x)|^2 \mathrm{d}x$, and where $\lambda \geqslant 0$, $m \geqslant 0$ are parameters,

$$S_{\mathrm{int}}(\varphi) = \int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x,$$

for some (non-linear) measurable real valued function V over $\mathbb{R}$. Note that $\mu(\mathrm{d}\varphi)$ is heuristically invariant under Euclidean transformations (translations and rotations and also additional reflections, in the case where V is an even function).

The associated stochastic quantization equation (first introduced in [65]) is of the form

$$\mathrm{d}X_\tau = (\Delta - m^2)X_\tau \mathrm{d}t - V'(X_\tau)\mathrm{d}t + \mathrm{d}W_\tau$$

where τ is an additional "computer time", Δ is the Laplacian in $\mathbb{R}^d$, $X_\tau = X_\tau(x)$, $x \in \mathbb{R}^d$, is a random variable taking values, for fixed τ, in the space of "(generalized) functions" of x, $\mathrm{d}W_\tau(x)$ is a Gaussian white noise in τ and x, and V' in the derivative of V.

Following a terminology used especially when V is a polynomial P, we call the above SPDE the $V(\varphi)_d$–stochastic quantization equation (SQE). Various detailed results have been obtained on $P(\varphi)_2$ SQEs, see the introduction of [7] and [45, 46] for references. In particular in [35] there were obtained a strong solution and a unique ergodic measure in the case where $\mathbb{R}^2$ is replaced by the 2-torus $\mathbb{T}^2$.

As for the case of the φ^4_3 SQE, a breakthrough for local in time solutions of the SQE on the 3-torus $\mathbb{T}^3$ came from [48], see also [33, 47]. More recently still for the case of the 3-torus $\mathbb{T}^3$ a construction of invariant solutions has been given in [7]. This can be looked upon as a construction of the φ^4_3-measure on $\mathbb{T}^3$, by methods different from the previous ones provided by mathematical physics approaches, see references in [43]. The methods in [7] have been simplified and extended to the case of $\mathbb{R}^3$ in [45], yielding a full alternative construction of the measures which qualify to be called φ^4_3 Euclidean measures. A previous approach following [48] is in [62]. A number of open problems has been mentioned in [7, 45], the main one concerns the uniqueness of the invariant measures.

Other approaches are however possible if one understands stochastic quantization in the broader sense of associating to a given heuristic "Euclidean measure" a stochastic process having this very measure as suitable marginal. A first approach consists in constructing a jump type process via infinite dimensional quasi-regular jump-type Dirichlet forms [5, 9, 22, 23] (for such Dirichlet forms

see also [8, 16]), that has by construction the given Euclidean measure as invariant measure. Another approach obtains the invariant measure of an SQE in dimension d by looking at solutions of a suitable associated elliptic SPDE in dimension $d+2$ and restricting the solution to the d dimensional Euclidean space obtained sending to zero the two additional variables. This latter procedure is known in the physical literature as Parisi-Sourlas dimensional reduction and has been implemented by using algebraic methods of supersymmetry (see [64]). For the case of a regularized non-linear term in the original SQE and regularized noise it has been studied from a mathematical point of view in [56].

Elliptic stochastic PDEs related to stochastic quantization of the φ^4_2 and φ^4_3 models respectively have been discussed both on the torus and in all space in [46]. A systematic mathematical investigation of the mechanism of dimensional reduction has been initiated in [10], for the $V(\varphi)_0$ model, for both the cases where V is a convex function and also the case where V is non-convex, where an interesting phenomenon of non-uniqueness of solutions has been put in evidence.

In fact in [10] an explicit formula for the law of the solution of a class of elliptic SPDE in $\mathbb{R}^2$ taken at the origin has been obtained, by means of a rigorous version of dimensional reduction. Besides proving an instance of elliptic stochastic quantization for scalar field and for the case of underlying Euclidean dimension $d = 0$, it also provided a realization of the relation between a supersymmetric quantum field model and an elliptic SPDE in 2 dimensions.

The present paper has two principal aims. The first one is to prove the dimensional reduction principle for elliptic SPDEs with non singular noise for equations in $d+2$ dimensions with $d > 0$. The second is to extend the result to elliptic singular SPDE having applications in stochastic quantization program and analyzing the corresponding dimensional reduction with respect to the removal of spatial cut-offs.

We restrict our attention to singular SPDEs with $d = 2$ and analyze polynomial and exponential interactions. In particular, in the case of potentials of the form $\exp(\sigma\varphi)_2$, for suitable real σ we are able to complete the dimensional reduction picture removing all the spatial cutoffs (i.e. in both the "fictitious" and "real" spatial variables).

Before passing to a more detailed description of our results, let us stress the importance of the $\exp(\sigma\varphi)_2$ model (and related ones) in relativistic and Euclidean quantum field theory. The $\exp(\sigma\varphi)_2$ model over $\mathbb{R}^2$ has been first introduced by Høegh-Krohn [52] (see also [13] and [74], p. 178 and 307-313) and constructed for an interaction of the form

$$U_g(\varphi) := \int : e^{\sigma\varphi(z)} : g(z)\mathrm{d}z\mathrm{d}\nu(\sigma)$$

for any positive space cut-off function $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and any positive finite measure ν with support in $(-\sqrt{4\pi}, \sqrt{4\pi})$, in the sense that the interacting measure

$$\mu_g(\mathrm{d}\varphi) := \frac{e^{-U_g(\varphi)}\mu_0(\mathrm{d}\varphi)}{\int e^{-U_g(\varphi)}\mu_0(\mathrm{d}\varphi)},$$

for $m > 0$, is absolutely continuous with respect to Nelson's free field Gaussian measure μ_0 with mean zero and covariance $(m^2 - \Delta_z)^{-1}$, $z \in \mathbb{R}^2$ (realized as a probability measure on, for example, $\mathcal{S}'(\mathbb{R}^2)$). See [13, 52] and [74] p. 178 for further properties of the measure μ_g. This model has been discussed in the infinite volume limit $g \to 1$, that is unique, in [13], under the more restrictive assumption that $\mathrm{supp}(\nu) \subset \left[-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}}\right]$ (see also [74] pp. 307-313 and [39]).

In fact in [13] the authors prove that the measure of the $\exp(\sigma\varphi)_2$ model for $|\sigma| < \sqrt{4\pi}$ (or more generally of suitable superpositions $\int \mathrm{Ch}(\sigma\varphi)_2\mathrm{d}\sigma$ of these models) satisfies all axioms of Euclidean quantum field theory and leads to relativistic quantum fields over two-dimensional Minkowski space-time satisfying all Wightman axioms with interaction and a positive, finite mass gap at the lower end of the spectrum of the corresponding Hamiltonian.

Further properties of such models were discussed in, e.g., [3, 4, 39, 42, 51, 74, 78] and [9, 14, 16, 21, 58]. The case $|\sigma| < \sqrt{8\pi}$ was discussed by S. Kusuoka in [57] and the relation with

independent work on multiplicative chaos by J. P. Kahane [55] (see also below) was pointed out by H. Sato. In [15] and also [74] estimates in L^p (for the interaction in a bounded region) were given for $|\sigma| < \sqrt{\frac{4\pi}{p-1}}$. The Gaussian character of the model for $|\sigma|$ sufficiently large or for any σ if the Euclidean space dimension d satisfies $d \geqslant 3$ has been pointed out in [15] and [2]. The relevance of the $\exp(\sigma\varphi)_2$ model for Polyakov string theory has been discussed in [17, 18, 19] ($|\sigma| < \sqrt{4\pi}$ corresponding to the embedding dimension $D < 13$) and rediscovered more recently in connection with topics like Liouville model, quantum gravity and multiplicative chaos. Let us mention in this connection [25, 37, 66] see also [29, 36, 44, 73] (for the literature for other approaches to Liouville type models not directly connected with probabilistic methods see, e.g. [26]).

Finally a connection of the $\exp(\sigma\varphi)_2$ model with the irreducibility of a unitary representation of the group of mappings of a manifold into a compact Lie group has been pointed out and studied in [24], see also the recent work [1].

The $\exp(\sigma\varphi)_2$ model is also closely connected with the $\sin(\sigma\varphi)_2$ model ("Sine-Gordon equation") as discussed in [12, 14, 40]. In the latter reference all Wightman axioms are proved again in the region $|\sigma| < \sqrt{4\pi}$.

Both in the $\exp(\sigma\varphi)_2$ model and the $\sin(\sigma\varphi)_2$ model a replacement of the exp (respectively sin) function by a Wick ordered version is needed (see [11]) but also suffices for the existence and non-triviality of the model for $|\sigma|$ up to $\sqrt{4\pi}$. For larger values of $|\sigma|$, in the case of the $\sin(\sigma\varphi)_2$ model, up to $\sqrt{8\pi}$, further renormalization by counter-terms is required (see [28]).

The study of the stochastic quantization equation associated with the latter class of models has been initiated in [11] (where strong solutions have been discussed and the necessity of renormalization has been pointed out). In the case where $\mathbb{R}^2$ (or $\mathbb{T}^2$) is replaced by $\mathbb{R}$ (respectively $\mathbb{T}$), i.e. for the model $\exp(\sigma\varphi)_1$, a deeper analysis is possible and has been pursued in [20], where the existence of solution and the strong uniqueness of the invariant measure are proven (for the corresponding stochastic quantization of $P(\varphi)_1$ see [54]).

The case of the SQE in $d = 2$ and with a regularized noise on the torus (and corresponding changes in the coefficients of the stochastic quantization equation needed to keep the same invariant measure) has been discussed in [60] (see also [6] for further development), where existence of solutions was proven using essentially the properties of the $\exp(\sigma\varphi)_2$ model established in [13, 39] and methods of [35]. Uniqueness problems are also discussed in [6] in conjunction with an approach using Dirichlet forms, for all $\sigma^2 < 4\pi$, in a setting similar to [60].

Hairer and Shen [50] introduced powerful methods to handle the dynamical $\sin(\sigma\varphi)_2$ model on $\mathbb{T}^2$, and via regularity structures local existence is shown up to $\sigma^2 < \frac{16}{3}\pi$. More recently these results has been extended in [34] to cover all the subcritical regime $\sigma^2 < 8\pi$.

Concerning the exponential interaction, the work of Garban [41] appeared recently in which the author studies the SQE on the torus $\mathbb{T}^2$ and on the sphere $\mathbb{S}^2$. After subtracting the solution to the linear equation Garban obtains an SPDE driven by a multi-fractal and intermitted multiplicative chaos. When $|\sigma| < \left(4 - 2\sqrt{3}\right)\sqrt{\pi}\left(< \sqrt{4\pi}\right)$ (a regime correspondingly called Da Prato-Debussche phase) he shows the existence of a strong solution and the convergence of the solution to the equation with regularized noise to the singular one both on the torus $\mathbb{T}^2$ and on the sphere $\mathbb{S}^2$ (see Theorem 1.7 and Theorem 1.9 in [41]). The method used is based on the Besov regularity of the Gaussian multiplicative chaos (related to the theory of [55]). The paper [41] also refers to other interesting relations to quantum gravity, conformal fields theory, strings theory, multiplicative chaos and exciting new results on random measures. A result on existence and uniqueness for solutions of SQE (without the proof of the convergence of the solution to the equation with regularized noise to the singular one) is also proved (Theorem 1.11 in [41]) for $|\sigma| < \left(4 - 2\sqrt{2}\right)\sqrt{\pi}\left(< \sqrt{4\pi}\right)$. Moreover a comparison with the SQE for the Sine-Gordon model is provided (cfr. Section 7 in [41]).[1]

[1] After the submission of this article for publication we have become aware of the papers [53, 63] which address the stochastic quantization of the exponential model in the parabolic setting for $\sigma^2 < 4\pi$.

Let us briefly review and comment the main results obtained in the present paper. In Section 2 we study the case of the following elliptic SPDE (equation (1) in Section 2)

$$(-\Delta_x + m^2 + \mathfrak{L})(\phi)(x,z) + \mathcal{A}^2[f(x')g(z')V'(\phi(x',z'))](x,z) = \xi^A(x,z)$$

on $\mathbb{R}^2 \times M$, with M a d-dimensional manifold with or without boundary, equipped with a positive Radon measure $\mathrm{d}z$ and a real-valued non-linearity given by the derivative of a positive smooth function V defined on the real line growing at most exponentially at infinity. We use the shorthand $f(x')g(z')V'(\phi(x',z'))$ with free variables (x',z') to mean the function $(x',z') \mapsto f(x')g(z')V'(\phi(x',z'))$. Additional assumptions are as follows. The function V is taken to the convex or "quasi convex" (see Hypothesis QC below). $\mathfrak{L}$ and $\mathcal{A}$ are two positive self-adjoint operators acting on $L^2(M)$, $\mathcal{A}$ is bounded, $\mathfrak{L}$ is not-necessarily bounded but has a dense domain in $L^2(M)$, $\mathcal{A}$ and $\mathfrak{L}$ commute and both have completely discrete spectrum with some additional hypotheses on the eigenfunctions and the eigenvalues of $\mathcal{A}$. The noise ξ^A is supposed to be Gaussian with mean zero and covariance corresponding to a regularization given by $\mathcal{A}$. Finally f is a C^2 real-valued cut-off function on $\mathbb{R}^2$, decaying exponentially at infinity and g is a smooth real-valued cut-off with compact support on M.

We introduce finite dimensional projections relative of (1) and prove the existence of a weak solution ν satisfying a form of the dimensional reduction principle described in our previous paper [10], extended to the d dimension case with the presence of a non-trivial operator $\mathfrak{L}$. See Theorem 7 for a precise statement.

Next we prove extensions of these results in two directions: in Section 2.2 we consider the case $M = \mathbb{R}^d$, $\mathfrak{L} = -\Delta_z$ (cfr. Theorem 14), in Section 2.3 we consider the problem of the removal of the cutoffs f, g in the case where V is a convex function (cfr. Theorem 16).

The first part of the present paper contains the first example of rigorous dimensional reduction for elliptic SPDE in $d+2$ dimension with $d > 0$. Indeed, the other only paper which to our knowledge study dimensional reduction from the rigorous point of view is [56] whose main result is a theorem about the dimensional reduction of an integral in a space of functions in $d+2$ variables to a Gibbs type measure in a space of functions in d variables. In particular the authors do not attack the problem of the relation with the elliptic SPDE. Furthermore in [56] only polynomial potentials V are considered and the regularization of the noise is chosen to be compactly supported in Fourier space. Ours Hypothesis QC on the potential V and Hypotheses H$\mathcal{A}$ and H$\mathcal{A}$1 for the regularization of the noise considered here are quite more general. Let us stress that the results obtained under these more general hypotheses uses in essential way the SPDE formulation of dimensional reduction.

In Section 3 we extend our result to singular SPDEs. In particular the elliptic stochastic quantization of the exponential interaction is proven, where V is a suitable renormalized exponential function $\exp(\alpha\phi - \infty)$, for $|\alpha| < 4\sqrt{2}\pi$ (corresponding to the reduced index $|\sigma| < \sqrt{8\pi}$), and the white noise is unregularized. Existence and uniqueness is first proven for a suitable regularized model using the results of the previous sections (cfr. Theorem 19). Subsequently the removal of the regularization in the noise is achieved in the Besov space $B^s_{p,p,\ell}(\mathbb{R}^4)$, where $-1 < s < 0$ and $1 < p \leqslant 2$ are suitable constants depending only on α. Using the existence, uniqueness and convergence results for the elliptic SPDEs over $\mathbb{R}^4$ proven in Section 2.3, the existence and uniqueness results for the elliptic stochastic quantization equation for exponential interaction follows in Theorem 23. The dimensional reduction result is then obtained (cfr. Theorem 33) with uniqueness for all $|\alpha| < 4\sqrt{2}\pi$. Our last result in this section, Theorem 35, shows the convergence and uniqueness of the equation when the spatial cut-off g is removed. As an easy corollary one obtains the full Euclidean invariance of the law of the solutions.

Finally in Section 4 we prove the existence for singular elliptic SPDE with Wick power non linearity in $d+2 = 4$ dimension. Furthermore we prove a dimensional reduction principle for some weak solutions to SPDEs of this form providing the first example of elliptic stochastic quantization for the polynomial interaction model $P(\varphi)_2$. Let us stress that for polynomial interactions the problem of uniqueness of weak solutions to the elliptic SPDE is open and this prevents a more detailed analysis of the dimensional reduction.

The results contained in this second part of the paper gives the first example of dimensional reduction for singular elliptic SPDE as originally formulated by Parisi and Sourlas in [64]. In particular with respect to [46], where the existence for singular elliptic SPDEs with polynomial type non-linearity is proven for both $d+2=4$ and, with cubic type non-linearity, $d+2=5$ dimension, in Section 4 we prove, at least in the $d+2=4$ dimensional case, that they can be used as stochastic quantization equation for quantum field theory. However many problems still remains open (for these polynomial models) with respect to the removal of the spatial cutoffs.

Let us stress that, in Section 3, we prove existence and uniqueness of solutions for elliptic SPDE and the relation with the Liouville measure in the full subcritical regime $|\sigma| < \sqrt{8\pi}$. This result is achieved using in an essential way two properties of the exponential model: the fact that the Wick exponential of a distribution is a positive measure (this fact is already exploited in [41]) and the multifractality of Wick exponentials (see Lemma 22 for a precise formulation of this property). In particular we prove the existence and uniqueness for the elliptic SPDE with Wick exponential non-linearity using the space $B^s_{p,p,\ell}(\mathbb{R}^4)$, instead of $B^{s'}_{\infty,\infty,\ell}(\mathbb{R}^4)$, and by only the Da Prato-Debussche trick. This is possible since for any exponent $|\sigma| < \sqrt{8\pi}$ the noise lives, beside that, inside $B^s_{p,p,\ell}(\mathbb{R}^4)$ for some $-1 < s < 0$ and $1 < p \leqslant 2$, instead for $|\sigma| \geqslant \left(4-2\sqrt{2}\right)\sqrt{\pi}\left(<\sqrt{4\pi}\right)$ the Wick power of the noise is in $B^s_{\infty,\infty,\ell}(\mathbb{R}^4)$ with $s < -2$. The other novelty is the fact that we are able to solve the equation in the full space and we are then easily able to prove the Euclidean invariance of the law.

The other best result, to our knowledge, concerning stochastic quantization of exponential interaction, is the paper [41], which studied the stochastic quantization in the parabolic setting for the charge parameter $|\sigma| < \left(4-2\sqrt{2}\right)\sqrt{\pi}\left(<\sqrt{4\pi}\right)$. We think that the analytic methods, developed here in Section 3.2, joined with the probabilistic results, typical of the parabolic setting obtained in [41], can also be applied to the parabolic case for the whole subcritical regime $|\sigma| < \sqrt{8\pi}$.

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2 Dimensional reduction with regularized noise

2.1 Discrete spectrum

Consider the following elliptic SPDE

$$(-\Delta_x + m^2 + \mathfrak{L})(\phi)(x,z) + \mathcal{A}^2[f(x')g(z')\partial V(\phi(x',z'))](x,z) = \xi^{\mathcal{A}}(x,z) \tag{1}$$

where $x \in \mathbb{R}^2$ and $z \in M$ (where M is a d dimensional manifold with or without boundary, equipped with a Radon positive measure $\mathrm{d}z$) and V is a smooth function on $\mathbb{R}$ growing at most exponentially at infinity, ∂V its gradient and $\phi : \mathbb{R}^2 \times M \to \mathbb{R}$ a scalar random field.

We now list the hypotheses and assumptions on the various elements of equation (1) and of its generalization to the vector case, where V is replaced by V_n being defined on $\mathbb{R}^n$ (when $\mathbb{R}^n = \mathbb{R}$ hereafter instead of writing V_1 we simply write V).

Hypothesis C. The potential $V_n : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function such that

$$y \in \mathbb{R}^n \mapsto V_n(y),$$

is strictly convex and it and its first and second partial derivatives grow at most exponentially at infinity.

Hypothesis QC. The potential $V_n : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and additionally

such that there exists a function $\mathfrak{H} : \mathbb{R} \to \mathbb{R}$ with exponential growth at infinity such that we have

$$-\langle \hat{n}, \partial V_n(y + r\hat{n})\rangle \leqslant \mathfrak{H}(y), \qquad \hat{n} \in \mathbb{S}^{n-1}, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}_+,$$

with $\mathbb{S}^{n-1}$ is the $n-1$ dimensional sphere.

Hypothesis Hf. The non-negative function $f : \mathbb{R}^2 \to \mathbb{R}$, is invariant with respect to rotations (i.e. there exists $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(x) = \tilde{f}(|x|^2)$), it has at least C^2 smoothness and in addition satisfies $f'(x) = \tilde{f}'(|x|^2) \leqslant 0$, it decays exponentially at infinity and fulfills $\Delta(f) \leqslant b^2 f$ for $b^2 \ll m^2$ (some examples of such functions are given in [56]).

Hypothesis Hg. The non-negative function $g : M \to \mathbb{R}$ is smooth and with compact support.

Hypothesis H$\mathfrak{L}$. The operator $\mathfrak{L}$ is closed (possibly unbounded) and defined on the vector space $D_{\mathfrak{L}} \subset L^2(M)$ (where $L^2(M)$ is the space of measurable L^2 functions defined on M with respect to the measure $\mathrm{d}z$ on M), which is dense in $L^2(M)$. The range of $\mathfrak{L}$ is a subset of $L^2(M)$. The operator $\mathfrak{L}$ is positive and self-adjoint and has a completely discrete spectrum. Furthermore we suppose that there exists at least an orthonormal basis $H_1, \ldots, H_k, \ldots$ in $L^2(M)$ composed by eigenfunctions of $\mathfrak{L}$ which are $C^0_{\mathfrak{w}}(M)$ functions (where $C^0_{\mathfrak{w}}(M)$ is the space of continuous functions h defined on M such that $\|h(z)\mathfrak{w}(z)\|_\infty < +\infty$ where $\mathfrak{w}$ is a positive continuous function uniformly bounded from above). Finally we denote by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ the eigenvalues of $\mathfrak{L}$ corresponding to the eigenfunctions $H_1, H_2, \ldots, H_k, \ldots$.

Hypothesis H$\mathcal{A}$. The operator $\mathcal{A} : L^2(M) \to L^2(M)$ is a linear, injective, continuous, positive self-adjoint operator commuting with $\mathfrak{L}$. If we denote by $\sigma_1, \ldots, \sigma_k, \ldots$ the eigenvalues of $\mathcal{A}$ corresponding to the basis $H_1, \ldots, H_k, \ldots$ we suppose that

$$\sum_{k=1}^{\infty} \sigma_k < +\infty \qquad \sum_{k=1}^{\infty} \sigma_k \|H_k\|^2_{\infty,\mathfrak{w}} < +\infty,$$

where $\|\cdot\|_{\infty,\mathfrak{w}}$ denotes the L^∞-norm in the weighted space $C^0_{\mathfrak{w}}(M)$.

Hypothesis Hξ. The noise $\xi^{\mathcal{A}}$ is Gaussian with mean zero and a covariance corresponding to a regularization provided by $\mathcal{A}$, namely such that if $h_1, h_2 \in C^\infty_0(\mathbb{R}^2 \times M)$ we have

$$\mathbb{E}[\langle h_1, \xi^{\mathcal{A}}\rangle\langle h_2, \xi^{\mathcal{A}}\rangle] = \int_{\mathbb{R}^2\times M} (I \otimes \mathcal{A})(h_1)(x,z)(I \otimes \mathcal{A})(h_2)(x,z)\mathrm{d}x\mathrm{d}z.$$

Remark 1 When $V_1 = V : \mathbb{R} \to \mathbb{R}$, Hypothesis QC can be reformulated as

$$\mp V'(y \pm r) \leqslant \mathfrak{H}(y),$$

where $r \geqslant 0$.

In the rest of this Subsection 2.1 we will always assume Hypothesis Hf, Hg, H$\mathfrak{L}$, H$\mathcal{A}$ and we specify the use of Hypothesis QC or C case by case. Moreover, hereafter we systematically use the following abuse of notation: we denote by $\mathfrak{L}$ and $\mathcal{A}$ both the operators with the same name defined on the space $L^2(M)$ and also the operators $I \otimes \mathfrak{L}$ and $I \otimes \mathcal{A}$ defined on $L^2(\mathbb{R}^2 \times M)$.

We describe now the setting for equation (1). We assume that $\xi^{\mathcal{A}}$ is defined on an abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu^{\mathcal{A}})$ where the Cameron-Martin space is

$$\mathcal{H} = L^2(\mathbb{R}^2) \otimes_H \mathcal{A}(L^2(M)),$$

(here $\otimes_H$ is the natural tensor product of Hilbert spaces) equipped with the following scalar product

$$\langle h_1, h_2\rangle = \int_{\mathbb{R}^2\times M} \mathcal{A}^{-1}(h_1)(x,z)\mathcal{A}^{-1}(h_2)(x,z)\mathrm{d}x\mathrm{d}z, \qquad h_1, h_2 \in \mathcal{H}.$$

The Wiener space $\mathcal{W}$ is given by

$$\mathcal{W} \quad = \quad (-\Delta_x + 1)(C^0_\ell(\mathbb{R}^2) \cap W^{1-,p}_\ell(\mathbb{R}^2)) \otimes_\epsilon \mathcal{A}^{1/2}(L^2(M)),$$

where $\ell \geqslant 0$, $C^0_\ell(\mathbb{R}^2)$ is the space of continuous functions on $\mathbb{R}^2$ such that

$$\|k\|_{\infty,\ell} = \sup_{x\in\mathbb{R}^2}(|k(x)|(1+|x|^2)^{-\ell/2}),$$

$W^{1-,p}_\ell(\mathbb{R}^2)$ denotes the Sobolev space of regularity $1-$ and weight

$$r_\ell(x) = (1+|x|^2)^{-\ell/2}, \tag{2}$$

and $\otimes_\epsilon$ denotes the injective tensor product whose norm is given by

$$\begin{aligned}
\|w\|_{\mathcal{W}} &= \sup\left\{\left|\sum_{i=1}^n \langle \mathfrak{b}, k_i\rangle_{\mathcal{O}_\ell}\langle \mathfrak{c}, h_i\rangle_{\mathcal{A}^{1/2}(L^2(M))}\right|, \mathfrak{b}\in\mathcal{B}^*_{\mathcal{O}_\ell} \quad \mathfrak{c}\in\mathcal{B}^*_{\mathcal{A}^{1/2}(L^2(M))}\right\}\\
&= \sup\left\{\left\|\sum_{i=1}^n \langle \mathfrak{b}, k_i\rangle_{\mathcal{O}_\ell} h_i\right\|_{\mathcal{A}^{1/2}(L^2(M))}, \mathfrak{b}\in\mathcal{B}^*_{\mathcal{O}_\ell}\right\}\\
&= \sup\left\{\left\|\sum_{i=1}^n \langle \mathfrak{c}, h_i\rangle_{\mathcal{A}^{1/2}(L^2(M))} k_i\right\|_{\mathcal{O}_\ell}, \mathfrak{c}\in\mathcal{B}^*_{\mathcal{A}^{1/2}(L^2(M))}\right\}
\end{aligned}$$

where we suppose that $w = \sum_{i=1}^n k_i \otimes h_i \in \mathcal{W}$, $\mathcal{O}_\ell = (-\Delta_x+1)(C^0_\ell(\mathbb{R}^2)\cap W^{1-,p}_\ell(\mathbb{R}^2))$, $\langle\cdot,\cdot\rangle_{\mathcal{O}_\ell}$ and $\langle\cdot,\cdot\rangle_{\mathcal{A}^{1/2}(L^2(M))}$ are the natural duality of the Banach spaces $\mathcal{O}_\ell$ and $\mathcal{A}^{1/2}(L^2(M))$ respectively and $\mathcal{B}^*_{\mathcal{O}_\ell}$ and $\mathcal{B}^*_{\mathcal{A}^{1/2}(L^2(M))}$ are the unitary balls of $\mathcal{O}^*_\ell$ and $(\mathcal{A}^{1/2}(L^2(M)))^*$ respectively (i.e., the natural dual Banach spaces of $\mathcal{O}_\ell$ and $\mathcal{A}^{1/2}(L^2(M))$; see [69] for more details). In particular if $w = \sum_{i=1}^\infty k_i \otimes h_i$ (where the series is supposed to converges in $\mathcal{W}$ with respect to its natural strong topology) we have

$$\|w\|_{\mathcal{W}} \leqslant \sup_{\tau\in\mathcal{B}_{\mathcal{A}^{-1/2}(L^2(M))}} \sqrt{\left(\sum_{i=1}^\infty\left(\int_M \tau(z)h_i(z)\mathrm{d}z\right)^2\right)}\sqrt{\sum_{k=1}^\infty \|k_i\|^2_{\mathcal{O}_\ell}}, \tag{3}$$

where $\mathcal{B}_{\mathcal{A}^{1/2}(L^2(M))}$ is the unitary ball of $\mathcal{A}^{-1/2}(L^2(M))$ (here we exploit the fact that $\mathcal{A}^{-1/2}(L^2(M))$ is the dual space of $\mathcal{A}^{1/2}(L^2(M))$ with respect the standard scalar product of $L^2(M)$), we use Cauchy-Schwarz inequality and the fact that $\mathcal{A}^{-1/2}(L^2(M))$ is the dual of $\mathcal{A}^{1/2}(L^2(M))$ with respect to the scalar product of $L^2(M)$.

The measure $\mu^{\mathcal{A}}$ is the centered Gaussian measure with Cameron-Martin space $\mathcal{H}$.

In order to prove that $(\mathcal{W},\mathcal{H},\mu^{\mathcal{A}})$ is actually an abstract Wiener space it is sufficient to prove that $\mathbb{E}[\|\xi^{\mathcal{A}}\|_{\mathcal{W}}] < +\infty$ where $\|\cdot\|_{\mathcal{W}}$ denotes the norm of $\mathcal{W}$. We have that

$$\xi^{\mathcal{A}}(x,z) = \sum_{k=1}^\infty \sigma_k \xi^k(x) H_k(z), \tag{4}$$

where $x\in\mathbb{R}^2$, $z\in M$, ξ^k are a sequence of independent Gaussian white noises defined on $\mathbb{R}^2$ and σ_k are defined as in Hypothesis H$\mathcal{A}$.

Using inequality (3) we have that

$$\begin{aligned}\|\xi^{\mathcal{A}}\|_{\mathcal{W}} &\leqslant \sup_{\tau\in\mathcal{B}_{\mathcal{A}^{-1/2}(L^2(M))}}\sqrt{\left(\sum_{k=1}^{\infty}\sigma_k\left(\int_M\tau(z)H_k(z)\mathrm{d}z\right)^2\right)}\cdot\\ &\quad\cdot\sqrt{\sum_{k=1}^{\infty}\sigma_k\|(-\Delta_x+1)^{-1}(\xi^k)\|^2_{C^0_\ell(\mathbb{R}^2)\cap W^{1-,p}_\ell(\mathbb{R}^2)}}\\ &\leq \sqrt{\sum_{k=1}^{\infty}\sigma_k\|(-\Delta_x+1)^{-1}(\xi^k)\|^2_{C^0_\ell(\mathbb{R}^2)\cap W^{1-,p}_\ell(\mathbb{R}^2)}},\end{aligned}\tag{5}$$

where in the second inequality we use the fact that $\sqrt{\left(\sum_{k=1}^{\infty}\sigma_k\left(\int_M\tau(z)H_k(z)\mathrm{d}z\right)^2\right)}=1$ whenever $\tau\in\mathcal{B}_{\mathcal{A}^{-1/2}(L^2(M))}$.

On the other hand, by Hypothesis H$\mathcal{A}$, we obtain

$$\mathbb{E}\left[\sum_{k=1}^{\infty}\sigma_k\|(-\Delta_x+1)^{-1}(\xi^k)\|^2_{C^0_\ell(\mathbb{R}^2)\cap W^{1-,p}_\ell(\mathbb{R}^2)}\right]\lesssim\sum_{k=1}^{\infty}\sigma_k<+\infty,$$

(where $\lesssim$ stands for $\leqslant$ modulo a multiplicative positive constant), and thus $\mathbb{E}[\|\xi^{\mathcal{A}}\|_{\mathcal{W}}]<+\infty$. This proves that $(\mathcal{W},\mathcal{H},\mu^{\mathcal{A}})$ is an abstract Wiener space when $\xi^{\mathcal{A}}:\mathcal{W}\to\mathcal{W}$, defined as $\xi^{\mathcal{A}}(w)=w$, has exactly $\mu^{\mathcal{A}}$ as probability distribution.

For later use, it is important to note that $\mathcal{W}$ is continuously embedded in

$$\mathcal{W}'=(-\Delta_x+1)(C^0_\ell(\mathbb{R}^2)\cap W^{1-,p}_\ell(\mathbb{R}^2))\otimes_\epsilon C^0_{\mathfrak{w}}(M),$$

since $\mathcal{A}^{1/2}(L^2(M))$ is continuously embedded in $C^0_{\mathfrak{w}}(M)$ by Hypothesis H$\mathcal{A}$. Furthermore by definition of injective tensor product we have $\mathcal{W}'\subset C^0_{(1+|x|)^{-\ell}\mathfrak{w}(z)}(\mathbb{R}^2\times M)$, indeed

$$\begin{aligned}\|w\|_{C^0_{(1+|x|)^{-\ell}\mathfrak{w}(z)}} &= \sup_{x_0\in\mathbb{R}^2,z\in M}|\langle r_\ell(x_0)\mathfrak{w}(z_0)\delta_{x_0}\otimes\delta_{z_0},w\rangle|\\ &\leqslant \sup\left\{|\langle\mathfrak{b}\otimes\mathfrak{c},w\rangle|,\mathfrak{b}\in\mathcal{B}^*_{\mathcal{O}_\ell}\quad\mathfrak{c}\in\mathcal{B}^*_{C^0_{\mathfrak{w}}(M)}\right\}=\|w\|_{\mathcal{W}'},\end{aligned}$$

where δ_{x_0} and δ_{z_0} are Dirac deltas in $x_0\in\mathbb{R}^2$ and $z_0\in M$ respectively (see [69] Section 3.2 for the details of the proof).

Equation (1) can be written as a problem defined on the abstract Wiener space $(\mathcal{W},\mathcal{H},\mu^{\mathcal{A}})$. In particular we define the map $U:\mathcal{W}\to\mathcal{H}$ by

$$U(w)=\mathcal{A}^2[f(x)g(z)V'(\mathcal{I}w)],$$

where $w\in\mathcal{W}$, $x\in\mathbb{R}^2$, $z\in M$, f and g as in equation (1), and $\mathcal{I}:\mathcal{W}\to(-\Delta_x+1)^{-1}(\mathcal{W})$ is the linear operator given by $\mathcal{I}=(-\Delta_x+m^2+\mathfrak{L})^{-1}$. We define the map $T:\mathcal{W}\to\mathcal{W}$ as $T(w)=w+U(w)$.

Definition 2 *It is clear that if S is any measurable map $S:\mathcal{W}\to\mathcal{W}$ satisfying $T\circ S=\mathrm{id}_{\mathcal{W}}$ $\mu^{\mathcal{A}}$-almost surely we have that $\phi(w,x,z)=\mathcal{I}(S(w))(x,z)$ is a (strong) solution to equation* (1)*. Furthermore if ν is a probability law on $\mathcal{W}$ such that $T_*(\nu)=\mu^{\mathcal{A}}$ then a random variable ϕ taking values on $\mathcal{I}(\mathcal{W})$ and having the same law as $\mathcal{I}_*(\nu)$ is a (weak) solution to equation* (1)*. For this reason in the following we call a map $S:\mathcal{W}\to\mathcal{W}$ such that $T\circ S=\mathrm{id}_{\mathcal{W}}$ $\mu^{\mathcal{A}}$-almost surely* a strong solution to equation (1) *and we call a measure ν on $\mathcal{W}$ such that $T_*(\nu)=\mu^{\mathcal{A}}$* weak solution to equation (1).

In order to solve equation (1) we need to introduce an approximation. Let P_n be the orthogonal projection in $L^2(M)$ onto the finite dimensional subspace generated by $H_1, \dots, H_n$. The restrictions $P_n|_{\mathcal{A}[L^2(M)]}$ and $P_n|_{\mathcal{A}^{1/2}[L^2(M)]}$ of P_n on $\mathcal{A}[L^2(M)]$ and $\mathcal{A}^{1/2}[L^2(M)]$ are the orthogonal projection on the subspace generated by $H_1, \dots, H_n$ too. This implies also that $I \otimes P_n$ (in the following denoted also simply by P_n) is a continuous linear operator on $\mathcal{W}$.

Let ϕ_n be the solution to the following approximated equation

$$(-\Delta_x + \mathfrak{L} + m^2)(\phi_n)(x,z) + P_n \mathcal{A}^2[f(x')g(z')V'(P_n(\phi_n)(x',z'))](x,z) = \xi^{\mathcal{A}}(x,z). \tag{6}$$

and let $U_n(w) = P_n(U(P_n(w)))$ and $T_n(w) = w + U_n(w)$ be the maps, analogous to U and T, related to equation (6). Using these objects we can define weak and strong solutions to equation (6) as in Definition 2.

Let us now study equation (6). We introduce the following function $V_n : \mathbb{R}^n \to \mathbb{R}$

$$V_n(y) = \int_M g(z) V\left(\sum_{k=1}^n y^k H_k(z)\right) \mathrm{d}z.$$

Then, since P_n commutes with $\mathfrak{L}$ and $\mathcal{A}$, being the projection on a subset of common eigenfunctions of both $\mathfrak{L}$ and $\mathcal{A}$, we have that ϕ_n is of the form

$$\phi_n(x,z) = \mathcal{I}\xi^{\mathcal{A}}(x,z) + \bar{\phi}_n(x,z) = \mathcal{I}\xi^{\mathcal{A}}(x,z) + \sum_{k=1}^n \bar{\psi}_n^k(x) H_k(z),$$

where $\bar{\psi}_n^k(x)$ solves the set of equations

$$\begin{aligned} 0 &= (-\Delta_x + m^2 + \lambda_k)(\bar{\psi}_n^k)(x) \\ &\quad + \sigma_k^2 f(x) \textstyle\int_M g(z)\, V'\Big(\sum_{j=1}^n \sigma_j \mathcal{I}_{\lambda_k}\xi^j(x) H_j(z) + \sum_{j=1}^n \bar{\psi}_n^j(x) H_j(z)\Big) H_k(z)\mathrm{d}z \\ &= (-\Delta_x + m^2 + \lambda_k)(\bar{\psi}_n^k)(x) + f(x)\sigma_k^2[\partial_{y_k}(V_n)(\tilde{\xi}_n(x) + \bar{\psi}_n)], \end{aligned} \tag{7}$$

with $\mathcal{I}_{\lambda_j} = (-\Delta_x + \lambda_j + m^2)^{-1}$, $\tilde{\xi}_n(x) = (\sigma_k \mathcal{I}_{\lambda_k}(\xi^k))_{k=1,\dots,n} \in \mathbb{R}^n$ and we used the fact that $\mathcal{A}$ is self-adjoint in $L^2(M)$. If we denote by $\mathcal{A}_n$ the $n \times n$ diagonal matrix such that $\mathcal{A}_n^{ij} = \sigma_j \delta^{ij}$, $i,j = 1, \dots, n$, and by $\mathfrak{L}_n$ the $n \times n$ matrix such that $\mathfrak{L}_n^{ij} = \lambda_i \delta^{ij}$ we can write equation (7) in the following way

$$(-\Delta_x + m^2 + \mathfrak{L}_n)(\psi_n) + f\mathcal{A}_n^2 \cdot \partial V_n(\psi_n) = \mathcal{A}_n \cdot \xi_n, \tag{8}$$

where $\psi_n = (\bar{\psi}_n^k + \tilde{\xi}_n^k)|_{k=1,\dots,n}$ and $\xi_n = (\xi^k)|_{k=1,\dots,n}$. Equation (8) is defined on the abstract Wiener space

$$\widehat{\mathcal{H}}_n := L^2(\mathbb{R}^2; \mathbb{R}^n),$$

with its natural scalar product and natural norm given by $\langle h, g\rangle = \sum_{i=1}^n \frac{1}{\sigma_i^2} \int_{\mathbb{R}^2} h^i(x) g^i(x) \mathrm{d}x$. Let $\widehat{\mathcal{W}}_n$ (in which $\widehat{\mathcal{H}}_n$ is densely embedded) be the space

$$\widehat{\mathcal{W}}_n := W_\ell^{-1-,p}(\mathbb{R}^2; \mathbb{R}^n) \cap (1 - \Delta_x)(C_\ell^0(\mathbb{R}^2; \mathbb{R}^n))$$

and $\hat{\mu}_n$ is the law of the white noise $\hat{\xi}_n = (\sigma_1 \xi^1, \dots, \sigma_n \xi^n)$. On $\widehat{\mathcal{W}}_n$ we can define the corresponding maps $\hat{U}_n(w)(x) = f(x)\partial V_n^{\mathcal{A}}(\mathcal{I}_{\mathfrak{L}_n}(w))$ and $\hat{T}_n(w) = w + \hat{U}_n(w)$ where

$$V_n^{\mathcal{A}}(y) := V_n(\sigma_1 y^1, \dots, \sigma_n y^n),$$

$y = (y^1, \dots, y^n)$, $y^i \in \mathbb{R}$, $i = 1, \dots, n$ and $\mathcal{I}_{\mathfrak{L}_n}$ is the matrix valued operator defined as $\mathcal{I}_{\mathfrak{L}_n}^{ij} = \delta^{ij}\mathcal{I}_{\lambda_j}$. Using $\hat{T}_n$ we can define the concept of strong and weak solutions to equation (8).

Remark 3 It is important to note that the relationship between ψ_n and the strong solution $\hat{S}_n$ (defined using the map $\hat{T}_n$) is given by $\psi_n(w,x) = \mathcal{I}_{\mathfrak{L}_n}(\hat{S}_n(w))(x)$, $x \in \mathbb{R}$. Furthermore if $\hat{\nu}_n$ is a weak solution to equation (8) (i.e. such that $\hat{T}_{n,*}(\hat{\nu}_n) = \hat{\mu}_n$) we have that $\hat{\nu}^n$ is the law of $(-\Delta_x + m^2 + \mathfrak{L}_n)(\psi_n)$.

In the following we shall denote by $\hat{\kappa}^n$ the probability law on $\mathbb{R}^n$ such that

$$\frac{\mathrm{d}\hat{\kappa}^n}{\mathrm{d}y} = \frac{\exp(-2\pi(|(m^2+\mathfrak{L}_n)^{1/2}(\mathcal{A}_n^{-1}\cdot y)|^2 + V_n(y)))}{Z_{\hat{\kappa}^n}},$$

where $Z_{\hat{\kappa}^n}$ is a suitable constant and $\mathrm{d}y$ is the Lebesgue measure on $\mathbb{R}^n$. Finally we define $\tilde{\Upsilon}_{f,n}$ as the random variable defined on $\widehat{\mathcal{W}}_n$ such that

$$\tilde{\Upsilon}_{f,n}(w) := \exp\left(4\int_{\mathbb{R}^2} f'(x)V_n(\mathcal{I}_{\mathfrak{L}_n}(w)(x))\mathrm{d}x\right) = \exp\left(4\int_{\mathbb{R}^2} f'(x)V_n(\psi_n(w,x))\mathrm{d}x\right).$$

Theorem 4 *If $V_n^{\mathcal{A}}$ satisfies Hypothesis QC then there exists at least a weak solution $\hat{\nu}^n$ to equation (7) such that for any bounded measurable function $F:\mathbb{R}^n\to\mathbb{R}$ we have*

$$\int_{\widehat{\mathcal{W}}_n} F(\mathcal{I}_{\mathfrak{L}_n}w(0))\tilde{\Upsilon}_{f,n}(w)\mathrm{d}\hat{\nu}_n(w) = Z_{f,n}\int_{\mathbb{R}^n} F(y)\mathrm{d}\hat{\kappa}^n \tag{9}$$

where

$$Z_{f,n} := \int_{\widehat{\mathcal{W}}_n} \tilde{\Upsilon}_{f,n}(w)\mathrm{d}\hat{\nu}_n(w). \tag{10}$$

Proof The proof is given in [10] Theorem 1 for the case $\mathfrak{L}_n = 0$. The case with $\mathfrak{L}_n$ a generic positive diagonal matrix is a trivial extension. □

It is simple to prove that if the potential V satisfies Hypothesis QC the potential $V_n^{\mathcal{A}}$ satisfies Hypothesis QC too, indeed the following proposition holds.

Proposition 5 *If $V:\mathbb{R}\to\mathbb{R}$ satisfies Hypothesis QC then $V_n^{\mathcal{A}}$ satisfies Hypothesis QC. Finally if V is strictly convex (i.e. it satisfies Hypothesis C) also $V_n^{\mathcal{A}}$ and V_n are convex.*

Proof First of all we note that

$$\partial_{y^k}V_n^{\mathcal{A}}(y) = \int_M g(z)V'\left(\mathcal{A}\left(\sum_{j=1}^n y^j H_j(z)\right)\right)\mathcal{A}(H_k)(z)\mathrm{d}z.$$

This means that if $\boldsymbol{n}\in\mathbb{S}^n$ we have

$$\begin{aligned}
-\boldsymbol{n}\cdot\partial V_n^A(y+t\boldsymbol{n}) &= -\sum_{k=1}^n n^k\partial_{y^k}(V_n^A)(y+t\boldsymbol{n}) \\
&= -\sum_{k=1}^n\int_M n^k g(z)V'\left(\mathcal{A}\sum_{j=1}^n y^j H_j(z) + \mathcal{A}\sum_{j=1}^n n^j H_j(z)\right)\mathcal{A}(H_k)(z)\mathrm{d}z \\
&\lesssim \int_M g(z)\mathfrak{H}\left(\mathcal{A}\sum_{j=1}^n y^j H_j(z)\right)\mathrm{d}z \\
&\lesssim \mathfrak{H}\left(|y|\sum_{k=1}^{+\infty}\sigma_k\left\|H_k\mathbb{1}_{g(z)\neq 0}\right\|_\infty\right)
\end{aligned}$$

where we use that g has compact support, the fact that V satisfies Hypothesis QC and the fact that $\mathfrak{H}$ is increasing. Since

$$\sum_{k=1}^{+\infty}\sigma_k\left\|H_k\mathbb{1}_{g(z)\neq 0}\right\|_\infty \leqslant \left\|\frac{1}{w}\mathbb{1}_{g(z)\neq 0}\right\|_\infty\sum_{k=1}^{+\infty}\sigma_k\|H_k\|_{\infty,w} < +\infty$$

by Hypothesis H$\mathcal{A}$, the thesis is proved. □

Using the weak solution $\hat{\nu}^n$ to equation (8), given by Theorem 4, we are able to construct a weak solution to equation (6) satisfying the dimensional reduction principle.

A weak solution ν_n to equation (6) is of the form $\nu_n = \tilde{\nu}_n \otimes \mu_n^{\mathcal{A}}$, where $\tilde{\nu}_n$ is the law of

$$(-\Delta_x + m^2 + \mathfrak{L})P_n(\phi_n)(x,z) = \sum_{k=1}^{n}(-\Delta_x + m^2 + \lambda_k)(\psi_n^k)(x)H_k(z),$$

on the subspace $\mathrm{Im}(P_n) \subset \mathcal{W}$ and $\mu_n^{\mathcal{A}}$ is the law of

$$Q_n(\phi_n) = (-\Delta_x + m^2 + \mathfrak{L})(I - P_n)(\phi_n)(x,z) = \sum_{k=n+1}^{+\infty} \sigma_k \xi^k(x)H_k(z),$$

which is the law of the Gaussian field on $\mathrm{Im}(Q_n) \subset \mathcal{W}$. Using the basis $H_1, \ldots, H_n$ we can identify the law $\tilde{\nu}_n$ on $\mathrm{Im}(P_n)$ with a probability measure $\hat{\nu}_n$ on $\widehat{\mathcal{W}}^n$. In this way it is evident that if the law $\hat{\nu}_n$ satisfies the dimensional reduction principle on $\widehat{\mathcal{W}}^n$ then the probability law $\nu_n = \hat{\nu}_n \otimes \mu_n^{\mathcal{A}}$ satisfies the dimensional reduction principle on $\mathcal{W}$, since it is the tensor product of two probability laws satisfying the dimensional reduction principle.

More precisely, we consider the following abstract Wiener space $(\mathbb{W}, \mathbb{H}, \mu^{\mathcal{A},\mathfrak{L}})$ where

$$\begin{aligned} \mathbb{W} &= \mathcal{A}^{1/2}(L^2(M)), \\ \mathbb{H} &= (\mathfrak{L} + m^2)^{-1/2}(\mathcal{A}(L^2(M))), \end{aligned}$$

(with $\mathcal{A}$ and $\mathfrak{L}$ as in (1)) and $\mathbb{W}$ is equipped with its natural norm while on $\mathbb{H}$ we define the following scalar product

$$\langle h_1, h_2 \rangle_{\mathbb{H}} = \frac{1}{4\pi}\int_M (\mathfrak{L} + m^2)^{1/2}(\mathcal{A}^{-1}(h_1))(z)(\mathfrak{L} + m^2)^{1/2}(\mathcal{A}^{-1}(h_2))(z)\mathrm{d}z. \tag{11}$$

Thus $\mu^{\mathcal{A},\mathfrak{L}}$ is the centered Gaussian measure on $\mathbb{W}$ with Cameron-Martin space given by $\mathbb{H}$. We now denote by κ_n the measure on $\mathbb{W}$ such that

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}\mu^{\mathcal{A},\mathfrak{L}}}(\omega) = \exp\left(-4\pi\int_M g(z)V(P_n(\omega)(z))\mathrm{d}z\right)\Big/ Z_{\kappa_n} \tag{12}$$

where $\omega \in \mathbb{W}$ and Z_{κ_n} is a renormalization constant.

We introduce also the following random variable

$$\Upsilon_{f,n}(w) := \exp\left(4\int_{\mathbb{R}^2\times M} f'(x)g(z)V(P_n(\mathcal{I}w)(x,z))\mathrm{d}x\mathrm{d}z\right) \tag{13}$$

$$\Upsilon_{f}(w) := \exp\left(4\int_{\mathbb{R}^2\times M} f'(x)g(z)V(\mathcal{I}w(x,z))\mathrm{d}x\mathrm{d}z\right) \tag{14}$$

where $w \in \mathcal{W}$. If we use the previous identification of $\mathrm{Im}(P_n)$ with $\widehat{\mathcal{W}}^n$ we have that

$$\tilde{\Upsilon}_{f,n}(P_n(w)) = \Upsilon_{f,n}(w).$$

Using the previous observations the next proposition (expressing the dimensional reduction principle) trivially follows.

Proposition 6 *If V satisfies Hypothesis QC then there exists a weak solution ν_n to equation* (6) *such that for any measurable function $F : \mathbb{W} \to \mathbb{R}$ we have that*

$$\int_{\mathcal{W}} F(\mathcal{I}w(0,\cdot))\Upsilon_{f,n}(w)\mathrm{d}\nu_n(w) = Z_{f,n}\int_{\mathbb{W}} F(\omega)\mathrm{d}\kappa_n(\omega), \tag{15}$$

with $Z_{f,n}$ as in (10).

The rest of the section concerns the generalization of Proposition 6 to solutions to equation (1). In particular we denote by κ the probability measure on $\mathbb{W}$ such that

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\mu^{\mathcal{A},\mathfrak{L}}}(\omega) = \exp\left(-4\pi\int_M g(z)V(\omega(z))\mathrm{d}z\right)\Big/ Z_\kappa\ , \tag{16}$$

where $\omega \in \mathbb{W}$.

Theorem 7 *Suppose that V satisfies Hypothesis QC then there exists at least a weak solution ν to equation (7) such that for any Borel measurable function $F : \mathbb{W} \to \mathbb{R}$ we have*

$$\int_{\mathcal{W}} F(\mathcal{I}w(0,\cdot))\Upsilon_f(w)\mathrm{d}\nu(w) = Z_f\int_{\mathbb{W}} F(\omega)\mathrm{d}\kappa(\omega), \tag{17}$$

where κ is defined as in (12), Υ_f is given by equation (14), and

$$Z_f = \int_{\mathcal{W}} \Upsilon_f(w)\mathrm{d}\nu(w).$$

Remark 8 Since κ_n converges weakly to κ, the right hand side of equation (15) converges to the right hand side of equation (17), as $n \to +\infty$.

In order to prove Theorem 7 we prove the following lemmas. In the space of functions $C^0_\ell(\mathbb{R}^2,\mathbb{R}^n)$ we denote by $\|\cdot\|^{\mathcal{A}}_{\ell,k}$ the following norm

$$\|h\|^{\mathcal{A}}_{\ell,k} \quad = \quad \sup_{x\in\mathbb{R}^2}\sqrt{\sum_{j=1}^n \sigma_j^{2k}(h^j(x))^2 r_\ell(x)^2},$$

where $h \in C^0(\mathbb{R}^2,\mathbb{R}^n)$ and $k,\ell \in \mathbb{R}$ (r_ℓ is the weight we introduced in (2)).

Lemma 9 *Let $\bar\psi_n$ be a solution to the equation (7) and let $\ell,\ell' > 0$, then we have*

$$\|\bar\psi_n\|^{\mathcal{A}}_{\ell,-1} \lesssim \left\|\exp(\alpha\Xi_{\ell',n}r_{-\ell'}(x)\mathfrak{w}^{-1}(z))\mathbb{1}_{g(z)\neq 0}f(x)r_\ell(x)\right\|_\infty \tag{18}$$

$$\begin{aligned}&\|(-\Delta_x + m^2 + \mathfrak{L}_n)(\bar\psi_n)\|^{\mathcal{A}}_{\ell,-1}\\ &\quad\lesssim \left\|\exp(\alpha(\alpha'(\Xi_{\ell',n}\mathfrak{w}^{-1}(z) + \|\bar\psi_n\|^{\mathcal{A}}_{\ell',-1})r_{-\ell'}(x))\mathbb{1}_{g(z)\neq 0}f(x)r_\ell(x)\right\|_\infty\end{aligned} \tag{19}$$

where $\Xi_{\ell',n} = \|P_n\mathcal{I}\xi^{\mathcal{A}}(x,z)\|_{C^0_{\ell'}(\mathbb{R}^2)\otimes_\epsilon C^0_{\mathfrak{w}}(M)}$, α,α' and the implicit constants do not depend on n.

Proof We write $\tilde\psi_n = \left(\frac{\bar\psi^k_n}{\sigma_k}\right)_{k=1,\dots,n}$. The equation for $\tilde\psi_n$ reads

$$(-\Delta_x + m^2 + \lambda_k)(\tilde\psi^k_n)(x) + \sigma_k f(x)\int_M g(z)H_k(z)V'(P_n\mathcal{I}\xi^A(x,z) + \bar\phi_n(x,z)\mathrm{d}z) = 0$$

where $\bar\phi_n(x,z)$ is related to $\tilde\psi_n$ by

$$\bar\phi_n(x,z) = \sum_{k=1}^n \sigma_k\tilde\psi_k(x)H_k(z).$$

Putting $\tilde{\Psi}_n^2(x) = (1+\theta|x|^2)^{-\ell}\sum_{k=1}^n(\tilde{\psi_n^k}(x))^2$, writing $r_{\ell,\theta}(x) = (1+\theta|x|^2)^{-\ell}$, for some $\theta > 0$, and denoting by $\bar{x}$ the maximum of $\tilde{\Psi}_n$ we have

$$\begin{aligned}
m^2\tilde{\Psi}_n^2(\bar{x}) &\leqslant -\frac{1}{2}\Delta(\tilde{\Psi}_n^2)(\bar{x}) + m^2\tilde{\Psi}_n^2(\bar{x}) + r_{\ell,\theta}(\bar{x})\tilde{\psi_n}(\bar{x})\cdot\mathfrak{L}_n(\tilde{\psi_n}(\bar{x}))\\
&\leqslant -r_{\ell,\theta}\tilde{\psi_n}\cdot\Delta\tilde{\psi_n} - r_{\ell,\theta}|\nabla\tilde{\psi_n}|^2 - \left(\frac{|\nabla r_{\ell,\theta}|^2 + \Delta r_{\ell,\theta}}{r_{\ell,\theta}^2}\right)\tilde{\Psi}_n^2(\bar{x})\\
&\quad + m^2\tilde{\Psi}_n^2(\bar{x}) + r_{\ell,\theta}\tilde{\psi_n}(\bar{x})\cdot\mathfrak{L}_n(\tilde{\psi_n}(\bar{x}))\\
&\leqslant -f(\bar{x})r_{\ell,\theta}(\bar{x})\int_M g(z)\bar{\phi}_n(\bar{x},z)V'(P_n\mathcal{I}\xi^A(\bar{x},z) + \bar{\phi}_n(\bar{x},z))\mathrm{d}z\\
&\quad - \left[\frac{|\nabla r_{\ell,\theta}|^2 + \Delta r_{\ell,\theta}}{r_{\ell,\theta}^2}\right]\tilde{\Psi}_n^2(\bar{x}),
\end{aligned} \tag{20}$$

where we used that

$$\nabla\left[\sum_{k=1}^n(\tilde{\psi_n^k}(\bar{x}))^2\right] = -\frac{\nabla r_{\ell,\theta}(\bar{x})}{r_{\ell,\theta}(\bar{x})}\sum_{k=1}^n(\tilde{\psi_n^k}(\bar{x}))^2$$

since $\bar{x}$ is a stationary point for $\tilde{\Psi}_n$. From equation (20) if, for any fixed ℓ, we choose a θ in such a way that

$$\left[\frac{|\nabla r_{\ell,\theta}|^2 + \Delta r_{\ell,\theta}}{r_{\ell,\theta}^2}\right] < m^2$$

we obtain

$$\tilde{\Psi}_n^2(\bar{x}) \lesssim \left\|\exp(\alpha\Xi_{\eta',n}r_{-\ell'}(x)\mathfrak{w}^{-1}(z))\mathbb{1}_{g(z)\neq 0}\right\|_\infty \cdot \left\|\bar{\phi}_n(x,z)\mathbb{1}_{g(z)\neq 0}\right\|_\infty$$

where all the constants do not depend on n. On the other hand

$$\begin{aligned}
|\bar{\phi}_n(x,z)| &\leqslant \sum_{k=1}^n|\bar{\psi_n^k}(x)|\cdot|H_k(z)|\\
&\lesssim \left\|\frac{1}{w(z)}\mathbb{1}_{g(z)\neq 0}\right\|_\infty\tilde{\Psi}_n(\bar{x})\sqrt{\sum_{k=1}^\infty\sigma_k\|H_k(z)w(z)\|_\infty^2},
\end{aligned}$$

for all $x\in\mathbb{R}^2$ and $z\in\mathrm{supp}(g)$. Using Hypothesis H$\mathcal{A}$ we deduce from this that

$$\tilde{\Psi}_n(\bar{x}) \lesssim \left\|\exp(\alpha\Xi_{\ell',n}r_{-\ell'}(x)w^{-1}(z))\mathbb{1}_{g(z)\neq 0}\right\|_\infty.$$

Since $\|\bar{\psi_n}\|_{\ell,-1}^{\mathcal{A}} \lesssim_\theta \tilde{\Psi}_n(\bar{x})$ inequality (18) is proved. Inequality (19) follows directly from inequality (18) and equation (7). □

Corollary 10 *Under the hypotheses and notations of Lemma 9, if $\ell > 1$, we have that there exists an increasing continuous function $K:\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\|(-\Delta_x + m^2 + \mathfrak{L})(\bar{\phi}_n)\|_{\mathcal{H}} \leqslant K(\Xi_{\ell,n}),$$

with $\Xi_{\ell,n}$ as in Lemma 9.

Proof The proof consists simply in noting that

$$\|(-\Delta_x + m^2 + \mathfrak{L})(\bar{\phi}_n)\|_{\mathcal{H}} \lesssim \|(-\Delta_x + m^2 + \mathfrak{L}_n)(\bar{\psi}_n)\|_{\ell,-1}^{\mathcal{A}},$$

indeed

$$\begin{aligned}
\|(-\Delta_x + m^2 + \mathfrak{L})(\bar{\phi}_n)\|_{\mathcal{H}}^2 &= \int_{M\times\mathbb{R}^2}(-\Delta_x + m^2 + \mathfrak{L})(\mathcal{A}^{-1}(\bar{\phi}_n)(x,z))^2\mathrm{d}x\mathrm{d}z\\
&= \sum_{k=1}^n\frac{1}{\sigma_k^2}\int_{\mathbb{R}^2}((-\Delta_x + m^2 + \lambda_k)(\bar{\psi}_n)(x))^2\mathrm{d}x\\
&\leqslant (\|(-\Delta_x + m^2 + \mathfrak{L}_n)(\bar{\psi}_n)\|_{\ell,-1}^{\mathcal{A}})^2\int_{\mathbb{R}^2}(1+|x|^2)^{-\ell}\mathrm{d}x
\end{aligned}$$

which is finite whenever $\ell > 1$. □

An important consequence of Corollary 10 is the following lemma.

Lemma 11 *Let $K \subset \mathcal{W}$ be a compact set, then $\mathfrak{K} = \cup_{n\in\mathbb{N}} T_n^{-1}(K)$ is precompact in $\mathcal{W}$.*

Proof We write $\mathcal{T}_n(\hat{\phi}, w) = \hat{\phi} + U_n(\hat{\phi} + w)$. If $w_{n,y} \in \mathcal{W}$ is solution to the equation $T_n(w_{n,y}) = y$ then $\hat{\phi}_{n,y} = w_{n,y} - y$ will be solution to the equation $\mathcal{T}_n(\hat{\phi}_{n,y}, y) = 0$. This will implies that $\bar{\phi}_{n,y} = \mathcal{I}(\hat{\phi}_{n,y})$ is a solution to equation (7) for the realization of the white noise $\xi^{\mathcal{A}}(y)$. By Corollary 10 we have that $\|\hat{\phi}_{n,y}\|_{\mathcal{H}} \leqslant K(\Xi_{n,\ell,y})$. On the other hand we have the estimate

$$\sup_{n\in\mathbb{N}}(\Xi_{\ell,n,y}) = \sup_{n\in\mathbb{N}} \|P_n \mathcal{I}\xi^{\mathcal{A}}(y)\|_{C^0_\ell(\mathbb{R}^2)\otimes_\epsilon C^0_{\mathfrak{w}}(M)} \lesssim \sup_{n\in\mathbb{N}} \|P_n \xi^{\mathcal{A}}(y)\|_{\mathcal{W}} \lesssim \|\xi^{\mathcal{A}}(y)\|_{\mathcal{W}},$$

which implies that

$$\sup_{n\in\mathbb{N}, y\in K} \|\hat{\phi}_{n,y}\|_{\mathcal{H}} \leqslant \sup_{n\in\mathbb{N}, y\in K} K(\Xi_{n,\ell,y}) \lesssim \sup_{y\in K} K(\|\xi^{\mathcal{A}}(y)\|_{\mathcal{W}}) = C_K < +\infty.$$

Letting $\tilde{K} = \{h \in \mathcal{H}, \|w\|_{\mathcal{H}} < C_K\}$ we have that $\tilde{K}$ is compact in $\mathcal{W}$ since the inclusion map $i : \mathcal{H} \to \mathcal{W}$ is compact. This fact implies that $\mathfrak{K} \subset K + \tilde{K}$ is precompact since it is contained in the sum of two compact sets. □

Lemma 12 *The family $(\nu_n)_n$ of measures such that $T_{n,*}(\nu_n) = \mu^{\mathcal{A}}$ is tight.*

Proof Let $\tilde{K}$ be a compact set such that $\mu^{\mathcal{A}}(\tilde{K}) \geqslant 1-\epsilon$ for a fixed $0 < \epsilon < 1$, then, by Lemma 11, $\mathfrak{K} := \overline{\cup_{n\in\mathbb{N}} T_n^{-1}(\tilde{K})}$ is a compact set in $\mathcal{W}$. This implies

$$\nu_n(\mathfrak{K}) \geqslant \nu_n(\cup_i T_i^{-1}(\tilde{K})) \geqslant \nu_n(T_n^{-1}(\tilde{K})) \geqslant \mu^{\mathcal{A}}(\tilde{K}) \geqslant 1 - \epsilon,$$

for any $n \in \mathbb{N}$. Thus the sequence ν_n is tight and the lemma is proven. □

Lemma 13 *Suppose that $(\nu_n)_n$ is a sequence of weak solutions to equation (6) weakly converging to ν, then ν is a solution to (1), i.e. $T_*(\nu) = \mu^{\mathcal{A}}$.*

Proof Proving the claim is equivalent to prove that for any function $h : \mathcal{W} \to \mathbb{R}$ which is bounded with continuous and bounded Fréchet derivative we have $\int h \circ T \mathrm{d}\nu = \int h \mathrm{d}\mu^{\mathcal{A}}$. In order to prove this we note that

$$\|h \circ T(w) - h \circ T_n(w)\|_{\mathcal{W}} \leqslant \|h\|_{C^1(\mathcal{W})} \cdot \|U(w) - P_n U(P_n(w))\|_{\mathcal{W}}.$$

On the other hand we have

$$\|U(w) - P_n U(P_n(w))\|_{\mathcal{W}} \leqslant \sqrt{\sum_{k=n}^{\infty} \sigma_k} \cdot \|U(w)\|_{\mathcal{H}} + \sup_{w\in[w,P_n(w)]} \|\nabla U(y)\|_{\mathcal{L}(\mathcal{W})} \|P_n(w) - w\|_{\mathcal{W}}.$$

Finally we observe that

$$\|P_n(w) - w\|_{\mathcal{W}} = \|(I - P_n)(T_n(w))\|_{\mathcal{W}}.$$

Let, for any fixed $\varepsilon > 0$, $K_\epsilon \subset \mathcal{W}$ be a compact set such that $\nu_n(\mathcal{W}\backslash K_\epsilon), \nu(\mathcal{W}\backslash K_\epsilon) < \epsilon$, then we have

$$\begin{aligned}
\left|\int_{\mathcal{W}} h \circ T \mathrm{d}\nu - \int_{\mathcal{W}} h \mathrm{d}\mu\right| &\leqslant \left|\int_{\mathcal{W}} h \circ T \mathrm{d}\nu - \int_{\mathcal{W}} h \circ T \mathrm{d}\nu_n\right| + \left|\int_{\mathcal{W}} (h \circ T - h \circ T_n) \mathrm{d}\nu_n\right| \\
&\quad + \left|\int_{\mathcal{W}} h \circ T_n \mathrm{d}\nu_n - \int_{\mathcal{W}} h \mathrm{d}\mu\right| \\
&\leqslant \left|\int_{\mathcal{W}} h \circ T \mathrm{d}\nu - \int_{\mathcal{W}} h \circ T \mathrm{d}\nu_n\right| + \epsilon \|h\|_{C^0(\mathcal{W})} \\
&\quad + \left|\int_{K_\epsilon} (h \circ T - h \circ T_n) \mathrm{d}\nu_n\right|. \qquad (21)
\end{aligned}$$

But

$$\left|\int_{K_\epsilon}(h\circ T-h\circ T_n)\mathrm{d}\nu_n\right| \lesssim_h \sqrt{\sum_{k=n}^{\infty}\sigma_k}\cdot \sup_{w\in K_\epsilon}\|U(w)\|_{\mathcal{H}}+\sup_{w\in K_\epsilon}\|\nabla U(w)\|_{\mathcal{L}(\mathcal{W})} \times\int_{\mathcal{W}}\|(I-P_n)(T_n(w))\|_{\mathcal{W}}\mathrm{d}\nu_n. \tag{22}$$

On the other hand

$$\int_{\mathcal{W}}\|(I-P_n)(T_n(w))\|_{\mathcal{W}}\mathrm{d}\nu_n=\int_{\mathcal{W}}\|(I-P_n)(w)\|_{\mathcal{W}}\mathrm{d}\mu^{\mathcal{A}}\lesssim\sum_{k=n}^{\infty}\sigma_k, \tag{23}$$

where we use that $\mathbb{E}[\|(-\Delta_x+1)^{-1}(\xi^k)\|^2_{C^0_\ell(\mathbb{R}^2)\cap W^{1-,p}_\ell(\mathbb{R}^2)}]$ is equal to a finite constant independent on k. Since $\sum\sigma_k$ is convergent, we have that the right hand side of (23) converges to 0 as $n\to+\infty$. Since $U(w),\nabla U(w)$ are continuous with respect to w, this implies that the right hand side of equation (22), converges to 0 as $n\to+\infty$. Exploiting this fact, the fact that ν_n weakly converges to ν and equation (21) we obtain that $\left|\int_{\mathcal{W}}h\circ T\mathrm{d}\nu-\int_{\mathcal{W}}h\mathrm{d}\mu^{\mathcal{A}}\right|\leqslant\epsilon\|h\|_{C^0(\mathcal{W})}$. Since $\epsilon>0$ is arbitrary the lemma is proved. □

Proof of Theorem 7 Thanks to Lemma 13 we have that the sequence of weak solutions ν_n, satisfying the dimensional reduction principle (whose existence is proven by Proposition 6) converges weakly, as $n\to+\infty$, to a solution ν of the equation (1). If we are able to prove that $\Upsilon_{n,f}\nu_n\to\Upsilon_f\nu$ weakly, then the theorem would follow.

First of all we note that $\Upsilon_{n,f}(w)=\Upsilon_f(P_n(w))$ and that Υ_f is a bounded Fréchet $C^1(\mathcal{W})$ function. We want to prove that $\nabla\Upsilon_f(w)$ is bounded when w is in a bounded set of $\mathcal{W}$. We have that

$$\nabla\Upsilon_f(w)[h]=-\Upsilon_f(w)\cdot\int_{\mathbb{R}^2\times M}f'(x)g(z)V'(\mathcal{I}(w)(x,z))(\mathcal{I}h)(x,z)\mathrm{d}x\mathrm{d}z.$$

In particular we have that

$$\|\nabla\Upsilon_f(w)\|_{\mathcal{W}^*}\lesssim\int_{\mathbb{R}^2\times M}|f'(x)g(z)|\exp(\alpha\|\mathcal{I}(w)\|_{C^0_\ell(\mathbb{R}^2)\otimes_\epsilon C^0_{\mathfrak{w}}(M)}(1+|x|)^\ell\mathfrak{w}^{-1}(z))\mathrm{d}x\mathrm{d}z,$$

for a suitable positive constant α. On the other hand the linear map $\mathcal{I}:\mathcal{W}\to C^0_\ell(\mathbb{R}^2)\otimes_\epsilon C^0_{\mathfrak{w}}(M)$ is continuous, which implies that if B is a bounded set of $\mathcal{W}$ given by $\sup_{w\in B}\|\mathcal{I}(w)\|_{C^0_\ell(\mathbb{R}^2)\otimes_\epsilon C^0_{\mathfrak{w}}(M)}<+\infty$. This observation proves that $\sup_{w\in B}\|\nabla\Upsilon_f(w)\|_{\mathcal{W}^*}<+\infty$ for any bounded set $B\subset\mathcal{W}$.

Let for any given $\epsilon>0$, $K_\epsilon\subset\mathcal{W}$ be a compact set such that $\nu_n(\mathcal{W}\backslash K_\epsilon),\nu(\mathcal{W}\backslash K_\epsilon)<\epsilon$, then there exists a ball $B_\epsilon\subset\mathcal{W}$ such that $K_\epsilon\cup(\cup_{n\in\mathbb{N}}P_n(K_\epsilon))\subset B_\epsilon$. Let F be a continuous and bounded function on $\mathcal{W}$, then

$$\begin{aligned}\left|\int_{\mathcal{W}}F(w)\Upsilon_{f,n}\mathrm{d}\nu_n-\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu\right| &\leqslant \left|\int_{\mathcal{W}}F(w)(\Upsilon_f(P_n(w))-\Upsilon_f(w))\mathrm{d}\nu_n\right|+\\ &\quad+\left|\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu_n-\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu\right|\\ &\lesssim \epsilon+\|\nabla\Upsilon_f(w)\|_{C^1(B_\epsilon)}\int_{\mathcal{W}}\|P_n(w)-w\|_{\mathcal{W}}\mathrm{d}\nu_n\\ &\quad+\left|\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu_n-\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu\right|\\ &\lesssim \epsilon+\|\nabla\Upsilon_f(w)\|_{C^1(B_\epsilon)}\int_{\mathcal{W}}\|P_n(T_n(w))-T_n(w)\|_{\mathcal{W}}\mathrm{d}\nu_n\\ &\quad+\left|\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu_n-\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu\right|\to\epsilon,\end{aligned}$$

as $n\to\infty$, where the constants implied in the symbol $\lesssim$ do not depend on ϵ. For this reason since ϵ is arbitrary positive, $\int_{\mathcal{W}}F(w)\Upsilon_{f,n}\mathrm{d}\nu_n\to\int_{\mathcal{W}}F(w)\Upsilon_f\mathrm{d}\nu$ and the theorem is proven. □

2.2 Dimensional reduction in the full space

In this section we consider the following equation

$$(-\Delta_x - \Delta_z + m^2)(\phi)(x,z) + \mathcal{A}^2[f(x')g(z')V'(\phi(x',z'))](x,z) = \xi^{\mathcal{A}}(x,z), \tag{24}$$

where $x, x' \in \mathbb{R}^2$ and $z, z' \in M$. We consider the following hypotheses on $\mathfrak{L}$ and $\mathcal{A}$ (in addition to the previous Hf and Hg on the cut-offs):

Hypothesis H$\mathfrak{L}$1 $\mathfrak{L} = -\Delta_z$ and $M = \mathbb{R}^d$ (for any fixed integer $d \geqslant 0$).

Hypothesis H$\mathcal{A}$1 $\mathcal{A}(h(z')) = \mathfrak{a} * h$ where $\mathfrak{a}(z)$ is a $\mathcal{C}^{1+d/2+}(\mathbb{R}^2)$ Hölder continuous function with compact support.

Hypothesis Hξ1 The noise $\xi^{\mathcal{A}}$ is such that if $h_1, h_2 \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^d)$ we have

$$\mathbb{E}[\langle h_1, \xi^{\mathcal{A}}\rangle \langle h_2, \xi^{\mathcal{A}}\rangle] = \int_{\mathbb{R}^2 \times \mathbb{R}^d} (I \otimes \mathcal{A})(h_1)(x,z)(I \otimes \mathcal{A})(h_2)(x,z)\mathrm{d}x\mathrm{d}z.$$

In this subsection we assume Hypotheses H$\mathfrak{L}$1, H$\mathcal{A}$1 and Hξ1 instead of the corresponding assumptions H$\mathfrak{L}$, H$\mathcal{A}$ and Hξ respectively we had before. The assumptions Hf and Hg on the space cut-offs however remain.

We define the abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu^{\mathcal{A}})$ relative to equation (24) as follows

$$\begin{aligned} \mathcal{H} &= L^2(\mathbb{R}^2) \otimes_H \mathcal{A}(L^2(\mathbb{R}^d)) \\ \mathcal{W} &= (-\Delta_x + 1)(C^0_\ell(\mathbb{R}^2) \cap W^{1-,p}_\ell(\mathbb{R}^2)) \otimes_\epsilon C^0_{\ell'}(\mathbb{R}^d) \end{aligned}$$

where we suppose $\ell, \ell' > 0$. The maps T and U, and so the concept of weak and strong solution to equation (24), are defined in the same way as in the previous section. We shall prove the following theorem.

Theorem 14 *Suppose that V satisfies Hypothesis QC (in Section 2.1), then there exists a weak solution ν to equation* (24) *such that for any Borel measurable function $F : \mathbb{W} \to \mathbb{R}$*

$$\int_{\mathcal{W}} F(\mathcal{I}w(0))\Upsilon_f(w)\mathrm{d}\nu(w) = Z_f \int_{\mathbb{W}} F(\omega)\mathrm{d}\kappa(\omega),$$

as in (17).

It is clear that the conditions H$\mathfrak{L}$1, H$\mathcal{A}$1 on $\mathfrak{L}, M, \mathcal{A}$ are incompatible with the previous Hypotheses H$\mathfrak{L}$ and H$\mathcal{A}$. Thus Theorem 14 does not follow directly from Theorem 7. The rest of this subsection will focus on the proof of Theorem 14.

In order to achieve the proof we first replace M by the manifolds $M_R = \mathbb{T}^d_R$, i.e. M_R is a torus of radius $R > 0$. If h is a real-valued function on $M = \mathbb{R}^d$ with sufficient decay at infinity we can easily define a function h_R on M_R in the following way

$$h_R(z_R) = \sum_{j \in \mathbb{Z}^d} h(z_R + Rj)$$

where $z_R \in [-R/2, R/2]^d$. There is also a standard way of defining a function $\tilde{h}_R$ on M_R given one on M, that is the following

$$\tilde{h}_R(z_R) = h(z_R).$$

If the function h has support contained in $[-R/2, R/2]^d$ then $\tilde{h}_R = h_R$.

On the other hand if we consider a function $k : M_R \to \mathbb{R}$ we can associated with it a function k^p defined on all of M in the following way

$$k^p(z) = k\left(-R + 2R\left(z - \left\lfloor \frac{z}{2R} \right\rfloor\right)\right),$$

$z \in M = \mathbb{R}^d$. In general we have that $\widetilde{(k^p)}_R = k$, and so if k has compact support contained in $(-R/2, R/2)^d$ we have $k = (k^p)_R$. Furthermore if $\mathfrak{b}$ is a function with compact support on $\mathbb{R}^d$ and k is a function on M_R we have $(\mathfrak{b}_R * k)^p = \mathfrak{b} * k^p$. Furthermore it is important to note that if $\mathfrak{G}_R$ is the Green function associated with the operator $(-\Delta_x - \Delta_z + m^2)^{-1}$ on $\mathbb{R}^2 \times M_R$ and $\mathcal{G}$ is the Green function of the same operator on $\mathbb{R}^2 \times M$ we have

$$\mathcal{G}_R = \mathfrak{G}_R$$

We define the operator $\mathcal{A}_R$ on $L^2(M_R)$ by $\mathcal{A}_R(k) = \mathfrak{a}_R * k$. In this way we can define the abstract Wiener space $(\mathcal{W}_R, \mathcal{H}_R, \mu_R^{\mathcal{A}_R})$ as follows

$$\begin{aligned} \mathcal{H}_R &= L^2(\mathbb{R}^2) \otimes_H \mathcal{A}_R(L^2(M_R)) \\ \mathcal{W}_R &= (-\Delta_x + 1)(C^0_\ell(\mathbb{R}^2) \cap W^{1-,p}_\ell(\mathbb{R}^2)) \otimes_\epsilon \mathcal{A}_R^{1/2}(L^2(M_R)) \end{aligned}$$

and $\mu_R^{\mathcal{A}_R}$ is the law of the noise $\xi_R^{\mathcal{A}}$ on $\mathcal{W}_R$ with covariance $\mathcal{A}_R$.

Using the map $(\cdot)^p$ we can extend the noise $\xi_R^{\mathcal{A}}$, defined on M_R, to the noise $\xi_R^{\mathcal{A},p}$ defined on all M. This means that the law $\mu_R^{\mathcal{A}_R}$ of $\xi_R^{\mathcal{A}}$ on $\mathcal{W}_R$, thanks to the map $(\cdot)^p$,induces a Gaussian measure $\mu_R^{\mathcal{A}_R,p}$ on $\mathcal{W}$, which is the, underling probability, measure of the noise $\xi_R^{\mathcal{A},p}$. It is simple to prove that $\mu_R^{\mathcal{A}_R,p}$ weakly converges to $\mu^{\mathcal{A}}$ when $R \to +\infty$ and the support of $\mathfrak{a}$ is compact.

Let $U_R : \mathcal{W}_R \to \mathcal{H}_R$ be the function defined by

$$U_R(w_R) := f(x)\mathcal{A}_R(g_R(z_R)V'(\mathfrak{G}_R * w_R)) = f(x)\mathcal{A}_R(g_R(z_R)V'(\mathcal{G}_R * w_R)).$$

We set $T_R(w_R) = w_R + U_R(w_R)$. The map U_R induces a map U_R^p on $\mathcal{W}$ in the following way

$$U_R^p(w) = f(x)\mathcal{A}_R(g_R^p(z_R)V'(\mathcal{G} * (\tilde{w}_R)^p)),$$

and $T_R^p(w) = w + U_R^p(w)$. Let ν_R be a probability law on $\mathcal{W}_R$ such that $T_{R,*}(\nu_R) = \mu_R^{\mathcal{A}_R}$, then it induces a probability law ν_R^p on $\mathcal{W}$ such that $T_R^p(\nu_R^p) = \mu_R^{\mathcal{A}_R,p}$.

Lemma 15 *Let $R_n \in \mathbb{R}_+$ be a sequence such that $R_n \to +\infty$, then the sequence $\nu_{R_n}^p$ is tight on $\mathcal{W}$ and if $\nu_{R_n}^p \to \nu$, as $n \to \infty$ and $R_n \to \infty$, then $T_*(\nu) = \mu^{\mathcal{A}}$.*

Proof We note that the support of $\nu_{R_n}^p$ is contained in the set of R_n periodic distributions contained in $\mathcal{W}$ and furthermore the map $T_{R_n}^p$ sends R_n periodic distributions into R_n periodic distributions.

Using the methods of Lemma 9 and Corollary 10, it is simple to prove that if $y_n \in \mathcal{W}$ is an R_n periodic distribution and if $w_{y_n} \in \mathcal{W}$ is an R_n periodic distribution such that $w_{y_n} \in T_{R_n}^{p,-1}(y_n)$ then

$$\|w_{y_n} - y_n\|_{\mathcal{H}} \leqslant K_{f,g}(\Xi_{\ell,\ell'}(y_n)), \tag{25}$$

where

$$\Xi_{\ell,\ell'}(y_n) = \sup_{(x,z)\in\mathbb{R}^2\times M} |\mathcal{G} * y_n(x,z)| r_\ell(x) r_{\ell'}(z),$$

and $K_{f,g}$ is a positive increasing continuous function depending only on the functions f and g. It is simple to prove that the map $\Xi_{\ell,\ell'}$ is continuous on $\mathcal{W}$. Using the fact that, as $n \to \infty$ and $R_n \to \infty$, $\mu_{R_n}^{\mathcal{A}_R,p}$ converges to $\mu^{\mathcal{A}}$ weakly and so the sequence $\mu_{R_n}^{\mathcal{A}_R,p}$ is tight, we can use the bound (25) and the same methods of Lemma 12 to prove that $\nu_{R_n}^p$ is tight.

Suppose that $\nu_{R_n}^p$ weakly converges to ν, we want to prove that $T_*(\nu) = \mu^{\mathcal{A}}$. Let F_{R_n} be a function of the form $F_{R_n}(w) = G(\langle h_1, w\rangle, \dots, \langle h_r, w\rangle)$, where $G : \mathbb{R}^r \to \mathbb{R}$ is a continuous and bounded function and $h_1, \dots, h_r$ are smooth functions with support in $(-R/2 + \mathfrak{r}, R/2 - \mathfrak{r})^n$ and $\mathfrak{r} = \mathrm{diam}(\mathrm{supp}(\mathfrak{a}))$. For this kind of function we have that $F_{R_n} \circ T_{R_k}^p = F_{R_n} \circ T$ for $k \geqslant n$. From this observation we get

$$\int F_{R_n} \circ T \mathrm{d}\nu = \lim_k \int F_{R_n} \circ T \mathrm{d}\nu_{R_k}^p = \lim_k \int F_{R_n} \circ T_{R_k}^p \mathrm{d}\nu_{R_k}^p = \lim_k \int F_{R_n} \mathrm{d}\mu_{R_k}^{\mathcal{A}_{R_k},p} = \int F_{R_n} \mathrm{d}\mu^{\mathcal{A}},$$

where the limit is taken for $k \to \infty$ and $R_k \to \infty$. Since the functions of the form F_{R_n}, for $n \in \mathbb{N}$, generate the space of all $\mathcal{W}$ Borel measurable functions the Lemma is proved. □

Proof of Theorem 14 By Theorem 7 there exists a probability law ν_{R_n} on $\mathcal{W}_{R_n}$ such that $T_{R_n,*}(\nu_{R_n}) = \mu_{R_n}^{\mathcal{A}_{R_n}}$. On the other hand this implies that $T_{R_n}^p(\nu_{R_n}^p) = \mu_{R_n}^{\mathcal{A}_{R_n},p}$ and so, by Lemma 15, there exists at last one probability measure ν on $\mathcal{W}$ such that $T_*(\nu) = \mu^{\mathcal{A}}$ and $\nu_{R_n}^p \to \nu$ weakly, as $n \to \infty$.

Then, using the notations of the proof of Lemma 15, we have that for any bounded continuous function $F_{R_n} : \mathbb{W} \to \mathbb{R}$

$$\int_{\mathcal{W}} F_{R_n}(\mathcal{G}*w(0,z))\Upsilon_f(w)\mathrm{d}\nu_{R_k}^p(w) = \int_{\mathbb{W}} F_{R_n}(\omega)\mathrm{d}\kappa_{R_k}^p(\omega), \tag{26}$$

where we used that

$$\int_{\mathcal{W}} F_{R_n}(\mathcal{G}_{R_k}*w_{R_k}(0,z))\Upsilon_f(w_{R_k})\mathrm{d}\nu_{R_k}(w_{R_k}) = \int_{\mathcal{W}} F_{R_n}(\mathcal{G}*w(0,z))\Upsilon_f(w)\mathrm{d}\nu_{R_k}^p(w),$$

for $k \geqslant n$ and a similar equality for κ_{R_k}. Since $F_{R_k}(\mathcal{G}*\cdot)$ and Υ_f are continuous on $\mathcal{W}$, the left hand side of (26), converges to $\int_{\mathcal{W}} F_{R_n}(\mathcal{G}*w(0,z))\Upsilon_f(w)\mathrm{d}\nu(w)$ as $k \to +\infty$.

Furthermore, since

$$\frac{\mathrm{d}\kappa_{R_k}^p}{\mathrm{d}\mu^{\mathcal{A}_{R_k},p}} = Z_{\kappa_{R_k}^p}^{-1} \exp\left(-4\pi \int_{\mathbb{R}^2} g(z)V(\omega(z))\mathrm{d}z\right)$$

and since $\mu^{\mathcal{A}_{R_k},p}$ weakly converges to $\mu^{\mathcal{A},\mathfrak{L}}$, we have that $\kappa_{R_k}^p$ weakly converges to κ, as $k, R_k \to \infty$. This proves that the right hand side of (26) converges to $\int_{\mathbb{W}} F_{R_n}(\omega)\mathrm{d}\kappa(\omega)$, as $k, R_k \to \infty$. Since the functions of the form F_{R_n}, for $n \in \mathbb{N}$, generate the space of $\mathbb{W}$ measurable functions, the theorem is proved. □

2.3 Cut-off removal with convex potential

Hereafter we denote by $\omega_\beta(x)$ the function

$$\omega_\beta(x) := \exp(-\beta\sqrt{(1+|x|^2)}), \tag{27}$$

$\beta > 0$, $x \in \mathbb{R}^2$, and introduce the space $\mathcal{W}_\beta$ in the following way

$$\mathcal{W}_\beta := (-\Delta + 1)C^0_{\exp\beta}(\mathbb{R}^2) \otimes_\epsilon \mathcal{A}^{1/2}(L^2(M))$$

where $C^0_{\exp\beta}$ is the space of continuous functions with respect to the weighted L^∞ norm

$$\|g\|_{\infty,\exp\beta} := \sup_{x\in\mathbb{R}^2} |\omega_\beta(x)g(x)|,$$

and M as before in Section 2.1.

In this section we want to prove the following theorem.

Theorem 16 *Suppose that V is a convex function, and suppose that $\mathcal{A}$, and $\mathfrak{L}$ satisfy Hypotheses H$\mathcal{A}$, H$\mathfrak{L}$ and Hξ (or Hypotheses H$\mathcal{A}$1, H$\mathfrak{L}$1 and Hξ1) then there exists a unique strong solution $\phi(x,z)$ to equation (1) (or to equation (24)) with $f \equiv 1$ taking values on $\mathcal{W}_\beta$ (for a $0 < \beta \leqslant \beta_0$ which depends only on m^2) such that for any $\mathbb{W}$ measurable bounded function F we have*

$$\mathbb{E}[F(\phi(0,z))] = \int_{\mathbb{W}} F(\omega)\mathrm{d}\kappa(\omega) \tag{28}$$

(κ as in equation (16)).

The proof is very similar to those of Theorem 7 and Theorem 14. For this reason we report here only the main differences. First of all we need a replacement for Proposition 6.

Proposition 17 *Let V (and so V_n) be a convex function then under assumptions H$\mathfrak{L}$, H$\mathcal{A}$ and Hξ there exists a unique strong solution ϕ_n to equation* (6) *such that for all $\mathbb{W}$ measurable bounded function F we have*

$$\mathbb{E}[F(\phi_n(0,z))] = \int_{\mathbb{W}} F(\omega)\mathrm{d}\kappa_n(\omega) \tag{29}$$

where κ_n is given by expression (12).

Proof The proof is given in [10], Theorem 2 in the case $\mathfrak{L}_n = 0$. The case considered here is a trivial extension. □

In order to pass from equation (29) to equation (28) we need a generalization of Lemma 9. We denote by $\|\cdot\|^{\mathcal{A}}_{\exp\beta,\ell}$ and $\|\cdot\|^{\mathcal{A}}_{U,k}$ the following norms

$$\begin{array}{rcl} \|h\|^{\mathcal{A}}_{\exp\beta,k} & := & \sup_{x\in\mathbb{R}^2}\sqrt{\sum_{j=1}^n \sigma_j^{2k}(h^j(x))^2\omega_\beta(x)^2}, \\ \|h\|^{\mathcal{A}}_{U,k} & := & \sup_{x\in U}\sqrt{\sum_{j=1}^n \sigma_j^{2k}(h^j(x))^2}. \end{array}$$

Lemma 18 *There exists a number $\beta_0 > 0$ such that for any $0 < \beta \leqslant \beta_0$, and for any open bounded set $U \subset \mathbb{R}^2$ and under Hypotheses C, Hg, H$\mathcal{A}$, H$\mathfrak{L}$, Hξ and $f \equiv 1$ we have*

$$\|\bar{\psi}_n\|^{\mathcal{A}}_{\exp\beta,-1} \lesssim \left\|\exp(\alpha\Xi_{\ell',n}r_{-\ell'}(x)\mathfrak{w}^{-1}(z))\mathbb{1}_{g(z)\neq 0}\omega_\beta(x)\right\|_\infty \tag{30}$$

$$\|(-\Delta_x + m^2 + \mathfrak{L}_n)(\bar{\psi}_n)\|^{\mathcal{A}}_{U,-1} \lesssim \|\exp(\alpha'(\Xi_{\ell',n}\mathfrak{w}^{-1}(z) + \|\bar{\psi}_n\|^{\mathcal{A}}_{\exp\beta,-1})\omega_{-\beta}(x))\|_{C^0(U_g)} \tag{31}$$

where $U_g = U \times \{g(z) \not= 0\} \subset \mathbb{R}^2 \times M$, uniformly in n (where $\Xi_{\ell,n}$ is defined as in Lemma 9).

Proof The proof is verbatim the same of Lemma 9 where we replace the function $r_{\theta,\ell}(x) = (1+\theta|x|^2)^\ell$ by the function ω_β, defined by (27), and we use the fact for β small enough we have

$$\left(\frac{|\nabla\omega_\beta|^2 + \Delta\omega_\beta}{\omega_\beta^2}\right) < m^2.$$

□

The inequality (31) implies that, for any bounded open subset U of $\mathbb{R}^{2+d}$ we have

$$\|(-\Delta_x + m^2 + \mathfrak{L})(\bar{\phi}_n)\|_{\mathcal{H}_U} \leqslant K_U(\Xi_{\ell,n}) \tag{32}$$

where $\|\cdot\|_{\mathcal{H}_U}$ is the natural norm of the Hilbert space $\mathcal{H}_U = L^2(U) \otimes_H \mathcal{A}(L^2(M))$, and K_U is a positive increasing continuous function depending only on U. Inequality (32) guarantees us enough compactness to generalize Lemma 11, Lemma 12 and Lemma 13 in order to prove the existence of a weak solution to equation (1) satisfying the dimensional reduction principle when $f \equiv 1$ and under Hypotheses C, Hg, H$\mathcal{A}$, H$\mathfrak{L}$, Hξ. We can generalize the described result under Hypotheses C, Hg, H$\mathcal{A}$1, H$\mathfrak{L}$1, Hξ1 using a similar strategy.

It remains to prove the uniqueness of the solution to equation (1) (or (24)) when V is convex. This can be done using the methods of Lemma 9 and Lemma 18. We give here only a sketch of the proof.

Let ϕ_1 and ϕ_2 be two strong solutions to equation (1). It is simple to prove that $\phi_1(x,\cdot) - \phi_2(x,\cdot) \in \mathcal{A}(L^2(M))$, so the following function is well defined

$$\Psi_\beta(x)^2 = \omega_{2\beta}(x)\int_M (\mathcal{A}^{-1}(\phi_1(x,z') - \phi_2(x,z')))^2\mathrm{d}z,$$

where $\omega_{2\beta}$ defined as in equation (27). The function Ψ_β^2 belongs to $C^2(\mathbb{R}^2)$ and goes to zero at infinity. This means that the maximum is realized in at least one point $\bar{x} \in \mathbb{R}^2$. Making some computations similar to the ones of Lemma 9 we obtain

$$\begin{aligned} m^2\Psi_\beta(\bar{x})^2 \quad &\leqslant \quad -\int_M g(z)(\phi_1(\bar{x},z)-\phi_2(\bar{x},z))(V'(\phi_1(\bar{x},z))-V'(\phi_2(\bar{x},z)))\mathrm{d}z \\ &\quad -\left(\tfrac{|\nabla\omega_\beta|^2+\Delta\omega_\beta}{\omega_\beta^2}\right)\Psi_\beta(\bar{x})^2. \end{aligned}$$

On the other hand there exists a point $\theta_{\bar{x},z} \in [\phi_1(\bar{x},z),\phi_2(\bar{x},z)]$ such that

$$V'(\phi_1(\bar{x},z))-V'(\phi_2(\bar{x},z)) \leqslant V''(\theta_{\bar{x},z})(\phi_1(\bar{x},z)-\phi_2(\bar{x},z)).$$

Choosing $\beta > 0$ small enough we obtain

$$\sup_{x\in\mathbb{R}^2}\Psi_\beta(x)^2 \lesssim -\int_M g(z)V''(\theta_{\bar{x},z})(\phi_1(\bar{x},z)-\phi_2(\bar{x},z))^2\mathrm{d}z \leqslant 0,$$

since V is convex. This implies that $\Psi_\beta(x) \equiv 0$ and so that $\phi_1(x,z) = \phi_2(x,z)$.

3 The exponential interaction on $\mathbb{R}^2$

In this section we want to consider the following elliptic SPDE

$$(-\Delta_x - \Delta_z + m^2)(\phi) + g(z)\alpha\exp(\alpha\phi - \infty) = \xi \tag{33}$$

where $x \in \mathbb{R}^2$, $z \in M = \mathbb{R}^2$ and $\xi = \xi(x,z)$ is a standard Gaussian white noise on $\mathbb{R}^4$, $|\alpha| < 4\sqrt{2}\pi$, and where $-\infty$ means that the equation should be properly renormalized. In order to give a meaning to the previous equation we formally subtract from the solution ϕ the solution to the linear equation (i.e. equation (33) with $g = 0$) which means that we consider the equation for the unknown $\bar{\phi}$:

$$(-\Delta_x - \Delta_z + m^2)(\bar{\phi}) + g(z)\alpha\exp(\alpha\bar{\phi})\eta(x,z) = 0, \tag{34}$$

where $\eta(x,z) = \exp^\diamond(\alpha\mathcal{I}\xi)$ is a renormalized version of the distribution $\exp(\alpha\mathcal{I}\xi - \infty)$, where $\exp^\diamond$ denotes the Wick exponential of the Gaussian distribution $\mathcal{I}\xi$. Hereafter we denote in general by $B^s_{p,q,\ell}(\mathbb{R}^{d+2})$ the weighted Besov space of indices $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ and weight given by $\bar{r}_\ell(x,z) = \left(\sqrt{|x|^2+|z|^2}+1\right)^{-\ell}$, where $\ell \in \mathbb{R}$ (see [77]). It is well known that $\mathcal{I}\xi \in B^{-\delta}_{p,p,\ell}$ for any $1 \leqslant p \leqslant \infty$, $\delta > 0$ and $\ell > 0$ (see e.g. [46]).

In the following we shall take $d = 2$ and give a rigorous meaning to equation (34) (and so to equation (33)) when the exponent $|\alpha| < 4\sqrt{2}\pi$ and, when $\alpha\bar{\phi} \leqslant 0$, prove that there exists only one solution to equation (33).

Furthermore we want to prove that dimensional reduction holds for the unique solution to equation (33), namely that, if we consider the measure κ_g given by

$$\frac{\mathrm{d}\kappa_g}{\mathrm{d}\mu} = \exp\left(-4\pi\int_{\mathbb{R}^2} g(z)\exp^\diamond(\alpha\omega)(\mathrm{d}z)\right), \tag{35}$$

where $\omega \in B^{-\delta}_{p,p,\ell}(\mathbb{R}^2)$ and where μ is the law of $(-\Delta_z + m^2)^{-1/2}(\xi)$ on $B^{-\delta}_{p,p,\ell}(\mathbb{R}^2)$, we have

$$\mathbb{E}[F(\phi(0,\cdot))] = \int F(\omega)\mathrm{d}\kappa_g(\omega),$$

for any bounded measurable function defined on $B^{-\delta}_{p,p,\ell}$.

In order to prove the existence and uniqueness of the solution $\bar{\phi}$ to equation (34) and dimensional reduction (3) for the random field $\phi = \bar{\phi} + \mathcal{I}\xi$, we need to introduce the following two approximate equations

$$(-\Delta_x - \Delta_z + m^2)(\phi_\epsilon) + \mathfrak{a}_\epsilon^{*2} * [g(z)\alpha\exp(\alpha\phi_\epsilon - C_\epsilon)] = \mathfrak{a}_\epsilon * \xi \tag{36}$$

$$(-\Delta_x - \Delta_z + m^2)(\bar{\phi}_\epsilon) + \mathfrak{a}_\epsilon^{*2} * [g(z)\alpha\exp(\alpha\bar{\phi}_\epsilon)\eta_\epsilon(x,z)] = 0 \tag{37}$$

where $\mathfrak{a}$ is a positive function satisfying Hypothesis H$\mathcal{A}$1 (see Section 2.2) and $C_\epsilon := \frac{\alpha^2}{2}\mathbb{E}[\mathcal{I}(\mathfrak{a}_\epsilon * \xi)]$, $\bar{\phi}_\epsilon := \phi_\epsilon - \mathcal{I}(\mathfrak{a}_\epsilon * \xi)$, $\eta_\epsilon := \exp^\diamond(\alpha\mathcal{I}(\mathfrak{a}_\epsilon * \xi))$ and $\bar{\phi}_\epsilon = \phi_\epsilon - \mathfrak{a}_\varepsilon * \mathcal{I}\xi$. The constant C_ϵ is chosen in such a way that $\eta_\epsilon \to \eta$ in $B^s_{p,p,\ell}$ for suitable $s < 0$, $1 < p \leqslant 2$ and $\ell > 0$ (see Section 3.1).

For equations (36) and (37) we have the following fundamental dimensional reduction result.

Theorem 19 *Equation* (37) *admits a unique solution in* $C^0_\ell(\mathbb{R}^4)$ *for* ℓ *big enough. Furthermore we have that that, for any* $\epsilon > 0$,

$$\mathbb{E}[F(\phi_\epsilon(0,z))] = \int_{\mathcal{C}^{0-}_\ell(\mathbb{R}^2)} F(\omega)\mathrm{d}\kappa_\epsilon(\omega), \tag{38}$$

for all $z \in \mathbb{R}^2$, *where*

$$\frac{\mathrm{d}\kappa_\epsilon}{\mathrm{d}\mu_\epsilon} = \exp\left(-4\pi\int_{\mathbb{R}^2} g(z)\exp(\alpha\omega(z) - \alpha C_\epsilon)\mathrm{d}z\right)$$

and μ_ϵ *is the Gaussian measure on* $\mathcal{C}^{0-}_\ell$ *with covariance* $\mathbb{E}[\omega(z)\omega(z')] = \mathfrak{a}_\epsilon^{*2} * \mathcal{G}_z(z - z')$, *where* $\mathcal{G}_z$ *is the Green function of the operator* $(-\Delta_z + m^2)$. *Finally the unique solution to equation* (37) *satisfies* $\alpha\bar{\phi}_\varepsilon \leqslant 0$.

Proof The proof is an application of Theorem 14 to equation (36) using the fact that $\exp(\alpha y - C_\epsilon)$ satisfies Hypothesis C in Section 2.

The fact that $\alpha\bar{\phi}_\varepsilon \leqslant 0$ follows from an application of the maximum principle to the function $\hat{z} \mapsto \bar{\phi}_\epsilon(\hat{z})\bar{r}_\ell(\lambda\hat{z})$ where $\hat{z} = (x, z) \in \mathbb{R}^4$ and $\lambda > 0$ small enough. □

In order to prove existence, uniqueness and the reduction principle for equation (33) we have now to study the behavior of the regularized noises η_ϵ and their convergence to η in the Besov spaces of the form $B^s_{p,p,\ell}$, as $\varepsilon \to 0$. Once we have established this convergence, we can give a meaning to equation (34) and we are able to prove that there exists a subsequence of $\epsilon_n \to 0$ such that $\phi_{\epsilon_n} \to \phi$ in probability in $B^{-\delta}_{p,p,\ell}$. This fact will permit us to prove equality (3).

3.1 Probabilistic analysis

In this subsection we propose an analysis of the regularity of the noise η and η_ϵ, for $\epsilon > 0$. First of all we note that

$$\eta_\epsilon = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}(\mathcal{I}\xi_\epsilon)^{\diamond k}, \tag{39}$$

$$\eta = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}(\mathcal{I}\xi)^{\diamond k}, \tag{40}$$

where

$$(\mathcal{I}\xi_\epsilon)^{\diamond k} = \underbrace{\mathcal{I}\xi_\epsilon \diamond \ldots \diamond \mathcal{I}\xi_\epsilon}_{k \quad \text{times}},$$

the symbol $\diamond$ denotes the Wick product, $\xi_\epsilon = \mathfrak{a}_\epsilon * \xi$ and $\mathcal{I}\xi_\epsilon$ is defined correspondingly. The previous expressions are well defined as $L^2(\mu)$ convergent series for $|\alpha| < 4\sqrt{2}\pi$. Indeed we have that for any smooth function g exponentially decaying at infinity

$$\mathbb{E}[|\langle\eta_\epsilon, g\rangle|^2] = \int_{\mathbb{R}^8} g(\hat{z})g(\hat{z}')\exp(\alpha^2\mathcal{G}_\epsilon(\hat{z} - \hat{z}'))\mathrm{d}\hat{z}\mathrm{d}\hat{z}' \tag{41}$$

where, hereafter, we write $\mathcal{G}_\epsilon = \mathfrak{a}_\epsilon^{*2} * \mathcal{G}$, where $\mathcal{G}$ is the Green function associated with the operator $(-\Delta_{\hat{z}} + m^2)^{-2}$, $\hat{z} = (x, z) \in \mathbb{R}^4$. It is well known (see Proposition A.1 of [9]) that for $\hat{z} \in \mathbb{R}^4$ such that $|\hat{z}| \leqslant 1$ there exists a constant $C_1 > 0$ for which the following inequality holds

$$\mathcal{G}(\hat{z}) \leqslant -\frac{2}{(4\pi)^2}\log_+(|\hat{z}|) + C_1. \tag{42}$$

Furthermore for $|\hat{z}| \geqslant 1$ there exists two constants $C_2, C_3 > 0$ for which we get

$$\mathcal{G}(\hat{z}) \leqslant C_2 \exp(-C_3|\hat{z}|). \tag{43}$$

An easy consequence of the inequalities (42) and (43) are the following inequalities

$$\mathfrak{a}_\epsilon^{*2} * \mathcal{G}(\hat{z}) \leqslant -\frac{2}{(4\pi)^2} \log_+(|\hat{z}|) + C_4, \tag{44}$$

when $4\,\mathrm{diam}(\mathrm{supp}(\mathfrak{a}))\epsilon \leqslant |\hat{z}| \leqslant 1$ and for some constant $C_4 > 0$;

$$\mathfrak{a}_\epsilon^{*2} * \mathcal{G}(\hat{z}) \leqslant -\frac{2}{(4\pi)^2} \log(\epsilon) + C_5, \tag{45}$$

when $|\hat{z}| < 2\,\mathrm{diam}(\mathrm{supp}(\mathfrak{a}))\epsilon$ and for some constant $C_5 > 0$; and finally

$$\mathfrak{a}_\epsilon^{*2} * \mathcal{G}(\hat{z}) \leqslant C_6 \exp(-C_7|\hat{z}|), \tag{46}$$

when $|\hat{z}| \geqslant 1$ and for some constants $C_6, C_7 > 0$. It is important to note that the constants C_4, C_5, C_6, C_7 are independent of ϵ. The previous inequalities and eq. (41) imply that $\mathbb{E}[|\langle \eta_\epsilon, g\rangle|^2] < +\infty$ for any $\epsilon \geqslant 0$ and $|\alpha| < 4\sqrt{2}\pi$ (with $\eta_0 = \eta$).

Let D_k, with $k \geqslant -1$, be the functions related to Littlewood-Paley block (see Appendix A for the definition of this concept). Using Remark 45, we can suppose that there exist some constants $\gamma_{-1}, \gamma > 0$ and $0 < \theta_{-1}, \theta < 1$ such that

$$|D_{-1}(\hat{z})| \lesssim \exp(-\gamma_{-1}|\hat{z}|^{\theta_{-1}}) \quad \text{and} \quad |D_k(\hat{z})| \lesssim 2^{4k} \exp(-\gamma 2^{k\theta}|\hat{z}|^\theta),$$

where $k > 0$. Using this decay at infinity, we get

$$\mathbb{E}[\|\langle \eta_\epsilon, D_k(\hat{z} - \cdot)\rangle\|^2_{L^2_{\rho_\ell}}] = \int_{\mathbb{R}^4} \mathbb{E}[\langle \eta_\epsilon, D_k\rangle^2](1 + |\hat{z}|)^{-2\ell} \mathrm{d}\hat{z} \lesssim \mathbb{E}[\langle \eta_\epsilon, D_k\rangle^2] < +\infty,$$

whenever $\alpha < 4\sqrt{2}\pi$, and where we use the invariance in law of η_ϵ with respect to translations. Since $D_k(x, z) = 2^{4k} D_0(2^k x, 2^k z)$ and using (44) and (46), we obtain

$$\begin{aligned}
\mathbb{E}[\langle \eta_\epsilon, D_k\rangle^2] &= \int_{\mathbb{R}^8} D_k(\hat{z}) D_k(\hat{z}') \exp(\alpha^2 \cdot \mathfrak{a}_\epsilon^{*2} * \mathcal{G}(\hat{z} - \hat{z}')) \mathrm{d}\hat{z}\mathrm{d}\hat{z}' \\
&\lesssim \int_{\mathbb{R}^8} |D_k(\hat{z}) D_k(\hat{z}')| |\hat{z} - \hat{z}'|^{-\frac{2\alpha^2}{(4\pi)^2}} \mathrm{d}\hat{z}\mathrm{d}\hat{z}' + \\
&\quad + \left(1 + \epsilon^{4 - \frac{2\alpha^2}{(4\pi)^2}}\right) \int_{\mathbb{R}^8} |D_k(\hat{z}) D_k(\hat{z}')| \mathrm{d}\hat{z}\mathrm{d}\hat{z}' \\
&\lesssim 2^{\frac{2\alpha^2}{(4\pi)^2} k} \left(\int_{\mathbb{R}^8} \left(1 + |\hat{z} - \hat{z}'|^{-\frac{2\alpha^2}{(4\pi)^2}}\right) |D_0(\hat{z}) D_0(\hat{z}')| \mathrm{d}\hat{z}\mathrm{d}\hat{z}' \right) \lesssim 2^{\frac{2\alpha^2}{(4\pi)^2} k}
\end{aligned}$$

where all the constants implied in the symbol $\lesssim$ are independent of ϵ. This means that

$$\mathbb{E}[\|\eta_\epsilon\|^2_{B^s_{2,2,\ell}}] \lesssim \sum_{k=1}^{+\infty} 2^{\frac{2\alpha^2}{(4\pi)^2} k + 2sk},$$

which is finite and uniformly bounded in ϵ whenever $s < -\frac{\alpha^2}{(4\pi)^2}$ and for $\ell > 0$ large enough. Furthermore we have that

$$\begin{aligned}
\mathbb{E}[\|\eta - \eta_\epsilon\|^2_{B^s_{2,2,\ell}}]^{1/2} &\lesssim \left(\sum_{k=-1}^{+\infty} 2^{2sk} \mathbb{E}[|\langle \eta - \eta_\epsilon, D_k\rangle|^2] \right)^{1/2} \\
&\lesssim \sum_{k=-1}^{+\infty} 2^{sk} \sum_{n=0}^{+\infty} \frac{|\alpha|^n}{n!} \mathbb{E}[|\langle \mathcal{I}\xi^{\diamond n} - \mathcal{I}\xi_\epsilon^{\diamond n}, D_k\rangle|^2]^{1/2}
\end{aligned}$$

for ℓ big enough. On the other hand, from the previous estimate, we get

$$\sum_{k=-1}^{+\infty} 2^{sk} \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} \mathbb{E}[|\langle \mathcal{I}\xi^{\diamond n} - \mathcal{I}\xi_\epsilon^{\diamond n}, D_k\rangle|^2]^{1/2} \leqslant \sum_{k=-1}^{+\infty} 2^{sk} \sum_{n=0}^{+\infty} \frac{|\alpha|^n}{n!} \mathbb{E}[|\langle \mathcal{I}\xi^{\diamond n}, D_k\rangle|^2]^{1/2} + \\ + \sum_{k=-1}^{+\infty} 2^{sk} \sum_{n=0}^{+\infty} \frac{|\alpha|^n}{n!} \mathbb{E}[|\langle \mathcal{I}\xi_\epsilon^{\diamond n}, D_k\rangle|^2]^{1/2} < C,$$

for some constant $C > 0$ independent of ϵ, whenever $s < -\frac{\alpha^2}{(4\pi)^2}$. Furthermore, since $\mathfrak{a}_\epsilon$ is a regular mollifier and by the properties of Wick product, we obtain

$$\mathbb{E}[|\langle \mathcal{I}\xi^{\diamond n} - \mathcal{I}\xi_\epsilon^{\diamond n}, D_k\rangle|^2] \to 0,$$

as $\varepsilon \to 0$ and for any $k \geq -1$. The above inequality, together with the Lebesgue dominated convergence theorem, implies that

$$\mathbb{E}[\|\eta - \eta_\epsilon\|^2_{B^s_{2,2,\ell}}]^{1/2} \to 0,$$

when $\epsilon \to 0$ for any $\ell > 0$ large enough. We have thus proven the following lemma.

Lemma 20 *For $|\alpha| < 4\sqrt{2}\pi$ and $s < -\frac{\alpha^2}{(4\pi)^2}$ and $\ell > 0$ large enough we have that $\eta_\epsilon \to \eta$ as $\epsilon \to 0$ in $L^2(\mathcal{W}; B^s_{2,2,\ell}(\mathbb{R}^4), \mathrm{d}\mu)$ and thus in probability in $B^s_{2,2,\ell}(\mathbb{R}^4)$.*

We want to use the previous lemma to prove the following theorem.

Theorem 21 *For $|\alpha| < 4\sqrt{2}\pi$, $1 < p \leqslant 2$, $s < -\frac{\alpha^2(p-1)}{(4\pi)^2}$ and $\ell > 0$ large enough we have that $\eta_\epsilon \to \eta$, as $\epsilon \to 0$, in $L^p(\mathcal{W}; B^s_{p,p,\ell}(\mathbb{R}^4), \mathrm{d}\mu)$ and thus in probability in $B^s_{p,p,\ell}(\mathbb{R}^4)$.*

In order to prove Theorem 21 we introduce the following lemma.

Lemma 22 *Let $B_r(\hat{z})$ be the ball of radius r and center in $\hat{z} \in \mathbb{R}^4$ then for any $r < R$, and for any $1 < p < 2$*

$$\mathbb{E}\left[\left(\int_{B_r(\hat{z})} \mathrm{d}\eta_\epsilon\right)^p\right] \lesssim r^{-\frac{\alpha^2}{(4\pi)^2}p(p-1)+4p}$$

where the constants depend only R and are uniform on $\epsilon \to 0$ and $|\alpha| < 4\sqrt{2}\pi$.

Proof The proof can be found in Proposition 2.7 of [67]. □

Proof of Theorem 21 First of all we have that, for any $\varepsilon > 0$ and $k \geqslant 0$,

$$\mathbb{E}[\|\langle \eta_\epsilon, D_k(\hat{z} - \cdot)\rangle\|^p_{L^p_\ell}] = \int_{\mathbb{R}^4} \mathbb{E}[\langle \eta_\epsilon, D_k\rangle^p](1 + |\hat{z}|)^{-p\ell} \mathrm{d}\hat{z} \lesssim \mathbb{E}[|\langle \eta_\epsilon, D_k\rangle|^p].$$

Then

$$\mathbb{E}[|\langle \eta_\epsilon, D_k\rangle|^p] \lesssim \mathbb{E}\left[\left(\int_{B_{r_k}(0)} |D_k(\hat{z})| \mathrm{d}\eta_\epsilon(\hat{z})\right)^p\right] + \mathbb{E}\left[\left(\int_{\mathbb{R}^4 \setminus B_{r_k}(0)} |D_k(\hat{z})| \mathrm{d}\eta_\epsilon(\hat{z})\right)^2\right]^{p/2}.$$

If we choose $r_k = 2^{-k}(\log(2^{\beta k}))^u$, for some $\beta, u > 0$, from Lemma 22 we have, for any $k \geqslant 0$,

$$\mathbb{E}\left[\left(\int_{B_{r_k}(0)} |D_k(\hat{z})| \mathrm{d}\eta_\epsilon(\hat{z})\right)^p\right] \lesssim 2^{4kp} 2^{\frac{\alpha^2}{(4\pi)^2}p(p-1)-4pk} \log(2^{\beta k})^{u\left(\frac{\alpha^2}{(4\pi)^2}p(p-1)-4pk\right)}$$

Furthermore, if we denote by $\gamma > 0$ and $0 < \theta < 1$ the real number such that $D_0(\hat{z}) \lesssim \exp(-\gamma|\hat{z}|^\theta)$ and choosing $u = \frac{1}{\theta}$, we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_{\mathbb{R}^4\setminus B_{r_k}(0)} |D_k(\hat{z})|\mathrm{d}\eta_\epsilon(\hat{z})\right)^2\right]^{2/p} &\lesssim 2^{4k-\gamma\beta k}\int_{\mathbb{R}^8} e^{-\gamma(|\hat{z}|^\theta+|\hat{z}'|^\theta)+\alpha^2\mathcal{G}_\epsilon(\hat{z}-\hat{z}')}\mathrm{d}\hat{z}\mathrm{d}\hat{z}' \\ &\lesssim 2^{4k-\gamma\beta k}.\end{aligned}$$

If we choose β such that $4-\gamma\beta < \frac{\alpha^2}{(4\pi)^2}p(p-1)$ we obtain that

$$\mathbb{E}[|\langle\eta_\epsilon, D_k\rangle|^p] \lesssim 2^{\frac{\alpha^2}{(4\pi)^2}p(p-1)}\log(2^{\beta k})^{u\left(\frac{\alpha^2}{(4\pi)^2}p(p-1)-4pk\right)} \tag{47}$$

uniformly in ϵ. This implies that $\mathbb{E}[\|\eta_\epsilon\|^p_{B^s_{p,p,\ell}}]$ is bounded for $s < -\frac{\alpha^2}{(4\pi)^2}(p-1)$ uniformly in $\epsilon > 0$.

Furthermore, by Lemma 20, $\mathbb{E}[|\langle\eta_\epsilon, D_k\rangle|^2]$ is uniformly bounded. This means that the random variables $|\langle\eta_\epsilon, D_k\rangle|^p$ (for $p < 2$) are uniformly integrable. On the other hand, by Lemma 20 and since $D_k \in B^2_{2,2,\ell}$, $|\langle\eta_\epsilon, D_k\rangle|^p$ converges to $|\langle\eta, D_k\rangle|^p$ in probability, we have that $\mathbb{E}[|\langle\eta_\epsilon, D_k\rangle|^p] \to \mathbb{E}[|\langle\eta, D_k\rangle|^p]$, as $\varepsilon \to 0$. Thus by the bound (47) we can use the Lebesgue dominated converge theorem for computing the limit of $\mathbb{E}[\|\eta_\epsilon\|^p_{B^s_{p,p,\ell}}]$, obtaining

$$\mathbb{E}[\|\eta_\epsilon\|^p_{B^s_{p,p,\ell}}] \to \mathbb{E}[\|\eta\|^p_{B^s_{p,p,\ell}}],$$

as $\varepsilon \to 0$. But, since $\mathbb{E}[\|\eta_\epsilon\|^p_{B^s_{p,p,\ell}}]$ is uniformly bounded and since, by Lemma 20, as $\varepsilon \to 0$, $\eta_\epsilon \to \eta$ in probability in $B^{-2}_{2,2,\ell}$, we have that η_ϵ converges weakly to η in $L^p(\mathcal{W}, B^s_{p,p,\ell}, \mathrm{d}\mu)$. Since $L^p(\mathcal{W}, B^s_{p,p,\ell}, \mathrm{d}\mu)$ is a uniformly convex space, being the space of L^p functions taking values in the uniformly convex space $B^s_{p,p,\ell}$ (see [32]), by the weak convergence $\eta_\epsilon \to \eta$ and the convergence of the L^p norm $\mathbb{E}[\|\eta_\epsilon\|^p_{B^s_{p,p,\ell}}] \to \mathbb{E}[\|\eta\|^p_{B^s_{p,p,\ell}}]$, we obtain that η_ϵ converges strongly to η in $B^s_{p,p,\ell}(\mathbb{R}^4)$, as $\varepsilon \to 0$. □

3.2 Analysis of the elliptic SPDE

In this section we want to prove the following theorem.

Theorem 23 *For any $|\alpha| < 4\sqrt{2}\pi$, there are some $p, s, \delta \in \mathbb{R}$ such that $1 < p \leqslant 2$ and $p < \frac{2(4\pi)^2}{\alpha^2}$, $-1 < s < \frac{\alpha^2(p-1)}{(4\pi)^2}$ and $0 < \delta < s+1$, for which equation (34) admits a unique solution in $B^{s+2}_{p,p,\ell}$ for $\ell > 0$ large enough and such that $\alpha\bar{\phi} \leqslant 0$. Furthermore we have that there exists a subsequence $\varepsilon_n \to 0$ such that $\bar{\phi}_{\epsilon_n} \to \bar{\phi}$ in $B^{s+2-\delta}_{p,p,\ell+\delta'}$ (for some $\delta' > 0$ small enough), as $n \to \infty$, almost surely.*

Hereafter, for $t > 0$, we write $\mathbb{B}^t_{p,p,\ell}(\mathbb{R}^4) = B^t_{p,p,\ell}(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4)$. The space $\mathbb{B}^t_{p,p,\ell}$ has a natural norm given by the sum of the norms of $B^t_{p,p,\ell}(\mathbb{R}^4)$ and $L^\infty(\mathbb{R}^4)$. Furthermore we can equip the space $\mathbb{B}^t_{p,p,\ell}$ with a different notion of convergence of sequences: we say that a sequence $h_n \in \mathbb{B}^t_{p,p,\ell}$ converges to $h \in \mathbb{B}^t_{p,p,\ell}$ if h_n converges to h strongly in $B^t_{p,p,\ell}$ and $\sup_n \|h_n\|_\infty < +\infty$ (it is important to note that $\|h\|_\infty \leqslant \liminf_{n\to+\infty}\|h_n\|_\infty$).

Lemma 24 *For any $t > 0$ and $1 < p \leqslant 2$, we have that $\mathbb{B}^t_{p,p,\ell} \subset B^{t(p-1)}_{q,q,\ell}$ (where $1/p + 1/q = 1$) and the immersion is continuous with respect to the natural convergence in $\mathbb{B}^t_{p,p,\ell}$. Furthermore $\mathbb{B}^t_{p,p,\ell}$ is a Banach algebra.*

Proof By Proposition 52, the interpolation space $(B^0_{\infty,\infty}, B^t_{p,p,\ell})_{\frac{p}{q},q}$ between $L^\infty \subset B^0_{\infty,\infty}$ and $B^t_{p,p,\ell}$ is exactly $B^{t\frac{p}{q}}_{q,q,\ell} = B^{t(p-1)}_{q,q,\ell}$, where $q = \frac{p-1}{p}$. This means that $\mathbb{B}^t_{p,p,\ell} = L^\infty \cap B^t_{p,p,\ell} \subset B^0_{\infty,\infty} \cap B^s_{p,p,\ell}$ is continuously embedded in $B^{t(p-1)}_{q,q,\ell}$. The fact that $\mathbb{B}^t_{p,p,\ell}$ is an algebra is proven in Corollary 2.86 of [27]. □

Lemma 25 *Under the hypotheses of Theorem 23 on α there are some p, s, δ such that $1 < p \leqslant 2$ and $p < \frac{2(4\pi)^2}{\alpha^2}$, $-1 < s < -\left(\frac{\alpha^2(p-1)}{(4\pi)^2} \vee (p-1)\right)$ and $0 < \delta < s+1$ and for which there is a $0 < \gamma < 1$ such that*

$$\delta < 2 + \frac{s}{p-1}, \qquad \frac{s+1}{1-\delta} < \gamma < 1 \qquad \text{and} \qquad \left(1 - \frac{s+1}{4}p\right)\gamma < p-1. \tag{48}$$

Proof Fix $|\alpha| < 4\sqrt{2}\pi$ and $\upsilon \in \left(\frac{\alpha^2}{(4\pi)^2} \vee 1, 2\right)$. We write $p = k+1$ and $s = -\upsilon k$. The thesis of the lemma is equivalent to prove that, for $\delta > 0$ small enough, there exists some $k¿0$ such that

$$\delta < 2 - \upsilon, \qquad \frac{-\upsilon k + 1}{1-\delta} < \frac{k}{\left(1 - \frac{-\upsilon k+1}{2}\right)}, \quad -\upsilon k > -1 \quad \text{and} \quad k < 1.$$

From the first two inequalities we obtain that

$$\frac{-1 + \delta + \sqrt{(1-\delta)^2 + \upsilon^2}}{\upsilon^2} < k < \frac{1}{\kappa}$$

which is a non-void inequality when $\delta < 2 - \upsilon$.

In order to finish the proof we have only to check that

$$\frac{-1 + \delta + \sqrt{(1-\delta)^2 + \upsilon^2}}{\upsilon^2} < 1$$

which is always true when $\delta < \frac{2-\sqrt{2}}{2}$ since it is equivalent to $(1-\delta)^2 + \upsilon^2 < (1-\delta)^2 + \upsilon^4 + 2\upsilon^2(1-\delta)^2$. □

We introduce the space $\mathfrak{M}^s_{p,p,\ell} \subset B^s_{p,p,\ell}$ (with s, p and ℓ as in Theorem 23) which is the set of Radon σ-finite measures μ contained in $B^s_{p,p,\ell}$ such that $|\mu| \in B^s_{p,p,\ell}$, where $|\mu| = \mu_+ + \mu_-$, and where μ_+ and μ_- are the unique positive measures such that $\mu = \mu_+ - \mu_-$. We can define a notion of convergence on $\mathfrak{M}^s_{p,p,\ell}$ in the following way: a sequence $\mu_n \in \mathfrak{M}^s_{p,p,\ell}$ converges to $\mu \in \mathfrak{M}^s_{p,p,\ell}$ if μ_n converges strongly in $B^s_{p,p,\ell}$ and $\sup_n |||\mu_n|||_{B^s_{p,p,\ell}} < +\infty$.

We now consider the natural product $\cdot$ between smooth functions which are in $\mathbb{B}^{s+2-\delta}_{r,r,\ell'}$ and and smooth measures in $\mathfrak{M}^s_{p,p,\ell}$.

Lemma 26 *If s, p, δ, γ satisfy the thesis of Lemma 25, then the product $\cdot$ can be extended in a unique and associative way from $\mathbb{B}^{s+2-\delta}_{r,r,\ell'} \times \mathfrak{M}^s_{p,p,\ell}$ into $\mathfrak{M}^s_{p,p,\ell}$, where*

$$r = \frac{p}{\left(\left(1 - \frac{s+1}{4}p\right)\gamma + 1\right)}.$$

This extension is (weakly) continuous with respect to the natural convergence in $\mathbb{B}^{s+2-\delta}_{r,r,\ell'}$ and $\mathfrak{M}^s_{p,p,\ell}$. Furthermore for any $h \in \mathbb{B}^{s+2-\delta}_{r,r,\ell'}$ and $\mu \in \mathfrak{M}^s_{p,p,\ell}$ we have

$$|||h \cdot \mu|||_{B^s_{p,p,\ell}} \lesssim \|h\|_\infty \cdot |||\mu|||_{B^s_{p,p,\ell}}. \tag{49}$$

Proof First of all we note that, by Proposition 51, the product $\cdot$ is well defined and (strongly) continuous as bilinear functional from $B^{\frac{(s+2-\delta)\frac{(p-1)r}{p}}}_{\frac{p}{p-1},\frac{p}{p-1},\ell'} \times B^s_{p,p,\ell}$ into $B^s_{1,1,\ell+\ell'}$, where $r = \frac{p}{\left(\left(1-\frac{s+1}{4}p\right)\gamma+1\right)}$ for some γ such that $\frac{s+1}{1-\delta} < \gamma < 1$ and $\left(1 - \frac{s+1}{4}p\right)\gamma < p-1$ (whose existence is proven in Lemma 25).

Indeed if γ satisfies the previous conditions, we have

$$\begin{aligned}(s+2-\delta)\frac{(p-1)r}{p} + s &\geq \left(\frac{s}{p-1}(p-1) + 2 - \delta\right)\frac{p-1}{p} + \frac{s}{p-1}(p-1) \\ &= \frac{\left(2 - \delta - \frac{s}{p-1}\right)(p-1)}{p} > 0\end{aligned}$$

since $\delta < 2 + \frac{s}{p-1}$ by hypothesis.

The only thing that remains to prove is inequality (49), since the other statements of the lemma easily follow from it. First of all, using the equivalent norm on Besov spaces of Proposition 47, we note that if μ is a Radon measure we have

$$\||\mu|\|_{B^s_{p,p,\ell}} \sim \left(\int_{\mathbb{R}^4 \times (0,1]} \left(\frac{|\mu|(B_\lambda(\hat{z}))}{\lambda^s} \right)^p \rho_\ell(\hat{z})^p \mathrm{d}\hat{z}\mathrm{d}\lambda \right)^{1/p}.$$

Using again Proposition 47, if h is a continuous bounded function we have that

$$\||h \cdot \mu|\|_{B^s_{p,p,\ell}} \lesssim \left(\int_{\mathbb{R}^4 \times (0,1]} \left(\frac{|h \cdot \mu|(B_\lambda(\hat{z}))}{\lambda^s} \right)^p \rho_\ell(\hat{z})^p \mathrm{d}\hat{z}\mathrm{d}\lambda \right)^{1/p} \lesssim \|h\|_\infty \cdot \||\mu|\|_{B^s_{p,p,\ell}}.$$

If h is a generic function $\mathbb{B}^{s+2-\delta}_{r,r,\ell'}$ there exists a sequence of smooth functions such that, as $n \to \infty$, $h_n \to h$ in $B^{s+2-\delta}_{r,r,\ell'}$ and $\|h_n\|_\infty \leqslant \|h\|_\infty$. By the first part of the lemma we have that $|h_n \cdot \mu|$ converges weakly in $\mathcal{S}'(\mathbb{R}^4)$ to $|h \cdot \mu|$ from which we obtain

$$\||h \cdot \mu|\|_{B^s_{p,p,\ell}} \leqslant \liminf \||h_n \cdot \mu|\|_{B^s_{p,p,\ell}} \lesssim \||\mu|\|_{B^s_{p,p,\ell}} \liminf \|h_n\|_\infty \lesssim \|h\|_\infty \||\mu|\|_{B^s_{p,p,\ell}}.$$

□

We introduce the following map

$$\mathcal{K}_g(\mu, \bar{\varphi}) := -\alpha(-\Delta_{\hat{z}} + m^2)^{-1}(g(z)G(\alpha\bar{\varphi}(\hat{z})) \cdot \mu(\hat{z})),$$

for any $|\alpha| < 4\sqrt{2}\pi$, where $G : \mathbb{R} \to \mathbb{R}_+$ is an increasing smooth bounded function with all bounded derivatives such that $G(x) = \exp(x)$ for $x \leqslant 0$. It is clear that if $\alpha\bar{\phi} \leqslant 0$ and if $\bar{\phi}$ solves equation (34) then we have

$$\bar{\phi} = \mathcal{K}_g(\eta, \bar{\phi}).$$

Hereafter we denote by $\mathfrak{M}^s_{+,p,p,\ell}$ the space formed by positive distribution contained in $\mathfrak{M}^s_{p,p,\ell}$.

Lemma 27 *Let G be the function defined by $\varphi \longmapsto G(\varphi)$, and let p, s, δ, γ be as in the thesis of Lemma 25 then, when $\ell > 0$ is big enough, we have that G is a continuous map from $B^{s+2-\delta}_{p,p,\ell}$ into $\mathbb{B}^{s+2-\delta}_{r,r,2\ell}$ where r is chosen as in Lemma 26.*

Proof For any $0 \leq k < 1$, $q > 1$ and ℓ big enough we have that G is a continuous (actually Lipschitz) map from $B^k_{q,q,\ell}$ into $\mathbb{B}^k_{q,q,\ell}$. We give a sketch of the proof for the weighted Besov spaces following the argument of [68] Section 5.5.

This fact can be proven using the equivalent definition of Besov norm using finite differences (79) for $N = 0$ and $M = 1$. Indeed we have that

$$\begin{aligned}
\|G(\varphi)\|_{B^k_{q,q,\ell}} &= \|G(\varphi)\|_{L^p_\ell} + \left(\int_{|h|<1} |h|^{-k} \|G(\varphi(x+h)) - G(\varphi(x))\|^q_{L^q_\ell} \frac{\mathrm{d}h}{|h|^n} \right)^{\frac{1}{q}} \\
&\lesssim (\|G\|_\infty + \|G'\|_\infty) \left(1 + \|\varphi\|_{L^p_\ell} + \left(\int_{|h|<1} |h|^{-k} \|\Delta^1_h \varphi\|^q_{L^q_\ell} \frac{\mathrm{d}h}{|h|^n} \right)^{\frac{1}{q}} \right) \\
&\lesssim (\|G\|_\infty + \|G'\|_\infty)(1 + \|\varphi\|_{B^k_{q,q,\ell}}).
\end{aligned}$$

This proves that the map G takes values in $\mathbb{B}^k_{q,q,\ell}$. By using the real interpolation of Sobolev spaces

$$(L^p_\ell, W^{1,p}_\ell)_{s,q} = B^s_{p,q,\ell},$$

which is a consequence of the interpolation of unweighted Sobolev spaces (see [30] Theorem 6.2.4) and the isomorphism between weighted and unweighted Besov and Sobolev spaces (see [77] Theorem 6.5), using the Lipschitz continuity of G in L^q_ℓ and in $W^{1,q}_\ell$ (see [31] for the proof in the

unweighted Sobolev spaces), and exploiting a standard interpolation argument (see [59]), we obtain that G is also continuous.

Consider $\varphi \in B^{s+2-\delta}_{p,p,\ell}$, then $\nabla G(\varphi) = G'(\varphi)\nabla\varphi$. By a Besov embedding theorem we have that $\varphi \in B^{1-\epsilon}_{p',p',\ell}$ where $p' = \left(1 - \frac{s+1}{4}p\right)$ for $\epsilon > \delta$ small enough. By what we say before this means that $G'(\varphi) \in \mathbb{B}^{1-\epsilon}_{p',p',\ell}$, and by Lemma 24, $G'(\varphi) \in B^{(1-\epsilon)\gamma}_{\frac{p'}{\gamma},\frac{p'}{\gamma},\ell}$ This implies that, for $\gamma > \frac{s+1}{1-\delta}$ and $\left(1 - \frac{s+1}{4}p\right)\gamma < p-1$, the product $G'(\varphi)\nabla\varphi$ is well defined and in $B^{s+1}_{r,r,\ell}$. By remark 49 below, this implies that $G(\varphi) \in \mathbb{B}^{s+2}_{r,r,\ell}$.

In a similar way, when $\varphi_n \in B^{s+2-\delta}_{p,p,\ell}$ converges to $\varphi \in B^{s+2-\delta}_{p,p,\ell}$, it is possible to prove that $G(\varphi_n) \to G(\varphi)$ in $\mathbb{B}^{s+2-\delta}_{r,r,\ell}$. Indeed, by Remark 49, we have

$$\begin{aligned}
\|G(\varphi) - G(\varphi_n)\|_{B^{s+2-\delta}_{r,r,2\ell}} &\lesssim \|G(\varphi) - G(\varphi_n)\|_{L^r_{2\ell}} + \|G'(\varphi)\nabla\varphi - G'(\varphi_n)\nabla\varphi_n\|_{B^{s+1-\delta}_{r,r,2\ell}} \\
&\lesssim \|G(\varphi) - G(\varphi_n)\|_{L^r_{2\ell}} + \|G'(\varphi)\|_{B^{(1-\epsilon)\gamma}_{\frac{p'}{\gamma},\frac{p'}{\gamma},\ell}} \|\nabla\varphi - \nabla\varphi_n\|_{B^{s+1-\delta}_{p,p,\ell}} + \\
&\quad + \sup_n(\|\nabla\varphi_n\|_{B^{s+1-\delta}_{p,p,\ell}})\|G'(\varphi) - G(\varphi_n)\|_{B^{(1-\epsilon)\gamma}_{\frac{p'}{\gamma},\frac{p'}{\gamma},\ell}} \to 0
\end{aligned}$$

since G' is continuous from $B^{s+2-\delta}_{p,p,\ell}$ into $B^{(1-\epsilon)\gamma}_{\frac{p'}{\gamma},\frac{p'}{\gamma},\ell}$ (as it was proven in the first part of the proof). □

Lemma 28 *For p, s, δ, γ satisfying the thesis of Lemma 25, for any $\ell > 0$ big enough, for any fixed $\mu \in \mathfrak{M}^s_{+,p,p,\ell}$ and small enough $\delta' > 0$, there exists at least a solution $\bar\varphi \in B^{s+2-\delta}_{p,p,\ell+\delta'}$ to the equation*

$$\bar\varphi = \mathcal{K}_g(\mu, \bar\varphi). \tag{50}$$

Proof We want to use Schaefer's fixed-point theorem (see Theorem 4 Section 9.2 Chapter 9 of [38]) to prove the lemma. In order to do this we have to prove that $\mathcal{K}_g$ is continuous in φ, that it maps any bounded set into a compact set and that the set of solutions to the equations

$$\bar\varphi = \lambda\mathcal{K}_g(\mu, \bar\varphi)$$

are bounded uniformly for all $0 \leqslant \lambda \leqslant 1$.

The continuity of $\mathcal{K}_g$ (in both μ and φ) is a consequence of Lemma 26 and Lemma 27. Indeed in the cone $\mathfrak{M}^s_{+,p,p,\ell}$ the natural convergence in $\mathfrak{M}^s_{+,p,p,\ell} \subset \mathfrak{M}^s_{p,p,\ell}$ coincides with the strong convergence in $B^s_{p,p,\ell}$ since if $\mu \in \mathfrak{M}^s_{+,p,p,\ell}$ then $|\mu| = \mu$.

Furthermore, by Lemma 27, the map $\varphi \to G(\varphi)$ is continuous from $B^{s+2-\delta}_{p,p,\ell+\delta'}$ into $\mathbb{B}^{s+2-\delta}_{r,r,\ell+\delta'}$ where $r = \frac{p}{\left(\left(1-\frac{s+1}{4}p\right)\gamma+1\right)}$. Thus we can apply Lemma 26, from which we obtain that the map $\varphi \longmapsto G(\varphi)\cdot\mu$ is continuous in φ. Finally the linear operator $(-\Delta_{\hat z} + m^2)^{-1}$ is continuous from $B^s_{p,p,\ell}$ into $B^{s+2}_{p,p,\ell}$ and thus compact from $B^s_{p,p,\ell}$ into $B^{s+2-\delta}_{p,p,\ell+\delta'}$ (since, by Proposition 50, the immersion $B^{s+2}_{p,p,\ell} \hookrightarrow B^{s+2-\delta}_{p,p,\ell+\delta'}$ is compact). Thus by Lemma 26 and Lemma 27, the map $\mathcal{K}_g$ is weakly continuous as a map into $B^{s+2}_{p,p,\ell}$ and strongly continuous as a map into $B^{s+2-\delta}_{p,p,\ell+\delta'}$.

The map $\mathcal{K}_g$ is compact since the following inequality holds

$$\|\mathcal{K}_g(\mu,\varphi)\|_{B^{s+2}_{p,p,\ell}} \lesssim \|G\|_\infty\|\mu\|_{B^s_{p,p,\ell}} \tag{51}$$

and, by Proposition 50, we have the compact immersion $B^{s+2}_{p,p,\ell} \hookrightarrow B^{s+2-\delta}_{p,p,\ell+\delta'}$.

Finally the uniform boundedness in λ follows from inequality (51). This proves the thesis of the lemma. □

Lemma 29 *Under the hypotheses of Theorem 23 the solution to equation* (50) *is unique in $B^{s+2-\delta}_{p,p,\ell+\delta'}$ for $\delta, \delta' \geqslant 0$ small enough.*

Proof Let $J:\mathbb{R}\to\mathbb{R}$ be a smooth, bounded, strictly increasing function such that $J(0)=0$ and let $\bar\varphi_1$ and $\bar\varphi_2$ be two solutions to equation (50). By Lemma 24 and Lemma 27, $J(\bar\varphi_1-\bar\varphi_2)\in B^{(s+2-\delta)\frac{p-1}{rp}}_{q,q,\ell+\delta'}$ which implies that $\bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2)\in(B^s_{p,p,\ell})^*$ for $\delta'\geqslant 0$ small enough, $\ell'>0$ large enough and any $\lambda>0$. This means that

$$\langle \bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2),(-\Delta_{\hat z}+m^2)(\bar\varphi_1-\bar\varphi_2-\mathcal{K}_g(\mu,\bar\varphi_1)+\mathcal{K}_g(\mu,\bar\varphi_2))\rangle=0.$$

We are going to see that the inequality

$$\langle \bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2),(-\Delta_{\hat z}+m^2)(\bar\varphi_1-\bar\varphi_2)\rangle\geqslant C\int \bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2)(\bar\varphi_1-\bar\varphi_2)\mathrm{d}\hat z$$

holds for $\lambda>0$ small enough and some constant $C>0$. Indeed let f_1,f_2 be two smooth functions then, for $\lambda>0$ small enough,

$$\begin{aligned}
&\langle \bar r_{\ell'}(\lambda\hat z)J(f_1-f_2),(-\Delta_{\hat z}+m^2)(f_1-f_2)\rangle\\
&\quad=\int \bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)(-\Delta_{\hat z}+m^2)(f_1-f_2)\mathrm{d}\hat z\\
&\quad=\int \bar r_{\ell'}(\lambda\hat z)J'(f_1-f_2)|\nabla f_1-\nabla f_2|^2\mathrm{d}\hat z+\\
&\qquad+\lambda\int\nabla\bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)\cdot(\nabla f_1-\nabla f_2)\mathrm{d}\hat z\\
&\qquad+m^2\int\bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)(f_1-f_2)\mathrm{d}\hat z\\
&\quad\geqslant-\lambda^2\int(\Delta\bar r_{\ell'}(\lambda\hat z))J^{-1}(f_1-f_2)\mathrm{d}\hat z+m^2\int\bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)(f_1-f_2)\mathrm{d}\hat z\\
&\quad\geqslant\int\left(m^2-\left|\tfrac{\lambda^2\Delta\bar r_{\ell'}}{\bar r_{\ell'}}\right|\right)\rho_{\ell'}(\lambda\hat z)J(f_1-f_2)(f_1-f_2)\mathrm{d}\hat z\\
&\quad\geqslant C\int\bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)(f_1-f_2)\mathrm{d}\hat z
\end{aligned}$$

where $J^{-1}(t)=\int_0^t J(\tau)\mathrm{d}\tau$ and we have used the fact that $J^{-1}(t)\leqslant J(t)t$ since J is increasing and the fact that we can choose λ small enough such that the last inequality holds. This proves that

$$\langle \bar r_{\ell'}(\lambda\hat z)J(f_1-f_2),(-\Delta_{\hat z}+m^2)(f_1-f_2)\rangle\geqslant C\int\bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)(f_1-f_2)\mathrm{d}\hat z,\tag{52}$$

for λ small enough, and some $C>0$, for smooth function $f_1,f_2\in B^{s+2-\delta}_{p,p,\ell+\delta'}$. Since the expressions $\langle \bar r_{\ell'}(\lambda\hat z)J(f_1-f_2),(-\Delta_{\hat z}+m^2)(f_1-f_2)\rangle$ and $\int\bar r_{\ell'}(\lambda\hat z)J(f_1-f_2)(f_1-f_2)\mathrm{d}\hat z$ are continuous for $f_1,f_2\in B^{2+2-\delta}_{p,p,\ell+\delta'}$ (with respect to the $B^{s+2-\delta}_{p,p,\ell+\delta}$-natural norm) we can extend inequality (52) to the case of general functions $f_1,f_2\in B^{s+2-\delta}_{p,p,\ell+\delta'}$ and so it holds in particular for $f_1=\bar\varphi_1$ and $f_2=\bar\varphi_2$. Furthermore we have that $\langle \bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2),(-\Delta_{\hat z}+m^2)(-\mathcal{K}_g(\mu,\bar\varphi_1)+\mathcal{K}_g(\mu,\bar\varphi_2))\rangle\geqslant 0$. Indeed by Lemma 26 the product is associative and thus we obtain

$$\begin{aligned}
&\langle \bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2),(-\Delta_{\hat z}+m^2)(-\mathcal{K}_g(\mu,\bar\varphi_1)+\mathcal{K}_g(\mu,\bar\varphi_2))\rangle=\\
&\quad=\int\bar r_{\ell'}(\lambda\hat z)(\alpha G(\alpha\bar\varphi_1)-\alpha G(\alpha\bar\varphi_1))J(\bar\varphi_1-\bar\varphi_2)\mathrm{d}\mu(\hat z)\geqslant 0,
\end{aligned}$$

where we use that $\mathrm{d}\mu(\hat z)$ is a positive measure and $(\alpha G(\alpha t_1)-\alpha G(\alpha t_2))\cdot J(t_1-t_2)$ is positive since both G and J are increasing functions and $J(0)=0$. Using the previous inequalities we obtain that

$$\int\bar r_{\ell'}(\lambda\hat z)J(\bar\varphi_1-\bar\varphi_2)(\bar\varphi_1-\bar\varphi_2)\mathrm{d}\hat z\leqslant 0$$

which holds only if $\bar\varphi_1-\bar\varphi_2=0$, since J is a strictly increasing function. □

Remark 30 Combining Lemma 28 and Lemma 29 we deduce that the map $\mu\longmapsto\bar\varphi$, associating with the measure μ the unique solution $\bar\varphi$ to equation (50), is continuous with respect to μ. Indeed suppose that $\mu_n\to\mu$ in $\mathfrak{M}^s_{+,p,p,\ell}$ (and so in $B^s_{p,p,\ell}$), then by Lemma 26, we have that $\|\bar\varphi_n\|_{B^{s+2}_{p,p,\ell}}\lesssim\sup\|\mu_n\|_{B^s_{p,p,\ell}}$, and so there exists a converging subsequence $\bar\varphi_{k_n}$, as $n\to\infty$. On the other hand, since $\mathcal{K}_g$ is continuous in both μ and φ, we get that $\bar\varphi_{k_n}$ converges to the unique solution φ to equation (50) associated with μ. Since the limit does not depend on the subsequence we have that $\varphi_n\to\varphi$ strongly in $B^{s+2-\delta}_{p,p,\ell+\delta'}$, which proves the continuity of the solution map.

This continuity of the solution map with respect to the measure μ is similar to the continuity result obtained for solutions of singular SPDEs defined by the methods of paracontrolled calculus or regularity structure theory.

Proof of Theorem 23 The existence and uniqueness of the solution to equation (34) are proved in Lemma 28 and Lemma 29 considering the equation $\bar{\phi} = \mathcal{K}(\eta, \bar{\phi})$. What remains to prove is the convergence of $\bar{\phi}_{\varepsilon_n}$ to $\bar{\phi}$, as $\varepsilon_n \to 0$. Let $\epsilon_n \to 0$ be a sequence of positive numbers such that $\eta_{\epsilon_n} \to \eta$ almost surely in $\mathfrak{M}^s_{+,p,p,\ell}$ and let $w \in \mathcal{W}$ be such that $\eta_{\epsilon_n}(w) \to \eta(w)$ in $\mathfrak{M}^s_{+,p,p,\ell}$, as $\varepsilon \to 0$. We note that

$$\bar{\phi}_{\epsilon_n}(w) = \mathfrak{a}^{*2}_{\epsilon_n} * (\mathcal{K}_g(\eta_{\epsilon_n}(w), \bar{\phi}_{\varepsilon_n}(w))). \tag{53}$$

From the equality (53) and Lemma 26 we obtain that

$$\|\bar{\phi}_{\epsilon_n}(w)\|_{B^{s+2}_{p,p,\ell}} \lesssim \sup_{n\in\mathbb{N}} \|\eta_{\varepsilon_n}\|_{B^s_{p,p,\ell}},$$

uniformly in n. This means that that there exists a subsequence $\bar{\phi}_{\epsilon_{k_n}}(w)$ converging to some $\bar{\varphi}$ in $B^{s+2-\delta}_{p,p,\ell+\delta'}$. On the other hand we have that

$$\bar{\varphi} = \lim_{n\to+\infty} \bar{\phi}_{\epsilon_{k_n}}(w) = \lim_{n\to+\infty} \mathfrak{a}^{*2}_{\epsilon_n} * \left(\mathcal{K}_g\left(\eta_{\epsilon_{k_n}}(w), \bar{\phi}_{\varepsilon_{k_n}}(w)\right)\right) = \mathcal{K}_g(\eta(w), \bar{\varphi}).$$

Since, by Lemma 29, equation (50) has a unique solution we have that all the subsequence $\bar{\phi}_{\epsilon_{k_n}}(w)$ converge to the same $\bar{\varphi}$ and so $\bar{\varphi} = \bar{\phi}(w)$ (which is the unique solution to equation (34) evaluated in $w \in \mathcal{W}$). Since the previous reasoning holds for almost every $w \in \mathcal{W}$ and since $\eta_{\epsilon_n} \to \eta$ almost surely, we have that $\bar{\phi}_{\epsilon_n} \to \bar{\phi}$ in $B^{s+2-\delta}_{p,p,\ell+\delta'}$ almost surely, as $\varepsilon_n \to 0$. □

3.3 Dimensional reduction

In this section we want to prove the reduction principle for equation (33), i.e. that the random field $\phi(0,z) = \mathcal{I}\xi(0,z) + \bar{\phi}(0,z)$ has the law κ_g given by expression (35).

Before proving the dimensional reduction for equation (33) we have to prove that the restriction of a solution to a two dimensional hyperplane is a well defined operation.

Lemma 31 *There exists a version of the functional $x \longmapsto \mathcal{I}\xi(x,\cdot)$ which is continuous as a function from $\mathbb{R}^2$ into $\mathcal{C}^{0-}_\ell(\mathbb{R}^2) \subset B^{0-}_{p,p,\ell+\delta'}$. Furthermore for any $h \in C^\infty_0(\mathbb{R}^2)$ the sequence of random variables $\int_{\mathbb{R}^2} \mathfrak{a}_\epsilon * (\mathcal{I}\xi)(z)h(z)\mathrm{d}z$ converges to $\langle \mathcal{I}\xi(0,\cdot), h\rangle$ in $L^2(\mathrm{d}\mu)$, as $\varepsilon \to 0$.*

Proof In order to prove that $x \longmapsto \mathcal{I}\xi(x,\cdot)$ admits a continuous version we prove that for any smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $(-\Delta_z + m^2)^{-1/2+\epsilon}(h) \in L^2(\mathbb{R}^2)$, with $\varepsilon > 0$, we have

$$\mathbb{E}[|\langle \mathcal{I}\xi(x,\cdot) - \mathcal{I}\xi(y,\cdot), h\rangle|^2] \lesssim |x-y|^\epsilon \|(-\Delta_z + m^2)^{-1/2+\epsilon}(h)\|_{L^2(\mathbb{R}^2)},$$

where the constants implied by $\lesssim$ do not depend on h, x, y. This result, exploiting hypercontractivity and a version of Kolmogorov continuity criterion for multidimensional random fields, implies the continuity of $\mathcal{I}\xi(x,\cdot)$ with respect to $x \in \mathbb{R}^2$. We note that, for any $x, y \in \mathbb{R}^2$:

$$\mathbb{E}[|\langle \mathcal{I}\xi(x,\cdot) - \mathcal{I}\xi(y,\cdot), h\rangle|^2] = 2\int_{\mathbb{R}^2} \frac{1 - e^{ik_1\cdot(x-y)}}{(k_1^2+k_2^2+m^2)^{1+2\epsilon}} \frac{\hat{h}(k_2)^2}{(k_1^2+k_2^2+m^2)^{1-2\epsilon}} \mathrm{d}k_1 \mathrm{d}k_2.$$

Now we observe that $|1 - e^{ik_1\cdot(x-y)}| \leqslant 2|x-y|^{\frac{p-1}{p}}|k_1|^{\frac{p-1}{p}}$ for any $p > 1$. If we choose $p = \frac{1}{1-\epsilon}$, $0 < \epsilon < 1$, we obtain the claim.

In order to prove that $\int_{\mathbb{R}^2} \mathfrak{a}_\epsilon * (\mathcal{I}\xi)(z)h(z)\mathrm{d}z$ converges, as $\varepsilon \to 0$, to $\langle \mathcal{I}\xi(0,\cdot), h\rangle$ in $L^2(\mathrm{d}\mu)$, we note that the distribution $\mathcal{I}(\delta_0(x)\cdot h(z))$ belongs to $L^2(\mathbb{R}^4)$, implying that

$$\int_{\mathbb{R}^2} \mathfrak{a}_\epsilon * (\mathcal{I}\xi)(z)h(z)\mathrm{d}z = \delta[\mathcal{I}(\delta_0(x)\cdot \mathfrak{a}_\epsilon * h(z))] \qquad \langle \mathcal{I}\xi(0,\cdot), h\rangle = \delta[\mathcal{I}(\delta_0(x)\cdot h(z))],$$

where δ denote the Skorohod integral with respect to the white noise ξ. Since $\mathcal{I}(\delta_0(x)\cdot \mathfrak{a}_\epsilon * h(z))$ converges to $\mathcal{I}(\delta_0(x)\cdot h(z))$ in $L^2(\mathbb{R}^4)$ also this claim follows. □

Lemma 32 *The operator $T_0 : C^0_\ell(\mathbb{R}^4) \to C^0_\ell(\mathbb{R}^2)$ given by $T_0(h)(\cdot) = h(0,\cdot)$, for $h \in C^0_\ell(\mathbb{R}^4)$, can be uniquely extended in a continuous way as an operator from $B^{s+2-\delta}_{p,p,\ell+\delta'}(\mathbb{R}^4)$ into $B^{s+2-2/p-\delta}_{p,p,\ell+\delta'}(\mathbb{R}^2)$, when s, p, δ satisfies the thesis of Lemma 25, $\ell > 0$ big enough and $\delta' > 0$ small enough.*

Proof If s, p, δ satisfies the thesis of Lemma 25 then $s + 2 - \frac{2}{p} - \delta > 0$. Under this condition the proof can be found in Section 4.4.1 and Section 4.4.2 of [75], see also Section 18.1 of [76]. □

Theorem 33 *Let ϕbe the unique solution to equation (33) then, if $|\alpha| < 4\sqrt{2}\pi$, there exists $\ell > 0$ such that for any measurable and bounded function F on $B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^2)$, for some $\delta' > 0$ small enough, we have*

$$\mathbb{E}[F(\phi(0,\cdot))] = \int_{B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^2)} F(\omega) \mathrm{d}\kappa_g(\omega). \tag{54}$$

Proof We prove equation (54) for the case in which $F(\omega) = \tilde{F}(\langle \omega, f_1\rangle, \dots, \langle \omega, f_n\rangle)$ where $\tilde{F}$ is a bounded continuous function, and $f_1, \dots, f_n \in C^\infty_0(\mathbb{R}^2)$. Since the functions of the previous form generate all the σ-algebra of Borel measurable functions on $B^{0-}_{p,p,\ell+\delta'}$, where $\delta' > 0$, proving the theorem for functions of the previous form is equivalent to prove the theorem in general.

Since equation (38) holds, we have only to prove that $\mathbb{E}[F(\phi_{\epsilon_n}(0,\cdot))] \to \mathbb{E}[F(\phi(0,\cdot))]$, as $\epsilon_n \to 0$ and where ϵ_n is any subsequence such that $\bar{\phi}_{\epsilon_n} \to \bar{\phi}$ almost surely (whose existence is proved in Theorem 23) and $\int_{B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^2)} F(\omega)\mathrm{d}\kappa_\epsilon(\omega) \to \int_{B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^2)} F(\omega)\mathrm{d}\kappa(\omega)$, as $\epsilon_n \to 0$.

The first convergence follows from the fact that $\bar{\phi}_\epsilon = \bar{\phi}_\epsilon + \mathfrak{a}_\epsilon(\mathcal{I}\xi)$. Indeed, by Theorem 23, as $\epsilon_n \to 0$,

$$\langle \bar{\phi}_{\epsilon_n}(0,\cdot), f_i\rangle = \langle \bar{\phi}_{\varepsilon_n}, T_0^*(f_i)\rangle \to \langle \bar{\phi}, T_0^*(f_i)\rangle = \langle \bar{\phi}(0,\cdot), f_i\rangle,$$

for any $i = 1, \dots, n$, almost surely and where the continuity of T_0, proved in Lemma 32, is used. On the other hand, by Lemma 31, we have $\langle \mathfrak{a}_\epsilon(\mathcal{I}\xi)(0,\cdot), f_i\rangle \to \langle \mathcal{I}\xi(0,\cdot), f_i\rangle$ in probability, as $\varepsilon_n \to 0$, and this implies that $\langle \phi_{\epsilon_n}(0,\cdot), f_i\rangle \to \langle \phi(0,\cdot), f_i\rangle$ in probability, for any $i = 1, \dots, n$.

Since the convergence in probability implies the one in distribution we get $\mathbb{E}[F(\phi_{\epsilon_n}(0,\cdot))] \to \mathbb{E}[F(\phi(0,\cdot))]$. Finally since κ_{ϵ_n} converges weakly to κ, as $\epsilon_n \to 0$, the thesis follows. □

Now we want to discuss what happens if we remove the cut-off g in equation (33) so we consider the equation

$$(-\Delta_{\hat{z}} + m^2)(\phi) + \alpha \exp(\alpha\phi - \infty) = \xi, \tag{55}$$

for the content of this equation see the discussion ate the beginning of the present section. In order to distinguish between the solution to equation (33) and equation (55) we denote by ϕ_g the solution of the former and by ϕ the solution of the latter. We use also the symbols $\bar{\phi}_g = \phi_g - \mathcal{I}\xi$ and $\bar{\phi} = \phi - \mathcal{I}\xi$, with $\mathcal{I} = (-\Delta + m^2)^{-1}$.

Proposition 34 *There exists an unique solution ϕ to equation (55). Furthermore, for any affine transformation $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ in the Euclidean group of $\mathbb{R}^4$, the random field $\Phi_*(\phi)(\hat{z}) = \phi(\Phi(\hat{z}))$ has the same law as $\phi(\hat{z})$.*

Proof The existence and uniqueness for the solution to equation (55) can be proven as in Lemma 28 and Lemma 29, since the estimates used to prove those lemmas do not depend on g.

The invariance of the law of the solutions with respect to affine transformations in the Euclidean group of $\mathbb{R}^4$ follows from the invariance (in law) of the white noise ξ and of the left-hand-side of equation (55) with respect to translations and rotations and from the uniqueness of the solution to equation (55). □

Theorem 35 *If g_n is an (increasing) sequence of cut-offs with compact support such that $g_n \uparrow 1$ then $\bar{\phi}_{g_n} \to \bar{\phi}$ in $B^{s+2-\delta}_{p,p,\ell+\delta'}$ (which means that $\phi_{g_n} \to \phi$ in $B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^4)$). This means that the sequence of probability measures κ_{g_n} on $B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^2)$ converges weakly, as $n \to \infty$, to a probability measure κ which is invariant with respect to the natural action of the Euclidean group of $\mathbb{R}^2$ on $B^{0-}_{p,p,\ell+\delta'}(\mathbb{R}^2)$ and does not depend on the sequence of cut-offs g_n.*

Proof The proof is essentially based on the fact that

$$\|\bar{\phi}_g\|_{B^{s+2}_{p,p,\ell}} \lesssim \|\eta\|_{B^s_{p,p,\ell}}$$

uniformly on g. From the previous inequality and some reasoning similar to the ones used in the proof of Theorem 23 the convergence follows. The properties of the limit measure κ follow from the same properties of the solution ϕ to equation (55) proved in Proposition 34. □

4 Elliptic quantization of the $P(\varphi)_2$ model

In this section we discuss the elliptic stochastic quantization of the $P(\varphi)_2$ model where P is a polynomial of even degree and satisfying Hypothesis QC. In order to avoid technical details we consider only the case $P(\varphi) = \frac{\varphi^{2n}}{2n}$ for $n \in \mathbb{N}$, the general case being then a straightforward generalization. We consider the equation

$$(-\Delta_x - \Delta_z + m^2)(\phi) + f(x)\phi^{\diamond 2n-1} = \xi \tag{56}$$

where $z \in M = \mathbb{T}^2$, $\diamond$ stands for the Wick product and ξ is a $\mathbb{R}^2 \times \mathbb{T}^2$ white noise. Equation (56) can be better understood if we consider the equation for $\bar{\phi} := \phi - \mathcal{I}\xi$, the usual Da Prato-Debussche trick (introduced in [35]), obtaining the equation that is

$$(-\Delta_x - \Delta_z + m^2)(\bar{\phi}) + \sum_{k=0}^{2n-1} \binom{2n-1}{k} f(x) \cdot \mathcal{I}\xi^{\diamond k} \cdot \bar{\phi}^{2n-1-k} = 0. \tag{57}$$

Equation (57) is expected to be well defined, since we expect that the solution $\bar{\phi}$ is in $H^1(\mathbb{R}^2 \times \mathbb{T}^2) = W^{1,2}(\mathbb{R}^2 \times \mathbb{T}^2)$, in $L^{2n}_{f^{1/2n}}(\mathbb{R}^2 \times \mathbb{T}^2)$ (where $L^{2n}_{f^{1/2n}}$ is the weighted L^{2n} space with respect to the space weight $f^{\frac{1}{2n}}$) and in $B^{2-\delta}_{\mathfrak{p},\mathfrak{p},\ell}(\mathbb{R}^2 \times \mathbb{T}^2)$, where $\mathfrak{p} = \frac{2n}{2n-1}$. Furthermore it is well known that $\mathcal{I}\xi^{\diamond k} \in \mathcal{C}^{-\delta}_\ell = B^{-\delta}_{\infty,\infty,\ell}(\mathbb{R}^2 \times \mathbb{T}^2)$ for any $\ell, \delta > 0$ (see [46]). In general we expect that equation (57), and so equation (56), for any realization of the noise ξ admits multiple solutions. So we need a notion of weak solution to equation (57).

First of all we consider a fixed probability space $\mathcal{W}^e = (\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \mathfrak{W}$, where

$$\mathfrak{W} = H^{1-\delta_1}(\mathbb{R}^2 \times \mathbb{T}^2) \cap L^{2n-\delta_2}_{f^{1/2n+\delta_3}}(\mathbb{R}^2 \times \mathbb{T}^2) \cap B^{2-\delta_4}_{\mathfrak{p},\mathfrak{p}}(\mathbb{R}^2 \times \mathbb{T}^2),$$

and where $\delta, \delta_1, \delta_2, \delta_3, \delta_4 > 0$ are small enough (we give some more precise conditions in what follows, see inequalities (62), (64), (66), (67) and (68)), that is the space where $(\mathcal{I}\xi, \mathcal{I}\xi^{\diamond 2}, \dots, \mathcal{I}\xi^{\diamond 2n}, \bar{\phi}) \in \mathcal{W}^e$ are defined. We consider a distinguished subspace of $\mathfrak{W}$ namely

$$\overline{\mathfrak{W}} = H^1(\mathbb{R}^2 \times \mathbb{T}^2) \cap L^{2n}_{f^{1/2n}}(\mathbb{R}^2 \times \mathbb{T}^2) \cap \mathfrak{W}.$$

Remark 36 Hereafter, we will always use the following additional hypothesis on the cut-off f:

Hypothesis Hf1. The function f satisfies Hypothesis Hf and furthermore f satisfy the following properties for any $x, y \in \mathbb{R}^2$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 2$

$$0 < f(x)^{\pm 1} < cf(x-y)^{\pm 1} \exp(d|y|)$$

$$|D^\alpha f(x)| \leqslant ef(x)$$

where $c, e > 0$ and $d \geqslant 0$ are some constants.

Examples of functions f satisfying Hypothesis Hf1 can be found in [56].
The importance of Hypothesis Hf1 is that an equivalent norm for $B^s_{p,q,f^\beta}(\mathbb{R}^2 \times \mathbb{T}^2)$, i.e. the

weighted Besov space with weight $(f(x))^\beta$ (see [71, 72] for the precise definitions), when $|s| < 2$, $-1 \leqslant \beta \leqslant 1$ and $p > 1$, is given by

$$\|u\|_{B^s_{p,q,f^\beta}} \sim \|f^\beta u\|_{B^s_{p,q}}$$

(see [72] Theorem 4.4). This means that an analogous version of Proposition 50 and Proposition 52 holds also for the space B^s_{p,q,f^β}.

Definition 37 *A probability measure ν^e on $\mathcal{W}^e$ is a weak solution to equation (57) if the projection of ν^e on $[\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2)]^{2n}$, $\delta > 0$, gives the law of $(\mathcal{I}\xi, \mathcal{I}\xi^{\diamond 2}, \dots, \mathcal{I}\xi^{\diamond 2n-1})$ (which is the law of a Gaussian noise with covariance $(-\Delta_x - \Delta_z + m^2)^{-2}$ and its Wick powers) and it is supported on the set of solutions to the equation*

$$(-\Delta_x - \Delta_z + m^2)(\bar\theta) + \sum_{k=0}^{2n-1} \binom{2n-1}{k} f(x) \cdot \sigma_k \cdot \bar\theta^{2n-1-k} = 0, \tag{58}$$

where $x \in \mathbb{R}^2$, $z \in \mathbb{T}^2$, $(\sigma_1, \dots, \sigma_{2n}, \bar\theta) \in \mathcal{W}^e$ and $\sigma_0 = 1$, defined on $\mathcal{W}^e$.

The weak solution ν to equation (56) associated with the weak solution ν^e to equation (57), is the probability law on $\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2) + \mathfrak{W}$, $\delta > 0$, given by the push-forward of ν^e with respect to the map $\sigma_1 + \theta$ defined on $\mathcal{W}^e$.

We introduce a modified equation on $\mathfrak{W}$ given by

$$(-\Delta_x - \Delta_z + m^2)(\bar\theta) + \mathfrak{a}^{*2}_\epsilon * \left[\sum_{k=0}^{2n-1} \binom{2n-1}{k} f(x) \cdot \sigma_k \cdot \bar\theta^{2n-1-k}\right] = 0 \tag{59}$$

where $\mathfrak{a}_\epsilon$ is some regular enough mollifier on $\mathbb{T}^2$ such that the operator $\mathcal{A}_\epsilon = \mathfrak{a}_\epsilon *$ satisfies Hypothesis H$\mathcal{A}$ (for example we can take $\mathfrak{a}_\epsilon$ as the Green function associated with the operator $(-\epsilon\Delta_z + 1)^{-k}$ for k large enough), $\varepsilon > 0$. Also in this case we consider a special subspace of $\mathfrak{W}$ given by

$$\overline{\mathfrak{W}}_\epsilon = \mathcal{A}_\epsilon(H^1(\mathbb{R}^2 \times \mathbb{T}^2)) \cap L^{2n}_{f^{1/2n}}(\mathbb{R}^2 \times \mathbb{T}^2) \cap \mathfrak{W}.$$

It is important to note that $\overline{\mathfrak{W}}_\epsilon \subset \overline{\mathfrak{W}}$.

Lemma 38 *Let $\bar\theta \in \overline{\mathfrak{W}}_\epsilon \subset \overline{\mathfrak{W}}$ be a solution to equation (59) then $\bar\theta \in B^{2-\delta''}_{\mathfrak{p},\mathfrak{p}}(\mathbb{R}^2 \times \mathbb{T}^2)$ (with $0 < \delta < \delta'' < \delta_4$ being δ_4 as in the definition of $\mathcal{W}^e$) and*

$$\|\bar\theta\|^2_{H^1} + \|\bar\theta\|^{2n}_{L^{2n}_{f^{1/2n}}} \lesssim \left[\sum_{k=1}^{2n} \|\sigma_k\|_{\mathcal{C}^{-\delta}_\ell}\right]^{\beta_1} \tag{60}$$

$$\|\bar\theta\|_{B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}}} \lesssim \|\mathcal{A}^{-2}_\epsilon \bar\theta\|_{B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}}} \lesssim \left[\|\theta\|_{H^1} + \|\theta\|_{L^{2n}_{f^{1/2n}}} + \sum_{k=1}^{2n} \|\sigma_k\|_{\mathcal{C}^{-\delta}_\ell}\right]^{\beta_2} \tag{61}$$

for any $0 \leqslant \beta < \frac{1}{2n} - (2n-1)\delta_3$, and where $\beta_1, \beta_2 \in \mathbb{R}_+$ and where β_1, β_2 and the constants implied by the symbol $\lesssim$ do not depend on ϵ, σ_k and β.

Proof Let $0 < \gamma < 1$ be such that

$$(2n-1)\left(\frac{\gamma}{2n} + \frac{1-\gamma}{2}\right) < 1. \tag{62}$$

Then by Remark 36, Proposition 50 and Proposition 52 it is simple to see that $\bar\theta \in B^{(1-\gamma)}_{p,2,f^{1/2n}}(\mathbb{R}^2 \times \mathbb{T}^2)$, where $p = \left(\frac{\gamma}{2n} + \frac{1-\gamma}{2}\right)^{-1}$, and

$$\|\bar\theta\|_{B^{(1-\gamma)}_{p,2,f^{1/2n}}} \lesssim \|\bar\theta\|^\gamma_{L^{2n}_{f^{1/2n}}} \cdot \|\bar\theta\|^{1-\gamma}_{H^1}.$$

Using the fact that $\bar\theta \in \mathcal{A}_\epsilon(H^1)$ we can multiply both sides of equation (59) by $\mathcal{A}_\epsilon^{-2}(\bar\theta)$ and take the integral over $\mathbb{R}^2 \times \mathbb{T}^2$ obtaining

$$\begin{aligned}\|\mathcal{A}_\epsilon^{-1}(\bar\theta)\|^2_{H^1} &\leqslant -\|\bar\theta\|^{2n}_{L^{2n}_{f^{1/2n}}} - \sum_{k=1}^{2n-1}\binom{2n-1}{k}\int f(x)\cdot\sigma_k(x,z)\cdot\bar\theta^{2n-k}(x,z)\mathrm{d}x\mathrm{d}z \\ &\leqslant -\|\bar\theta\|^{2n}_{L^{2n}_{f^{1/2n}}} + C\sum_{k=1}^{2n-1}\|f^{1/2n}\sigma_k\|_{\mathcal{C}^{-\delta}}\left[\|\bar\theta\|^{\gamma}_{L^{2n}_{f^{1/2n}}}\|\bar\theta\|^{1-\gamma}_{H^1}\right]^{2n-k}\end{aligned}$$

where we choose $\delta < 1-\gamma$ and for some constant $C>0$ depending only on n. Using the condition (48), the fact that $\|\bar\theta\|_{H^1} \lesssim \|\mathcal{A}_\varepsilon^{-1}(\bar\theta)\|_{H^1}$ and Young's inequality for products we obtain

$$\|\bar\theta\|^2_{H^1} + \|\bar\theta\|^{2n}_{L^{2n}_{f^{1/2n}}} \lesssim \sum_{k=1}^{2n-1}\|f^{1/2n}\sigma_k\|^{1/\epsilon}_{\mathcal{C}^{-\delta}}. \tag{63}$$

for $\epsilon = \left(1-(2n-1)\left(\frac{\gamma}{2n}+\frac{1-\gamma}{2}\right)\right)$. Let

$$0<\beta<\frac{1}{2n} \quad \text{and} \quad \delta<\delta''<(1-\gamma), \tag{64}$$

we have

$$\begin{aligned}\|(\mathcal{A}_\epsilon^{-2}\bar\theta)f^{-\beta}\|_{B^{2-\delta''}_{\mathfrak{p},\mathfrak{p}}} &\lesssim \left\|(f(x))^{-\beta}\left[\textstyle\sum_{k=0}^{2n-1}\binom{2n-1}{k}f(x)\cdot\sigma_k\cdot\bar\theta^{2n-1-k}\right]\right\|_{B^{-\delta''}_{\mathfrak{p},\mathfrak{p}}} \\ &\lesssim \textstyle\sum_{k=0}^{2n-1}\|(f(x))^{1-\beta}\cdot\sigma_k\cdot\bar\theta^{2n-1-k}\|_{B^{-\delta''}_{\mathfrak{p},\mathfrak{p}}} \\ &\lesssim \textstyle\sum_{k=0}^{2n-1}\|f^{1/2n-\beta}\cdot\sigma_k\|_{\mathcal{C}^{-\delta}}\times \\ &\quad \times\left[\|\bar\theta\|^{\gamma}_{L^{2n}_{f^{1/2n}}}\cdot\|\bar\theta\|^{1-\gamma}_{H^1}\right]^{2n-1-k} \\ &\lesssim \|\bar\theta\|_{H^1} + \|\bar\theta\|_{L^{2n}_{f^{1/2n}}} + \textstyle\sum_{k=1}^{2n-1}\|f^{1/2n-\beta}\sigma_k\|^{\beta_2}_{\mathcal{C}^{-\delta}}\end{aligned} \tag{65}$$

where $\delta>0$ such that $\delta>\delta''$. Inserting now inequality (63) into inequality (65), and using the fact that $\|\bar\theta\|_{B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}}} \lesssim \|(\mathcal{A}_\epsilon^{-2}\bar\theta)f^{-\beta}\|_{B^{2-\delta''}_{\mathfrak{p},\mathfrak{p}}}$ we obtain the thesis. □

An easy consequence of the previous lemma is the following one.

Lemma 39 *Let $\mathcal{F}_\epsilon : (\mathcal{C}_\ell^{-\delta})^{2n-1} \to \mathcal{P}(\bar{\mathfrak{W}}_\varepsilon)$, $\varepsilon>0$, be the set-valued map associating $(\sigma_1,\dots,\sigma_{2n-1})$ with the set of solutions to equation (59) in $\overline{\mathfrak{W}}_\varepsilon$. If $K \subset (\mathcal{C}_\ell^{-\delta})^{2n-1}$ is a compact set then $\overline{\bigcup_{\epsilon<1}\mathcal{F}_\epsilon(K)}$ is compact is a subset of $\bar{\mathfrak{W}}$ and it is compact in $\mathfrak{W}$.*

Proof The proof of the lemma consists only in noticing that a consequence of Proposition 50 is that the inclusion map $i : B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}} \cap L^{2n}_{f^{1/2n}} \to \mathfrak{W}$ is compact, when β satisfies condition (64) and it is large enough and $\delta''>\delta>0$ is small enough. Indeed using Proposition 50 and Proposition 52, it is simple to prove that $B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}}$ is compactly imbedded in $B^{2-\delta_4}_{\mathfrak{p},\mathfrak{p}}$ when

$$\delta<\delta''<\delta_4 \tag{66}$$

since $f^{-\beta}\to+\infty$ as $x\to+\infty$.

Recalling that, by Remark 46 and Proposition 50, $L^p_{f^{1/2n}} \subset B^0_{p,p,f^{1/2n}}$ when $p \geqslant 2$, we obtain using Proposition 52, we have that $B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}} \cap L^{2n}_{f^{1/2n}}$ is compactly embedded in $B^{(2-\delta'')\theta-\epsilon}_{p_\theta-\epsilon,p_\theta-\epsilon,f^{\beta\theta+\epsilon}}$ where

$$\begin{aligned}p_\theta &= \frac{2n}{(2n-2)\theta+1} \\ \beta_\theta &= -\theta\left(\frac{1}{2n}+\beta\right)+\frac{1}{2n}\end{aligned}$$

for any $0 \leqslant \theta \leqslant 1$. Taking $0 < \theta < \frac{\delta''}{(2n-\delta'')(2n-2)}$, $\beta = 0$ and ϵ small enough we have that $B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}} \cap L^{2n}_{f^{1/2n}}$ is compactly embedded in $L^{2n-\delta_2}_{f^{1/2n+\delta_3}}$ for any $\delta_2 < 1$ and $\delta_3 > 0$.

Furthermore if we take

$$\frac{1-\delta_1}{2-\delta''} < \theta < \frac{n-1}{2n-2}$$

$$\left(\frac{2-\delta''}{1-\delta_1}\right)\frac{1}{2n} - \frac{1}{2n} < \beta < \frac{1}{2n},$$

which are a non-void conditions whenever

$$\delta < \delta'' < 1 \tag{67}$$

$$0 < \delta < \delta'' < \delta_1 \frac{2n-2}{n-1}, \tag{68}$$

we obtain that $B^{(2-\delta'')\theta-\epsilon}_{p_\theta-\epsilon,p_\theta-\epsilon,f^{\beta_\theta+\epsilon}}$ compactly embeds in $H^{1-\delta_1}$ for ϵ small enough.

Using inequalities (60) and fact that the balls of $i(B^{2-\delta''}_{\mathfrak{p},\mathfrak{p},f^{-\beta}} \cap L^{2n}_{f^{1/2n}})$ are closed sets of $\mathfrak{W}$, $\overline{\bigcup_{\epsilon<1}\mathcal{F}_\epsilon(K)}$ is contained in $\overline{\mathfrak{W}}$. □

Hereafter we denote by $\mathfrak{P} : \mathcal{W}^e \to (\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2))^{2n}$ the natural projection of the Cartesian product $\mathcal{W}^e = (\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \mathfrak{W}$ on the first component.

Lemma 40 *Let ν^e_ϵ be a sequence of weak solutions to equation (59) such that $\nu^e_\epsilon((\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2\times\mathbb{T}^2))^{2n}\times \bar{\mathfrak{W}}_\epsilon) = 1$, and suppose that $\mathfrak{P}_*(\nu^e_\epsilon)$ is tight. Then ν^e_ϵ is tight. Furthermore suppose that $\mathfrak{P}_*(\nu^e_\epsilon)$ weakly converges to the law of $(\mathcal{I}\xi, \mathcal{I}\xi^{\diamond 2}, \dots, \mathcal{I}\xi^{\diamond 2n-1})$ as $\epsilon \to 0$, then any convergent subsequence $\nu^e_{\epsilon_n}$ converges, as $\varepsilon_n \to 0$, to a weak solution ν^e to equation (57) such that $\nu^e((\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \overline{\mathfrak{W}}) = 1$.*

Proof The proof of the tightness of ν^e_ϵ is similar to the one of Lemma 12, where Lemma 11 is replaced by Lemma 39. The hypothesis that $\nu^e_\epsilon((\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \bar{\mathfrak{W}}_\epsilon) = 1$ enters in the proof in the following way: let $\widetilde{\mathcal{F}}_\epsilon : (\mathcal{C}^{-\delta}_\ell)^{2n-1} \to \mathcal{P}(\mathfrak{W})$, $\varepsilon > 0$, be the set-valued map associating $(\sigma_1, \dots, \sigma_{2n-1})$ with the set of solutions to equation (59) in $\mathfrak{W}$, let $\tilde{K} \subset (\mathcal{C}^{-\delta}_\ell)^{2n}$ be a compact such that $\mu_\varepsilon(\tilde{K}) > 1 - \tilde{\epsilon}$, where μ_ε are the measure of $(\mathcal{I}\xi_\varepsilon, \mathcal{I}\xi^{\diamond 2}_\varepsilon, \dots, \mathcal{I}\xi^{\diamond 2n}_\varepsilon)$ which are tight since μ_ε converges to μ the measure of $(\mathcal{I}\xi_\varepsilon, \mathcal{I}\xi^{\diamond 2}_\varepsilon, \dots, \mathcal{I}\xi^{\diamond 2n}_\varepsilon)$. Furthermore denote by $\mathfrak{K} := \overline{\cup_\epsilon \mathcal{F}_\epsilon(\tilde{K})}$, which is compact by Lemma 39, then

$$\nu^e_\varepsilon(\tilde{K} \times \mathfrak{K}) \geqslant \nu^e_\varepsilon(\tilde{K} \times \cup_\epsilon \mathcal{F}_\epsilon(\tilde{K})) \geqslant \nu^e_\varepsilon(\tilde{K} \times \mathcal{F}_\epsilon(\tilde{K})) = \nu^e_\varepsilon(\tilde{K} \times \widetilde{\mathcal{F}_\epsilon}(\tilde{K})) = \mu_\varepsilon(\tilde{K}) \geqslant 1 - \tilde{\epsilon},$$

where in the fourth passage we use that $\nu_\varepsilon(\tilde{K} \times \mathcal{F}_\epsilon(\tilde{K})) = \nu_\varepsilon(\tilde{K} \times \widetilde{\mathcal{F}_\epsilon}(\tilde{K}))$ since $\nu^e_\epsilon((\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \bar{\mathfrak{W}}_\epsilon) = 1$.

Since $\mathfrak{P}_*(\nu^e_\epsilon)$ weakly converges to the law of $(\mathcal{I}\xi, \mathcal{I}\xi^{\diamond 2}, \dots, \mathcal{I}\xi^{\diamond 2n-1})$ and so $\mathfrak{P}_*(\nu^e)$ has the same law as $(\mathcal{I}\xi, \mathcal{I}\xi^{\diamond 2}, \dots, \mathcal{I}\xi^{\diamond 2n-1})$, the proof of the fact that ν^e is a weak solution to equation (57) is equivalent to prove that ν^e is such that for any C^1 bounded function F from $B^{-\delta_4}_{\mathfrak{p},\mathfrak{p}}$ into $\mathbb{R}$ with bounded derivative we have

$$\int F\left((-\Delta_x - \Delta_z + m^2)(\bar{\theta}) + \sum_{k=0}^{2n-1}\binom{2n-1}{k} f(x)\sigma_k \bar{\theta}^{2n-1-k}\right) \mathrm{d}\nu^e = \int F(E)\mathrm{d}\nu^e = F(0). \tag{69}$$

We know that $\int F(E_\epsilon)\mathrm{d}\nu^e_\epsilon = F(0)$ with

$$E_\epsilon = (-\Delta_x - \Delta_z + m^2)(\bar{\theta}) + \mathfrak{a}^{*2}_\epsilon * \left(\sum_{k=0}^{2n-1}\binom{2n-1}{k} f(x) \cdot \sigma_k \cdot \bar{\theta}^{2n-k}\right).$$

On the other hand E_ϵ converges to E uniformly on compact sets (since $\mathfrak{a}_\epsilon^{*2}*$ strongly converges as an operator to the identity on $B^{-\delta_4}_{\mathfrak{p},\mathfrak{p}}$), which implies that $F\circ E_\epsilon$ converges to $F\circ E$ uniformly on compact sets, as $\varepsilon\to 0$.

Finally since the sets of the form $\overline{\bigcup_{\epsilon<1}\mathcal{F}_\epsilon(K)}$ are subsets of $\bar{\mathfrak{W}}$, they are closed and they have an arbitrary measure we have that any limit ν^e is such that $\nu^e((\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2\times\mathbb{T}^2))^{2n}\times\overline{\mathfrak{W}})=1$. □

This fact and the boundedness of F and of its derivatives implies (using a reasoning similar to the one of Lemma 13, see also Lemma 2 in [10]) that $F\circ E_\epsilon \mathrm{d}\nu^e_\epsilon$ weakly converges to $F\circ E\mathrm{d}\nu^e$, as $\varepsilon\to 0$. This implies equation (69) and thus the thesis of the lemma.

Theorem 41 *There exists at least one weak solution ν^e to equation (57) such that, for any measurable bounded function $F: B^{-\delta}_{\mathfrak{p},\mathfrak{p},\ell}(\mathbb{T}^2)\to\mathbb{R}$ (where $\delta>0$ and $\ell>0$ as in the definition of $\mathcal{W}^e$ and $\mathfrak{p}=\frac{2n-1}{2n}$),*

$$\int_{\mathcal{W}^e} F(\phi(0,\cdot))\tilde{\Upsilon}_f(\sigma,\bar\theta)\mathrm{d}\nu^e(\sigma,\bar\theta)=Z_f\int_{B^{-\delta}_{\mathfrak{p},\mathfrak{p},\ell}(\mathbb{T}^2)}F(\omega)\mathrm{d}\kappa(\omega) \tag{70}$$

where $\phi=\sigma_1+\bar\theta$ (where σ_1 and $\bar\theta$ are as in Definition 37),

$$\tilde{\Upsilon}_f(\sigma,\bar\theta)=\exp\left(\sum_{k=0}^{2n}\binom{2n}{k}\langle\sigma_k\bar\theta^{2m-k},f'\rangle\right),$$

$$Z_f=\int_{\mathcal{W}^e}\tilde{\Upsilon}_f(\sigma,\bar\theta)\mathrm{d}\nu^e(\sigma,\bar\theta)$$

and κ is given by

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\mu^{-\Delta_z}}=\frac{\exp\left(\int_{\mathbb{T}^2}\frac{\omega^{\diamond 2m}(z)}{2m}\mathrm{d}z\right)}{Z_\kappa},$$

where $\mu^{-\Delta_z}$ is the Gaussian with covariance given by (11) with $\mathfrak{L}=-\Delta_z$ and $\mathcal{A}=\mathrm{id}$.

Lemma 42 *Let ν^e_ϵ be a sequence of weak solutions to equation (59) such that $\mathfrak{P}_*(\nu^e_\epsilon)\sim(\mathfrak{a}_\epsilon*\mathcal{I}\xi,\dots,(\mathfrak{a}_\epsilon*\mathcal{I}\xi)^{\diamond 2n-1})$ (where $\sim$ means "with the same law") and $\nu^e_\epsilon((\mathcal{C}^{-\delta}_\ell(\mathbb{R}^2\times\mathbb{T}^2))^{2n}\times\bar{\mathfrak{W}}_\epsilon)=1$ then*

$$\int\exp\left(p\sum_{k=0}^{2n}\binom{2n}{k}\langle\sigma_k,f'\bar\theta^{2n-k}\rangle\right)\mathrm{d}\nu^e_\epsilon<C_p$$

form some constant C_p uniform in ϵ small enough and depending on $p\geq 1$.

Proof We want to use Nelson's trick to prove the theorem (see Chapter V of [74]). Then we put $E(\sigma_k,\bar\theta)=\sum_{k=0}^{2n}\binom{2n}{k}\langle\sigma_k,f'\bar\theta^{2n-k}\rangle$ and introduce the following expression

$$E_{\epsilon,N}(\sigma_1,\bar\theta)=\begin{cases}\int_{\mathbb{R}^2\times\mathbb{T}^2}\sum_{k=0}^{2n}\binom{2n}{k}\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat z)\right)^{\diamond k}f'(x)\bar\theta^{2n-k}(\hat z)\mathrm{d}\hat z, & \text{for } \epsilon<\frac{1}{N},\\ \int_{\mathbb{R}^2\times\mathbb{T}^2}\sum_{k=0}^{2n}\binom{2n}{k}(\sigma_1(\hat z))^{\diamond k}f'(x)\bar\theta^{2n-k}(\hat z)\mathrm{d}\hat z, & \text{for } \epsilon\geqslant\frac{1}{N}.\end{cases}$$

We have to prove that, for any $p\geq 1$,

$$\left(\int|E-E_{\epsilon,N}|^p\mathrm{d}\nu^e_\epsilon\right)^{1/p}\lesssim(p-1)^uN^{-\alpha} \tag{71}$$

for some $u\in\mathbb{N}$ and $\alpha>0$ and

$$E_{\epsilon,N}\lesssim -f'(x)(\log(N))^{\alpha'}, \tag{72}$$

for some $\alpha' > 0$, and that both inequalities are uniform in ϵ. Inequality (71) is obvious when $\epsilon \geqslant \frac{1}{N}$, since then $E - E_{\epsilon,N} = 0$ ν_ϵ^e-almost surely. Consider the case $\epsilon < \frac{1}{N}$ then

$$\begin{aligned}|E - E_{\epsilon,N}| &\lesssim \left|\sum_{k=0}^{2n}\int_{\mathbb{R}^2\times\mathbb{T}^2} f'(x)\left(\sigma_k(z) - \left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z})\right)^{\diamond k}\right)\cdot\bar{\theta}^{2n-k}(\hat{z})\mathrm{d}\hat{z}\right| \\ &\lesssim \sum_{k=1}^{2n}\left\|\left|f'(x)\right|^{1/2n}\left(\sigma_k(z) - \left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z})\right)^{\diamond k}\right)\right\|_{\mathcal{C}^{-\delta}}\cdot\left\||f'(x)|^{1-1/2n}\bar{\theta}\right\|^{2n-k}_{B^\delta_{\mathfrak{p},\mathfrak{p}}}\end{aligned}$$

where $\mathfrak{p} = \frac{2n-1}{2n}$. On the other hand, by Lemma 38, we have:

$$\left\||f'(x)|^{1-1/2n}\bar{\theta}\right\|^{2n-k}_{B^\delta_{\mathfrak{p},\mathfrak{p}}} \lesssim \left(\|\bar{\theta}\|_{L^{2n}_{f^{1/2n}}} + \|\bar{\theta}\|_{H^1}\right)^{2n-k} \lesssim \left(\sum_{k=1}^{2n}\|\sigma_k\|_{\mathcal{C}_\ell^{-\delta}}\right)^{\beta_1(2n-1)},$$

ν_ϵ^e-almost surely and uniformly in ϵ (since $\nu_\epsilon^e((\mathcal{C}_\ell^{-\delta}(\mathbb{R}^2\times\mathbb{T}^2))^{2n}\times\bar{\mathfrak{W}}_\epsilon) = 1$). Using Hölder's inequality we obtain, for any $p \geq 1$,

$$\begin{aligned}\left(\int|E - E_{\epsilon,N}|^p\mathrm{d}\nu_\epsilon^e\right)^{\frac{1}{p}} &\lesssim \sum_{k=1}^{2n}\left[\int\left\|\left|f'(x)\right|^{1/2n}\left[\sigma_k(z) - \left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z})\right)^{\diamond k}\right]\right\|^{2p}_{\mathcal{C}^{-\delta}}\mathrm{d}\nu_\epsilon^e\right]^{\frac{1}{2p}}\times \\ &\quad\times\left(\int\left(\sum_{k=1}^{2n}\|\sigma_k\|_{\mathcal{C}_\ell^{-\delta}}\right)^{2\beta_1(2n-1)p}\mathrm{d}\nu_\epsilon^e\right)^{\frac{1}{2p}}\end{aligned}$$

Since the law of σ_k is a multilinear functional of a Gaussian random field, by hypercontractivity we have that there exists $m_1, m_2 \in \mathbb{N}$ and $\alpha > 0$ such that

$$\begin{aligned}\int\left\|\left|f'(x)\right|^{1/2n}\left(\sigma_k(z) - \left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z})\right)^{\diamond k}\right)\right\|^{2p}_{\mathcal{C}^{-\delta}}\mathrm{d}\nu_\epsilon^e &\lesssim (p-1)^{pu_1}N^{-\alpha} \\ \int\left(\sum_{k=1}^{2n}\|\sigma_k\|_{\mathcal{C}_\ell^{-\delta}}\right)^{2\beta_1(2N-1)p}\mathrm{d}\nu_\epsilon^e &\lesssim (p-1)^{pu_2}\end{aligned}$$

uniformly in ϵ. This proves inequality (71). We note that

$$\sum_{k=0}^{2n}\binom{2n}{k}\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z})\right)^{\diamond k}f'(x)\bar{\theta}^{2n-k}(\hat{z}) = f'(x)H_{2n}\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}); c_{N,\epsilon}\right)$$

where $H_{2n}(x;\lambda)$ is the Hermite polynomial of degree $2n$ of a Gaussian random variable with variance λ and $c_{N,\epsilon} = \mathbb{E}\left[\left(\mathfrak{a}_{\frac{1}{N}} * (\mathcal{I}\xi)\right)^2\right] \sim \log(N)$, as $N \to +\infty$ for any $\epsilon > 0$. It is simple to see that, for any $x \in \mathbb{R}^2$ and $\hat{z} = (x, z) \in \mathbb{R}^2\times\mathbb{T}^2$,

$$\begin{aligned}f'(x)H_{2n}\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z}); c_{N,\epsilon}\right) &\lesssim a_n f'(x)\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z})\right)^{2n} \\ &\quad + f'(x)\sum_{k=1}^{m}a_k\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z})\right)^{2N-2k}c_{N,\epsilon}^k \\ &\lesssim \tilde{a}_n f'(x)\left(\mathfrak{a}_{\frac{1}{N}-\epsilon}(\sigma_1)(\hat{z}) + \bar{\theta}(\hat{z})\right)^{2N} - f'(x)c_{N,\epsilon}^n \\ &\lesssim -f'(x)c_{N,\epsilon}^n \lesssim -f'(x)(\log(N))^n,\end{aligned}$$

where we used that $f' \leqslant 0$ and $a_N, \tilde{a}'_N > 0$. This proves inequality (72).

Now we use Lemma V.5 of [74] and the fact that inequality (71) and (72) are uniform in ϵ, to prove that there exist two constants $b, \alpha'' > 0$ depending only on f' and m but not on $\epsilon > 0$ such that

$$\nu_\epsilon^e(E(\sigma,\bar{\theta}) > b\log(K)) \leqslant e^{-K^{\alpha''}}. \tag{73}$$

Inequality (73) implies that $\exp(E) \in L^p(\nu_\epsilon^e)$ for any $p \geq 1$, and we have also a uniform bound on the L^p norm of $\exp(E)$ with respect to $\epsilon > 0$. This concludes the proof of the lemma. □

Proof of Theorem 41 First we introduce the equation

$$(-\Delta_x - \Delta_z + m^2)(\phi) + \mathfrak{a}_\varepsilon^{*2} * (H_{2n}(\phi; c_\epsilon)) = \mathfrak{a}_\epsilon * \xi, \tag{74}$$

where $c_\epsilon = \mathbb{E}[(\mathfrak{a}_\epsilon * \xi)^2]$. Equation (74) is of the form (1), and the potential $V_\epsilon(y) = H_{2n}(y; c_\epsilon)$, where $y \in \mathbb{R}$, satisfies Hypothesis QC since it is an even polynomial with positive leading coefficient. This means that, for any $\epsilon > 0$, there exists a weak solution ν_ϵ to equation (74), such that

$$\int_{\mathcal{W}} F(\phi(0,\cdot)) \Upsilon_f^\epsilon(\phi) \mathrm{d}\nu_\epsilon = \int_{B^{-\delta}_{\mathfrak{p},\mathfrak{p},\ell}(\mathbb{T}^2)} F(\omega) \mathrm{d}\kappa_\epsilon(\omega)$$

where $\Upsilon_f^\epsilon(\phi) = \int_{\mathbb{R}^2 \times \mathbb{T}^2} f'(x) H_{2n}(\phi(\hat{z}); c_\epsilon) \mathrm{d}\hat{z}$. On the other hand if by ν_ϵ^e we denote the law of $(\mathfrak{a}_\epsilon * \mathcal{I}\xi, \dots, (\mathfrak{a}_\epsilon * \mathcal{I}\xi)^{\diamond 2n}, \phi - \mathfrak{a}_\epsilon * \mathcal{I}\xi)$ (where ϕ has law ν and $\mathfrak{a}_\epsilon * \mathcal{I}\xi$ is given by equation (74)), we have that ν_ϵ^e is a solution to equation (59) and

$$\int_{\mathcal{W}} F(\phi(0,\cdot)) \Upsilon_f^\epsilon(\phi) \mathrm{d}\nu_\epsilon = \int_{\mathcal{W}^e} F(\phi(0,\cdot)) \tilde{\Upsilon}_f(\sigma, \bar{\theta}) \mathrm{d}\nu_\epsilon^e(\sigma, \bar{\theta}).$$

Using the Galerkin approximation of Section 2.1 and a method similar to the one of Lemma 40, it is possible to prove that $\nu_\epsilon^e((\mathcal{C}_\ell^{-\delta}(\mathbb{R}^2 \times \mathbb{T}^2))^{2n} \times \overline{\mathfrak{W}}_\varepsilon) = 1$.

By Lemma 31 and Lemma 32 we have that the measure $F(\phi(0,\cdot))\mathrm{d}\nu_\epsilon^e$ weakly converges to $F(\phi(0,\cdot))\mathrm{d}\nu^e$, as $\varepsilon \to 0$. Let $G_r : \mathcal{W}^e \to \mathbb{R}$, $r \in \mathbb{N}$, be a sequence of continuous functions such that $0 \leqslant G_r \leqslant 1$, $G_r = 1$ when $\tilde{\Upsilon}_f(\sigma, \bar{\theta}) \leqslant r$ and $G_r = 0$ when $\tilde{\Upsilon}_f(\sigma, \bar{\theta}) \geqslant 2r$ (the existence of this kind of functions follows from the fact that $\tilde{\Upsilon}_f(\sigma, \bar{\theta})$ is continuous on $\mathcal{W}^e$). Since $F(\phi(0,\cdot))\mathrm{d}\nu_\epsilon^e$ weakly converges to $F(\phi(0,\cdot))\mathrm{d}\nu^e$ then $\int_{\mathcal{W}^e} G_r F \tilde{\Upsilon}_f \mathrm{d}\nu_\epsilon^e \to \int_{\mathcal{W}^e} G_r F \tilde{\Upsilon}_f \mathrm{d}\nu^e$, as $\varepsilon \to 0$. On the other hand, by Lemma 42, for all $\varepsilon > 0$ and any $r > 0$,

$$\left| \int_{\mathcal{W}^e} (G_r - 1) F \tilde{\Upsilon}_f \mathrm{d}\nu_\epsilon^e \right| \leqslant \int_{\tilde{\Upsilon}_f \geqslant r} |F| \tilde{\Upsilon}_f \mathrm{d}\nu_\epsilon^e \leqslant \frac{C_p \|F\|_\infty}{r^{p-1}}$$

for any $p \geq 1$ (and a similar inequality is true for $\epsilon = 0$). Taking $r \to +\infty$ the thesis follows. □

A Besov spaces

In this appendix we recall some results on weighted Besov spaces used in this paper. We consider only the case of Besov spaces defined on $\mathbb{R}^n$ but all what follows holds also for Besov spaces on $\mathbb{T}^n$ or on $\mathbb{R}^{n_1} \times \mathbb{T}^{n_2}$ (like the ones used in Section 4).

First we recall the definition of Littewood-Paley block: let χ, φ be smooth non-negative functions from $\mathbb{R}^n$ into $\mathbb{R}$ such that

- $\mathrm{supp}(\chi) \subset B_{\frac{4}{3}}(0)$ and $\mathrm{supp}(\varphi) \subset B_{\frac{8}{3}}(0) \setminus B_{\frac{3}{4}}(0)$,
- $\chi, \varphi \leq 1$ and $\chi(y) + \sum_{j \geq 0} \varphi(2^{-j} y) = 1$ for any $y \in \mathbb{R}^n$,
- $\mathrm{supp}(\chi) \cap \mathrm{supp}(\varphi(2^{-i}\cdot)) = \emptyset$ for $i \geqslant 1$,
- $\mathrm{supp}(\varphi(2^{-j}\cdot)) \cap \mathrm{supp}(\varphi(2^{-i}\cdot)) = \emptyset$ if $|i - j| > 1$,

where by $B_r(x)$ we denote the ball of center $x \in \mathbb{R}^n$ and radius $r > 0$.

We introduce the following notations: $\varphi_{-1} = \chi$, $\varphi_j(\cdot) = \varphi(2^{-j}\cdot)$, $D_j = \hat{\varphi}_j$ and for any $f \in \mathcal{S}'(\mathbb{R}^n)$ we put $\Delta_j(f) = D_j * f$. Furthermore we write, for any $\ell > 0$, $r_\ell(y) = (1 + |y|^2)^{-\ell/2}$, $L^p_\ell(\mathbb{R}^d)$ is the L^p space with respect to the norm

$$\|f\|_{p,\ell} = \left(\int_{\mathbb{R}^n} (f(y) r_\ell(y))^p \mathrm{d}y \right)^{1/p},$$

where $p \in [1, +\infty]$.

Definition 43 *Consider $s \in \mathbb{R}, p, q \in [1, +\infty]$ and $\ell \geq 0$. If $f \in \mathcal{S}(\mathbb{R}^n)$ we define the norm*

$$\|f\|_{B^s_{p,q,\ell}} = \left(\sum_{j \geq -1} \|\Delta_j(f)\|^q_{p,\ell} \right)^{1/q}. \tag{75}$$

The space $B^s_{p,q,\ell}$ is the subset of $\mathcal{S}'(\mathbb{R}^n)$ such that the norm (75) is finite.

Remark 44 It is important to note that $B^s_{2,2,\ell}$ is equal to the weighted Sobolev space $W^{s,2}_\ell = H^s_\ell$ of L^2 distributions, and that for $s > 0$, $s \not\in \mathbb{N}$, $B^s_{\infty,\infty}$ is equal to $\mathcal{C}^s$, the space of Hölder functions of regularity s.

Remark 45 It is always possible to choose χ and φ in such a way that there exists some constants $\gamma_{-1}, \gamma > 0$ and $0 < \theta_{-1}, \theta < 1$ such that

$$|D_{-1}(y)| \lesssim \exp(-\gamma_{-1}|y|^{\theta_{-1}}) \quad \text{and} \quad |D_0(y)| \lesssim \exp(-\gamma|y|^\theta),$$

(see, i.e., [70] Section 1.2.2 Proposition 1 or [61]).

Remark 46 In this appendix we treat only the case of polynomial weight. More generally all the statements here presented can be extended to the case of regular enough weights ω satisfying the following conditions

$$0 < \omega(y)^{\pm 1} < c\omega(w - y)^{\pm 1} \exp(d|w|) \tag{76}$$

$$|D^\alpha \omega(y)| \leqslant e\omega(y) \tag{77}$$

where $c, d, e > 0$ and $y, w \in \mathbb{R}^n$. Here we have decided to present the case with polynomial weights because for finite distributions with respect to weights satisfying only the conditions (76) and (77) the Fourier transform is not defined. This means that we cannot use directly Definition 43 and a more technical discussion is needed. We refer to [71, 72] for the study of weighted Besov spaces with a weight satisfying the conditions (76) and (77).

We denote by $\mathfrak{F}_r$, with $r \in \mathbb{N}$, the space of functions$f \in C^r(\mathbb{R}^n)$ with support in $B_1(0)$ and norm $\|f\|_{C^r(\mathbb{R}^n)} \leq 1$. If $f \in \mathfrak{F}_r$ we write $f_{y,\lambda}(\cdot) = \lambda^{-n} f\left(\frac{\cdot - y}{\lambda}\right)$, $y \in \mathbb{R}^n$ and $\lambda > 0$.

Proposition 47 *For any $s \in \mathbb{R}, p, q \in [1, +\infty]$ and $\ell \in \mathbb{R}$ an equivalent norm in the space $B^s_{p,q,\ell}$ is given by the following expression*

$$\|f\|_{B^s_{p,q,\ell}} = \left(\int_0^1 \frac{\| \sup_{g \in \mathfrak{F}_r} |\langle f, g_{\cdot,\lambda} \rangle| \|^q_{p,\ell}}{\lambda^{sq}} \right)^{1/q}. \tag{78}$$

where $r \in \mathbb{N}$ is the first integer bigger than $-s$.

Proof Theorem 6.15, in [77], proves the equivalence between norm (75) and the norm of $B^s_{p,p,\ell}$ built using wavelets, while Proposition 2.4 in [49] proves the equivalence between the norm of $B^s_{p,p,\ell}$ built using wavelets and the norm (78). Combining the two results we obtain the thesis. □

If f is a measurable function and $N \in \mathbb{N}$ we define

$$\Delta^N_h(f)(y) = \sum_{i=0}^N (-1)^{N-i} \begin{pmatrix} N \\ i \end{pmatrix} f(y + hi)$$

for $x, h \in \mathbb{R}^n$.

Proposition 48 *For any $s > 0$, $p, q \geqslant 1$, $\ell \in \mathbb{R}$, and$N, M \in \mathbb{N}_0$ such that $N < s < M$ an equivalent norm in the space $B^s_{p,q,\ell}$ is given by the following expression*

$$\|f\|_{B^s_{p,q,\ell}} = \|f\|_{L^p_\ell} + \sum_{\alpha\in\mathbb{N}^n,|\alpha|=N} \left(\int_{|h|<1} |h|^{-(s-N)} \|\Delta_h^N D^\alpha f\|^q_{L^p_\ell} \frac{\mathrm{d}h}{|h|^n} \right)^{\frac{1}{q}}. \tag{79}$$

Proof The proof is given in [77] Theorem 6.9 for the case $N = 0$. The generic case $N > 0$ can be proved applying the techniques of Corollary 2 Section 2.6.1 of [75] (where formula (79) is proven for Besov spaces without weight). □

Remark 49 An important consequence of Proposition 48 is that for any $N \in \mathbb{N}$ such that $N < s$ we have that an equivalent norm of $B^s_{p,q,\ell}$ is given by

$$\|f\|_{B^s_{p,q,\ell}} \sim \|f\|_{L^p_\ell} + \sum_{\alpha\in\mathbb{N}^n,|\alpha|=N} \|D^\alpha f\|_{B^{s-N}_{p,q,\ell}}. \tag{80}$$

Proposition 50 *Consider $p_1, p_2, q_1, q_2 \in [1, \infty]$, $\ell_1, \ell_2 \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$ such that $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$ and $\ell_1 > \ell_2$ then $B^{s_2}_{p_2,q_2,\ell_2} \subset B^{s_1}_{p_1,q_1,\ell_1}$ and the immersion is compact.*
Furthermore for $1 \leq p \leq 2$

$$B^s_{p,p,\ell} \subset W^{s,p}_\ell \subset B^s_{p,2,\ell};$$

for $2 \leq p < \infty$

$$B^s_{p,2,\ell} \subset W^{s,p}_\ell \subset B^s_{p,p,\ell}$$

and for $p = \infty$ $W^{s,\infty}_\ell \subset B^s_{\infty,\infty}$. Each of above immersions is continuous.

Proof The proof of the first part of the proposition can be found in Theorem 6.7 in [77].
The second part of the proposition is proved in Theorem 6.4.4 and Theorem 6.2.4 of [30] for unweighted spaces. The result for weighted spaces follows from the fact that $f \in B^s_{p,q,\ell}$ and $g \in W^{s,p}_\ell$ if and only if $f \cdot r_\ell \in B^s_{p,q}$ and $g \cdot r_\ell \in W^{s,p}_\ell$ (see [77] Theorem 6.5 point ii)). □

Proposition 51 *Consider $p_1, p_2, p_3, q_1, q_2, q_3 \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and $q_1 = q_3$ and $q_2 = \infty$. Moreover consider $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$ with $\ell_1 + \ell_2 = \ell_3$ and consider $s_1 < 0$ $s_2 \geq 0$ and $s_3 = s_1 + s_2 > 0$. Finally, consider the bilinear functional $\Pi(f, g) = f \cdot g$ defined on $\mathcal{S}(\mathbb{R}^n)$ taking values in $\mathcal{S}(\mathbb{R}^n)$. Then there exists a unique continuous extension of Π as the map*

$$\Pi : B^{s_1}_{p_1,q_1,\ell_1} \times B^{s_2}_{p_2,q_2,\ell_2} \to B^{s_3}_{p_3,q_3,\ell_3}$$

and we have, for any f, g for which the norms are defined:

$$\|\Pi(f, g)\|_{B^{s_3}_{p_3,q_3,\ell_3}} \lesssim \|f\|_{B^{s_1}_{p_1,q_1,\ell_1}} \|g\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Proof The proof can be found in [61] in Section 3.3 for Besov spaces with exponential weights. The proof for polynomial weights is similar. □

Proposition 52 *Consider $p_1, p_2, q_1, q_2 \in [1, \infty]$, $\ell_1, \ell_2 \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$. For any $\theta \in [0, 1]$ we write $\frac{\theta}{p_1} + \frac{1-\theta}{p_2} = \frac{1}{p_\theta}$, $\frac{\theta}{q_1} + \frac{1-\theta}{q_2} = \frac{1}{q_\theta}$, $\theta\ell_1 + (1 - \theta)\ell_2 = \ell_\theta$ and $\theta s_1 + (1 - \theta)s_2 = s_\theta$. If $f \in B^{s_1}_{p_1,q_1,\ell_1} \cap B^{s_2}_{p_2,q_2,\ell_2}$ then $f \in B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}$, and furthermore*

$$\|f\|_{B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}} \leq \|f\|^\theta_{B^{s_1}_{p_1,q_1,\ell_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Furthermore if $f \in W^{s_1,p_1}_{\ell_1} \cap W^{s_2,p_2}_{\ell_2}$ then $f \in W^{s_\theta,p_\theta}_{\ell_\theta}$, and furthermore

$$\|f\|_{W^{s_\theta,p_\theta}_{\ell_\theta}} \leq \|f\|^\theta_{W^{s_1,p_1}_{\ell_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}_{\ell_2}}.$$

Proof The proof is based on the fact that the complex interpolation $(B^{s_1}_{p_1,q_1,\ell_1}, B^{s_2}_{p_2,q_2,\ell_2})_\theta$ of the two spaces $B^{s_1}_{p_1,q_1,\ell_1}$, $B^{s_2}_{p_2,q_2,\ell_2}$ is given by $B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}$. This interpolation is proved in [30] Theorem 6.4.5 point (6) for unweighted space. For weighted space it follows from the fact that $f \in B^{s_i}_{p_i,q_i,\ell_i}$ if and only if $f \cdot r_{\ell_i} \in B^{s_i}_{p_i,q_i}$.

A similar reasoning, using the interpolation proved in [30] Theorem 6.4.5 point (7), holds also for the second inequality. □

References

[1] S. Albeverio, B. K. Driver, M. Gordina, and A. M. Vershik. Equivalence of the Brownian and energy representations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 441(Veroyatnostc i Statistika. 22):17–44, 2015. Reprinted in J. Math. Sci. (N.Y.) **2**19 (2016), no. 5, 612–630.

[2] S. Albeverio, G. Gallavotti, and R. Hø egh Krohn. Some results for the exponential interaction in two or more dimensions. *Comm. Math. Phys.*, 70(2):187–192, 1979.

[3] S. Albeverio, T. Hida, J. Potthoff, M. Röckner, and L. Streit. Dirichlet forms in terms of white noise analysis. I. Construction and QFT examples. *Rev. Math. Phys.*, 1(2-3):291–312, 1989.

[4] S. Albeverio, T. Hida, J. Potthoff, M. Röckner, and L. Streit. Dirichlet forms in terms of white noise analysis. II. Closability and diffusion processes. *Rev. Math. Phys.*, 1(2-3):313–323, 1989.

[5] S. Albeverio, T. Kagawa, Y. Yahagi, and M.W. Yoshida. Non-local markovian symmetric forms on infinite dimensional spaces i, ii, 2019.

[6] S. Albeverio, H. Kawabi, S. Mihalache, and M. Röckner. Dirichlet form approach to stochastic quantization under exponential interaction in finite volume. in preparation, 2016.

[7] S. Albeverio and S. Kusuoka. The invariant measure and the flow associated to the Φ^4_3-quantum field model. *arXiv:1711.07108*, November 2017. to appear in Ann. Scuola Normale di Pisa.

[8] S. Albeverio and B. Rüdiger. Infinite-dimensional stochastic differential equations obtained by subordination and related Dirichlet forms. *J. Funct. Anal.*, 204(1):122–156, 2003.

[9] S. Albeverio and M. W. Yoshida. $H - C^1$ maps and elliptic spdes with polynomial and exponential perturbations of Nelson's euclidean free field. *Journal of Functional Analysis*, 196(2):265–322, 2002.

[10] Sergio Albeverio, Francesco C. De Vecchi, and Massimiliano Gubinelli. Elliptic stochastic quantization. *arXiv preprint arXiv:1812.04422*, 2018.

[11] Sergio Albeverio, Zbignew Haba, and Francesco Russo. A two-space dimensional semilinear heat equation perturbed by (Gaussian) white noise. *Probab. Theory Related Fields*, 121(3):319–366, 2001.

[12] Sergio Albeverio and Raphael Hø egh Krohn. Uniqueness of the physical vacuum and the Wightman functions in the infinite volume limit for some non polynomial interactions. *Comm. Math. Phys.*, 30:171–200, 1973.

[13] Sergio Albeverio and Raphael Hø egh Krohn. The Wightman axioms and the mass gap for strong interactions of exponential type in two-dimensional space-time. *J. Functional Analysis*, 16:39–82, 1974.

[14] Sergio Albeverio and Raphael Hø egh Krohn. Uniqueness and the global Markov property for Euclidean fields. The case of trigonometric interactions. *Comm. Math. Phys.*, 68(2):95–128, 1979.

[15] Sergio Albeverio and Raphael Hø egh Krohn. Martingale convergence and the exponential interaction in $\mathbf{R}^n$. In *Quantum fields—algebras, processes (Proc. Sympos., Univ. Bielefeld, Bielefeld, 1978)*, pages 331–353. Springer, Vienna, 1980.

[16] Sergio Albeverio, Raphael Hø egh Krohn, Jens Erik Fenstad, and Tom Lindstrø m. *Nonstandard methods in stochastic analysis and mathematical physics*, volume 122 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.

[17] Sergio Albeverio, Raphael Hø egh Krohn, Sylvie Paycha, and Sergio Scarlatti. Path space measure for the Liouville quantum field theory and the construction of relativistic strings. *Phys. Lett. B*, 174(1):81–86, 1986.

[18] Sergio Albeverio, Raphael Hø egh Krohn, Sylvie Paycha, and Sergio Scarlatti. A global and stochastic analysis approach to bosonic strings and associated quantum fields. *Acta Appl. Math.*, 26(2):103–195, 1992.

[19] Sergio Albeverio, Jürgen Jost, Sylvie Paycha, and Sergio Scarlatti. *A mathematical introduction to string theory*, volume 225 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1997. Variational problems, geometric and probabilistic methods.

[20] Sergio Albeverio, Hiroshi Kawabi, and Michael Röckner. Strong uniqueness for both Dirichlet operators and stochastic dynamics to Gibbs measures on a path space with exponential interactions. *J. Funct. Anal.*, 262(2):602–638, 2012.

[21] Sergio Albeverio and Song Liang. A new lattice approximation for the Høegh-Krohn quantum field model. *Rep. Math. Phys.*, 54(2):149–157, 2004.

[22] Sergio Albeverio and Minoru W. Yoshida. Reflection positive random fields and Dirichlet spaces. In *Proceedings of RIMS Workshop on Stochastic Analysis and Applications*, RIMS Kôkyûroku Bessatsu, B6, pages 15–29. Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[23] Sergio Albeverio and Minoru W. Yoshida. Some abstract considerations on the homogenization problem of infinite dimensional diffusions. In *Applications of renormalization group methods in mathematical sciences*, RIMS Kôkyûroku Bessatsu, B21, pages 183–192. Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.

[24] Sergio A. Albeverio, Raphael J. Hø egh Krohn, Jean A. Marion, Daniel H. Testard, and Bruno S. Torrésani. *Noncommutative distributions*, volume 175 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1993. Unitary representation of gauge groups and algebras.

[25] Sebastian Andres and Naotaka Kajino. Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *Probab. Theory Related Fields*, 166(3-4):713–752, 2016.

[26] Dorothea Bahns and Kasia Rejzner. The quantum sine-Gordon model in perturbative AQFT. *Comm. Math. Phys.*, 357(1):421–446, 2018.

[27] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[28] G. Benfatto, G. Gallavotti, and Francesco Nicolò. On the massive sine-Gordon equation in the first few regions of collapse. *Comm. Math. Phys.*, 83(3):387–410, 1982.

[29] Nathanaël Berestycki, Scott Sheffield, and Xin Sun. Equivalence of liouville measure and gaussian free field. *arXiv preprint arXiv:1410.5407*, 2014.

[30] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[31] Gérard Bourdaud and Winfried Sickel. Composition operators on function spaces with fractional order of smoothness. In *Harmonic analysis and nonlinear partial differential equations*, RIMS Kôkyûroku Bessatsu, B26, pages 93–132. Res. Inst. Math. Sci. (RIMS), Kyoto, 2011.

[32] Nataliia Boyko and Vladimir Kadets. Uniform G-convexity for vector-valued L_p spaces. *Serdica Math. J.*, 35(1):1–14, 2009.

[33] Rémi Catellier and Khalil Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *Ann. Probab.*, 46(5):2621–2679, 2018.

[34] Ajay Chandra, Martin Hairer, and Hao Shen. The dynamical sine-gordon model in the full subcritical regime. *arXiv preprint arXiv:1808.02594*, 2018.

[35] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. *The Annals of Probability*, 31(4):1900–1916, 2003.

[36] Julien Dubédat and Hao Shen. Stochastic Ricci Flow on Compact Surfaces. *arXiv preprint arXiv:1904.10909*, 2019.

[37] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Log-correlated Gaussian fields: an overview. In *Geometry, analysis and probability*, volume 310 of *Progr. Math.*, pages 191–216. Birkhäuser/Springer, Cham, 2017.

[38] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[39] Jürg Fröhlich and Yong Moon Park. Remarks on exponential interactions and the quantum sine-Gordon equation in two space-time dimensions. *Helv. Phys. Acta*, 50(3):315–329, 1977.

[40] Jürg Fröhlich and Erhard Seiler. The massive Thirring-Schwinger model (qed2): convergence of perturbation theory and particle structure. *Helv. Phys. Acta*, 49(6):889–924, 1976.

[41] Christophe Garban. Dynamical Liouville. *J. Funct. Anal.*, 278(6):108351, 54, 2020.

[42] Roman Gielerak. Verification of the global Markov property in some class of strongly coupled exponential interactions. *J. Math. Phys.*, 24(2):347–355, 1983.

[43] James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.

[44] Hanno Gottschalk and Horst Thaler. A triviality result in the AdS/CFT correspondence for Euclidean quantum fields with exponential interaction. *Comm. Math. Phys.*, 324(1):63–75, 2013.

[45] Massimiliano Gubinelli and Martina Hofmanová. A PDE construction of the Euclidean ϕ^4_3 quantum field theory. *arXiv preprint arXiv:1810.01700*, October 2018.

[46] Massimiliano Gubinelli and Martina Hofmanová. Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space. *Communications in Mathematical Physics*, Mar 2019.

[47] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[48] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[49] Martin Hairer and Cyril Labbé. The reconstruction theorem in Besov spaces. *J. Funct. Anal.*, 273(8):2578–2618, 2017.

[50] Martin Hairer and Hao Shen. The dynamical sine-Gordon model. *Comm. Math. Phys.*, 341(3):933–989, 2016.

[51] Takeyuki Hida, Hui-Hsiung Kuo, Jürgen Potthoff, and Ludwig Streit. *White noise*, volume 253 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993. An infinite-dimensional calculus.

[52] Raphael Hø egh Krohn. A general class of quantum fields without cut-offs in two space-time dimensions. *Comm. Math. Phys.*, 21:244–255, 1971.

[53] Masato Hoshino, Hiroshi Kawabi, and Seiichiro Kusuoka. Stochastic quantization associated with the $\exp(\phi)_2$-quantum field model driven by space-time white noise on the torus. *arXiv preprint arXiv:1907.07921*, 2019.

[54] Koichiro Iwata. Reversible measures of a $P(\phi)_1$-time evolution. In *Probabilistic methods in mathematical physics (Katata/Kyoto, 1985)*, pages 195–209. Academic Press, Boston, MA, 1987.

[55] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985.

[56] A. Klein, L. J. Landau, and J. F. Perez. Supersymmetry and the Parisi-Sourlas dimensional reduction: a rigorous proof. *Comm. Math. Phys.*, 94(4):459–482, 1984.

[57] Shigeo Kusuoka. Høegh-Krohn's model of quantum fields and the absolute continuity of measures. In *Ideas and methods in quantum and statistical physics (Oslo, 1988)*, pages 405–424. Cambridge Univ. Press, Cambridge, 1992.

[58] Song Liang. Different lattice approximations for Høegh-Krohn's quantum field model. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pages 493–499. Springer, Berlin, 2007.

[59] Lech Maligranda. Interpolation of locally Hölder operators. *Studia Math.*, 78(3):289–296, 1984.

[60] S. Mihalache. Stochastische Quantisierung bei exponentieller Wechselwirkung. Master's thesis, Universiät Bonn, 2006.

[61] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, July 2017.

[62] Jean-Christophe Mourrat and Hendrik Weber. The dynamic Φ^4_3 model comes down from infinity. *Comm. Math. Phys.*, 356(3):673–753, 2017.

[63] Tadahiro Oh, Tristan Robert, and Yuzhao Wang. On the parabolic and hyperbolic liouville equations. *arXiv preprint arXiv:1908.03944*, 2019.

[64] G. Parisi and N. Sourlas. Random Magnetic Fields, Supersymmetry, and Negative Dimensions. *Physical Review Letters*, 43(11):744–745, September 1979.

[65] G. Parisi and Y. S. Wu. Perturbation theory without gauge fixing. *Scientia Sinica. Zhongguo Kexue*, 24(4):483–496, 1981.

[66] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probab. Surv.*, 11:315–392, 2014.

[67] Raoul Robert and Vincent Vargas. Gaussian multiplicative chaos revisited. *Ann. Probab.*, 38(2):605–631, 2010.

[68] Thomas Runst and Winfried Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.

[69] Raymond A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[70] Hans-Jürgen Schmeisser and Hans Triebel. *Topics in Fourier analysis and function spaces*. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1987.

[71] Thomas Schott. Function spaces with exponential weights. I. *Math. Nachr.*, 189:221–242, 1998.

[72] Thomas Schott. Function spaces with exponential weights. II. *Math. Nachr.*, 196:231–250, 1998.

[73] Alexander Shamov. On Gaussian multiplicative chaos. *J. Funct. Anal.*, 270(9):3224–3261, 2016.

[74] B. Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory*. Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.

[75] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[76] Hans Triebel. *Fractals and spectra*, volume 91 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1997. Related to Fourier analysis and function spaces.

[77] Hans Triebel. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.

[78] Bogusł aw Zegarliński. Uniqueness and the global Markov property for Euclidean fields: the case of general exponential interaction. *Comm. Math. Phys.*, 96(2):195–221, 1984.