Parabolic stochastic quantisation of the fractional Φ^4_3 model in the full subcritical regime

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Abstract

We present a construction of the measure of the fractional Φ^4 Euclidean quantum field theory on $\mathbb{R}^3$ in the full subcritical regime via parabolic stochastic quantisation. Our approach is based on the use of a truncated flow equation for the effective description of the model at sufficiently small scales and on coercive estimates for the nonlinear stochastic partial differential equation describing the interacting field. The constructed measure is invariant under translations, reflection positive and has quartic exponential tails.

Keywords: stochastic quantisation, renormalisation group, fractional Laplacian, singular SPDEs.

A.M.S. subject classification: 60H17, 81T08, 81T17, 35B45, 60H30

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1 Introduction

For $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$ let

$$\mathbb{R}^d_\varepsilon := (\varepsilon\mathbb{Z})^d, \qquad \mathbb{T}^d_{\varepsilon,M} := (\varepsilon\mathbb{Z})^d/(M\mathbb{Z})^d, \qquad \Omega_{\varepsilon,M} := \{\varphi : \mathbb{T}^d_{\varepsilon,M} \longrightarrow \mathbb{R}\}.$$

Define a probability measure $\nu_{\varepsilon,M}$ on $\Omega_{\varepsilon,N}$ by

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) := \frac{\exp(-S_{\varepsilon,M}(\varphi))}{Z_{\varepsilon,M}} \prod_{x \in \mathbb{T}^d_{\varepsilon,M}} \mathrm{d}\varphi(x), \tag{1.1}$$

where

$$S_{\varepsilon,M}(\varphi) := 2\varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon,M}} \left[\frac{1}{2}\varphi(x)\,(-\Delta_\varepsilon)^s \varphi(x) + \frac{m^2}{2}\,\varphi(x)^2 + \frac{\lambda}{4}\,\varphi(x)^4 - \frac{r_{\varepsilon,M}}{2}\,\varphi(x)^2 \right]. \tag{1.2}$$

The normalisation constant $Z_{\varepsilon,M} > 0$ is chosen so that the total mass of $\nu_{\varepsilon,M}$ equals one. The parameters $m > 0, \lambda > 0, r_\varepsilon \in \mathbb{R}$ are referred to as *mass*, *coupling constant* and *mass renormalisation*, respectively. The operator $(-\Delta_\varepsilon)^s$ is the discrete fractional Laplacian of order $s \in (0,1]$ defined via functional calculus as the s-th power of the discrete nearest-neighbour Laplacian $-\Delta_\varepsilon$. In order to pass to the limit, we define a probability measure $\hat{\nu}_{\varepsilon,M}$ on $\mathcal{S}'(\mathbb{R}^3)$ by embedding sample paths of $\nu_{\varepsilon,M}$ in $\mathcal{S}'(\mathbb{R}^3)$ via a suitable Fourier multiplier, see (2.35) below.

For the sake of clarity, we restrict our considerations to $d = 3$. Then $s = 1$ corresponds to the standard Φ^4_3 model, while for fractional exponents $s > s_c := 3/4$ the model is subcritical, meaning that the nonlinear part can be treated as a perturbation of the Gaussian measure at small scales. Since reflection positivity is expected to hold only for $s \leqslant 1$, so we restrict our analysis to $s \in (s_c, 1]$. For further discussion of the measure in (1.1), we refer to [GH19]. In what follows, we present a detailed analysis of the fractional regime $s \in (3/4, 1)$. The case $s = 1$, corresponding to the classical Laplacian, can be handled by the same strategy with substantial simplifications. The main result of this paper is a proof of the following:

Theorem 1.1. *Let $d = 3$ and fix $s \in (3/4, 1]$, $m > 0$, $\lambda > 0$. There exists a choice of mass renormalisation $(r_{\varepsilon,M})_{\varepsilon \in 2^{-\mathbb{N}_0}, M \in \mathbb{N}_+}$ such that $(\hat{\nu}_{\varepsilon,M})_{\varepsilon \in 2^{-\mathbb{N}_0}, M \in \mathbb{N}_+}$ is a tight family of probability measures on $\mathcal{S}'(\mathbb{R}^3)$. Any accumulation point ν of this family is non-Gaussian, invariant under translations, reflection positive, and satisfies*

$$\int e^{\theta \|(1+|\bullet|)^{-b}(1-\Delta)^{-a}\phi\|^4_{L^2}}\, \nu(\mathrm{d}\phi) < \infty, \tag{1.3}$$

for sufficiently large $a, b > 0$ and sufficiently small $\theta > 0$.

Proof. Tightness is established in Sect. 2.4 and the bound (1.3) is proved in Sec. 2.5. The reflection positivity and translation invariance of any accumulation point follow as in [GH19], since these properties are inherited from the corresponding properties of the approximate measures $(\nu_{\varepsilon,M})_{\varepsilon,M}$. Non-Gaussianity follows from (1.3), as Gaussian measures cannot integrate functions exhibiting super-exponential growth. □

We expect that any accumulation point is invariant under all Euclidean transformations. Unfortunately, a direct characterisation of the limiting measures remains unavailable. As a result, we must deduce the properties of the accumulation points indirectly, relying on certain features of the approximating measures. Notably, since the measures on lattices are not rotationally invariant, establishing rotational invariance of the continuum limit poses a particularly challenging problem. It is conceivable that this issue could be addressed using the technique developed in [DDJ24] in the context of $P_2(\Phi)$ model. However, pursuing this direction lies beyond the scope of the present paper.

Let us also mention the recent work [DHYZ25], where uniqueness of the limiting measure is established in small coupling regime for the standard Φ^4_3 model. A similar strategy might extend to the fractional variant, providing a potential route toward a full verification of the OS axioms, at least in the regime of small coupling.

Our proof strategy introduces a novel combination of renormalisation group ideas and PDE techniques which we believe can be useful more widely in the context of the theory of subcritical singular SPDEs. The proof also applies to the vector version of the model where the field takes values in $\mathbb{R}^n$ for $n > 1$ and the functional $S_{\epsilon,M}(\varphi)$ depends on φ in an $O(n)$ symmetric way. We briefly discuss how to adapt the proof to the vector case in Sec. 2.6.

Theorem 1.1 gives a construction of a model of Euclidean quantum field theory (EQFT) commonly referred in the literature as the *fractional* Φ^4_3 model. This terminology stems from the form of the density appearing in the formula for the measure (1.1) and the fact that the underlying space is three-dimensional. In the case $s = 1$, the model reduces to the classical Φ^4_3 theory, which has long been regarded as a crucial benchmark in constructive quantum field theory. Foundational results by Glimm and Jaffe [GJ73], Feldman and Osterwalder [Fel74, FO76] and other pioneers of EQFT laid the groundwork for demonstrating the existence of models satisfying the Wightman axioms for local relativistic QFT through probabilistic methods based on the Euclidean framework [GJ87]. In the fractional regime, that is, for $s \in (s_c, 1)$, the model we consider was introduced by Brydges, Mitter, and Scoppola [BMS03] as a rigorous foundation for proving the existence of a non-trivial infrared fixed point via an ϵ-expansion for sufficiently small $\epsilon = s - s_c > 0$. See also [BDH98] for an analogous result concerning a related four-dimensional model.

Our proof also extends to the cases $d = 1$ or $d = 2$ and $s \in (d/4, 1]$ without requiring any substantial modifications. We stress that we crucially need the condition of subcriticality (*i.e.* super-renormalizability), which translates into $s > d/4$, where s is the power of the fractional Laplacian. In particular, if $d \geq 4$ this would require $s > 1$. However, it is known that fractional Laplacian $(-\Delta)^s$ with $s > 1$ does not satisfy the maximum principle, and as a result, our a priori estimate does not apply in this regime. Moreover, for $s > 1$ the measure is not expected to be reflection positive—since the Gaussian part itself lacks this property—and is therefore of limited interest from the perspective of constructive quantum field theory. Finally, we note that for $d \leq 4$ it was shown in [Pan25] that at the criticality $s = d/4$ as well as in the supercritical regime $s < d/4$, any possible continuum limit is trivial, in the sense that it is Gaussian.

In recent years there has been a renewed interest in EQFTs due to the development of an alternative approach to the proof of theorems like Theorem 1.1. This new approach is grounded in the basic ideas of stochastic calculus and it is usually called *stochastic quantisation*. This term was introduced by Parisi and Wu [PW81] to describe the quantisation of gauge theories via the construction of a stochastic process evolving in fictitious time and whose stationary distribution is the target Euclidean QFT. This stochastic evolution is a nonlinear stochastic partial differential equation of a singular kind, for which a particular procedure of renormalisation is needed to give it a precise meaning. The analysis of such equations requires a mix of probabilistic and analytic arguments that escape the usual approach of Itō's stochastic differential equations. For this reason it took some time before the SPDE community learned how to handle such singular equation and discovered theories like regularity structures [Hai14] and paracontrolled calculus [GIP15, CC18] or renormalisation group [Kup16] which finally allowed to tackle the problem of the stochastic quantisation of the Φ^4_3 model. Gubinelli and Hofmanová [GH19] obtained the equivalent of Theorem 1.1 with $s = 1$ and a small range of values below that using a mix of paracontrolled calculus for the small scale singularities of the equation and coercive estimates to tame the large scale fluctuations. We refer the reader to the introduction to [GH19] for a deeper review of the literature and the history of constructive QFT and also to contextualise the meaning and consequences of Theorem 1.1.

The probability measure $\nu_{\varepsilon,M}$ in (1.1) is the equilibrium distribution of the Langevin dynamics governed by the finite system of SDEs

$$\mathcal{L}_\varepsilon \phi^{(\varepsilon,M)} + \lambda (\phi^{(\varepsilon,M)})^3 - r_{\varepsilon,M}\, \phi^{(\varepsilon,M)} = \xi^{(\varepsilon,M)}, \tag{1.4}$$

on $\Lambda_{\varepsilon,M} := \mathbb{R} \times \mathbb{T}^d_{\varepsilon,M}$, where

$$\mathcal{L}_\varepsilon := \partial_t + (-\Delta_\varepsilon)^s + m^2$$

and $\xi^{(\varepsilon,M)}$ is a spacetime white noise satisfying

$$\mathbb{E}[\xi^{(\varepsilon,M)}(t,x)\, \xi^{(\varepsilon,M)}(s,y)] = \delta(t-s)\, \mathbb{1}_{x=y}, \qquad (t,x),(s,y) \in \Lambda_{\varepsilon,M}.$$

The constants $m > 0, \lambda > 0, r_{\varepsilon,M} \in \mathbb{R}$ coincide with the parameters appearing in (1.2). By standard stochastic analysis arguments, there exists a unique stationary solution $\phi^{(\varepsilon,M)}$ of (1.4) and we have $\mathrm{Law}(\phi^{(\varepsilon,M)}(t)) = \nu_{\varepsilon,M}$ for all $t \in \mathbb{R}$. In what follows, we identify $\phi^{(\varepsilon,M)}$ and $\xi^{(\varepsilon,M)}$ with periodic functions on $\Lambda_\varepsilon := \mathbb{R} \times \mathbb{R}^d_\varepsilon$.

The nontrivial step is now to control the solutions to the stochastic quantisation equation (1.4) uniformly as $\varepsilon \longrightarrow 0$ and $M \longrightarrow \infty$. At small scales, it is expected that the nonlinear term in the dynamics is a perturbation of the linearised equation driven by spacetime white noise. Consequently, in the continuum limit, the solutions $\phi^{(\varepsilon,M)}$ are expected to converge to random distributions belonging to Besov spaces of negative regularity slightly worse than $s - d/2$, which is the regularity of the Gaussian free field. This low regularity presents a major analytical challenge, as it complicates the control of the nonlinear term.

Inspired by the works of Wilson [Wil71, WK74] and Polchinski [Pol84] on the continuous renormalisation group and by the more recent approach introduced by one of the authors in [Duc25a, Duc22], we use a flow equation to effectively describe the solution $\phi^{(\varepsilon,M)}$ of the SPDE at some spatial scale much larger than ε (see Kupiainen [Kup16] for a discrete counterpart). Let $\phi^{(\varepsilon,M)}_\sigma$ denote a description of the solution at a scale of order

$$[\![\sigma]\!] := (1-\sigma) \gg \varepsilon > 0$$

for some $\sigma \in (0,1)$. The flow equation approach consists in deriving a parabolic equation for $\phi^{(\varepsilon,M)}_\sigma$ of the form

$$\mathcal{L}_\varepsilon \phi^{(\varepsilon,M)}_\sigma = F^{(\varepsilon,M)}_\sigma\big(\phi^{(\varepsilon,M)}_\sigma\big). \tag{1.5}$$

Here, $\psi \mapsto F^{(\varepsilon,M)}_\sigma(\psi)$ is an analytic functional depending on the noise $\xi^{(\varepsilon,M)}$, called the *effective force*, such that

$$F^{(\varepsilon,M)}_1(\psi) = -\lambda\, \psi^3 + r_{\varepsilon,M}\, \psi + \xi^{(\varepsilon,M)}. \tag{1.6}$$

In particular, (1.4) can be recovered from (1.5) for $\sigma = 1$ and $\phi^{(\varepsilon,M)}_1 = \phi^{(\varepsilon,M)}$. The functional $F^{(\varepsilon,M)}_\sigma$ can be obtained by solving a *flow equation*

$$\partial_\sigma F^{(\varepsilon,M)}_\sigma = \mathbb{B}_\sigma\big(F^{(\varepsilon,M)}_\sigma, F^{(\varepsilon,M)}_\sigma\big), \tag{1.7}$$

backwards for $\sigma \in (\mu, 1]$ with the final condition (1.6) and where $\mathbb{B}_\sigma$ is an appropriate bilinear operator. The parameter $\sigma \in [0,1]$ does not have any specific physical meaning and the spatial scale of the decomposition is fixed conventionally to be of order $[\![\sigma]\!]$, that is $\phi^{(\varepsilon,M)}_\sigma$ is expected to fluctuate at spatial scales of order $[\![\sigma]\!]$ or larger, and in particular to be a locally bounded function on $\Lambda_0 := \mathbb{R} \times \mathbb{R}^d$, when extended in some reasonable way from the lattice Λ_ε to the continuum. A key ingredient is the control of the stochastic process $\big(F^{(\varepsilon,M)}_\sigma\big)_\sigma$ solving the flow equation (1.7). Following a simple but crucial observation of [Duc25a, Duc22], this control can be obtained by studying the evolution in the scale parameter of the cumulants $\big(\mathcal{F}^{(\varepsilon,M)}_\sigma\big)_\sigma$ of the process $\big(F^{(\varepsilon,M)}_\sigma\big)_\sigma$, which themselves satisfy a kind of higher-order deterministic flow equation

$$\partial_\sigma \mathcal{F}^{(\varepsilon,M)}_\sigma = \mathcal{A}_\sigma\big(\mathcal{F}^{(\varepsilon,M)}_\sigma\big) + \mathcal{B}_\sigma\big(\mathcal{F}^{(\varepsilon,M)}_\sigma, \mathcal{F}^{(\varepsilon,M)}_\sigma\big), \tag{1.8}$$

with prescribed initial condition $\mathcal{F}_1^{(\varepsilon,M)}$. Upon choosing appropriately this initial condition by tuning the parameter $r_{\varepsilon,M}$ in (1.6) one can prove uniform in ε and M estimates for the cumulants $\big(\mathcal{F}_\sigma^{(\varepsilon,M)}\big)_\sigma$ and therefore, by a Kolmogorov-type argument, suitable bounds on the effective force $\big(F_\sigma^{(\varepsilon,M)}\big)_\sigma$ uniform as $\varepsilon \longrightarrow 0$ and $M \longrightarrow \infty$.

The flow equation (1.7) is bilinear and therefore solvable in general only in a perturbative regime, e.g. in a small interval $I = [\bar\sigma, 1]$ around the initial condition at $\sigma = 1$ or for small data (or small time). The size of this perturbative region depends crucially on the size of the noise $\xi^{(\varepsilon,M)}$ and while this dependence can be made uniform in ε, M there could be large fluctuations in the noise which make the region arbitrarily small and reduce the available proof of existence of solutions to *local* results. A similar limitation is present in the work of Kupiainen [Kup16] who, instead, uses a discrete renormalisation group (RG) iteration, and more generally in all the other approaches which use an expansion of solutions in order to resolve the singular terms and control the limit as $\varepsilon \longrightarrow 0$, e.g. in regularity structures and also in paracontrolled calculus. This difficulty is the signal of the "large field problem", well known in constructive EQFT.

From the point of view of the stochastic quantisation equation, the large-field problem can be addressed by exploiting the coercivity of the nonlinear term which pulls the solution back from infinity. While this observation is standard in PDE theory, it still requires some nontrivial adaptation to be effective for singular SPDEs. The first to solve the problem have been Mourrat and Weber [MW17] in their proof of global existence for the Φ^4_2 dynamics on the full space with the usual Laplacian diffusion term and subsequently Gubinelli and Hofmanová in the context of paracontrolled analysis of Φ^4 models [GH19, GH21] including the parabolic three dimensional setting. Moinat and Weber [MW20] proved the so called spacetime localisation property for the dynamic Φ^4_3 model in the framework of regularity structures. This result was further extended by Chandra, Moinat, and Weber [CMW23] to cover the full subcritical regime. In the latter work, the authors modify the covariance of the noise, rather than the diffusion term, in order to explore regularities arbitrarily close to the critical threshold. Although their estimates suffice to establish tightness of the invariant measures associated with their SPDE, these measures are not explicit, and is not cleat if they are reflection positive. Consequently, their role in the stochastic quantisation of Euclidean QFTs is, at present, not fully understood.

In the broader context of global solutions for singular SPDEs, we also mention the recent preprint by Chandra, Feltes, and Weber [CFW24], which establishes results for the stochastic quantisation of the two-dimensional sine-Gordon model on a periodic domain for parameter values slightly above the first threshold. This has since been extended up to the second threshold in [BC25]. Moreover, [BC24] proves long-time well-posedness of the two-dimensional Abelian Higgs model.

During the revision of the present paper, a new preprint [EW24] appeared in which a priori bounds for the fractional Φ^4_3 model on the three-dimensional torus were established in the full subcritical regime. The main distinctions between our results and those of [EW24] are that we employ the flow-equation framework and work on an infinite lattice, whereas [EW24] use the regularity structures approach and study the continuum model on a torus with mollified noise. As far as we know, mollification of the noise is not a feasible strategy for establishing stochastic quantisation of an EQFT, in particular with respect to reflection positivity.

The main contribution of our work is the identification of a framework in which the flow equation method is combined with PDE estimates for the dynamics. This hybrid approach yields a powerful variant of the renormalisation group (RG). Instead of requiring an exact solution to the flow equation (1.7), it suffices to construct a suitable approximate solution $(F_\sigma)_\sigma = \big(F_\sigma^{(\varepsilon,M)}\big)_\sigma$ satisfying (1.6) for which the quantity

$$H_\sigma := \partial_\sigma F_\sigma - \mathbb{B}_\sigma(F_\sigma, F_\sigma), \tag{1.9}$$

is small enough in an appropriate sense. The price to pay for this approximation is a *remainder* term $R_\sigma^{(\varepsilon,M)}$ in the SPDE which now reads as a *system* of two equations:

$$\begin{cases} \mathscr{L}_\varepsilon \phi_\sigma = \mathscr{J}_\sigma[F_\sigma(\phi_\sigma) + R_\sigma], \\ \partial_\sigma R_\sigma = H_\sigma(\phi_\sigma) + \mathrm{D}F_\sigma(\phi_\sigma)((\partial_\sigma G_\sigma) R_\sigma), \qquad R_1 = 0, \end{cases} \tag{1.10}$$

for the pair of scale-dependent functions

$$(\phi_\sigma, R_\sigma)_\sigma = \big(\phi_\sigma^{(\varepsilon,M)}, R_\sigma^{(\varepsilon,M)}\big)_\sigma.$$

Here $G := \mathscr{L}_\varepsilon^{-1}$ is the fractional heat kernel, with $\mathscr{L}_\varepsilon^{-1}$ denoting the inverse of $\mathscr{L}_\varepsilon$ as defined in (1.19) below, $(\mathscr{J}_\sigma)_\sigma$ is a family of smoothing operators (see Def. 1.13) and

$$(G_\sigma)_{\sigma\in(0,1)} := (\mathscr{J}_\sigma G)_{\sigma\in(0,1)},$$

is a scale decomposition of G. Moreover, $\mathrm{D}F(\psi)\tilde{\psi}$ denotes the functional derivative of a functional F at ψ in the direction of $\tilde{\psi}$.

One can prove that the term $F_\sigma(\phi_\sigma)$ retains the coercive structure of (1.6), that is,

$$\mathscr{J}_\sigma F_\sigma(\phi_\sigma) = -\lambda \phi_\sigma^3 + Q_\sigma(\phi_\sigma),$$

where $Q_\sigma(\phi_\sigma)$ is "smaller" than the cubic contribution provided $[\![\sigma]\!] \ll 1$. This together with the linearity in $R_\sigma^{(\varepsilon,M)}$ of the second equation of (1.10) make this system amenable to standard PDE techniques: by choosing $[\![\sigma]\!] \ll 1$ one can control the non-coercive part $Q_\sigma(\phi_\sigma)$ of the effective force using the coercive part $-\lambda\phi_\sigma^3$, thereby resolving the large-field problem. At the same time, for $[\![\sigma]\!] > 0$, we have uniform estimates for $Q_\sigma, \mathrm{D}F_\sigma, H_\sigma$ as $\varepsilon \longrightarrow 0$ and $M \longrightarrow \infty$, provided the renormalisation constant $r_{\varepsilon,M}$ appearing in the boundary condition (1.6) for the effective force is chosen appropriately. This allows the full control of (1.4) and the proof of tightness of the laws of the processes $(\phi^{(\varepsilon,M)})_{\varepsilon,M}$, and therefore of the family of measures $(\nu_{\varepsilon,M})_{\varepsilon,M}$.

The implementation of this plan has to deal with two main technical difficulties:

a) The scale-by-scale decomposition $(\partial_\sigma G_\sigma)_\sigma$ of the fractional heat kernel G (cf. (1.20)) produces kernels with limited spacetime decay (see Lemma A.7), reflecting the restricted smoothness of G away from the origin. This algebraic decay of $\partial_\sigma G_\sigma$ propagates to the kernels of the effective force, necessitating a careful choice of weighted spaces for both the solutions and the kernels. To handle this limited decay, spacetime localisations of various operators and kernels are employed in several places—most notably in the localisation procedure for the relevant cumulants, which governs the flow of the renormalisation constants, see Appendix B.2. It would be interesting to devise an alternative strategy to bypass this problem with some other scale decomposition (or an additional localisation procedure).

b) The natural setting of the analysis provides only negative spacetime regularity for the solutions of the SPDE. Such regularity is insufficient for stochastic quantisation, since one must be able to compute the marginal of the solution at a given time. To address this, it is necessary to work with distributional norms defined via smoothing operators $(K_\mu)_\mu$ that provide only limited time regularisation. Suitable Schauder estimates for these operators can be established (see Lemma A.21). These smoothing operators also influence the definition of the norms for the kernels of the effective force (cf. Def. 4.6) and require a careful adaptation of the Kolmogorov-type argument used in Lemma 4.19.

Comparison with other approaches. The possibility of working with an approximate flow equation makes it easier to compare the RG approach advocated in this paper (and originally proposed in [Kup16] and [Duc25a, Duc22]) with regularity structures [Hai14] and paracontrolled distributions [GIP15]. There is a clear parallel among the various approaches. The flow equation constructs a random object $F_\sigma^{(\varepsilon,M)}$ — the scale-dependent *effective force* — which encapsulates the influence of the noise and takes the form of a finite polynomial built from the noise and the linear part of the equation. This object corresponds respectively to the *model* in the theory of regularity structures, the *enhanced noise* in paracontrolled calculus, or the *rough path* in rough path theory.

While $F_\sigma^{(\varepsilon,M)}$ is obtained through a probabilistic construction, the remainder term $R_\sigma^{(\varepsilon,M)}$ is defined analytically in terms of $F_\sigma^{(\varepsilon,M)}$. This deterministic component mirrors the analytic machinery of regularity structures, the paracontrolled operators in the paracontrolled calculus, and the sewing lemma in rough path analysis.

When the parameter s is near its critical value of $s_c = 3/4$ the number of terms which have to be accounted for in the approximation $F_\sigma^{(\varepsilon,M)}$ of the solution of the flow equation grows in an unbounded manner. A notable advantage of the flow equation approach, however, lies in its relative insensitivity to this increasing complexity: the analysis remains compact and largely independent of the distance to criticality. This analytical efficiency — the ability to capture the nonlinear propagation of randomness with minimal combinatorial overhead — was first observed by Polchinski [Pol84] in his proof of perturbative renormalizability of the Euclidean ϕ_4^4 QFT. For a modern account of this approach to perturbation theory of QFTs, the reader can consult the book of Salmhofer [Sal07] or Kopper [Kop07].

As we already noted, the application of RG ideas to SPDEs is made efficient by the observation of one of the authors [Duc25a, Duc22] that flow equations can also be used to estimate cumulants directly, thereby avoiding explicit and cumbersome inductive arguments on trees — much as Polchinski's method circumvents the inductive structure of BPHZ renormalisation. There are further conceptual similarities with recent developments by Otto, Weber, and collaborators [OSSW21, LOTT21], who use PDE-based arguments to derive probabilistic estimates for the modes in regularity structures. The flow equation framework, however, offers an additional advantage: renormalisation conditions naturally appear as boundary conditions for the corresponding flow equation.

The combination of the flow equation approach with stochastic quantisation in the context of the construction of EQFT has been recently exploited by Meyer and one of the authors in [GM24] to study the sine–Gordon model in the full space up to the second threshold and, by De Vecchi, Fresta and one of the authors, in [DFG22], to develop a new approach to Euclidean Fermionic theories. In both papers the stochastic quantisation is obtained by using a forward-backwards stochastic differential equation together with an approximate analysis of Polchinski's flow equation. Finally, we mention the work [Duc24], where the Polchinski equation was employed to construct the Gross–Neveu model, a critical fermionic model of Euclidean quantum field theory.

Conclusions. Despite the technical difficulties due to the analysis of the fractional heat equation, we would like to stress that the present paper is self-contained and presents complete arguments for all proofs. Compared with the few existing works on singular SPDEs in the full subcritical regime, we believe our work is the first to address, in a unified framework, several intricate aspects of the problem:

a) we present the entire argument — both analytic and probabilistic — in full detail;

b) we work with an extremely nonlocal SPDE posed on a (semi-)discrete space;

c) we establish an a priori bound in the full subcritical regime, valid globally in spacetime;

d) we obtain optimal tail estimates for the solutions.

We hope that this work illustrates the strength and flexibility of the stochastic quantisation approach in tackling, in a genuinely nonperturbative manner, the construction of Euclidean quantum field models in the subcritical regime.

Plan of the paper. In Sec. 2 we introduce the main objects of our analysis: the scale decomposition, the spacetime weighted norms which will be used to control the large fields and all the intermediate results which are needed in the proof of Theorem 1.1. The coercive estimates will be proven in Sec. 3, while the approximate flow equation for the effective force will be analysed in Sec. 4 via the flow equation for the cumulants. Appendix A contains some technical lemmas and Schauder estimates tailored to our setting while Appendix B contains the detailed definition of the various contributions to the flow equations for the cumulants and their analytic estimates.

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1.1 Preliminaries and notation

In this section we shall introduce the main notations we are going to use throughout the paper.

We let $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{R}_\varepsilon := \varepsilon\mathbb{Z}$, $\mathbb{T}_{\varepsilon,M} := (\varepsilon\mathbb{Z})/(M\mathbb{Z})$ and define the corresponding spacetime domains

$$\Lambda_\varepsilon := \mathbb{R}\times\mathbb{R}_\varepsilon^d, \qquad \Lambda_{\varepsilon,M} := \mathbb{R}\times\mathbb{T}_{\varepsilon,M}^d, \qquad d=3, \quad \varepsilon\in 2^{-\mathbb{N}_0}, \quad M\in\mathbb{N}_+.$$

The assumption that $\varepsilon\in 2^{-\mathbb{N}_0}$ ensures that $M\mathbb{Z}\subset\varepsilon\mathbb{Z}$ for every $M\in\mathbb{N}_+$. These domains are continuous in the time direction and discrete in the d spatial dimensions. For a measurable function $f:\Lambda_\varepsilon\longrightarrow\mathbb{R}^n$, we write

$$\|f\| := \|f\|_{L^\infty(\Lambda_\varepsilon)}.$$

Given a nonnegative weight $w\in C(\Lambda_0)$ we denote by $C(\Lambda_\varepsilon, w)$ the space of continuous functions $f:\Lambda_\varepsilon\longrightarrow\mathbb{R}$ such that

$$\|f\|_{L^\infty(w)} := \sup_{z\in\Lambda_\varepsilon} |w(z)f(z)| < \infty.$$

We always identify functions $f:\Lambda_{\varepsilon,M}\longrightarrow\mathbb{R}$ with their spatially periodic extensions $f:\Lambda_\varepsilon\longrightarrow\mathbb{R}$. In particular, we write $C(\Lambda_{\varepsilon,M}, w)$ for the subspace of $C(\Lambda_\varepsilon, w)$ consisting of functions that are periodic in space with period M. We denote by $\mathcal{S}(\Lambda_\varepsilon)$ the space of smooth rapidly decreasing functions over Λ_ε. The Fourier transform of $f\in\mathcal{S}(\Lambda_\varepsilon)$ is defined as

$$\hat f(\omega,k) := \int_{\Lambda_\varepsilon} e^{-i(\omega t+k\cdot x)} f(t,x)\,\mathrm{d}t\,\mathrm{d}x, \qquad (\omega,k)\in\Lambda_\varepsilon^* := \mathbb{R}\times(\mathbb{R}_\varepsilon^d)^*,$$

where

$$(\mathbb{R}_\varepsilon^d)^* := (-\pi/\varepsilon, \pi/\varepsilon]^d$$

is the dual of the group $\mathbb{R}_\varepsilon^d$. The integral over $\mathbb{R}_\varepsilon^d$ is understood with respect the counting measure (still written by $\mathrm{d}x$) with normalisation

$$\int_{\mathbb{R}_\varepsilon^d} f(x)\,\mathrm{d}x := \varepsilon^d \sum_{x\in\mathbb{R}_\varepsilon^d} f(x),$$

which ensures weak convergence to the Lebesgue measure as $\varepsilon \longrightarrow 0$. As usual, the Fourier transform extends to $\mathcal{S}'(\Lambda_\varepsilon)$ by duality. The inverse Fourier transform is given by

$$f(t,x) = \int_{\Lambda_\varepsilon^*} \hat{f}(\omega,k)\, e^{i(\omega t + k\cdot x)} \frac{\mathrm{d}\omega\, \mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x) \in \Lambda_\varepsilon.$$

For $\varepsilon \geqslant 0$, we define the Laplacian Δ_ε as the Fourier multiplier on $\mathcal{S}'(\Lambda_\varepsilon)$ with symbol

$$(\mathbb{R}^d_\varepsilon)^* \ni k \longmapsto -q_\varepsilon^2(k) \in \mathbb{R},$$

where

$$q_0(k) = |k|, \qquad q_\varepsilon(k) := \left[\sum_{i=1}^d \left(\frac{1}{\varepsilon} \sin(\varepsilon k_i) \right)^2 \right]^{1/2}, \qquad \varepsilon > 0. \tag{1.11}$$

Note that for $\varepsilon > 0$, the operator Δ_ε coincides with the standard nearest-neighbour discrete Laplacian.

The fractional Laplacian. For $s \in (0,1)$ the fractional Laplacian $(-\Delta_\varepsilon)^s$ is defined as the Fourier multiplier with symbol $(\mathbb{R}^d_\varepsilon)^* \ni k \longmapsto q_\varepsilon^{2s}(k)$. In particular it is self-adjoint and positive in $L^2(\mathbb{R}^d_\varepsilon)$ and for $s \in (0,1)$ it has the (discrete, when $\varepsilon > 0$) heat–kernel representation [Kwa17]

$$(-\Delta_\varepsilon)^s f = C_s \int_{\mathbb{R}_+} (f - e^{\theta \Delta_\varepsilon} f)\, \theta^{-1-s} \mathrm{d}\theta, \tag{1.12}$$

with the constant $C_s = |\Gamma(-s)|^{-1}$. In the continuum, the fractional Laplacian has, for $s \in (0,1)$, the integral representation [Kwa17]:

$$(-\Delta_0)^s f(x) = C_{d,s}\, \mathrm{PV} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+2s}} \mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{1.13}$$

where $C_{d,s}$ is an universal constant. In the discrete setting a similar formula holds [CRS+15]:

$$(-\Delta_\varepsilon)^s f(x) = \varepsilon^d \sum_{y \in \mathbb{R}^d_\varepsilon | y \neq x} H_s^{(\varepsilon)}(x-y)\, (f(x) - f(y)), \qquad x \in \mathbb{R}^d_\varepsilon, \tag{1.14}$$

where the kernel $H_s^{(\varepsilon)} \colon \mathbb{R}^d_\varepsilon \longrightarrow \mathbb{R}$ is positive and such that $H_s^{(\varepsilon)}(0) = 0$, $H_s^{(\varepsilon)}(x) = H_s^{(\varepsilon)}(-x)$ and

$$\left| H_s^{(\varepsilon)}(x) \right| \leqslant C'_{d,s} |x|^{-d-2s}, \qquad x \in \mathbb{R}^d_\varepsilon,$$

uniformly in $\varepsilon > 0$ for some constant $C'_{d,s} > 0$. Note that in our notation (1.14) can be equivalently written as

$$(-\Delta_\varepsilon)^s f(x) = \int_{\mathbb{R}^d_\varepsilon} H_s^{(\varepsilon)}(x-y)\, (f(x) - f(y))\, \mathrm{d}y.$$

For $\varepsilon > 0$, we can encode the representation (1.14) of the fractional Laplacian, via a positive measure $\mu_s^{(\varepsilon)}$ on $\Lambda_\varepsilon \times \Lambda_\varepsilon$ for which

$$\langle f, (-\Delta_\varepsilon)^s g \rangle = \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} f(z)\, (g(z) - g(z'))\, \mu_s^{(\varepsilon)}(\mathrm{d}z\, \mathrm{d}z').$$

We also define the kernel $\mu_s^{(\varepsilon)}(z, \mathrm{d}z')$ arising from the disintegration of $\mu_s^{(\varepsilon)}$ with respect to the measure $\mathrm{d}z$ on Λ_ε. Specifically, we have

$$\mu_s^{(\varepsilon)}(\mathrm{d}z\, \mathrm{d}z') = \delta(t-u) H_s^{(\varepsilon)}(x-y)\, \mathrm{d}z\, \mathrm{d}z', \qquad \mu_s^{(\varepsilon)}(z, \mathrm{d}z') = \delta(t-u) H_s^{(\varepsilon)}(x-y)\, \mathrm{d}z', \tag{1.15}$$

where $z = (t,x)$ and $z' = (u,y)$. The kernel $\mu_s^{(\varepsilon)}$ is symmetric, that is,

$$\mu_s^{(\varepsilon)}(z, \mathrm{d}z')\, \mathrm{d}z = \mu_s^{(\varepsilon)}(z', \mathrm{d}z)\, \mathrm{d}z'.$$

With this notation in place, the following Leibniz-type formula with remainder holds:

$$I_s^{(\varepsilon)}(f,g) := (-\Delta_\varepsilon)^s(fg) - f\,(-\Delta_\varepsilon)^s g - g\,(-\Delta_\varepsilon)^s f = \int_{\Lambda_\varepsilon} (f(\bullet)-f(z))\,(g(\bullet)-g(z))\,\mu_s^{(\varepsilon)}(\bullet,\mathrm{d}z). \tag{1.16}$$

Let us also introduce the fractional difference operator

$$\mathfrak{D}_s^{(\varepsilon)}(f)(z) := I_s^{(\varepsilon)}(f,f)^{1/2}(z) = \Big[\int_{\Lambda_\varepsilon} [f(z')-f(z)]^2 \mu_s^{(\varepsilon)}(z,\mathrm{d}z')\Big]^{1/2}, \tag{1.17}$$

which behaves like a derivative of order s. In particular using that, for all $\delta>0$, one has the estimates

$$\int_{B(z,\delta)} [f(z')-f(z)]^2 \mu_s^{(\varepsilon)}(z,\mathrm{d}z') \lesssim \|\nabla_\varepsilon f\|^2 \delta^{2-2s}, \quad \int_{B(z,\delta)^c} [f(z')-f(z)]^2 \mu_s^{(\varepsilon)}(z,\mathrm{d}z') \lesssim \|f\|^2 \delta^{-2s},$$

where

$$B(z,\delta) := \big\{z'\in\Lambda_\varepsilon \,\big|\, |z-z'|_s \le \delta\big\}$$

denotes the ball of radius $\delta>0$. The discrete gradient is defined as

$$\nabla_\varepsilon f = (\partial^{1+}, \partial^{1-}, \dots, \partial^{d+}, \partial^{d-}),$$

where $\partial^{i\pm}$ stands for the discrete forward/backward derivative on the lattice $\mathbb{R}_\varepsilon^d$ and $\|\nabla_\varepsilon f\|$ is the supremum norm. Choosing $\delta = \|f\|/\|\nabla_\varepsilon f\|$ yields

$$\mathfrak{D}_s^{(\varepsilon)}(f)(z) \lesssim \|\nabla_\varepsilon f\|\delta^{1-s} + \|f\|\delta^{-s} \lesssim \|\nabla_\varepsilon f\|^s \|f\|^{1-s}. \tag{1.18}$$

Remark 1.2. Although the kernel representation of the fractional Laplacian fails in the continuum case $\varepsilon=0$ (due to the presence of the principal value), the above considerations and the results presented below extend to the continuum with only minor modifications in the proofs, or simply by taking the $\varepsilon\longrightarrow 0$ limit in the relevant inequalities. We emphasise, however, that the main results of the paper do not rely on these continuum extensions. In particular, the limit $\varepsilon\longrightarrow 0$ in Theorem 1.1 is obtained via tightness, using the a priori estimates established uniformly for $\varepsilon\in 2^{-\mathbb{N}_0}$.

Remark 1.3. A basic observation is that the fractional Laplacian (whether in the continuum or on the lattice) satisfies an inequality under the action of convex functions. Let $s\in(0,1]$, $\Phi:\mathbb{R}\longrightarrow\mathbb{R}$ be a convex function and Φ' one of its sub-differentials, then, for any $\varepsilon\ge 0$ and $u\in C(\Lambda_\varepsilon)$, we have

$$(-\Delta_\varepsilon)^s \Phi(u) \le \Phi'(u)\,(-\Delta_\varepsilon)^s u.$$

Indeed, let $\Phi:\mathbb{R}\longrightarrow\mathbb{R}$ be a convex function, then

$$\Phi(a)-\Phi(b) \le \Phi'(a)\,(a-b), \qquad a,b\in\mathbb{R},$$

so if $u:\Lambda_\varepsilon\longrightarrow\mathbb{R}$ is a continuous and bounded function, we have

$$\Phi(u) - e^{\theta\Delta_\varepsilon}\Phi(u) \le \Phi'(u)\,(u - e^{\theta\Delta_\varepsilon}u),$$

since $e^{\theta\Delta_\varepsilon}$ has a positive definite probability kernel. The claimed inequality follows now from (1.12) in the case $s\in(0,1)$. The case $s=1$ is elementary. For $s>1$ the result is not true. It is clear that the same proof works for $\varepsilon=0$ with some additional regularity assumption. We will incorporate this idea in the proof of the key Lemma 3.2, below.

The operator $G=G^{(\varepsilon)}=\mathcal{L}_\varepsilon^{-1}$ is defined as

$$(\mathcal{L}_\varepsilon^{-1} f)(t,\bullet) := \int_{-\infty}^t e^{-(m^2+(-\Delta_\varepsilon)^s)(t-u)} f(u,\bullet)\mathrm{d}u, \qquad t\in\mathbb{R}, \tag{1.19}$$

and will be applied to continuous function on Λ_ε with at most a limited polynomial growth in spacetime. Indeed, on account of Lemma 5.4 of [Gri03] together with the argument of Sec. 1 of [GT01], the kernel $G(t,x)$ of G satisfies

$$G(t,x) \lesssim \mathbb{1}_{t\geq 0}\, e^{-m^2 t} \min\left\{ t^{-\frac{d}{2s}}, \frac{t}{|x|^{d+2s}} \right\} \lesssim \frac{\mathbb{1}_{t\geq 0}\, t\, e^{-cm^2 t}}{(|t|^{1/2s}+|x|)^{d+2s}}, \tag{1.20}$$

uniformly in $\varepsilon \geq 0$. Here $\mathbb{1}_{t\geq 0} = 0$ if $t<0$ and 1 if $t\geq 0$. If $s=1$, then the above estimate is not optimal and the following bound

$$G(t,x) \lesssim \mathbb{1}_{t\geq 0}\, t^{-d/2}\, e^{-m^2 t - c|x|^2/t} \tag{1.21}$$

holds true uniformly in $\varepsilon \geq 0$.

Parameters. During the subsequent analysis, we shall introduce several parameters

$$\alpha, \beta, \gamma, \delta, \theta, \vartheta, \varrho, \nu, \kappa, \bar{\kappa}, \kappa_0, \bar{\ell}, \bar{k}, a, \flat. \tag{1.22}$$

Although their precise values are not specified at this stage, all these parameters are to be regarded as fixed once and for all. Their choice depends solely on the power $s \in (3/4, 1)$ of the fractional Laplacian. Since the specific constraints determining these values will emerge later in the analysis, we postpone their detailed specification to Sec. 4.4 and 4.9 below.

Space-time weights.

Definition 1.4. *We define the fractional parabolic distance by*

$$|z|_s := |z_0|^{1/2s} + |\bar{z}|, \qquad z = (z_0, \bar{z}) \in \Lambda_0 = \mathbb{R}\times\mathbb{R}^3, \tag{1.23}$$

where $|\bullet|$ denotes the usual Euclidean distance on $\mathbb{R}$ and $\mathbb{R}^3$. For $\varkappa > 0$ and $z = (z_0, \bar{z}) \in \Lambda_0$ we denote by

$$\varkappa.z := (\varkappa^{2s} z_0, \varkappa \bar{z}) \in \Lambda_0$$

the fractional parabolic rescaling, satisfying

$$|[\![\mu]\!]^a.z|_s = [\![\mu]\!]^a |z|_s.$$

Definition 1.5. *We introduce the following Japanese brackets:*

$$\langle z\rangle_s := (1+|z_0|^{1/s}+|\bar{z}|^2)^{1/2} \qquad \text{and} \qquad \langle \bar{z}\rangle := (1+|\bar{z}|^2)^{1/2}, \qquad z=(z_0,\bar{z})\in\Lambda_0, \tag{1.24}$$

where, as in the previous definition, $|\bullet|$ denotes the usual Euclidean distance on $\mathbb{R}$ and $\mathbb{R}^3$.

a) Let $a>1$ and $\nu \in (0,1/3)$.

b) Let $(\chi_i : \Lambda_0 \longrightarrow \mathbb{R}_+)_{i\geq -1}$ be a dyadic partition of unity on Λ_0 with χ_i supported on an annulus of radius $\sim 2^{ai}$ for $i\geq 0$, χ_{-1} supported in a ball of radius ≈ 1 and $\sum_{i\geq -1}\chi_i = 1$.

c) Define $(\mu_i)_{i\geq -1} \subseteq [1/2, 1)$ by $[\![\mu_i]\!] = 2^{-i-2}$, $i \geq -1$.

d) Let $\zeta : \Lambda_0 \longrightarrow \mathbb{R}$ be a weight defined by

$$\zeta(z) := \langle z\rangle_s^{-1}, \qquad z\in\Lambda_0.$$

For $\mu \in [0,1]$, we also introduce the associated rescaled weights

$$\zeta_\mu(z) := \langle [\![\mu]\!]^a.z\rangle_s^{-1}, \qquad \rho_\mu := \zeta_\mu^\nu = \langle [\![\mu]\!]^a.z\rangle_s^{-\nu}, \qquad z\in\Lambda_0.$$

Remark 1.6. Concerning the parameters ν and a introduced above, we impose the relation

$$a\nu = \gamma,$$

where the parameter $\gamma \in (0, 2s)$ will be fixed in Sec. 4.9. In particular, throughout our analysis we shall be concerned with very small values of $\nu > 0$ and with $a > 1$.

Remark 1.7. The form of the Japaneese bracket (1.24) is motivated by the requirement that the weight function ζ (and its powers) be C^1 in the time variable and C^2 in the spatial variables. This regularity is essential in the a priori estimates established below (see Sec. 3). Indeed, one readily verifies that

$$\partial_t \zeta(t,x) = -\frac{|t|^{(1-s)/s}\partial_t|t|}{2s(1+|t|^{1/s}+|x|^2)^{3/2}}, \qquad (t,x)\in\Lambda_0,$$

which is continuous for $s<1$. An analogous argument applies to all spatial derivatives.

Remark 1.8. Our weights satisfy the following properties.

a) We have

$$\zeta(z)\zeta^{-1}(z_1) \lesssim \zeta^{-1}(z-z_1),$$

uniformly over $z, z_1 \in \Lambda_0$. Consequently,

$$\begin{aligned} \zeta_\mu(z)\zeta_\mu^{-1}(z_1) &\lesssim \zeta_\mu^{-1}(z-z_1) \leq \zeta^{-1}(z-z_1), \\ \rho_\mu(z)\rho_\mu^{-1}(z_1) &\lesssim \rho_\mu^{-1}(z-z_1) \lesssim \rho_0^{-1}(z-z_1), \end{aligned}$$

uniformly over $z, z_1 \in \Lambda_0$ and $\mu \in [0,1]$.

b) We have

$$\chi_i \zeta_\mu^{-1} \lesssim (1+([\![\mu]\!]^a [\![\mu_i]\!]^{-a})^2)^{1/2} \lesssim 1,$$

uniformly in $i \geq -1$ and $\mu \geq \mu_i$.

Scale decomposition. Let us introduce a scale decomposition of spacetime functions parametrised by $\sigma \in [0,1]$ and where we let

$$[\![\sigma]\!] := (1-\sigma)$$

for convenience. The value $\sigma = 1$ corresponds to allowing fluctuations at all scales while $\sigma < 1$ only at spatial scales $\gtrsim [\![\sigma]\!]$ or equivalently at Fourier scales $\lesssim [\![\sigma]\!]^{-1}$.

Definition 1.9. *Consider a smooth and compactly supported function $j: \mathbb{R} \longrightarrow \mathbb{R}_+$ such that*

$$j(\eta) = \begin{cases} 1 & \text{if} \quad |\eta| \leq 1, \\ 0 & \text{if} \quad |\eta| \geq 2. \end{cases}$$

For $\ell = 0, 1, 2, \dots$ denote

$$j_{\sigma,\ell}(\eta) := j(2^{-\ell}\sigma^{-1}[\![\sigma]\!]\eta), \qquad \eta \in \mathbb{R},$$

and let $j_\sigma := j_{\sigma,0}$ and $\tilde{j}_\sigma := j_{\sigma,1}$.

Remark 1.10. Note that $j_{\sigma,\ell}(\eta) j_{\sigma,\ell'}(\eta) = j_{\sigma,\ell}(\eta)$ for $0 \leq \ell < \ell'$.

Definition 1.11. *The family $(\mathcal{J}_\sigma)_{\sigma\in(0,1)} = (\mathcal{J}_\sigma^{(\varepsilon)})_{\sigma\in(0,1)}$ of Fourier multipliers acting on distributions is defined as*

$$\mathcal{J}_\sigma f(t,x) := \int_{\Lambda_\varepsilon^*} j_\sigma(|\omega|^{1/2s})\, j_\sigma(q_\varepsilon(k))\, \hat{f}(\omega,k)\, e^{i(\omega t + k\cdot x)} \frac{\mathrm{d}\omega\, \mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x)\in\Lambda_\varepsilon, \quad f \in \mathcal{S}'(\Lambda_\varepsilon), \tag{1.25}$$

where $q_\varepsilon^2(k)$ is the symbol of the Laplacian introduced in (1.11). In addition, we define $(\tilde{\mathcal{J}}_{\sigma,\ell})_{\sigma\in(0,1),\ell=1,2,\dots}$ by (1.25) with the function j_σ replaced by $j_{\sigma,\ell}$. Moreover, we let $\tilde{\mathcal{J}}_\sigma := \tilde{\mathcal{J}}_{\sigma,1}$ and $\dot{\mathcal{J}}_\sigma := \partial_\sigma \mathcal{J}_\sigma$.

Note that $\mathcal{J}_\sigma f \longrightarrow f$ as $\sigma \nearrow 1$ in $\mathcal{S}'(\Lambda_\varepsilon)$. We let $q_0(k) := |k|$, so that $q_\varepsilon(k) \longrightarrow q_0(k)$ pointwise for $k \in \mathbb{R}^d$ as $\varepsilon \longrightarrow 0$. We observe that, on account of the above definitions, for any $\ell \in \mathbb{N}_+$ and $\sigma \in (0,1)$, it holds that

$$\tilde{\mathcal{J}}_{\sigma,\ell}\mathcal{J}_\sigma = \mathcal{J}_\sigma, \qquad \tilde{\mathcal{J}}_{\sigma,\ell+1}\tilde{\mathcal{J}}_{\sigma,\ell} = \tilde{\mathcal{J}}_{\sigma,\ell}. \tag{1.26}$$

Furthermore, since $\mathcal{J}_\sigma$, $\tilde{\mathcal{J}}_{\sigma,\ell}$ as well as $\mathcal{L}_\varepsilon$ are Fourier multipliers, they all commute. The operators $(\mathcal{J}_\sigma)_\sigma$ are used to define the scale decomposition

$$\sigma \mapsto G_\sigma := \mathcal{J}_\sigma G$$

of the Green function G. The operators $(\tilde{\mathcal{J}}_\sigma)_\sigma$ play an auxiliary role and will be used frequently in the estimates.

Remark 1.12. We shall use the fact that if $\sigma < \mu_i$, where μ_i is as in Def. 1.5, then $\mathcal{J}_\sigma \mathcal{J}_{\mu_{i+1}} = \mathcal{J}_\sigma$. This is a consequence of the definitions of $\mathcal{J}_\sigma$ and μ_i, and of the fact that, for $\sigma < \mu_i$,

$$2\,\sigma\,[\![\sigma]\!]^{-1} \leqslant 2\,\mu_i\,[\![\mu_i]\!]^{-1} = 2(2^{i+2}-1) < (2^{i+3}-1) = \mu_{i+1}\,[\![\mu_{i+1}]\!]^{-1},$$

which implies $j_\sigma(\eta)\, j_{\mu_{i+1}}(\eta) = j_\sigma(\eta)$ for all $\eta \in \mathbb{R}$.

Smoothing operators. To establish a suitable Schauder estimate, we will employ smoothing operators $(K_\sigma)_\sigma$ that possess only limited regularising effects in time. It is convenient to choose these operators as inverses of differential operators.

Definition 1.13. *For $\sigma, \eta \in (0,1)$ let*

$$L_\sigma := (1 + [\![\sigma]\!]^{2s}\partial_t)(1 - [\![\sigma]\!]^2\Delta)^2, \qquad K_\sigma := L_\sigma^{-1} = (1 + [\![\sigma]\!]^{2s}\partial_t)^{-1}(1 - [\![\sigma]\!]^2\Delta)^{-2}, \qquad K_{\eta,\sigma} := L_\sigma K_\eta.$$

In the tightness argument, we will also employ a Littlewood–Paley decomposition acting solely on the spatial variables.

Definition 1.14. *(Spatial LP blocks) Let $(\hat{\Delta}_i \colon [0,\infty) \to \mathbb{R}_+)_{i \geqslant -1}$ be a dyadic partition of unity on $[0,\infty)$, where $\hat{\Delta}_{-1}$ is supported in $[0,1]$, $\hat{\Delta}_0$ is supported in $[1/2, 3/2]$ and $\hat{\Delta}_i(\cdot) := \hat{\Delta}_0(2^{-i}\cdot)$ for $i \geqslant 1$. We define spatial Littlewood-Paley blocks $(\bar{\Delta}_i)_{i \geqslant -1}$ as the Fourier multipliers on $\mathcal{S}'(\mathbb{R}^d_\varepsilon)$ associated with the symbols $k \mapsto \hat{\Delta}_i(q_\varepsilon(k))$.*

Remark 1.15. By extension, we identify the fractional Laplacian $(-\Delta_\varepsilon)^s$ and the spatial Littlewood-Paley blocks $(\bar{\Delta}_i)_{i \geqslant -1}$ with the corresponding operators on $C(\Lambda_\varepsilon, \zeta)$ acting trivially on the time variable.

Convolution operators. We call T a convolution operator on $C(\Lambda_\varepsilon, w)$ if there exists a signed measure $m(\mathrm{d}z)$ such that

$$(Tf)(z) = \int_{\Lambda_\varepsilon} m(\mathrm{d}z')\, f(z - z').$$

By slight abuse of notation, we usually denote the measure $m(\mathrm{d}z)$ associated to T by $T(\mathrm{d}z)$. We call $T(\mathrm{d}z)$ the kernel of the operator T. We denote by $|T(\mathrm{d}z)|$ the variation of the measure $T(\mathrm{d}z)$. Given a nonnegative weight $w \in C(\Lambda_0)$ we write

$$\|T\|_{\mathrm{TV}(w)} := \int_{\Lambda_\varepsilon} w(z)\,|T(\mathrm{d}z)|$$

for the weighted total variation norm of the kernel of T. If $T(\mathrm{d}z)$ is absolutely continuous with respect to the measure $\mathrm{d}z$ on Λ_ε, we write $T(z)$ for its density, *i.e.* $T(\mathrm{d}z) = T(z)\,\mathrm{d}z$. For $p \in [0,\infty]$, we denote by $\|T\|_{L^p(w)}$ the weighted L^p norm of the density of the kernel of T, *i.e.*

$$\|T\|_{L^p(w)} := \|z \mapsto w(z)T(z)\|_{L^p},$$

where $\|\bullet\|_{L^p}$ is the usual L^p norm on Λ_ε with respect to the measure $\mathrm{d}z$. Given a weight w, a convolution operator T and a function f we write

$$(wTf)(z) := w(z)\,(Tf)(z). \tag{1.27}$$

Note that the operators

$$G, G_\sigma, K_\sigma, K_{\eta,\sigma}, \mathcal{J}_\sigma, \tilde{\mathcal{J}}_\sigma, (-\Delta_\varepsilon)^s, \bar{\Delta}_i$$

introduced above are all convolution operators on $C(\Lambda_\varepsilon, \zeta^\alpha)$ for every $\alpha \in [0, 2s)$. The kernels of the operators $G_\sigma, \mathcal{J}_\sigma, \tilde{\mathcal{J}}_\sigma$ have smooth densities. The kernels of G, K_σ have densities of limited regularity. The kernels of $K_{\eta,\sigma}, (-\Delta_\varepsilon)^s, \bar{\Delta}_i$ do not posses densities. In particular, the kernels of $(-\Delta_\varepsilon)^s, \bar{\Delta}_i$ are proportional to the Dirac delta in time.

The following lemmas collect the fundamental properties of these convolution operators, which will be used repeatedly throughout our analysis.

Definition 1.16. *For $\omega \in \mathbb{R}$ and $\mu \in [0,1)$, we write*

$$w_\mu^\omega(z) := (1 + [\![\mu]\!]^{-1}|z|_s)^\omega, \qquad z \in \Lambda_0.$$

Lemma 1.17. *For all $\flat \in (0, 2s)$, we have*

a) $\|K_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \vee \|K_{\sigma,\eta}\|_{\mathrm{TV}(w_\sigma^2)} \lesssim 1$,

b) $\|\mathcal{J}_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \vee \|\tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \vee \|L_\sigma^2 \mathcal{J}_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \vee \|L_\sigma^2 \tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \lesssim 1$,

c) $\|\partial_t \tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \lesssim [\![\sigma]\!]^{-2s}$ *and* $\|\nabla_\varepsilon \tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(w_\sigma^2)} \lesssim [\![\sigma]\!]^{-1}$,

d) $\|G\|_{\mathrm{TV}(w_0^\flat)} \lesssim 1$ *and* $\|\dot{G}_\sigma\|_{\mathrm{TV}(w_\sigma^\flat)} \vee \|L_\sigma^3 \dot{G}_\sigma\|_{\mathrm{TV}(w_\sigma^\flat)} \lesssim [\![\sigma]\!]^{2s-1}$,

uniformly over $\varepsilon \in 2^{-\mathbb{N}_0}$ and $1/2 \le \sigma \le \eta < 1$.

Remark 1.18. The kernels of the operators $K_\sigma, K_{\sigma,\eta}, \mathcal{J}_\sigma$ and $\tilde{\mathcal{J}}_\sigma$ exhibit rapid decay in spacetime. In particular, these operators have finite total variation norms $\|\bullet\|_{\mathrm{TV}(w_\sigma^\omega)}$ for all $\omega \ge 0$. In contrast, for fractional Laplacians of order $s \in (0,1)$, the kernels of the operators G and $\dot{G}_\sigma$ decay much more slowly. Their total variation norms $\|\bullet\|_{\mathrm{TV}(w_\sigma^\omega)}$ is finite only for $\omega \in [0, 2s)$. This slow decay of G and $\dot{G}_\sigma$ constitutes a major analytical difficulty and is one of the main reasons why PDEs involving fractional Laplacians are particularly challenging to study.

Remark 1.19. Note that all of the above bounds remain valid if one replaces the weights by $\zeta^{-\alpha}$ or $\zeta_\mu^{-\alpha}$ with any $\mu \in (0,1)$ and $\alpha \in [0,1]$, since $1 < 2s < 2$ and

$$\zeta_\mu^{-\alpha} \le \zeta^{-\alpha} \lesssim w_\sigma^1 \le w_\sigma^2,$$

for all $\mu, \sigma \in (0,1)$.

Proof. Item a) is proved in Lemma A.3. Items b) and c) follow from Lemma A.2. Item d) follows from the estimates for $\dot{G}_\sigma$ established in Lemma A.7. □

Lemma 1.20. *We have*

$$\|\mathcal{L}_\varepsilon \tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-2s}$$

uniformly over $\varepsilon \in 2^{-\mathbb{N}_0}$ and $1/2 \le \sigma < 1$.

Proof. First note that

$$\|(-\Delta_\varepsilon)^s \mathcal{J}_\mu\|_{\mathrm{TV}(\zeta^{-1})} \le \sum_{i \ge -1} \|(-\Delta_\varepsilon)^s \bar{\Delta}_i \mathcal{J}_\mu\|_{\mathrm{TV}(\zeta^{-1})},$$

where we set $\bar{\Delta}_{-2} = 0$. Next, observe that $\bar{\Delta}_i \mathcal{J}_\mu = 0$, unless $[\![\mu_i]\!] \geqslant 2^{-10}[\![\mu]\!]$. Consequently, we have

$$\|(-\Delta_\varepsilon)^s \mathcal{J}_\mu\|_{\mathrm{TV}(\zeta^{-1})} \leqslant \sum_{i:[\![\mu_i]\!] \geqslant 2^{-10}[\![\mu]\!]} \|(-\Delta_\varepsilon)^s \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \|\mathcal{J}_\mu\|_{\mathrm{TV}(\zeta^{-1})} \lesssim \sum_{i:[\![\mu_i]\!] \geqslant 2^{-10}[\![\mu]\!]} [\![\mu_i]\!]^{-2s} \lesssim [\![\mu]\!]^{-2s},$$

where the second estimate follows from Lemmas 1.17 and A.19. □

The following lemma combines the estimates from the preceding lemmas with Young's convolution inequality. The resulting bounds will be used frequently in the subsequent analysis, often without explicit reference.

Lemma 1.21. *For all $\alpha \in [0,1]$ we have*

a) $\|\zeta_\mu^\alpha K_\sigma f\| \lesssim \|\zeta_\mu^\alpha K_\eta f\| \lesssim \|\zeta_\mu^\alpha f\|$,

b) $\|\zeta_\mu^\alpha \mathcal{J}_\sigma f\| \lesssim \|\zeta_\mu^\alpha \tilde{\mathcal{J}}_\sigma f\| \lesssim \|\zeta_\mu^\alpha K_\sigma f\| \lesssim \|\zeta_\mu^\alpha f\|$,

c) $\|\zeta_\mu^\alpha \mathcal{L}_\varepsilon \tilde{\mathcal{J}}_\sigma f\| \lesssim [\![\sigma]\!]^{-2s} \|\zeta_\mu^\alpha f\|$, $\|\zeta_\mu^\alpha \partial_t \tilde{\mathcal{J}}_\sigma f\| \lesssim [\![\sigma]\!]^{-2s} \|\zeta_\mu^\alpha f\|$ *and* $\|\zeta_\mu^\alpha \nabla_\varepsilon \tilde{\mathcal{J}}_\sigma f\| \lesssim [\![\sigma]\!]^{-1} \|\zeta_\mu^\alpha f\|$,

d) $\|\zeta_\mu^\alpha G f\| \lesssim \|\zeta_\mu^\alpha f\|$ *and* $\|\zeta_\mu^\alpha \dot{G}_\sigma f\| \vee \|\zeta_\mu^\alpha L_\sigma^3 \dot{G}_\sigma f\| \lesssim [\![\sigma]\!]^{2s-1} \|\zeta_\mu^\alpha f\|$,

uniformly over $\varepsilon \in 2^{-\mathbb{N}_0}$, $0 \leqslant \mu \leqslant 1$, $1/2 \leqslant \sigma \leqslant \eta < 1$ and $f \in \mathcal{S}'(\Lambda_\varepsilon)$, where $\|\bullet\|$ denotes the L^∞ norm over Λ_ε, and we use the notation introduced in (1.27).

Proof. The stated bounds are consequences of the previous lemmas, together with Young's inequality for convolution, the identities

$$K_\sigma = K_{\sigma,\eta} K_\eta, \qquad \mathcal{J}_\sigma = \mathcal{J}_\sigma \tilde{\mathcal{J}}_\sigma, \qquad \tilde{\mathcal{J}}_\sigma = L_\sigma^2 \tilde{\mathcal{J}}_\sigma K_\sigma K_\sigma$$

and Remarks 1.8 a) and 1.19. For example, to prove the first bound, we note that

$$\begin{aligned}
\|\zeta_\mu^\alpha K_\sigma f\| &\leqslant \sup_z \int_{\Lambda_\varepsilon} \zeta_\mu^\alpha(z) |(K_\eta f)(z - z_1)| \, |K_{\sigma,\eta}(\mathrm{d}z_1)| \\
&\lesssim \sup_z \int_{\Lambda_\varepsilon} \zeta_\mu^\alpha(z - z_1) |(K_\eta f)(z - z_1)| \, \zeta^{-1}(z_1) \, |K_{\sigma,\eta}(\mathrm{d}z_1)| \\
&\leqslant \|\zeta_\mu^\alpha (K_\eta f)\| \, \|K_{\sigma,\eta}\|_{\mathrm{TV}(\zeta^{-1})} \\
&\lesssim \|\zeta_\mu^\alpha f\|.
\end{aligned}$$

The proof of the remaining bounds proceeds analogously. □

2 Stochastic quantisation

In this section we lay out the main steps in the proof of Theorem 1.1, starting from the effective equation at (fractional, parabolic) spacetime scale $[\![\sigma]\!] := 1 - \sigma$, $\sigma \in [0,1]$, obtained through a scale decomposition and the introduction of the approximate effective force. Since our analysis primarily concerns scales close to 1, we shall restrict all scale parameters, such as σ, to the interval $[1/2, 1]$.

2.1 Scale decomposition

Let $\phi^{(\varepsilon,M)}$ be a stationary solution to the finite system of SDEs (1.4) and define the scale-dependent field

$$\phi_\sigma^{(\varepsilon,M)} := \mathcal{J}_\sigma \phi^{(\varepsilon,M)}$$

localised at (fractional, parabolic) spacetime scales $\gtrsim [\![\sigma]\!]$. Then we have

$$\mathcal{L}_\varepsilon \phi_\sigma^{(\varepsilon,M)} = \mathcal{J}_\sigma F^{(\varepsilon,M)}(\phi^{(\varepsilon,M)}), \qquad \sigma \in [1/2,1],$$

where

$$F^{(\varepsilon,M)}(\phi) := -\lambda \phi^3 - r_{\varepsilon,M}\phi + \xi^{(\varepsilon,M)}. \tag{2.1}$$

We call $F^{(\varepsilon,M)}$ the force. Let $\mathcal{E} := \bigcap_{\alpha>0} C(\Lambda_{\varepsilon,M}, \zeta^\alpha)$ be the space of continuous periodic in space functions on Λ_ε and exhibiting subpolynomial growth in time at infinity. Denote by $\hat{\mathcal{E}} \subset \mathcal{S}'(\Lambda_{\varepsilon,M})$ the image of $\mathcal{E}$ under ∂_t, interpreted as the time derivative operator on $\mathcal{S}'(\Lambda_{\varepsilon,M})$. Note that the white noise $\xi^{(\varepsilon,M)}$ on $\Lambda_{\varepsilon,M}$ belongs a.s. to $\hat{\mathcal{E}}$ and the stochastic convolution $G\xi^{(\varepsilon,M)}$ belongs a.s. to $\mathcal{E}$. Consider a family of functionals, referred to as the *effective force*,

$$\big(F_\sigma^{(\varepsilon,M)} \colon \mathcal{E} \to \hat{\mathcal{E}}\big)_{\sigma\in[1/2,1]},$$

which is differentiable in $\sigma \in (1/2,1)$ and satisfies the final condition

$$F_1(\psi) = F^{(\varepsilon,M)}(\phi). \tag{2.2}$$

Using the identity $\phi_1^{(\varepsilon,M)} = \phi^{(\varepsilon,M)}$, we obtain

$$F^{(\varepsilon,M)}(\phi^{(\varepsilon,M)}) = F_1\big(\phi_1^{(\varepsilon,M)}\big) = F_\mu\big(\phi_\mu^{(\varepsilon,M)}\big) + R_\mu^{(\varepsilon,M)},$$

for all $\mu \in [1/2,1]$, where

$$R_\mu^{(\varepsilon,M)} := \int_\mu^1 \big[\partial_\sigma F_\sigma^{(\varepsilon,M)}\big(\phi_\sigma^{(\varepsilon,M)}\big) + \mathrm{D}F_\sigma^{(\varepsilon,M)}\big(\phi_\sigma^{(\varepsilon,M)}\big)\big(\partial_\sigma \phi_\sigma^{(\varepsilon,M)}\big)\big]\,\mathrm{d}\sigma.$$

Here $\mathrm{D}F(\psi)\psi'$ denotes the Fréchet derivative of a functional F in the direction of $\psi' \in \mathcal{E}$ at the point $\psi \in \mathcal{E}$. Moreover, we have

$$\partial_\sigma \phi_\sigma^{(\varepsilon,M)} = \dot{G}_\sigma(F^{(\varepsilon,M)}(\phi^{(\varepsilon,M)})) = \dot{G}_\sigma\big(F_\sigma^{(\varepsilon,M)}\big(\phi_\sigma^{(\varepsilon,M)}\big) + R_\sigma^{(\varepsilon,M)}\big),$$

where $\dot{G}_\sigma := \mathcal{L}_\varepsilon^{-1}\dot{\mathcal{J}}_\sigma$ and $\dot{\mathcal{J}}_\sigma := \partial_\sigma \mathcal{J}_\sigma$. Consequently, we deduce that for any choice of the effective force

$$(F_\sigma)_{\sigma\in[1/2,1]} = \big(F_\sigma^{(\varepsilon,M)}\big)_{\sigma\in[1/2,1]},$$

satisfying the conditions specified above, the pair

$$(\phi_\mu, R_\mu)_{\mu\in[1/2,1]} = \big(\phi_\mu^{(\varepsilon,M)}, R_\mu^{(\varepsilon,M)}\big)_{\mu\in[1/2,1]}$$

satisfies the system of equations

$$\begin{cases} \mathcal{L}_\varepsilon \phi_\mu = \mathcal{J}_\mu(F_\mu(\phi_\mu) + R_\mu) \\ R_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\,\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F_\sigma(\phi_\sigma)(\dot{G}_\sigma R_\sigma)]\,\mathrm{d}\sigma, \end{cases} \tag{2.3}$$

where

$$H_\sigma(\psi) := \partial_\sigma F_\sigma(\psi) + \mathrm{D}F_\sigma(\psi)(\dot{G}_\sigma F_\sigma(\psi)).$$

Our main goal will be to show that this system allows for good a priori estimates for a suitably chosen effective force $(F_\sigma)_{\sigma\in[1/2,1]}$.

Remark 2.1. Except for Sec. 2.4 and 2.5, we shall almost always suppress the explicit dependence on $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$. To avoid repetition, we will not restate this each time, but **all estimates should be understood as uniform in $\boldsymbol{\varepsilon \in 2^{-\mathbb{N}_0}}$ and $\boldsymbol{M \in \mathbb{N}_+}$**, unless explicitly stated otherwise. In particular, we shall write Λ for Λ_ε and Λ_M for $\Lambda_{\varepsilon,M}$. Likewise, we write $(-\Delta)^s$ for the discrete fractional Laplacian.

2.2 Overview of the strategy

The goal of this section is to provide a blueprint that guides the reader through the technical aspects of the proofs and highlights the underlying heuristics. We will not present rigorous arguments here. Our intention is instead to convey intuition and the global structure of the analysis. Readers who find the informal reasoning confusing can safely skip this section without loss of logical continuity.

Our approach to obtaining a priori global spacetime estimates for solutions to the system (2.3) relies on several new conceptual and technical ideas:

a) the introduction of a remainder term R_σ, which circumvents the need to explicitly solve the flow equation for F_σ (unlike in the original approach of [Duc25a, Duc22]);

b) the use of new weighted norms that control the solution across the entire space, inspired by the spatial decomposition introduced in [GH19]);

c) a stopping argument for the effective force, allowing us to close nonlinear estimates;

d) a collection of technical innovations addressing the difficulties arising from the limited spacetime decay of the fractional heat kernel.

In what follows, we focus primarily on the new ideas required to handle the large-field problem. For a pedagogical overview of the flow equation approach, we refer the reader to the lecture notes [Duc25b].

To keep the exposition clear, let us ignore the remainder R_σ and model the original equation by

$$\mathcal{L}\phi_\sigma - \lambda\phi_\sigma^3 \approx F_\sigma(\phi_\sigma) - \lambda\phi_\sigma^3, \tag{2.4}$$

where $F_\sigma(\phi_\sigma)$ is a polynomial in the field ϕ_σ. To measure the spacetime growth of fields, we use the weight

$$\zeta_\sigma(z) \approx (1 + [\![\sigma]\!]^a |z|_s)^{-1}, \qquad z \in \Lambda,$$

where $|z|_s$ is the fractional parabolic distance on Λ and $a > 1$ is an exponent chosen to balance the scale behavior at large distances. Its precise value will later be crucial to closing our nonlinear estimates.

Pathwise bounds on the random effective force $(F_\sigma)_\sigma$ are obtained from a flow-equation analysis of its probabilistic cumulants. The outcome is that $F_\sigma(\phi_\sigma)$ behaves as a random nonlocal polynomial in ϕ_σ and its coefficients are localized in regions of size $\approx [\![\sigma]\!]$ and scale roughly as

$$[\![\sigma]\!]^{(k-3)\beta+\delta\ell},$$

where $k \in \mathbb{N}_0$ is the monomial degree, $\beta > \gamma$ encodes the field scaling, $\delta > 0$ measures the distance to criticality, and ℓ is the perturbative order. Due to a Kolmogorov-type argument needed to extract the almost sure behaviour of the force F_σ from its moments, we loose also a bit in the spacetime growth, which will be modelled by a weight $\zeta_\sigma^{-\kappa_0(\ell+1)}$, where $\kappa_0 > 0$ is an arbitrarily small exponent. Overall we have, schematically,

$$F_\sigma(\phi_\sigma) \approx \xi + \bar{r}\,\phi_\sigma - \lambda\phi_\sigma^3 + \sum_{\ell=1}^{\bar{\ell}} \sum_{k=0}^{\bar{k}} \zeta_\sigma^{-\kappa_0(\ell+1)} [\![\sigma]\!]^{(k-3)\beta+\delta\ell} \phi_\sigma^k, \tag{2.5}$$

where the sums over k and ℓ are finite. The cutoffs $\bar{k}$ and $\bar{\ell}$ are chosen so that the equation for the remainder R_σ can be solved, a technical aspect we omit in this heuristic discussion.

To estimate the size of the solution to (2.4), we introduce a constant $C_\Phi = |||\phi|||$ (cf. Def. 2.2) such that

$$|\phi_\sigma(z)| = |(\mathcal{J}_\sigma\phi)(z)| \leq C_\Phi\, \zeta_\sigma^{-1/3}(z)\, [\![\sigma]\!]^{-\gamma}, \qquad z \in \Lambda, \tag{2.6}$$

valid for all $\sigma \geq \bar{\mu}$, where $\bar{\mu}$ such that $[\![\bar{\mu}]\!] \ll 1$ is a random scale to be chosen later.

For moderate distances $|z| \lesssim [\![\sigma]\!]^{-a}$, the spatial weight $\zeta_\sigma^{-1/3}(z)$ is of order one and we are describing the distributional nature of the solution, growing like $[\![\sigma]\!]^{-\gamma}$ as $[\![\sigma]\!] \searrow 0$ for some $\gamma > 0$. For large distances $|z| \gg [\![\sigma]\!]^{-a}$, the spatial growth can be improved as follows. Let $\hat{\mu} \ll \sigma$ be such that $|z| \approx [\![\hat{\mu}]\!]^{-a}$. Observing that $\phi_\sigma = \mathcal{J}_\sigma \phi_{\hat{\mu}}$ due to the properties of the smoothing operators, we obtain

$$\begin{aligned} |\phi_\sigma(z)| &\approx |(\mathcal{J}_\sigma \phi_{\hat{\mu}})(z)| \approx |\phi_{\hat{\mu}}(z)| \\ &\lesssim C_\Phi (1 + [\![\hat{\mu}]\!]^a |z|)^{1/3} [\![\hat{\mu}]\!]^{-\gamma} \\ &\approx C_\Phi |z|^{\gamma/a} \approx C_\Phi (1 + [\![\sigma]\!]^a |z|)^{\gamma/a} [\![\sigma]\!]^{-\gamma}. \end{aligned} \tag{2.7}$$

This improved spatial growth replaces $\zeta_\sigma^{-1/3}$ with $\zeta_\sigma^{-\gamma/a}$, and by taking a large we can make the effective growth arbitrarily mild, an essential feature for closing nonlinear bounds.

Since higher-order monomials ($k > 3$) in (2.5) are accompanied by small coefficients $[\![\sigma]\!]^{(k-3)\beta+\delta\ell} \ll 1$, a coercive bound for (2.4) yields, schematically,

$$|\phi_\sigma|^3 \approx |F_\sigma(\phi_\sigma) + \lambda \phi_\sigma^3| \lesssim \zeta_\sigma^{-(\bar{\ell}+1)\kappa_0} \sum_{k,\ell} [\![\sigma]\!]^{(k-3)\beta+\delta\ell} |\phi_\sigma|^k. \tag{2.8}$$

We estimateded the spacetime growth by taking the worst possible weight $\zeta_\sigma^{-(\bar{\ell}+1)\kappa_0}$, where $\bar{\ell}$ denotes the largest perturbative order that needs to be considered. Substituting (2.7) in (2.8), and ignoring the mild nonlocality of the effective force, we obtain

$$|\phi_\sigma|^3 \approx |F_\sigma(\phi_\sigma) + \lambda \phi_\sigma^3| \lesssim \zeta_\sigma^{-(\bar{\ell}+1)\kappa_0} \Bigg[\sum_{k,\ell} [\![\sigma]\!]^{(k-3)\beta - 3k\gamma + \delta\ell} \zeta_\sigma^{-k\gamma/a} C_\Phi^k \Bigg].$$

The terms with $\ell = 0$, that is the first two terms on the right-hand side of (2.5), are explicit and yield improved estimates, so that in total we arrive at

$$|\phi_\sigma|^3 \lesssim [\![\sigma]\!]^{-3\beta+\delta} \zeta_\sigma^{-(\bar{\ell}+1)\kappa_0 - \bar{k}\gamma/a} (1 + C_\Phi)^{\bar{k}}, \tag{2.9}$$

where $\bar{k}$ is the maximal degree of the monomials. Here we used the fact that $\beta > \gamma$, which allows the field amplitude to be compensated by the kernel size. The constant C_Φ can then be estimated as

$$C_\Phi \approx \sup_{\sigma \geq \bar{\mu}} [\![\sigma]\!]^\gamma \zeta_\sigma^{1/3} |\phi_\sigma| \lesssim [\![\sigma]\!]^{\gamma-\beta+\delta/3} \zeta_\sigma^{(1-(\bar{\ell}+1)\kappa_0 - \bar{k}\gamma/a)/3} (1 + C_\Phi)^{\bar{k}/3}.$$

Choosing $\gamma \leq \beta$ such that $\gamma - \beta + \delta/3 \geq \vartheta > 0$ and then taking a large enough such that

$$1 - (\bar{\ell}+1)\kappa_0 - \bar{k}\gamma/a \geq 0,$$

we obtain the bound

$$C_\Phi \lesssim [\![\bar{\mu}]\!]^\vartheta (1 + C_\Phi)^{\bar{k}/3}.$$

It follows that, for sufficiently small $[\![\bar{\mu}]\!] \ll 1$, the nonlinear estimate closes and yields $C_\Phi \approx 1$.

Let us now address the treatment of the nonlocality of the effective force kernels. The main technical challenges in this work stem from the limited decay of the slice propagator $\dot{G}_\sigma$ associated with the fractional parabolic operator $\mathcal{L}$. Roughly speaking, we only have algebraic decay of the form (see Lemma A.7):

$$|\dot{G}_\sigma(z)| \lesssim [\![\sigma]\!]^{-d-1} (1 + |z|_s / [\![\sigma]\!])^{-d-2s+\epsilon}, \qquad z \in \Lambda, \tag{2.10}$$

where $\epsilon > 0$ is arbitrarily small. This behaviour contrasts sharply with the stretched-exponential decay of the standard heat kernel, and also with the fractional Laplacian case appearing in the usual ("static") renormalisation group analysis of the fractional Φ^4 model [BMS03]. The difference arises from the limited smoothness of the symbol of the fractional heat operator $\mathcal{L}$.

The main consequence of (2.10) is that the effective force kernels, obtained by solving a flow equation driven by $\dot{G}_\sigma$, inherit a similar algebraic decay. The monomials which appear in $F_\sigma(\phi_\sigma)$ take the schematic form

$$F_\sigma(\phi_\sigma)(z) \approx \sum_{k,\ell} \int F_\sigma^{[\ell],(k)}(z; z_1, \dots, z_k) \prod_{j=1}^{k} \phi_\sigma(z_j)\, \mathrm{d}z_j,$$

where $F_\sigma^{[\ell],(k)}$ are random kernels. Ignoring their distributional nature and thinking of them as bona-fide functions, their spatial nonlocality and spatial growth can be modelled as (cf. Def. 4.3 and 4.6)

$$F_\sigma^{[\ell],(k)}(z, z_1, \dots, z_k) \approx \zeta_\sigma^{-(\ell+1)\kappa_0}(z)\,(1 + [\![\sigma]\!]^{-1}\mathrm{St}(z, z_1, \dots, z_k))^{-(\mathfrak{b}-\ell\kappa_0)},$$

where $\mathrm{St}(z, z_1, \dots, z_k)$ measures the diameter of the set $\{z, z_1, \dots, z_k\}$. The initial decay exponent $\mathfrak{b} \approx 2s$ follows from (2.10), while the additional loss $\ell\kappa_0$ reflects the growth at spacetime infinity of the kernel in its output variable, induced by the similar growth of the noise. It is now clear that, to proceed as in (2.9), we must be able to compensate for the spatial growth of the fields ϕ_σ by exploiting the limited decay of the kernels away from the diagonal.

The remainder of this section makes this argument rigorous and establishes further properties of the solution. Sec. 3 develops the coercive estimate required in (2.8), while Sec. 4 contains the detailed analysis of the random force coefficients and the derivation of the precise form of estimate (2.5).

2.3 Main estimate

In this section we introduce a family of weighted norms that will be used to measure the size of fields over spacetime uniformly across scales. These norms depend on an exponent $\gamma > 0$, to be fixed in Sec. 4.9, and a terminal scale $\bar{\mu} \in [1/2, 1)$, which will later be tuned according to the noise amplitude.

Definition 2.2. *For $\psi \in \mathcal{S}'(\Lambda)$ and $f \in C([1/2,1), C(\Lambda, \zeta))$, we set*

$$|||\psi||| = |||\psi|||_{\bar{\mu}} := \sup_{\sigma \geq \bar{\mu}} [\![\sigma]\!]^\gamma \, \|\zeta_\sigma^{1/3} \mathcal{J}_\sigma \psi\|, \tag{2.11}$$

$$|||f|||_\# = |||f|||_{\#,\bar{\mu}} := \sup_{\sigma \geq \bar{\mu}} [\![\sigma]\!]^{3\gamma} \, \|\zeta_\sigma f_\sigma\|. \tag{2.12}$$

The above "triple norms" exhibit a specific behaviour with respect to the weight. In particular, the following lemma shows that changing the weight yields equivalent norms.

Lemma 2.3. *For $\alpha \in [3\nu, 1]$ the following bounds*

$$\left[\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}} [\![\sigma]\!]^\gamma \, \|\zeta_\mu^{\alpha/3}\phi_\sigma\|\right] \lesssim |||\phi|||_{\bar{\mu}} \lesssim \left[\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}} [\![\sigma]\!]^\gamma \, \|\zeta_\mu^{\alpha/3}\phi_\sigma\|\right], \tag{2.13}$$

$$\left[\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}} [\![\sigma]\!]^{3\gamma} \, \|\zeta_\mu^{\alpha}\mathcal{L}\phi_\sigma\|\right] \lesssim |||\sigma \mapsto \mathcal{L}\phi_\sigma|||_{\#,\bar{\mu}} \lesssim \left[\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}} [\![\sigma]\!]^{3\gamma} \, \|\zeta_\mu^{\alpha}\mathcal{L}\phi_\sigma\|\right], \tag{2.14}$$

hold uniformly in $\bar{\mu} \in [1/2, 1)$ and $\phi \in \mathcal{S}'(\Lambda)$, where $\phi_\sigma := \mathcal{J}_\sigma\phi$ for $\sigma \in [1/2, 1)$.

Proof. Let us prove the first inequality in (2.13). We start by observing that, on account of the support properties of $\{\chi_i\}_i$, we have the following decomposition

$$\|\zeta_\mu^{\alpha/3}\phi_\sigma\| \leq \sup_i \|\zeta_\mu^{\alpha/3}(\chi_i + \chi_{i-1} + \chi_{i+1})\phi_\sigma\| \lesssim \sup_i \|\zeta_\mu^{\alpha/3}\chi_i\phi_\sigma\|.$$

Let us consider separately the cases $\mu_i \leq \sigma$ and $\mu_i \geq \sigma$. In the first case, thanks to $\|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_i}^{-1/3}\| \lesssim 1$, we have

$$\sup_{i|\mu_i\leq\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\phi_\sigma\| \lesssim \sup_{i|\mu_i\leq\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_i}^{-1/3}\|\,\|\zeta_{\mu_i}^{1/3}\phi_\sigma\| \lesssim [\![\sigma]\!]^\gamma \sup_{i|\mu_i\leq\sigma} \|\zeta_{\mu_i}^{1/3}\phi_\sigma\| \leq [\![\sigma]\!]^\gamma \|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim |\!|\!|\phi|\!|\!|.$$

Let us turn to the case $\mu_i > \sigma$. Note that

$$\|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_{i+1}}^{-1/3}\| \lesssim (1+[\![\mu]\!]^a[\![\mu_{i+1}]\!]^{-a})^{-\alpha/3} \lesssim [\![\mu]\!]^{-a\alpha/3}[\![\mu_{i+1}]\!]^{\alpha a/3}.$$

Moreover, on account of Remarks 1.12, 1.8 and Lemma 1.17, we have

$$\|\zeta_{\mu_{i+1}}^{1/3}\phi_\sigma\| = \|\zeta_{\mu_{i+1}}^{1/3}\mathfrak{J}_\sigma\phi_{\mu_{i+1}}\| \lesssim \|\mathfrak{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\| \lesssim \|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\|.$$

Combining the above estimates, we deduce that

$$\begin{aligned}
\sup_{i|\mu_i>\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\phi_\sigma\| &\lesssim \sup_{i|\mu_i>\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_{i+1}}^{-1/3}\|\,\|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\| \\
&\lesssim [\![\sigma]\!]^\gamma [\![\mu]\!]^{-a\alpha/3}\Big[\sup_{i|\mu_i>\sigma}[\![\mu_{i+1}]\!]^{\alpha a/3}[\![\mu_{i+1}]\!]^{-\gamma}\Big]\Big[\sup_{i|\mu_i>\sigma}[\![\mu_{i+1}]\!]^\gamma\|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\|\Big] \\
&\lesssim [\![\sigma]\!]^{\alpha a/3}[\![\mu]\!]^{-a\alpha/3}\,|\!|\!|\phi|\!|\!| \\
&\lesssim |\!|\!|\phi|\!|\!|,
\end{aligned}$$

where we used that $\alpha a/3 \geq \nu a = \gamma$ and $\sigma \geq \mu$. This proves the first inequality in (2.13). The second inequality in (2.13) is a direct consequence of the fact that $\zeta_\sigma^{1/3} \leq \zeta_\sigma^{\alpha/3}$. The inequalities (2.14) are proved similarly. □

The two norms $|\!|\!|\phi|\!|\!|, |\!|\!|f|\!|\!|_\#$ fix the analytical setting for the global analysis of the SPDE (2.3). In Sec. 3, we will prove a suitable coercive estimates for fractional parabolic equation with cubic nonlinearity, which allows us to control the large values of the fields. Below we state a direct consequence of this estimate.

Theorem 2.4. *(Coercive estimate) For $\phi \in \mathcal{S}'(\Lambda)$ and $\sigma \in [1/2, 1)$ define*

$$f_\sigma := \mathcal{L}\phi_\sigma + \lambda\,\phi_\sigma^3, \qquad \phi_\sigma := \mathfrak{J}_\sigma\phi, \qquad \mathcal{L} := \partial_t + (-\Delta)^s + m^2.$$

Then the following bound

$$|\!|\!|\phi|\!|\!|_{\bar\mu} \lesssim \lambda^{-1/2}[\![\bar\mu]\!]^\gamma + \lambda^{-1/3}|\!|\!|f|\!|\!|_{\#,\bar\mu}^{1/3}$$

holds uniformly in $\phi \in \mathcal{S}'(\Lambda)$, $\bar\mu \in [1/2, 1)$ and $\lambda \in (0,\infty)$.

Proof. We apply the a priori estimates from Theorem 3.1 to $u = \phi_\sigma$ and $\rho = \zeta_\sigma^{1/3}$. The constants A and B appearing in the theorem can be bounded as follows:

$$A = \|(-\Delta)^s\zeta_\sigma^{2/3}\| + \|\zeta_\sigma^{1/3}(\partial_t\zeta_\sigma^{1/3})\| \lesssim [\![\sigma]\!]^{2sa},$$

and

$$\begin{aligned}
B &= \|\zeta_\sigma^{1/3}u\|\,(\|\zeta_\sigma^{1/3}(\partial_t\zeta_\sigma^{1/3})\| + \|\zeta_\sigma^{1/3}(-\Delta)^s\zeta_\sigma^{1/3}\| + \|\zeta_\sigma^{2/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})\mathfrak{D}_s(\zeta_\sigma^{1/3})\|) + \|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{1/3})\mathfrak{D}_s(\zeta_\sigma^{1/3}u)\| \\
&\lesssim Q_\sigma^2\|\zeta_\sigma^{1/3}\phi_\sigma\| + Q_\sigma\|\mathfrak{D}_s(\zeta_\sigma^{1/3}\phi_\sigma)\|.
\end{aligned}$$

Here we define

$$Q_\sigma^2 := \|\partial_t\zeta_\sigma^{1/3}\| + \|(-\Delta)^s\zeta_\sigma^{1/3}\| + \|\mathfrak{D}_s(\zeta_\sigma^{1/3})\|^2 + \|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})\|^2 \lesssim [\![\sigma]\!]^{2sa}.$$

The bound $\|(-\Delta)^s\zeta_\sigma^\alpha\| \lesssim [\![\sigma]\!]^{2sa}$ is proved in Lemma A.9. To estimate $\|\mathfrak{D}_s(\zeta_\sigma^{1/3})\|^2$ we used (1.18), while the bound on $\|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})\|^2$ follows from Lemma A.10. Moreover, using the bound (1.18), we obtain

$$\|\mathfrak{D}_s(\zeta_\sigma^{1/3}\phi_\sigma)\| \lesssim \|\nabla_\varepsilon(\zeta_\sigma^{1/3}\phi_\sigma)\|^s\,\|\zeta_\sigma^{1/3}\phi_\sigma\|^{1-s}. \tag{2.15}$$

We note that

$$\|\nabla_\varepsilon(\zeta_\sigma^{1/3}\,\phi_\sigma)\| \lesssim \|(\nabla_\varepsilon\zeta_\sigma^{1/3})\,\phi_\sigma\| + \|\zeta_\sigma^{1/3}(\nabla_\varepsilon\phi_\sigma)\|.$$

Hence, using the fact that $|\nabla_\varepsilon\zeta_\sigma^{1/3}| \lesssim \zeta_\sigma^{1/3}$ and $\phi_\sigma = \tilde{\mathcal{J}}_\sigma\phi_\sigma$ as well as $\|\nabla_\varepsilon\tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-1}$, we arrive at

$$\|\nabla_\varepsilon(\zeta_\sigma^{1/3}\,\phi_\sigma)\| \lesssim \|\zeta_\sigma^{1/3}\,\phi_\sigma\| + \|\zeta_\sigma^{1/3}(\nabla_\varepsilon\tilde{\mathcal{J}}_{\sigma,1})\phi_\sigma\| \lesssim \|\zeta_\sigma^{1/3}\,\phi_\sigma\| + [\![\sigma]\!]^{-1}\|\zeta_\sigma^{1/3}\,\phi_\sigma\| \lesssim [\![\sigma]\!]^{-1}\|\zeta_\sigma^{1/3}\,\phi_\sigma\|.$$

Together with (2.15), this proves that

$$\|\mathfrak{D}_s(\zeta_\sigma^{1/3}\,\phi_\sigma)\| \lesssim [\![\sigma]\!]^{-s}\,\|\zeta_\sigma^{1/3}\,\phi_\sigma\|,$$

uniformly in $\sigma \geq \bar{\mu}$. Therefore,

$$B \lesssim ([\![\sigma]\!]^{2sa} + [\![\sigma]\!]^{sa}[\![\sigma]\!]^{-s})\|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim [\![\sigma]\!]^{sa-s}\,\|\zeta_\mu^{1/3}\phi_\sigma\|.$$

By Theorem 3.1 and Young's inequality, we obtain

$$\begin{aligned}\|\zeta_\sigma^{1/3}\phi_\sigma\| &\leqslant \lambda^{-1/2}C\,[\![\sigma]\!]^{as} + \lambda^{-1/3}(\|\zeta_\sigma f_\sigma\| + C[\![\sigma]\!]^{as-s}\|\zeta_\sigma^{1/3}\phi_\sigma\|)^{1/3}\\ &\leqslant \lambda^{-1/2}C\,[\![\sigma]\!]^{as} + \lambda^{-1/3}\|\zeta_\sigma f_\sigma\|^{1/3} + C\lambda^{-1/3}[\![\sigma]\!]^{(as-s)/3}\|\zeta_\sigma^{1/3}\phi_\sigma\|^{1/3}\\ &\leqslant \lambda^{-1/2}C\,[\![\sigma]\!]^{as} + \lambda^{-1/3}\|\zeta_\sigma f_\sigma\|^{1/3} + C'\lambda^{-1/2}[\![\sigma]\!]^{(as-s)/2} + \frac{1}{2}\|\zeta_\sigma^{1/3}\phi_\sigma\|.\end{aligned}$$

As a result, we arrive at

$$\|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim \lambda^{-1/2}[\![\sigma]\!]^{(as-s)/2} + \lambda^{-1/3}\|\zeta_\sigma f_\sigma\|^{1/3}.$$

This allows us to deduce that

$$|||\phi|||_{\bar{\mu}} = \sup_{\sigma\geq\bar{\mu}}[\![\sigma]\!]^{\gamma}\,\|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim \sup_{\sigma\geq\bar{\mu}}[\![\sigma]\!]^{\gamma+(as-s)/2}\lambda^{-1/2} + \lambda^{-1/3}\sup_{\sigma\geq\bar{\mu}}([\![\sigma]\!]^{3\gamma}\,\|\zeta_\sigma f_\sigma\|)^{1/3}.$$

Hence, provided $\gamma + (as-s)/2 \geqslant \gamma \geqslant 0$, which follows from $a > 1$, we conclude our claim. □

The coercivity estimate derived above provides the key ingredient for deriving a priori bounds for solutions to system (2.3), as stated in the following theorem.

Theorem 2.5. *Let $\phi = \phi^{(\varepsilon,M)}$ be a solution of the stochastic quantisation equation (1.4) and*

$$\big(F_\sigma = F_\sigma^{(\varepsilon,M)}\colon \mathcal{E} \longrightarrow \hat{\mathcal{E}}\big)_{\sigma\in[1/2,1]}$$

be a family of functionals differentiable in $\sigma \in (1/2,1)$ and satisfying the final condition (2.2). For $\sigma \in [1/2,1]$ define

$$\phi_\sigma := \mathcal{J}_\sigma\phi, \qquad R_\sigma := F(\phi) - F_\sigma(\phi_\sigma).$$

Suppose that there exist constants $S \in \mathbb{N}_+$, $C_F \geqslant 1$ and $\vartheta, \bar{\kappa} > 0$ such that

$$\bar{\kappa} \in [\nu, 1), \qquad \vartheta \leqslant \gamma \wedge (2s-\gamma), \qquad (s+\gamma)\,\bar{\kappa}/(1-\bar{\kappa}) \leqslant \vartheta/4, \tag{2.16}$$

and $(F_\sigma)_\sigma$ satisfies the following estimates

$$\begin{aligned}\|\zeta_\mu[\mathcal{J}_\sigma F_\sigma(\psi_\sigma) - (-\lambda\,\psi_\sigma^3)]\| &\leqslant [\![\sigma]\!]^{-3\gamma+\vartheta}\Big[C_F(1+|||\psi|||_{\bar{\mu}})^S + (1+|||\psi|||_{\bar{\mu}})^2\,|||\mathcal{L}\psi_\bullet|||_{\#,\bar{\mu}}\Big],\\ \|\zeta_\mu K_\sigma H_\sigma(\psi_\sigma)\| &\leqslant C_F[\![\sigma]\!]^{\vartheta-1}(1+|||\psi|||_{\bar{\mu}})^S,\\ \|\zeta_\mu^{\bar{\kappa}}K_\sigma F_\sigma(\psi_\sigma)\| &\leqslant C_F[\![\sigma]\!]^{-3\gamma}(1+|||\psi|||_{\bar{\mu}})^S,\\ \|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot{G}_\sigma\hat{\psi}))\| &\leqslant C_F[\![\sigma]\!]^{\vartheta-1}(1+|||\psi|||_{\bar{\mu}})^S\,\|\zeta_\mu^{1-\bar{\kappa}}\tilde{\mathcal{J}}_\sigma^2\hat{\psi}\|,\end{aligned} \tag{2.17}$$

for all $\bar{\mu} \in [1/2,1), \mu \in [\bar{\mu},1), \sigma \in [\mu,1)$ and $\psi, \hat{\psi} \in \mathcal{S}'(\Lambda)$, where we denote $\psi_\sigma := \mathcal{J}_\sigma\psi$ and $(\mathcal{L}\psi)_\sigma := \mathcal{L}\psi_\sigma$, and the functional H is defined by (2.3).

Then there exists a universal constant $\hat{C}>0$ such that for all $\bar{\mu}\in[1/2,1)$ satisfying

$$[\![\bar{\mu}]\!]^{\vartheta}\leq\hat{C}\,((\lambda^{-1}+1)C_F)^{-2}, \tag{2.18}$$

we have

$$|||\phi|||_{\bar{\mu}}\leq 1,\qquad |||\mathcal{L}\phi_\bullet|||_{\#,\bar{\mu}}\leq\lambda\,[\![\bar{\mu}]\!]^{-\vartheta/2},\qquad |||K_\bullet R_\bullet|||_{\#,\bar{\mu}}\leq\lambda\,[\![\bar{\mu}]\!]^{\vartheta/2}.$$

Remark 2.6. By the elementary argument presented in Sec. 2.1, our assumptions on ϕ and $(F_\mu)_\mu$ imply that $(\phi_\mu,R_\mu)_\mu$ is a solution of (2.3).

Proof. Define

$$\Phi=\Phi_{\bar{\mu}}:=1+|||\phi|||_{\bar{\mu}}+\theta_{\mathcal{L}}^{-1}\,|||\mathcal{L}\phi_\bullet|||_{\#,\bar{\mu}}+\theta_R^{-1}\,|||K_\bullet R_\bullet|||_{\#,\bar{\mu}}, \tag{2.19}$$

for constants $\theta_{\mathcal{L}},\theta_R>0$ to be fixed later. Our goal is to bound each term on the right-hand side by an increasing function of Φ itself and then apply a continuity argument to establish the uniform bounds claimed. For now, we omit the index $\bar{\mu}\in[1/2,1)$, as the estimates hold uniformly in this parameter.

Let $C_Q=C_F(1+|||\phi|||)^S$. We first prove a bound for $|||K_\bullet R_\bullet|||_\#$. To this end, we use Lemma 2.8 below, which shows that we can control $\|\zeta_\mu K_\sigma R_\sigma\|$ in terms of $\|\zeta_\mu^{\bar{\kappa}}\tilde{\mathcal{J}}_\sigma^2 R_\sigma\|$. Note that

$$\tilde{\mathcal{J}}_\sigma^2 R_\sigma=\tilde{\mathcal{J}}_\sigma^2(\mathcal{L}\phi-F_\sigma(\phi_\sigma)),$$

which follows from (1.4) and (2.2). Since $\bar{\kappa}\in[\nu,1)$ and $\vartheta\in(0,\gamma]$, by Lemmas 1.20 and 2.3, we have

$$\|\zeta_\mu^{\bar{\kappa}}\tilde{\mathcal{J}}_\sigma^2\mathcal{L}\phi\|=\|\zeta_\mu^{\bar{\kappa}}(\mathcal{L}\tilde{\mathcal{J}}_\sigma)\tilde{\mathcal{J}}_\sigma\phi\|\lesssim\|\mathcal{L}\tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\,\|\zeta_\mu^{\bar{\kappa}}\tilde{\mathcal{J}}_\sigma\phi\|\lesssim[\![\sigma]\!]^{-2s}[\![\sigma]\!]^{-\gamma}|||\phi|||\lesssim C_Q[\![\bar{\mu}]\!]^{\vartheta}[\![\sigma]\!]^{-2(s+\gamma)}.$$

Since $\vartheta\in(0,2s-\gamma]$, by Lemma 1.17 and the estimates (2.17),we have

$$\|\zeta_\mu^{\bar{\kappa}}\tilde{\mathcal{J}}_\sigma^2F_\sigma(\phi_\sigma)\|\lesssim\|\zeta_\mu^{\bar{\kappa}}K_\sigma F_\sigma(\phi_\sigma)\|\lesssim C_Q[\![\sigma]\!]^{-3\gamma}\lesssim C_Q[\![\bar{\mu}]\!]^{\vartheta}[\![\sigma]\!]^{-2(s+\gamma)}.$$

Altogether, this yields

$$\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}}[\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\mu^{\bar{\kappa}}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|\lesssim C_Q[\![\bar{\mu}]\!]^{\vartheta}. \tag{2.20}$$

Using (2.17) and Lemma 1.17 we also have

$$\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}}\left\|\zeta_\mu K_\mu\int_\sigma^1 H_\eta(\phi_\eta)\mathrm{d}\eta\right\|\lesssim\int_{\bar{\mu}}^1\|\zeta_\mu K_\eta H_\eta(\phi_\eta)\|\,\mathrm{d}\sigma\lesssim C_Q[\![\bar{\mu}]\!]^{\vartheta}.$$

By Lemma 2.8, it then follows that

$$\sup_{\sigma,\mu|\sigma\geq\mu\geq\bar{\mu}}\|\zeta_\mu K_\sigma R_\sigma\|\lesssim\exp(CC_Q[\![\bar{\mu}]\!]^{\vartheta/2}),$$

possibly after adjusting the constant C. Hence, by (2.12),

$$|||K_\bullet R_\bullet|||_\#\lesssim[\![\bar{\mu}]\!]^{3\gamma}\exp(CC_F[\![\bar{\mu}]\!]^{\vartheta/2}(1+|||\phi|||)^S)\lesssim[\![\bar{\mu}]\!]^{3\gamma}\exp(CC_F[\![\bar{\mu}]\!]^{\vartheta/2}\Phi^S). \tag{2.21}$$

The control of $|||\mathcal{L}\phi_\bullet|||_\#$ is obtained from (2.17)

$$\begin{aligned}|||\mathcal{L}\phi_\bullet+\lambda\phi_\bullet^3|||_\# &\leq |||(\mathcal{J}_\bullet F_\bullet(\phi_\bullet)+\lambda\phi_\bullet^3)|||_\#+|||\mathcal{J}_\bullet R_\bullet|||_\#\\ &\lesssim [\![\bar{\mu}]\!]^{\vartheta}C_F(1+|||\phi|||)^S+[\![\bar{\mu}]\!]^{\vartheta}(1+|||\phi|||)^2\,|||\mathcal{L}\phi_\bullet|||_\#+|||K_\bullet R_\bullet|||_\#,\end{aligned}$$

using in particular Lemma 1.17 to bound the contribution from R. From this it follows that

$$|||\mathcal{L}\phi_\bullet+\lambda\phi_\bullet^3|||_\#\lesssim([\![\bar{\mu}]\!]^{\vartheta}C_F+\theta_{\mathcal{L}}[\![\bar{\mu}]\!]^{\vartheta}+\theta_R)\Phi^S \tag{2.22}$$

and

$$|||\mathcal{L}\phi_\bullet|||_\#\leq|||\lambda\phi_\bullet^3|||_\#+|||\mathcal{L}\phi_\bullet+\lambda\phi_\bullet^3|||_\#\lesssim(\lambda+[\![\bar{\mu}]\!]^{\vartheta}C_F+\theta_{\mathcal{L}}[\![\bar{\mu}]\!]^{\vartheta}+\theta_R)\Phi^S, \tag{2.23}$$

by triangular inequality and

$$|||\lambda\phi_\bullet^3|||_{\#} = \lambda \sup_{\sigma\geqslant\bar\mu} [\![\sigma]\!]^{3\gamma}\, \|\zeta_\sigma \phi_\sigma^3\| \lesssim \lambda \left[\sup_{\sigma\geqslant\bar\mu} [\![\sigma]\!]^{\gamma}\, \|\zeta_\sigma^{1/3}\, \phi_\sigma\|\right]^3 = \lambda\, |||\phi|||_{\bar\mu}^3.$$

Next, the a priori estimates of Theorem 2.4 and (2.22) give

$$\begin{aligned} |||\phi||| &\lesssim \lambda^{-1/2}[\![\bar\mu]\!]^{\gamma} + \lambda^{-1/3}\,|||\mathcal{L}\phi_\bullet + \lambda\phi_\bullet^3|||_{\#}^{1/3} \\ &\lesssim \lambda^{-1/2}[\![\bar\mu]\!]^{\gamma} + \lambda^{-1/3}([\![\bar\mu]\!]^{\vartheta} C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^{\vartheta} + \theta_R)^{1/3}\Phi^{S/3}. \end{aligned} \tag{2.24}$$

Gathering (2.21), (2.23) and (2.24), we obtain

$$\Phi_{\bar\mu} \leqslant 1 + C\left[\tau(\bar\mu) + \tau(\bar\mu)^{1/3}\Phi_{\bar\mu}^{S/3} + \tau(\bar\mu)\,\Phi_{\bar\mu}^{S} + \tau(\bar\mu)\exp(C\tau(\bar\mu)\Phi_{\bar\mu}^{S})\right], \tag{2.25}$$

where

$$\begin{aligned} \tau(\bar\mu) \;:=\; & \lambda^{-1/2}[\![\bar\mu]\!]^{\gamma/2} + \lambda^{-1}([\![\bar\mu]\!]^{\vartheta} C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^{\vartheta} + \theta_R) \\ & + \theta_{\mathcal{L}}^{-1}(\lambda + [\![\bar\mu]\!]^{\vartheta} C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^{\vartheta} + \theta_R) + \theta_R^{-1}[\![\bar\mu]\!]^{3\gamma} + C_F[\![\bar\mu]\!]^{\vartheta/2}. \end{aligned}$$

Choose

$$\theta_R = [\![\bar\mu]\!]^{\vartheta/2}\lambda, \qquad \theta_{\mathcal{L}} = [\![\bar\mu]\!]^{-\vartheta/2}\lambda. \tag{2.26}$$

Then $\bar\mu \mapsto \tau(\bar\mu)$ is a decreasing function and

$$\tau(\bar\mu) \leqslant [\![\bar\mu]\!]^{\vartheta/2}(\lambda^{-1/2}[\![\bar\mu]\!]^{(\gamma-\vartheta)/2} + (2\lambda^{-1}+1)C_F + 5 + \lambda^{-1}[\![\bar\mu]\!]^{3\gamma-\vartheta/2}) \leqslant 10\,[\![\bar\mu]\!]^{\vartheta/2}(\lambda^{-1}+1)C_F.$$

Fix $\tau_* > 0$ small enough such that

$$C\,[\,\tau_* + \tau_*^{1/3}4^{S/3} + \tau_*\,4^{S} + \tau_*\exp(C\tau_*\,4^{S})] \leqslant 1, \tag{2.27}$$

and define $\bar\mu_* = \bar\mu_*(\lambda, C_F) \in (0,1)$ as the (unique) solution to

$$10\,[\![\mu_*]\!]^{\vartheta/2}(\lambda^{-1}+1)C_F = \tau_*.$$

Then for all $\bar\mu \in [\bar\mu_*, 1)$ we have $\tau(\bar\mu) \leqslant \tau_*$ and as a consequence of (2.25) and (2.27),

$$\Phi_{\bar\mu} \leqslant 4 \Longrightarrow \Phi_{\bar\mu} \leqslant 2.$$

Define the set

$$A := \{\bar\mu \in [\bar\mu_*, 1)\,|\,\Phi_{\bar\mu} \leqslant 4\} \subseteq [\bar\mu_*, 1).$$

Note that $A \neq \emptyset$ since for $\bar\mu \nearrow 1$ we have $\Phi_{\bar\mu} \searrow 1$. As the map $\bar\mu \mapsto \Phi_{\bar\mu}$ is continuous, the set A is closed in $[\bar\mu_*, 1)$. Hence, to prove that $A = [\bar\mu_*, 1)$, it is enough to show that A is open in $[\bar\mu_*, 1)$. If $\mu \in A$, then $\Phi_\mu \leqslant 2$ and by continuity, there exists a neighbourhood of μ within $[\bar\mu_*, 1)$ on which the function $\bar\mu \mapsto \Phi_{\bar\mu}$ takes values not exceeding 4. We conclude that $A = [\bar\mu_*, 1)$ and therefore that $\Phi_{\bar\mu_*} \leqslant 2$. Using (2.26) this implies that for $\bar\mu = \bar\mu_*$, we have

$$|||\phi|||_{\bar\mu} \leqslant 1, \qquad |||\mathcal{L}\phi_\bullet|||_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{-\vartheta/2}, \qquad |||K_\bullet R_\bullet|||_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{\vartheta/2},$$

with

$$[\![\bar\mu]\!]^{-1} = \big(\tau_*^{-1}10\,(\lambda^{-1}+1)C_F\big)^{2/\vartheta} \lesssim \big((\lambda^{-1}+1)C_F\big)^{2/\vartheta},$$

uniformly in C_F and $\lambda > 0$. □

To apply the previous theorem to the analysis of (2.3), we need to construct suitable approximate solution to the flow equation

$$\partial_\sigma F_\sigma + \mathrm{D}F_\sigma(\dot G_\sigma F_\sigma) = 0 \tag{2.28}$$

for the effective force

$$(F_\sigma)_{\sigma\in[1/2,1]} = \big(F_\sigma^{(\varepsilon,M)}\big)_{\sigma\in[1/2,1]},$$

subject to the final condition (2.2). Sec.-4 will be devoted to the construction of such an effective force $(F_\sigma)_\sigma$, as formulated in the following theorem.

Theorem 2.7. *There exists a choice of the deterministic parameters*

$$(r_{\varepsilon,M})_{\varepsilon\in(0,1],M\in\mathbb{N}_+}$$

and a family of random, scale-dependent functionals

$$\big(F_\sigma = F_\sigma^{(\varepsilon,M)}\colon \mathcal{E}\to\hat{\mathcal{E}}\big)_{\sigma\in[1/2,1]}$$

such that:

a) The map $\sigma\mapsto F_\sigma$ is differentiable for $\sigma\in(1/2,1)$.

b) The final condition (2.2) holds.

c) The estimates (2.17) are satisfied with $C_F = 1+\|F^{\mathfrak{A}}\|^2$ and $\|F^{\mathfrak{A}}\|$ introduced in Def. 4.6.

d) For all $N\geqslant 1$, it holds that

$$\sup_{\varepsilon\in 2^{-\mathbb{N}_0},M\in\mathbb{N}_+}\mathbb{E}\Big[\|F^{\mathfrak{A}}\|^N\Big]<\infty. \tag{2.29}$$

We complete this section with the proof of the following auxiliary lemma, used in Theorem 2.5.

Lemma 2.8. *Suppose that there exist constants $S\in\mathbb{N}_+, C_F\geqslant 1$ and $\vartheta,\bar\kappa>0$ such that*

$$\bar\kappa\in[\nu,1),\qquad (s+\gamma)\,\bar\kappa/(1-\bar\kappa)\leqslant\vartheta/4$$

and $(F_\sigma)_\sigma$ satisfies the following estimate

$$\|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot G_\sigma\hat\psi))\|_{L^\infty}\ \leqslant\ C_F\,[\![\sigma]\!]^{\vartheta-1}(1+|||\psi|||_{\bar\mu})^S\,\|\zeta_\mu^{1-\bar\kappa}\tilde{\mathcal{J}}_\sigma^2\hat\psi\|_{L^\infty}, \tag{2.30}$$

for all $\bar\mu\in[1/2,1),\mu\in[\bar\mu,1),\sigma\in[\mu,1)$ and $\psi,\hat\psi\in\mathcal{S}'(\Lambda)$, where we denote $\psi_\sigma:=\mathcal{J}_\sigma\psi$. Fixed a function $[\bar\mu,1)\ni\mu\mapsto\mathcal{H}_\mu\in\mathcal{S}'(\Lambda)$ and consider the linear equation

$$R_\mu=\mathcal{H}_\mu+\int_\mu^1\mathrm{D}F_\sigma(\phi_\sigma)(\dot G_\sigma R_\sigma)\,\mathrm{d}\sigma,\qquad \mu\geqslant\bar\mu. \tag{2.31}$$

Then there exists a universal constant $C>0$ such that

$$\|\zeta_\eta K_\mu R_\mu\|\leqslant\exp(CC_Q[\![\bar\mu]\!]^{\vartheta/2})\Big[\sup_{\sigma\geqslant\mu}\|\zeta_\eta K_\sigma\mathcal{H}_\sigma\|+\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{2(s+\gamma)}\,\|\zeta_\eta^{\bar\kappa}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|\Big],\qquad \mu\geqslant\eta\geqslant\bar\mu, \tag{2.32}$$

where $C_Q=C_F(1+|||\phi|||)^S$.

Proof. Start by observing that, by interpolation, for any $\alpha\in[0,1]$ and $\beta\in\mathbb{R}$, we have by Young's inequality

$$[\![\sigma]\!]^{\beta(1-\alpha)}\,\|\zeta_\eta^{\alpha+\bar\kappa(1-\alpha)}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|\lesssim[\![\sigma]\!]^{\beta(1-\alpha)}\|\zeta_\eta\tilde{\mathcal{J}}_\sigma^2R_\sigma\|^\alpha\,\|\zeta_\eta^{\bar\kappa}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|^{1-\alpha}\lesssim\|\zeta_\eta\tilde{\mathcal{J}}_\sigma^2R_\sigma\|+[\![\sigma]\!]^\beta\,\|\zeta_\eta^{\bar\kappa}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|.$$

Choose $\alpha=(1-2\bar\kappa)/(1-\bar\kappa)$. Then $\alpha+\bar\kappa(1-\alpha)=1-\bar\kappa$ and $1-\alpha=\bar\kappa/(1-\bar\kappa)$. Consequently,

$$[\![\sigma]\!]^{\beta\bar\kappa/(1-\bar\kappa)}\|\zeta_\mu^{1-\bar\kappa}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|\lesssim\|\zeta_\mu\tilde{\mathcal{J}}_\sigma^2R_\sigma\|+[\![\sigma]\!]^\beta\,\|\zeta_\mu^{\bar\kappa}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|. \tag{2.33}$$

Then, by (2.30), we have

$$\|\zeta_\eta K_\sigma(\mathrm{D}F_\sigma(\phi_\sigma)(\dot G_\sigma R_\sigma))\|\lesssim[\![\sigma]\!]^{\vartheta-1}C_Q\,\|\zeta_\eta^{1-\bar\kappa}\tilde{\mathcal{J}}_\sigma^2R_\sigma\|,$$

and by (2.33) with $\beta = 2(s+\gamma)$, and Lemma 1.17, we have

$$\|\zeta_\eta K_\sigma(\mathrm{D}F_\sigma(\phi_\sigma)(\dot G_\sigma R_\sigma))\| \lesssim C_Q[\![\sigma]\!]^{\vartheta-1}[\![\sigma]\!]^{-2(s+\gamma)\bar\kappa/(1-\bar\kappa)}(\|\zeta_\eta K_\sigma R_\sigma\| + [\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\eta^{\bar\kappa}\tilde J_\sigma^2 R_\sigma\|). \tag{2.34}$$

At this point, to deduce the desired estimate, we apply Gronwall's inequality to

$$\|\zeta_\eta K_\mu R_\mu\| \lesssim \|\zeta_\eta K_\mu \mathcal{H}_\mu\| + CC_Q\int_\mu^1 [\![\sigma]\!]^{\vartheta-1-2(s+\gamma)\bar\kappa/(1-\bar\kappa)}(\|\zeta_\eta K_\sigma R_\sigma\| + [\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\eta^{\bar\kappa}\tilde J_\sigma^2 R_\sigma\|)\mathrm{d}\sigma,$$

which follows from (2.31) and (2.34), and use the inequality $\vartheta - 2(s+\gamma)\bar\kappa/(1-\bar\kappa) \geq \vartheta/2$. □

2.4 Tightness

In this section, we apply Theorem 2.5, together with the effective force constructed in Theorem 2.7, to prove tightness of the sequence of measures $(\hat\nu_{\varepsilon,M})_{\varepsilon,M}$. This constitutes the first step in the proof of our main result, Theorem 1.1. In order to pass to the limit $\varepsilon \to 0$ and $M \to \infty$ we embed all the random spatially periodic fields $\phi = \phi^{(\varepsilon,M)}$ in the same space by extending them from Λ_ε to Λ_0. Let

$$\phi^{[\varepsilon,M]}(t,x) := \int_{\Lambda_\varepsilon^*} \hat\theta(\varepsilon|k|)\,\hat\phi^{(\varepsilon,M)}(\omega,k)\,e^{i(\omega t + k\cdot x)}\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x)\in\Lambda_0, \tag{2.35}$$

where $\hat\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ that $\hat\theta(0) = 1$ and $\hat\theta(\eta) = 0$ for $|\eta| > 1$. The random fields $\phi^{[\varepsilon,M]}$ all live now in the continuum domain Λ_0 for any $\varepsilon \in 2^{-\mathbb{N}_0}$. To extract information about the EQFT, we must evaluate the marginal at a fixed time of the SPDE solution. This is not possible when controlling only spacetime distributional norms such as $|||\bullet|||_{\bar\mu}$. The necessary temporal regularity can, however, be recovered by means of a Schauder estimate adapted to our weighted norms. Indeed, by Lemma A.21 and using $\rho \lesssim \rho_{\bar\mu}$, we obtain

$$\sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi\| \lesssim \sup_i 2^{-i\gamma}\|\rho_{\bar\mu}\bar\Delta_i\phi\| \lesssim [\![\bar\mu]\!]^{-\gamma}\Big[|||\phi|||_{\bar\mu} + |||K_\bullet\mathcal{L}\phi|||_{\#,\bar\mu}\Big].$$

By Theorem 2.5 there exists $\bar\mu \in [1/2,1)$ such that $[\![\bar\mu]\!]^{-\vartheta/2}$ is of order $C_F = 1 + \|F^{\mathfrak{A}}\|^2$ and

$$|||\phi|||_{\bar\mu} \lesssim 1, \qquad |||\mathcal{L}\phi_\bullet|||_{\#,\bar\mu} \lesssim \lambda[\![\bar\mu]\!]^{-\vartheta/2}, \qquad |||K_\bullet R_\bullet|||_{\#,\bar\mu} \lesssim \lambda[\![\bar\mu]\!]^{\vartheta/2}.$$

Recalling that $\mathcal{L}\phi = F_\mu(\phi_\mu) + R_\mu$, we deduce from the above bounds that

$$|||K_\bullet\mathcal{L}\phi|||_{\#,\bar\mu} \lesssim |||K_\bullet F_\bullet(\phi_\bullet)|||_{\#,\bar\mu} + |||K_\bullet R_\bullet|||_{\#,\bar\mu} \lesssim C_F(1+|||\phi|||_{\bar\mu})^S + \lambda[\![\bar\mu]\!]^{\vartheta/2} \lesssim C_F. \tag{2.36}$$

Combining this with the previous Schauder estimate yields

$$\sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi^{(\varepsilon,M)}\|_{L^\infty(\Lambda_\varepsilon)} = \sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi\| \lesssim [\![\bar\mu]\!]^{-\gamma}C_F \lesssim C_F^{2\gamma/\vartheta+1}. \tag{2.37}$$

To derive a meaningful estimate, we need to control a suitable weighted Besov norm of $\phi^{[\varepsilon,M]}$ in terms of the left-hand side of the bound above. Indeed, we have

$$\begin{aligned}\|\rho\bar\Delta_i\phi^{[\varepsilon,M]}\|_{L^\infty(\Lambda_0)} &= \sup_{t\in\mathbb{R}}\|\rho(t,\bullet)(\theta_\varepsilon *_\varepsilon \bar\Delta_i\phi^{(\varepsilon,M)}(t,\bullet))\|_{L^\infty(\mathbb{R}^d)}\\ &\lesssim \sup_{x\in\mathbb{R}^d}\|(\theta_\varepsilon/\bar\rho)(x-\bullet)\|_{L^1(\mathbb{R}^d_\varepsilon)}\|\rho\bar\Delta_i\phi^{(\varepsilon,M)}\|_{L^\infty(\Lambda_\varepsilon)}\\ &\lesssim \|\rho\bar\Delta_i\phi^{(\varepsilon,M)}\|_{L^\infty(\Lambda_\varepsilon)},\end{aligned}$$

where $\bar\rho(x) := \rho(0,x)$ and $*_\varepsilon$ denotes the convolution on the lattice $\mathbb{R}^d_\varepsilon$. Here we used the fact that

$$\phi^{[\varepsilon,M]}(t,x) = (\theta_\varepsilon *_\varepsilon \phi^{(\varepsilon,M)}(t,\bullet))(x), \qquad \theta_\varepsilon(x) := \int_{\mathbb{R}^d}\hat\theta(\varepsilon|k|)e^{ik\cdot x}\frac{\mathrm{d}k}{(2\pi)^d}.$$

As a result, we have

$$\sup_{t\in\mathbb{R}} \|\phi^{[\varepsilon,M]}(t)\|_{C^{-\gamma}(\mathbb{R}^d,\rho(t))} = \sup_i 2^{-i\gamma} \|\rho\bar{\Delta}_i\phi^{[\varepsilon,M]}\|_{L^\infty(\Lambda_0)} \lesssim C_F^{2\gamma/\vartheta+1} \lesssim 1 + \|F^{\mathfrak{A}}\|^{4\gamma/\vartheta+2},$$

where $C^\alpha(\mathbb{R}^d, w)$ stands for the usual Hölder–Besov norm with the weight w and the regularity index α. Combining the above estimate with Theorem 2.7, we arrive at

$$\begin{aligned}\sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \int \|\varphi\|^N_{C^{-\gamma}(\mathbb{R}^d,\bar{\rho})} \hat{\nu}_{\varepsilon,M}(\mathrm{d}\varphi) &= \sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \mathbb{E}\Big[\sup_{t\in\mathbb{R}} \|\phi^{[\varepsilon,M]}(t)\|^N_{C^{-\gamma}(\mathbb{R}^d,\rho(t))}\Big] \\ &\lesssim \sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \mathbb{E}\Big[1 + \|F^{\mathfrak{A}}\|^{4\gamma N/\vartheta+2N}\Big] < \infty,\end{aligned} \tag{2.38}$$

for all $N\in\mathbb{N}_+$. The bound (2.38) implies the tightness of the family $(\hat{\nu}_{\varepsilon,M})_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+}$ in $\mathcal{S}'(\Lambda_0)$. This proves the first part of Theorem 1.1.

2.5 Integrability

In order to complete the proof of Theorem 1.1 it remains to establish the integrability property of the measures $\tilde{\nu}_{\varepsilon,M}$ uniformly in $\varepsilon\in(0,1)$ and $M\in\mathbb{N}_+$ and obtain the bound (1.3) for any limit points. We look to estimate quantities of the form

$$Z_{\varepsilon,M,\theta} := \int \exp\Big[\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big] \nu_{\varepsilon,M}(\mathrm{d}\varphi),$$

where $\theta>0$ is a small parameter,

$$Q_\varepsilon := (1-\Delta_\varepsilon)^{-A/2} \tag{2.39}$$

is a regularising kernel and

$$h(x) := (1+|x|)^{-B} \tag{2.40}$$

is a polynomially decaying weight in the spatial variables. The constants A and B are chosen big enough according to Lemma 2.10 below.

The main tool for this purpose is the Hairer–Steele argument [HS22], which provides optimal estimates with respect to the growth of the function. We introduce a new *tilted* probability measure

$$\nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi) := \frac{\exp\Big[\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big] \nu_{\varepsilon,M}(\mathrm{d}\varphi)}{Z_{\varepsilon,M,\theta}}. \tag{2.41}$$

To prove that this measure is well defined we take advantage of the presence of the coercive term $\|\varphi\|^4_{L^4(\mathbb{T}^d_{\varepsilon,M})}$ in the action functional (1.2) defining the original measure $\nu_{\varepsilon,M}$ and use the inequality

$$\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \le \frac{\lambda}{4} \|\varphi\|^4_{L^4(\mathbb{T}^d_{\varepsilon,M})},$$

for all ε and M as soon as $\theta>0$ is small enough and A, B are big enough. Observe that Jensen's inequality gives

$$1 = \int \nu_{\varepsilon,M}(\mathrm{d}\varphi) = Z_{\varepsilon,M,\theta} \int \exp\Big[-\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big] \nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi) \ge Z_{\varepsilon,M,\theta} \exp\Big[-\theta \int \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi)\Big].$$

Hence,

$$\log \int \exp\Big[\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big] \nu_{\varepsilon,M}(\mathrm{d}\varphi) = \log Z_{\varepsilon,M,\theta} \le \theta \int \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi). \tag{2.42}$$

The problem of controlling the size of $Z_{(\varepsilon,M,\theta)}$ is, by virtue of (2.42), reduced to estimating certain polynomial moments of φ under the tilted measure $\nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi)$. Through stochastic quantisation, this measure can be identified with the marginal law of a stationary solution to the SPDE

$$\mathscr{L}_\varepsilon \phi + \lambda \phi^3 - r_{\varepsilon,M}\phi = O(\phi) + \xi^{(\varepsilon,M)}, \tag{2.43}$$

where the additional perturbation $O(\phi)$ is given by

$$O(\phi)(t,\bullet) = -\theta \frac{\delta}{\delta \phi} \Big\| h Q_\varepsilon \phi(t,\bullet) \Big\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} = -2\theta \Big\| h Q_\varepsilon \phi(t,\bullet) \Big\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (Q_\varepsilon h^2 Q_\varepsilon \phi)(t,\bullet).$$

We use the notation

$$Q_\varepsilon h^2 Q_\varepsilon \phi = Q_\varepsilon (h^2 Q_\varepsilon \phi), \qquad (h^i Q_\varepsilon \phi)(t,x) = (h(x))^i (Q_\varepsilon \phi)(t,x), \quad i \in \{1,2\}.$$

We exploited above the fact that the operator Q_ε is self-adjoint and bounded. Note that using the fact that Q_ε has a fast-decaying continuous kernel, one shows that $Q_\varepsilon \phi$ is continuous and has a mild polynomial growth. Hence, $hQ_\varepsilon\phi(t,\bullet)$ is in $L^2(\mathbb{T}^d_{\varepsilon,M})$.

Estimates for this new equation uniform in $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$ can be obtained by modifying our previous arguments. As before we rewrite (2.43) as a system of equations

$$\begin{cases} \mathscr{L}_\varepsilon \phi_\mu = \mathscr{J}_\mu (F_\mu(\phi_\mu) + R^O_\mu) \\ R^O_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\,\mathrm{d}\sigma + O(\phi) + \int_\mu^1 \mathrm{D}F_\sigma(\phi_\sigma)\,(\dot{G}_\sigma R^O_\sigma)\mathrm{d}\sigma \end{cases} \tag{2.44}$$

for

$$(\phi_\mu, R^O_\mu)_\mu = \big(\phi^{(\varepsilon,M)}_\mu, R^{(\varepsilon,M),O}_\mu\big)_\mu.$$

Applying estimate (2.32) with $\eta = \mu$ to the equation for $R^O_\bullet$, we obtain

$$[\![\mu]\!]^{3\gamma} \,\|\zeta_\mu K_\mu R^O_\mu\| \lesssim \exp(CC_Q [\![\bar\mu]\!]^{\vartheta/2}) \Big[\sup_{\sigma \geq \mu} [\![\mu]\!]^{3\gamma} \,\|\zeta_\mu K_\sigma \mathscr{H}_\sigma\| + [\![\bar\mu]\!]^{3\gamma} \sup_{\sigma \geq \mu} [\![\sigma]\!]^{2(s+\gamma)} \,\|\zeta^{\bar\kappa}_\mu \tilde{\mathscr{J}}^2_\sigma R^O_\sigma\| \Big],$$

with

$$\mathscr{H}_\mu = O(\phi) + \int_\mu^1 H_\sigma(\phi_\sigma)\mathrm{d}\sigma.$$

The term with H_σ can be estimated as previously from (2.17), while to estimate $O(\phi)$ we use the uniform boundedness of $\|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})}$ and Lemma 2.10 below. We obtain that

$$\sup_{\sigma | \sigma \geq \mu} \|\zeta_\mu K_\sigma \mathscr{H}_\sigma\| \lesssim C_F [\![\bar\mu]\!]^\vartheta (1 + |||\phi|||)^S + \theta [\![\mu]\!]^{-3\gamma} \Big[|||\phi||| + |||K_\bullet \mathscr{L}\phi|||_\sharp \Big]^3,$$

uniformly in $\mu \geq \bar\mu$. We also observe that

$$\sup_{\sigma,\mu | \sigma \geq \mu \geq \bar\mu} [\![\sigma]\!]^{2(s+\gamma)} \,\|\zeta^{\bar\kappa}_\mu \tilde{\mathscr{J}}^2_\sigma R^O_\sigma\| \lesssim [\![\bar\mu]\!]^\vartheta C_F (1 + |||\phi|||)^S,$$

uniformly in $\mu \geq \bar\mu$, by an argument analogous to the one leading to (2.20). Therefore, combining the above estimates, we obtain

$$|||K_\bullet R^O_\bullet|||_\sharp \lesssim \exp(CC_Q [\![\bar\mu]\!]^{\vartheta/2}) \Big[C_F [\![\bar\mu]\!]^{3\gamma+\vartheta} (1 + |||\phi|||)^S + \theta \Big[|||\phi||| + |||K_\bullet \mathscr{L}\phi|||_\sharp \Big]^3 \Big].$$

We need a good bound for $|||K_\bullet \mathscr{L}\phi|||_\sharp$ in terms of $\Phi = \Phi_{\bar\mu}$ defined by (2.19) with R replaced by R^O. Observe that by (2.17), we have

$$\begin{aligned} |||K_\bullet \mathscr{L}\phi|||_\sharp &\lesssim |||K_\bullet F(\phi_\bullet)|||_\sharp + |||K_\bullet R^O_\bullet|||_\sharp \\ &\lesssim \lambda \Phi^3 + [\![\bar\mu]\!]^\vartheta C_F \Phi^S + |||K_\bullet R^O_\bullet|||_\sharp \\ &\lesssim \lambda \Phi^3 + [\![\bar\mu]\!]^\vartheta C_F \Phi^S + \theta_R \Phi. \end{aligned} \tag{2.45}$$

As a result, for $\theta\in[0,1]$, we obtain

$$\begin{aligned}|||K_\bullet R^O_\bullet|||_\sharp &\lesssim \exp(CC_F\Phi^S[\![\bar\mu]\!]^{\vartheta/2})\Big[C_F\Phi^S[\![\bar\mu]\!]^{3\gamma+\vartheta}+\theta(1+\lambda)^3\Phi^9+([\![\bar\mu]\!]^\vartheta C_F\Phi^S)^3+\theta\theta_R^3\Phi^3\Big]\\ &\lesssim \exp(CC_F\Phi^S[\![\bar\mu]\!]^{\vartheta/2})\Big[[\![\bar\mu]\!]^{\vartheta/2}+\theta(1+\lambda)^3+\theta\theta_R^3\Big]\Phi^9,\end{aligned}\tag{2.46}$$

with possibly different constants C. Gathering (2.46), (2.23) and (2.24) we arrive at

$$\Phi_{\bar\mu}\le 1+C\Big[\tau(\bar\mu)+\tau(\bar\mu)^{1/3}\Phi_{\bar\mu}^{S/3}+\tau(\bar\mu)\,\Phi_{\bar\mu}^S+\tau(\bar\mu)\Phi_{\bar\mu}^9\exp(C\tau(\bar\mu)\Phi_{\bar\mu}^S)\Big],\tag{2.47}$$

where

$$\begin{aligned}\tau(\bar\mu) :=\ & \lambda^{-1/2}[\![\bar\mu]\!]^{\gamma/2}+\lambda^{-1}([\![\bar\mu]\!]^\vartheta C_F+\theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta+\theta_R)+\theta_{\mathcal{L}}^{-1}(\lambda+[\![\bar\mu]\!]^\vartheta C_F+\theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta+\theta_R)\\ &+\theta_R^{-1}\Big[[\![\bar\mu]\!]^{\vartheta/2}+\theta(1+\lambda)^3+\theta\theta_R^3\Big]+C_F[\![\bar\mu]\!]^{\vartheta/2}.\end{aligned}$$

Let

$$\theta_R=\lambda\theta^{1/2},\qquad \theta_{\mathcal{L}}=[\![\bar\mu]\!]^{-\vartheta/2}\lambda.\tag{2.48}$$

Then $\bar\mu\mapsto\tau(\bar\mu)$ is a decreasing function and, for $\theta\in[0,1]$, we have

$$\tau(\bar\mu)\le[\![\bar\mu]\!]^{\vartheta/2}[(3+\lambda^{-1/2})+(1+2\lambda^{-1})C_F+\lambda^{-1}\theta^{-1/2}]+\theta^{1/2}[2+\lambda^{-1}(1+\lambda)^3+\lambda^2].$$

Fix $\tau_*>0$ so that

$$C\Big[\tau_*+\tau_*^{1/3}4^{S/3}+\tau_*4^S+\tau_*4^9\exp(C\tau_*4^S)\Big]\le 1.$$

Let $\theta_*=\theta_*(\lambda)$ be such that

$$\theta_*^{1/2}[2+\lambda^{-1}(1+\lambda)^3+\lambda^2]=\tau_*/2,$$

and define $\bar\mu_*=\bar\mu_*(\lambda,\theta_*,C_F)$ by

$$[\![\bar\mu_*]\!]^{\vartheta/2}[(3+\lambda^{-1/2})+(1+2\lambda^{-1})C_F+\lambda^{-1}\theta^{-1/2}]=\tau_*/2.$$

Then, for $\theta\le\theta_*$ and $\bar\mu\ge\bar\mu_*$, we have $\tau(\bar\mu_*)\le\tau_*$, and the continuity argument from Theorem 2.5 can be applied to obtain the desired estimates. We conclude that $\Phi_{\bar\mu_*}\le 2$. By (2.48) this implies that for $\bar\mu=\bar\mu_*$, we have

$$|||\phi|||_{\bar\mu}\le 1,\qquad |||\mathcal{L}\phi_\bullet|||_{\sharp,\bar\mu}\le\lambda[\![\bar\mu]\!]^{-\vartheta/2},\qquad |||K_\bullet R^O_\bullet|||_{\sharp,\bar\mu}\le\lambda\theta^{1/2},$$

with

$$[\![\bar\mu]\!]^{-1}=\Big[2\tau_*^{-1}\Big((3+\lambda^{-1/2})+(1+2\lambda^{-1})C_F+\lambda^{-1}\theta^{-1/2}\Big)\Big]^{2/\vartheta}\lesssim\Big[(1+\lambda^{-1})C_F+\lambda^{-1}\theta^{-1/2}\Big]^{2/\vartheta},$$

uniformly in $C_F=1+\|F^{\mathfrak{A}}\|^2$, $\theta\le\theta_*(\lambda)$ and $\lambda>0$. By Lemma 2.10, (2.45) and (2.48), we also have

$$\begin{aligned}\sup_{t\in\mathbb{R}}\rho(t,0)^4\,\|h\,Q_\varepsilon\phi^{(\varepsilon)}(t)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} &\lesssim [\![\bar\mu]\!]^{-4\gamma}\Big[|||\phi|||+|||K_\bullet\mathcal{L}\phi|||_\sharp\Big]^4\\ &\lesssim [\![\bar\mu]\!]^{-4\gamma}\Big[\Phi+\lambda\Phi^3+[\![\bar\mu]\!]^\vartheta C_F\Phi^S+\theta_R\Phi\Big]^4\\ &\lesssim_\lambda [\![\bar\mu]\!]^{-4\gamma}C_F\lesssim 1+\|F^{\mathfrak{A}}\|^{16\gamma/\vartheta+2}.\end{aligned}$$

As a consequence of Theorem 2.7, we obtain

$$\begin{aligned}\sup_{\varepsilon\in 2^{-\mathbb{N}_0},M\in\mathbb{N}_+}\int\|h\,Q_\varepsilon\varphi\|^{4N}_{L^2(\mathbb{T}^d_{\varepsilon,M})}\nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi) &= \sup_{\varepsilon\in 2^{-\mathbb{N}_0},M\in\mathbb{N}_+}\mathbb{E}\Big[\|h\,Q_\varepsilon\phi^{(\varepsilon,M)}(t)\|^{4N}_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big]\\ &\lesssim \sup_{\varepsilon\in 2^{-\mathbb{N}_0},M\in\mathbb{N}_+}\mathbb{E}\Big[1+\|F^{\mathfrak{A}}\|^{16\gamma N/\vartheta+2N}\Big]<\infty,\end{aligned}$$

for all $N\in\mathbb{N}_+$ and $t\in\mathbb{R}$. From this and (2.42), we deduce easily that any accumulation point ν of the sequence $(\hat\nu_{\varepsilon,M})_{\varepsilon,M}$ satisfies (1.3) provided $\theta>0$ is small enough. This proves the exponential integrability, required for the Osterwalder–Schrader reconstruction, completing the proof of Theorem 1.1.

Remark 2.9. The choice of the norm to verify the exponential integrability is quite arbitrary. Since we need to determine an SPDE for it, we want a differentiable norm. In general, we could replace the L^2 norm by any L^{2n} norm, as long as n is finite and similarly use a different weight h in space and smoothing operator Q_ε, as long as they remain compatible with our Schauder estimate.

Lemma 2.10. *Suppose that the parameters A and B in (2.39) and (2.40) satisfy $A > \gamma$ and $B \geq d+1$. Then it holds that*

$$\|\zeta_\sigma O(\phi)\| \lesssim \theta [\![\sigma]\!]^{-3\gamma} \Big[|\!|\!| \phi |\!|\!| + |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp \Big]^3$$

and

$$\sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|h\, Q_\varepsilon \phi(t)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim [\![\bar\mu]\!]^{-4\gamma} \Big[|\!|\!| \phi |\!|\!| + |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp \Big]^4.$$

Proof. Let us first observe that

$$\begin{aligned}
\|\zeta_\sigma O(\phi)\| &\lesssim \theta \Big\| (t,x) \mapsto \zeta_\sigma(t,x) \|h (Q_\varepsilon \phi)(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (Q_\varepsilon h^2 Q_\varepsilon \phi)(t,x) \Big\| \\
&= \theta \Big\| (t,x) \mapsto \|h \rho_\sigma^{-1}(t,\bullet) \rho_\sigma(t,\bullet) (Q_\varepsilon \phi)(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (\zeta_\sigma \rho_\sigma^{-1} \rho_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi)(t,x) \Big\| \\
&\lesssim \theta \Big\| (t,x) \mapsto \|h \rho_\sigma^{-1}(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (\zeta_\sigma \rho_\sigma^{-1})(t,x) \Big\| \, \|\rho_\sigma Q_\varepsilon \phi\|^2 \, \|\rho_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi\|.
\end{aligned}$$

By Remark (1.8) (a) we have

$$\rho_\sigma^{-1}(t,x) \lesssim \rho_\sigma^{-1}(t,0) \rho_\sigma^{-1}(0,x).$$

Hence, using the fact that $h(x) = (1+|x|)^{-B}$ with $B \geq d+1$, we arrive at

$$\|h \rho_\sigma^{-1}(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim \rho_\sigma^{-2}(t,0) \|h \rho_\sigma^{-1}(0,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim \rho_\sigma^{-2}(t,0).$$

Since $\rho_\sigma^{-2}(t,0) \leq \rho_\sigma^{-2}(t,x)$ for any $x \in \Lambda$ and $\rho_\sigma = \zeta_\sigma^\nu$ with $\nu \in (0,1/3)$ we obtain

$$\Big\| (t,x) \mapsto \|h \rho_\sigma^{-1}(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (\zeta_\sigma \rho_\sigma^{-1})(t,x) \Big\| \lesssim \Big\| (t,x) \mapsto \rho_\sigma^{-2}(t,0) (\zeta_\sigma \rho_\sigma^{-1})(t,x) \Big\| \lesssim \|\zeta_\sigma \rho_\sigma^{-3}\| \lesssim 1.$$

Since $A > \gamma$ by Lemma A.21, we have

$$\begin{aligned}
\|\rho_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi\| &\lesssim \|Q_\varepsilon h^2 Q_\varepsilon \phi\| \lesssim \|h^2 Q_\varepsilon \phi\| \lesssim \|\rho_\sigma Q_\varepsilon \phi\| \\
&\lesssim \sum_i 2^{-iA} \|\rho_\sigma \bar\Delta_i \phi\| \lesssim [\![\sigma]\!]^{-\gamma} \Big[|\!|\!| \phi |\!|\!| + [\![\bar\mu]\!]^{2s-2\gamma} |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp \Big],
\end{aligned}$$

uniformly over $\sigma \geq \bar\mu$. Combining the above estimates we obtain

$$\|\zeta_\sigma O(\phi)\| \lesssim \theta [\![\sigma]\!]^{-3\gamma} \Big[|\!|\!| \phi |\!|\!| + |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp \Big]^3.$$

Using Remark (1.8) a) and $B \geq d+1$ we get

$$\sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|h \rho_0^{-1}(t,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim \|h \rho_0^{-1}(0,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim 1.$$

Since

$$\|\rho_0 Q_\varepsilon \phi\| \lesssim \|\rho_{\bar\mu} Q_\varepsilon \phi\| \lesssim \sum_i 2^{-iA} \|\rho \bar\Delta_i \phi\| \lesssim [\![\bar\mu]\!]^{-\gamma} \Big[|\!|\!| \phi |\!|\!| + [\![\bar\mu]\!]^{2s-2\gamma} |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp \Big],$$

we obtain

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|h\, Q_\varepsilon \phi(t)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} &\lesssim \sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|h \rho_0^{-1}(t,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \|\rho_0 Q_\varepsilon \phi\|^4 \\
&\lesssim [\![\bar\mu]\!]^{-4\gamma} \Big[|\!|\!| \phi |\!|\!| + |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp \Big]^4.
\end{aligned}$$

This finishes the proof. □

2.6 The vector model

In this final subsection, we discuss the modifications required to extend our results to the vector-valued model, where the field $\phi = \phi^{(\varepsilon,M)}$ takes values in the Euclidean space $\mathbb{R}^n$ for some $n > 1$. We denote by $(\phi^a)_{a\in\{1,\dots,n\}}$ the components of the field ϕ in the canonical basis. The dynamics reads

$$\mathcal{L}_\varepsilon \phi^a + \lambda |\phi|^2 \phi^a - r\phi^a = \xi^a, \qquad a = 1,\dots,n, \tag{2.49}$$

on $\Lambda_{\varepsilon,M}$, where

$$|\phi|^2 := \sum_a (\phi^a)^2.$$

Here

$$\xi = (\xi^a)_{a\in\{1,\dots,n\}} = (\xi^{(\varepsilon,M),a})_{a\in\{1,\dots,n\}}$$

is a vector-valued, spacetime white noise on $\Lambda_{\varepsilon,M}$ such that

$$\mathbb{E}[\xi^a(t,x)\,\xi^b(s,y)] = \delta(t-s)\,\delta_{a,b}\,\mathbb{1}_{x=y}, \qquad (t,x),(s,y)\in\Lambda_{\varepsilon,M}, \quad a,b = 1,\dots,n. \tag{2.50}$$

As before, we identify ϕ and ξ with spatially periodic functions on Λ_ε.

Our proof of the main result in the scalar-valued case extends straightforwardly to the vector-valued case. The main subtlety lies in obtaining the analogue of Theorem 2.4 in this setting, which, however, follows directly from Theorem 3.1 below. Beyond this adjustment, the subsequent analysis carries over with minimal changes, as it does not depend on the scalar nature of the equation until the classification of the relevant cumulants in Sec. 4.5. At that stage, one must exploit the $O(n)$-symmetry of the noise (2.50) to deduce that the cumulants are likewise symmetric. Consequently, the only contribution to the mass renormalisation is diagonal in the vector indices, and can therefore be absorbed into a redefinition of the renormalisation constant $r = r_{(\varepsilon,M)} \in \mathbb{R}$.

3 A priori estimates

This section is devoted to establishing weighted estimates for classical solutions of a fractional parabolic equation with a cubic coercive term. To the best of our knowledge, this result is new, although the proof closely follows the argument developed in the case of the standard Laplacian (see, for instance, [GH19]). Furthermore, we present the estimates directly in a vector-valued setting, as no additional difficulties arise in this more general formulation.

Theorem 3.1. *Let $\lambda > 0$, $\bar\nu < 2s$ and $u \in C^1_t C^2_x(\Lambda_\varepsilon, \mathbb{R}^n)$ be such that $|\zeta^{\bar\nu} u|$ is a bounded function, where the weight ζ was introduced in Def. 1.5. Then u belongs to the domain of the operator $\partial_t + (-\Delta_\varepsilon)^s$. Define $f \in C(\Lambda_\varepsilon, \mathbb{R}^n)$ by*

$$f^a := \partial_t u^a + (-\Delta_\varepsilon)^s u^a + m^2 u^a + \lambda |u|^2 u^a, \qquad a \in \{1,\dots,n\}, \tag{3.1}$$

where $|u|^2 := \sum_a (u^a)^2$.

For any positive weight $\rho \in C^1_t C^2_x(\Lambda_0)$ such that for some $\tilde\nu \in (\bar\nu, 2s)$ the functions $\zeta^{-\tilde\nu}\rho$, $\zeta^{\tilde\nu}\rho^{-1}$ are bounded we have

$$\|\rho u\| \le 2\lambda^{-1/2} A^{1/2} + \lambda^{-1/3}\big(\|\rho^3 f\| + B\big)^{1/3}, \tag{3.2}$$

where

$$A := \|(-\Delta)^s \rho^2\| + \|\rho(\partial_t \rho)\|$$

and

$$B := \|\rho u\|\big(\|\rho(\partial_t\rho)\| + \|\rho(-\Delta)^s\rho\| + \|\rho^2 \mathfrak{D}_s(\rho^{-1})\mathfrak{D}_s(\rho)\|\big) + \|\rho\mathfrak{D}_s(\rho)\mathfrak{D}_s(\rho u)\|.$$

Proof. We assume that $\|\rho^3 f\| + A + B < \infty$ as otherwise there is nothing to prove. For $N \in \mathbb{N}_+$ and $L > 0$, we define a convex function $\Phi = \Phi_{L,N} \in C(\mathbb{R})$ by

$$\Phi(\eta) := (\eta - L)_+^N, \qquad \Phi'(\eta) := N(\eta - L)_+^{N-1}, \qquad \eta \in \mathbb{R},$$

where Φ' denotes the derivative of Φ and $(\eta)_+ := \mathbb{1}_{\eta \geq 0}\eta$. We fix the parameter $N > 1$ such that $2N(\tilde{\nu} - \bar{\nu}) > d + 2s$. The parameter L will be fixed later. Let $\hat{u} := \rho u$. After testing (3.1) with $\Phi'(|\hat{u}|^2)\hat{u}^a\rho^3$ and summing over $a \in \{1, \dots, n\}$, we obtain

$$0 = \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)\,[\rho^3\,\hat{u}\cdot\partial_t u + \rho^3\,\hat{u}\cdot(-\Delta)^s u + m^2\rho^2|\hat{u}|^2 + \lambda(|\hat{u}|^2)^2 - \rho^3 f\cdot\hat{u}].$$

Observe that

$$\begin{aligned}
\int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)\,\rho^3\,\hat{u}\cdot\partial_t u &= \frac{1}{2}\int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)\,[\rho^2\partial_t|\hat{u}|^2 - (\rho\,\partial_t\rho)\,|\hat{u}|^2] \\
&= \frac{1}{2}\int_{\Lambda_\varepsilon} [\rho^2\partial_t\Phi(|\hat{u}|^2) - (\rho\,\partial_t\rho)\,\Phi'(|\hat{u}|^2)\,|\hat{u}|^2] \\
&= -\frac{1}{2}\int_{\Lambda_\varepsilon} [\partial_t(\rho^2)\,\Phi(|\hat{u}|^2) + (\rho\,\partial_t\rho)\,\Phi'(|\hat{u}|^2)\,|\hat{u}|^2] \\
&\geq -\|\rho(\partial_t\rho)\|\int_{\Lambda_\varepsilon}\Phi(|\hat{u}|^2) - \frac{1}{2}\|\hat{u}\|^2\,\|\rho(\partial_t\rho)\|\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2).
\end{aligned}$$

By Lemma 3.2 below,

$$\sum_a \langle\Phi'(|\hat{u}|^2)\,\hat{u}^a, \rho^3(-\Delta)^s u^a\rangle \geq -\frac{1}{2}\|(-\Delta)^s\rho^2\|\int_{\Lambda_\varepsilon}\Phi(|\hat{u}|^2) - \bar{B}\,\|\hat{u}\|\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2).$$

We also have

$$\begin{aligned}
\langle\Phi'(|\hat{u}|^2), m^2\rho^2|\hat{u}|^2 + \lambda(|\hat{u}|^2)^2\rangle &\geq \langle\Phi'(|\hat{u}|^2), \lambda(|\hat{u}|^2)^2\rangle \\
&\geq \lambda N\langle(|\hat{u}|^2 - L)_+^{N-1}, ((|\hat{u}|^2 - L)_+ + L)\,L\rangle \\
&\geq \lambda L\int_{\Lambda_\varepsilon}\Phi(|\hat{u}|^2) + \lambda L^2\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2)
\end{aligned}$$

and

$$-\langle\Phi'(|\hat{u}|^2), \rho^3 f\cdot\hat{u}\rangle \geq -\|\rho^3 f\|\,\|\hat{u}\|\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2).$$

We conclude that

$$0 \geq [\lambda L - A]\int_{\Lambda_\varepsilon}\Phi(|\hat{u}|^2) + \left[\lambda L^2 - (\|\rho^3 f\| + B)\,\|\hat{u}\|\right]\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2).$$

Taking

$$L > L_* := \max(\lambda^{-1}A, \lambda^{-1/2}[\|\rho^3 f\| + B]^{1/2}\,\|\hat{u}\|^{1/2}),$$

we deduce that

$$\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2) = \int_{\Lambda_\varepsilon}\Phi(|\hat{u}|^2) = 0,$$

which implies that $|\hat{u}|^2 \leq L$ a.e. on Λ_ε. Thus,

$$\|\hat{u}\|^2 \leq \inf_{L > L_*} L = L_* \leq \lambda^{-1}A + \lambda^{-1/2}[\|\rho^3 f\| + B]^{1/2}\|\hat{u}\|^{1/2}.$$

This implies that

$$\|\hat{u}\| \leq \lambda^{-1/2}A^{1/2} + \lambda^{-1/4}\,\|\hat{u}\|^{1/4}\,[\|\rho^3 f\| + B]^{1/4}.$$

By Young's inequality, we have

$$\lambda^{-1/4}\,\|\hat{u}\|^{1/4}\,[\|\rho^3 f\| + B]^{1/4} \leq \frac{\|\hat{u}\|}{4} + \frac{3}{4}\lambda^{-1/3}\,[\|\rho^3 f\| + B]^{1/3},$$

and hence

$$\|\hat{u}\| \leq 2\lambda^{-1/2}A^{1/2} + \lambda^{-1/3}\,[\|\rho^3 f\| + B]^{1/3},$$

as claimed. □

The following lemma completes the proof.

Lemma 3.2. *Suppose that $u \in C^1_t C^2_x(\Lambda_\varepsilon, \mathbb{R}^n)$ is such that $|\zeta^{\bar\nu} u|$ is a bounded function for some $\bar\nu < 2s$ and $\rho \in C^1_t C^2_x(\Lambda_\varepsilon)$ is a positive weight such that for some $\tilde\nu \in (\bar\nu, 2s)$ the functions $\zeta^{-\tilde\nu}\rho$, $\zeta^{\tilde\nu}\rho^{-1}$ are bounded. Let $\Phi = \Phi_{L,N} \in C(\mathbb{R})$ with $N > 1$ fixed as in the proof of the above lemma and arbitrary $L > 0$. Then we have*

$$\sum_a \langle \Phi'(|\hat u|^2)\, \hat u^a, \rho^3 (-\Delta)^s u^a \rangle \geq -\frac{1}{2} \|(-\Delta)^s \rho^2\| \int_{\Lambda_\varepsilon} \Phi(|\hat u|^2) - \bar B \, \|\hat u\| \int_{\Lambda_\varepsilon} \Phi'(|\hat u|^2).$$

where $\hat u := \rho u$ and

$$\bar B := \|\hat u\| \left(\|\rho(-\Delta)^s \rho\| + \|\rho^2 \mathfrak{D}_s(\rho^{-1}) \mathfrak{D}_s(\rho)\| \right) + \|\rho \mathfrak{D}_s(\rho) \mathfrak{D}_s(\hat u)\|.$$

Proof. Leaving the sum over a implicit we have

$$\begin{aligned}
(\mathbb{X}) &:= \langle \Phi'(|\hat u|^2)\, \hat u^a, \rho^3 (-\Delta)^s u^a \rangle \\
&= \int_{\Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z)\, \rho^3(z)\, (u^a(z) - u^a(z')) \\
&= \int_{\Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z)\, \rho^2(z)\, (\hat u^a(z) - \hat u^a(z')) \qquad (=:(\mathbb{I})) \\
&\quad + \int_{\Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z)\, \rho^2(z)\, (\rho(z') - \rho(z))\, u^a(z') \quad (=:(\mathbb{III}))
\end{aligned}$$

Let $V_\tau := \tau \hat u(z') + (1-\tau)\, \hat u(z)$. Since $\tau \in [0,1] \mapsto \Phi(|V_\tau|^2)$ is a convex function (as a composition of a convex function with an affine one),

$$\Phi(|V_1|^2) - \Phi(|V_0|^2) \geq \partial_\tau \Phi(|V_\tau|^2)|_{\tau=0},$$

that is,

$$\Phi(|\hat u(z')|^2) - \Phi(|\hat u(z)|^2) \geq 2\, \Phi'(|\hat u(z)|^2) \sum_a \hat u^a(z)\, (\hat u^a(z') - \hat u^a(z)).$$

Using the above inequality we obtain

$$\begin{aligned}
\mathbb{I} &\geq \frac{1}{2} \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \rho^2(z)\, [\Phi(|\hat u(z)|^2) - \Phi(|\hat u(z')|^2)] \\
&= \frac{1}{2} \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi(|\hat u(z)|^2)\, [\rho^2(z) - \rho^2(z')] \\
&\geq -\frac{1}{2} \|(-\Delta)^s \rho^2\| \int_{\Lambda_\varepsilon} \Phi(|\hat u|^2),
\end{aligned}$$

where the equality in the second line follows by an integration by parts of the fractional Laplacian, which is a symmetric operator in L^2. Let us now consider $(\mathbb{III})$ and split it as follows

$$\begin{aligned}
(\mathbb{III}) &= \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z) \rho^2(z)\, (\rho(z') - \rho(z))\, \rho^{-1}(z')\, \hat u^a(z') \\
&= \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z) \rho^2(z)\, (\rho(z') - \rho(z))\, (\rho^{-1}(z') - \rho^{-1}(z))\, \hat u^a(z') \\
&\quad + \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z)\, \rho^2(z)\, (\rho(z') - \rho(z))\, \rho^{-1}(z)\, \hat u^a(z') \\
&= \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, |\hat u(z)|^2 \rho(z)\, (\rho(z') - \rho(z)) \\
&\quad + \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z) \rho^2(z)\, (\rho(z') - \rho(z))\, (\rho^{-1}(z') - \rho^{-1}(z))\, \hat u^a(z') \\
&\quad + \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z\, \mathrm{d}z')\, \Phi'(|\hat u(z)|^2)\, \hat u^a(z) \rho^2(z) \rho^{-1}(z)\, (\rho(z') - \rho(z))\, (\hat u^a(z') - \hat u^a(z)) \\
&= (\mathbb{III}_1) + (\mathbb{III}_2) + (\mathbb{III}_3).
\end{aligned}$$

For $(\mathbb{III}_1)$ we have

$$(\mathbb{III}_1) \geq -\|\hat u\|^2 \, \|\rho\, (-\Delta)^s \rho\| \int_{\Lambda_\varepsilon} \Phi'(|\hat u(z)|^2).$$

Next,

$$\begin{aligned}(\mathbb{III}_2) &\geqslant -\|\hat{u}\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}(z)|^2)\, \rho^2(z) \Big(\int_{\Lambda_\varepsilon} |\rho(z') - \rho(z)|\, |\rho^{-1}(z') - \rho^{-1}(z)| \mu_s(z, \mathrm{d}z') \Big) \\ &\geqslant -\|\hat{u}\|^2 \, \|\rho^2 \mathfrak{D}_s(\rho^{-1}) \mathfrak{D}_s(\rho)\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).\end{aligned}$$

Finally, using the Cauchy–Schwarz inequality, we obtain

$$(\mathbb{III}_3) \geqslant -\|\hat{u}\| \, \|\rho \mathfrak{D}_s(\rho) \mathfrak{D}_s(\hat{u})\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).$$

Since $(\mathbb{X}) \geqslant (\mathbb{I}) + (\mathbb{III}_1) + (\mathbb{III}_2) + (\mathbb{III}_3)$, our claim is proved. □

4 Analysis of the flow equation

In this section, we prove Theorem 2.7, which asserts the existence of an approximate solution to the flow equation (2.28) for the effective force $(F_\sigma)_\sigma$, with well-controlled bounds encoded by (2.17). These bounds are achievable only because we can "tune" the boundary condition (2.2) using the ε- and M-dependent renormalisation term $r_{\varepsilon,M}$.

Conceptually, we are dealing with a random bilinear equation whose solution is analysed via the evolution equation for its cumulants. The flow equation for cumulants has a similar structure to the flow equation for the effective force and propagates comparable bounds backwards from the final condition at $\sigma = 1$, except in a low-dimensional (so-called relevant) subspace, where the bounds must be propagated forwards from small to large σ. This procedure requires tuning an appropriate final condition so that the solution lies on a trajectory with controlled bounds. To simplify this tuning, we decompose the flow equation to reduce the relevant subspace to one dimension.

Once bounds for the cumulants are established, a Kolmogorov-type argument allows us to deduce pathwise bounds on the effective force. The section concludes with a technical "post-processing" step, which extracts the coercive term essential for the global a priori estimates and verifies the conditions (2.17).

4.1 Random flow equation

To study approximate solutions $(F_\sigma)_{\sigma \in [1/2,1]}$ of the flow equation (2.28) we need to set up the appropriate spaces. Recall that

$$\mathcal{E} := \bigcap_{\alpha > 0} C(\Lambda_M, \zeta^\alpha), \qquad \hat{\mathcal{E}} := \partial_t \mathcal{E}.$$

As we shall argue, the flow equation can be approximatively solved in the space $\mathcal{P}(\mathcal{E})$ of polynomial functionals on $\mathcal{E}$ with values in $\hat{\mathcal{E}}$. We say that a functional F belongs to $\mathcal{P}(\mathcal{E})$ if for some $\bar{k} \in \mathbb{N}_+$ there exist kernels $(F^{(k)})_{k \in \{0,\dots,\bar{k}\}}$ of operators $\mathcal{E}^{\otimes k} \to \hat{\mathcal{E}}$ such that

$$F(\phi) = \sum_{k=0}^{\bar{k}} F^{(k)}(\phi) = \sum_{k=0}^{\bar{k}} \int_{\Lambda^k} F^{(k)}(\bullet; \mathrm{d}z_1, \dots, \mathrm{d}z_k)\, \phi(z_1) \cdots \phi(z_k) \in \hat{\mathcal{E}},$$

for all $\phi \in \mathcal{E}$. In order to construct a suitable approximate solution of (2.28) we introduce a formal parameter $\hbar$ and make the ansatz

$${}^{\hbar}F_\sigma = \sum_{\ell \geqslant 0} \hbar^\ell F_\sigma^{[\ell]}.$$

Moreover, we assume that the finial condition is of the form

$$^{\hbar}F(\phi) = -\lambda\,\phi^3 - {}^{\hbar}r_{\varepsilon,M}\phi + \xi^{(\varepsilon,M)}, \qquad {}^{\hbar}r_{\varepsilon,M} = \bar{r} + \sum_{\ell\geq 0}\hbar^{\ell}\,r_{\varepsilon,M}^{[\ell]}.$$

We are led to look for solutions of the *perturbative flow equation*

$$\partial_\sigma {}^{\hbar}F_\sigma + \hbar \mathrm{D}\,{}^{\hbar}F_\sigma(\dot{G}_\sigma\,{}^{\hbar}F_\sigma) = 0, \qquad {}^{\hbar}F_1 = {}^{\hbar}F, \tag{4.1}$$

in the space of $\mathscr{P}(\mathscr{E})[\![\hbar]\!]$ of formal power series in $\hbar$ with coefficients in $\mathscr{P}(\mathscr{E})$. This setup has the advantage that now the flow equation has a unique global solution which can be determined by induction on the degree $\hbar$. An approximate solution to (4.25) is obtained by fixing an integer $\bar{\ell}\geq 0$ and letting

$$F_\sigma := \sum_{\ell=0}^{\bar{\ell}} F_\sigma^{[\ell]}.$$

The choice of value for $\bar{\ell}$ will be discussed in Sec. 4.9 below. We observe that, thanks to (4.1), this truncation implies the existence of a maximal polynomial order $\bar{k}\in\mathbb{N}_+$ in the fields for the kernels. We decompose the force as

$$F_\mu(\phi) = \sum_{\ell=0}^{\bar{\ell}} F_\mu^{[\ell]}(\phi) = \sum_{\ell=0}^{\bar{\ell}}\sum_{k=0}^{\bar{k}} F_\mu^{[\ell](k)}(\phi),$$

where ℓ measures the perturbative order in $\hbar$ while k the polynomial degree in ϕ.

Let us now introduce a condensed notation to manipulate these kernels. Let

$$\mathfrak{A} := \{(\ell,k) | 0\leq\ell\leq\bar{\ell}, 0\leq k\leq\bar{k}\}.$$

For $\mathfrak{a}\in\mathfrak{A}$ with $\mathfrak{a}=(\ell,k)$ we let $k(\mathfrak{a}) := k$, $\ell(\mathfrak{a}) := \ell$ and write

$$[\mathfrak{a}] := -\alpha + \delta\ell(\mathfrak{a}) + \beta k(\mathfrak{a}), \tag{4.2}$$

for suitable positive parameters α,δ and β whose value will be fixed later. We say that a kernel

$$F^{\mathfrak{a}}(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k) := F^{[\ell],(k)}(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k)$$

is *relevant* if $[\mathfrak{a}]<0$, *marginal* if $[\mathfrak{a}]=0$ and *irrelevant* if $[\mathfrak{a}]>0$. We refer to z as the *output* variable, and to $z_1,\dots,z_k$ as the *input* variables of the kernel $F^{\mathfrak{a}}(z;\mathrm{d}z_1,\dots,\mathrm{d}z_k)$. To simplify the notation, we usually ignore the fact that the kernel $F^{\mathfrak{a}}(z;\mathrm{d}z_1,...,\mathrm{d}z_k)$ is generally not well-defined pointwise in the output variable z.

4.2 Norms for kernels

To introduce suitable norms for the effective force kernels

$$(F_\sigma^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A},\sigma\in[1/2,1]},$$

we first need some notations and preliminary definitions for weights and smoothing operators.

Definition 4.1. *We denote by* $\mathrm{St}(z_1,\dots,z_n)$ *the Steiner diameter of the set* $\{z_1,\dots,z_n\}$ *with respect to the parabolic distance introduced in Def. 1.4, i.e. the minimum over lengths of trees with nodes at the points* $\{z_1,\dots,z_n\}$ *and possibly other points* $\{z_1',\dots,z_m'\}$.

Remark 4.2. Our fractional parabolic distance (1.23) satisfies the triangle inequality

$$|z_1+z_2|_s \leq |z_1|_s + |z_2|_s, \qquad z_1,z_2\in\Lambda_0.$$

Moreover, for $n \in \mathbb{N}_+$, $m \in \{1, \dots n-1\}$ and $z, z_1, \dots, z_n \in \Lambda_0$, the Steiner diameter satisfies

$$\mathrm{St}(z_1, \dots, z_n) \leqslant \mathrm{St}(z_1, \dots, z_m, z) + \mathrm{St}(z, z_{m+1}, \dots, z_n),$$

which, noting that $\mathrm{St}(z_1, z_2) = |z_1 - z_2|_s$, can be seen as a generalisation of the triangle inequality for the fractional parabolic distance.

Definition 4.3. *Let $\mathfrak{b} \in (1, 2s)$ be a constant close to $2s$ and $\kappa_\mathfrak{o} \in (0, \mathfrak{b}/(1+\bar{\ell}))$ be a small constant, to be fixed later.*

a) For $m \in \mathbb{N}_+$ and $\omega \in \mathbb{R}$, the weight $w_\mu^{(1+m),\omega} \in C\big(\Lambda_0^{(1+m)}\big)$ is defined by

$$w_\mu^{(1+m),\omega}(z, z_1, \dots, z_m) := (1 + [\![\mu]\!]^{-1}\mathrm{St}(z, z_1, \dots, z_m))^\omega, \qquad z, z_1, \dots, z_m \in \Lambda_0.$$

We write $w_\mu^{(1),\omega} = 1 \in C(\Lambda_0)$.

b) For $m \in \mathbb{N}_+$, the weight $v_\mu^{(1+m)} \in C^\infty\big(\Lambda_0^{(1+m)}\big)$ is defined by

$$v_\mu^{(m+1)}(z, z_1, \dots, z_m) := v^{(m+1)}([\![\mu]\!]^{-1}.z, [\![\mu]\!]^{-1}.z_1, \dots, [\![\mu]\!]^{-1}.z_m),$$

where $v^{(m+1)} \in C^\infty\big(\Lambda_0^{(1+m)}, [0,1]\big)$ is a fixed function such that

$$v^{(m+1)}(z, z_1, \dots, z_m) = \begin{cases} 1 & \text{if } \mathrm{St}(z, z_1, \dots, z_m) \leqslant 1, \\ 0 & \text{if } \mathrm{St}(z, z_1, \dots, z_m) \geqslant 2. \end{cases}$$

We write $v_\mu^{(1)} = 1 \in C(\Lambda_0)$.

c) The weight $\mathfrak{o} \in C(\Lambda_0)$ is defined by

$$\mathfrak{o}(z) := \zeta(z)^{\kappa_\mathfrak{o}} = \langle z \rangle_s^{-\kappa_\mathfrak{o}}, \qquad z \in \Lambda_0.$$

d) For $\mathfrak{a} \in \mathfrak{A}$, we write

$$w_\mu^\mathfrak{a} := w_\mu^{(1+k(\mathfrak{a})),\mathfrak{b}}, \qquad \tilde{w}_\mu^\mathfrak{a} := w_\mu^{(1+k(\mathfrak{a})),\mathfrak{b}-\ell(\mathfrak{a})\kappa_\mathfrak{o}}, \qquad v_\mu^\mathfrak{a} := v_\mu^{(1+k(\mathfrak{a}))}, \qquad \mathfrak{o}^\mathfrak{a} := \mathfrak{o}^{1+k(\mathfrak{a})}.$$

e) For $\mu \in [0,1]$, the weight $h_\mu \in C(\Lambda_0^2)$ is defined by

$$h_\mu(z, z') := (1 + [\![\mu]\!]^{-2}|z - z_1|_s^2)^{-1}.$$

Recall that

$$\langle z \rangle_s := (1 + |z_0|^{1/s} + |\bar{z}|^2)^{1/2}.$$

The weights $w_\mu^\mathfrak{a}$ and $\tilde{w}_\mu^\mathfrak{a}$ will control the approximate localisation of the effective force kernels $F_\mu^\mathfrak{a}(z; \mathrm{d}z_1, \dots, \mathrm{d}z_k)$ near the diagonal $\{z = z_1 = \cdots = z_k\}$. The weights $\mathfrak{o}^\mathfrak{a}$ will instead be used to control the growth at spacetime infinity of the kernels in the output variable z, reflecting the corresponding growth inherited from the white noise. These three families of weights are constructed to be compatible with the estimates for the bilinear terms appearing in the flow equation, as detailed in the lemma below. The additional weights $v_\mu^{(m+1)}$ and h_μ will be used only in intermediate estimates and play an auxiliary role. Some of their useful properties are summarised in Appendix A.2. Finally, observe that

$$w_\mu^{(2),\mathfrak{b}}(z, z') = w_\mu^\mathfrak{b}(z - z') = (1 + [\![\mu]\!]^{-1}|z - z'|_s)^\mathfrak{b},$$

where w_μ^ω is the weight introduced in Def. 1.16. The exponent $\mathfrak{b} - \ell(\mathfrak{a})\kappa_\mathfrak{o}$ in the definition of the weight $\tilde{w}_\mu^\mathfrak{a}$ is designed in such a way that the bound (4.4) below holds true.

Lemma 4.4. *For all $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathfrak{A}$ be such that $k(\mathfrak{b})+k(\mathfrak{c})=k(\mathfrak{a})+1$ and $\ell(\mathfrak{b})+\ell(\mathfrak{c})=\ell(\mathfrak{a})-1$, the bounds*

$$w_\sigma^{\mathfrak{a}}(z,z_{1,},\dots,z_{k(\mathfrak{a})})\lesssim w_\sigma^{\mathfrak{b}}(z,z_{1,},\dots,z_{k(\mathfrak{b})-1},\hat{z})\,w_\sigma^{\mathfrak{b}}(\hat{z}-\tilde{z})\,w_\sigma^{\mathfrak{c}}(\tilde{z},z_{k(\mathfrak{b})},\dots,z_{k(\mathfrak{a})}),\tag{4.3}$$

$$\mathfrak{o}^{\mathfrak{a}}(z)\,\tilde{w}_\sigma^{\mathfrak{a}}(z,z_{1,},\dots,z_{k(\mathfrak{a})})\lesssim \mathfrak{o}^{\mathfrak{b}}(z)\tilde{w}_\sigma^{\mathfrak{b}}(z,z_{1,},\dots,z_{k(\mathfrak{b})-1},\hat{z})\,w_\sigma^{\mathfrak{b}}(\hat{z}-\tilde{z})\,\mathfrak{o}^{\mathfrak{c}}(\tilde{z})\,\tilde{w}_\sigma^{\mathfrak{c}}(\tilde{z},z_{k(\mathfrak{b})},\dots,z_{k(\mathfrak{a})}),\tag{4.4}$$

hold uniformly in $\mu\in(0,1)$ and $z,\hat{z},\tilde{z},z_{1,},\dots,z_{k(\mathfrak{a})}\in\Lambda_0$. For all $\mathfrak{a}\in\mathfrak{A}$ such that $\ell(\mathfrak{a})\leq\bar{\ell}$, the bounds

$$\mathfrak{o}^{\mathfrak{a}}(z)\lesssim(\mathfrak{o}^{\mathfrak{a}}(z-z'))^{-1}\,\mathfrak{o}^{\mathfrak{a}}(z')\lesssim w_\mu^{2-\mathfrak{b}}(z-z')\,\mathfrak{o}^{\mathfrak{a}}(z'),\tag{4.5}$$

$$w_\mu^{\mathfrak{a}}(z,z_1,\dots,z_{k(\mathfrak{a})})\lesssim w_\mu^{\mathfrak{b}}(z-z')w_\mu^{\mathfrak{b}}(z_1-z_1')\dots w_\mu^{\mathfrak{b}}(z_{k(\mathfrak{a})}-z'_{k(\mathfrak{a})})\,w_\mu^{\mathfrak{a}}(z',z_1',\dots,z'_{k(\mathfrak{a})}),\tag{4.6}$$

$$\mathfrak{o}^{\mathfrak{a}}(z)\,\tilde{w}_\mu^{\mathfrak{a}}(z,z_1,\dots,z_{k(\mathfrak{a})})\lesssim w_\mu^{2}(z-z')w_\mu^{\mathfrak{b}}(z_1-z_1')\dots w_\mu^{\mathfrak{b}}(z_{k(\mathfrak{a})}-z'_{k(\mathfrak{a})})\,\mathfrak{o}^{\mathfrak{a}}(z')\,\tilde{w}_\mu^{\mathfrak{a}}(z',z_1',\dots,z'_{k(\mathfrak{a})}),\tag{4.7}$$

hold uniformly in $\mu\in(0,1)$ and $z,z_{1,},\dots,z_{k(\mathfrak{a})},z',z_{1,}',\dots,z'_{k(\mathfrak{a})}\in\Lambda_0$.

Proof. To prove the claim we apply iteratively the inequalities from Remark 4.2 and use the bounds

$$1+\sum_{i\in\{1,\dots,n\}}a_i\leq\prod_{i\in\{1,\dots,n\}}(1+a_i),\qquad(1+a_i)^{\omega}\leq(1+a_i)^{\bar{\omega}},$$

valid for all $a_1,\dots,a_n\geq 0$ and $0\leq\omega\leq\bar{\omega}$. To prove (4.5) and (4.6), we use furthermore that

$$(1+\ell(\mathfrak{a}))\kappa_0\leq(1+\bar{\ell})\kappa_0\leq\mathfrak{b}\leq 2-\mathfrak{b},$$

which holds because $\mathfrak{b}>1$. □

Definition 4.5. *For $n,m\in\mathbb{N}_0$ and $\sigma,\eta\in(1/2,1)$ we set*

$$K_\sigma^{n,m}:=1^{\otimes n}\otimes(K_\sigma)^{\otimes m},\qquad L_\sigma^{n,m}:=1^{\otimes n}\otimes(L_\sigma)^{\otimes m},\qquad K_{\eta,\sigma}^{n,m}:=L_\sigma^{n,m}K_\eta^{n,m},$$

$$\tilde{K}_\sigma^{n,m}:=K_\sigma^{\otimes n}\otimes(K_\sigma^2)^{\otimes m},\qquad \tilde{L}_\sigma^{n,m}:=L_\sigma^{\otimes n}\otimes(L_\sigma^2)^{\otimes m},\qquad \tilde{K}_{\eta,\sigma}^{n,m}:=\tilde{L}_\sigma^{n,m}\tilde{K}_\eta^{n,m},$$

where $L_\sigma,K_\sigma,K_{\eta,\sigma}$ are introduced in Def. 1.13. Given $\mathfrak{a}\in\mathfrak{A}$, we write

$$\tilde{K}_\mu^{\mathfrak{a}}:=\tilde{K}_\mu^{1,k(\mathfrak{a})}=K_\mu^{\otimes(1+k(\mathfrak{a}))}K_\mu^{1,k(\mathfrak{a})}.$$

Definition 4.6. *The norm of a kernel F associated with an operator $\mathcal{E}^{\otimes k}\longrightarrow\mathcal{E}$ is defined by*

$$\|F\|:=\sup_{z\in\Lambda}\int_{\Lambda^k}|F(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k)|.\tag{4.8}$$

Given weights $\mathfrak{o}\in C(\Lambda_0)$ and $w\in C(\Lambda_0^{1+k})$, we write

$$(\mathfrak{o}\cdot F\cdot w)(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k):=\mathfrak{o}(z)\,F(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k)\,w(z,z_1,\cdots,z_k).$$

Let $\mathfrak{a}\in\mathfrak{A}$. The norm of a kernel $F^{\mathfrak{a}}$ of an operator $\mathcal{E}^{\otimes k(\mathfrak{a})}\longrightarrow\hat{\mathcal{E}}$ at scale $\sigma\in(1/2,1)$ is defined by

$$\|F^{\mathfrak{a}}\|_\sigma:=\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde{K}_\sigma^{\mathfrak{a}}F^{\mathfrak{a}}]\cdot\tilde{w}_\sigma^{\mathfrak{a}}\|,$$

where $\tilde{K}_\sigma^{\mathfrak{a}}F^{\mathfrak{a}}$ denotes the convolution of $F^{\mathfrak{a}}$, viewed as an element of $\mathcal{S}'(\Lambda^{1+k(\mathfrak{a})})$, with the kernel of $\tilde{K}_\sigma^{\mathfrak{a}}$. We further introduce a norm for the family of effective force kernels

$$F^{\mathfrak{A}}:=(F_\sigma^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A},\sigma\in(1/2,1)},$$

defined by

$$\|F^{\mathfrak{A}}\|:=\left[\sup_{\mathfrak{a}\in\mathfrak{A}}\sup_{\sigma\in(1/2,1)}[\![\sigma]\!]^{-[\mathfrak{a}]}\,\|F_\sigma^{\mathfrak{a}}\|_\sigma\right]\vee\left[\sup_{\sigma\in(1/2,1)}[\![\sigma]\!]^{d/2+s+2\kappa}\,\|F_\sigma^{[0],(0)}\|_\sigma\right],\tag{4.9}$$

where $[\mathfrak{a}]$ *is defined by* (4.2).

Remark 4.7. An inspection of our parameter choices discussed in Sec. 4.4 and 4.9, together with (4.2), shows that $[(0,0)] < -d/2 - s - 2\kappa$. Consequently, the second term on the right-hand side of (4.9) provides a stronger bound on $F_\sigma^{[0],(0)} = \xi^{(\varepsilon,M)}$ than the first term. For technical reasons, this stronger control will be required in Sec. 4.9 to establish the estimates stated in (2.17).

Remark 4.8. By Young's inequality, (4.7) and Lemma 1.17, we have

$$\begin{aligned} \|F^{\mathfrak{a}}\|_\mu = \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}_\mu^{1,k(\mathfrak{a})} F^{\mathfrak{a}}] \cdot \tilde{w}_\mu^{\mathfrak{a}}\| &= \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}_{\mu,\sigma}^{1,k(\mathfrak{a})} \tilde{K}_\sigma^{1,k(\mathfrak{a})} F^{\mathfrak{a}}] \cdot \tilde{w}_\mu^{\mathfrak{a}}\| \\ &\le \|K_{\mu,\sigma}\|_{\mathrm{TV}(w_\mu^2)} \|K_{\mu,\sigma}^2\|_{\mathrm{TV}(w_\mu^b)}^{k(\mathfrak{a})} \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}_\sigma^{1,k(\mathfrak{a})} F^{\mathfrak{a}}] \cdot \tilde{w}_\mu^{\mathfrak{a}}\| \\ &\lesssim \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}_\sigma^{1,k(\mathfrak{a})} F^{\mathfrak{a}}] \cdot \tilde{w}_\sigma^{\mathfrak{a}}\| \\ &= \|F^{\mathfrak{a}}\|_\sigma, \end{aligned} \tag{4.10}$$

uniformly in $\mu \in [1/2, 1)$ and $\sigma \in [\mu, 1)$.

Given kernels F' and F'' of operators $\mathcal{E}^{\otimes k'} \to \mathcal{E}$ and $\mathcal{E}^{\otimes k''} \to \mathcal{E}$ respectively, we denote by

$$F = \mathfrak{C}(\dot{G}_\sigma)(F' \otimes F'') \tag{4.11}$$

the kernel of the operator $\mathcal{E}^{\otimes k} \to \mathcal{E}$, with $k = k' + k'' - 1$, defined by

$$\begin{aligned} &F(z; \mathrm{d}z_1, \cdots, \mathrm{d}z_k) \\ &:= \frac{1}{k!} \sum_{\pi \in \mathcal{P}_k} \int_{\Lambda^2} F'(z; \mathrm{d}z_{\pi(1)}, \cdots, \mathrm{d}z_{\pi(k'-1)}, \mathrm{d}z')\, \dot{G}_\sigma(z' - z'')\, F''(z''; \mathrm{d}z_{\pi(k')}, \cdots, \mathrm{d}z_{\pi(k)})\, \mathrm{d}z' \mathrm{d}z'', \end{aligned} \tag{4.12}$$

where $\mathcal{P}_k$ denotes the set of permutations of $\{1, \dots, k\}$ and the integral is over the variables z' and z''. Note that $\mathfrak{C}(\dot{G}_\sigma)$ applies $\dot{G}_\sigma$ to the output variable of the kernel F'' and plugs the result to the last input variable of the kernel F'. The above definition extends naturally to kernels of operators $\mathcal{E}^{\otimes k'} \to \hat{\mathcal{E}}$ and $\mathcal{E}^{\otimes k''} \to \hat{\mathcal{E}}$.

The flow equation (4.1) for the effective force can be rewritten as the following flow equation for the kernels

$$\partial_\sigma F_\sigma^{\mathfrak{a}} = \sum_{\mathfrak{b},\mathfrak{c}} B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot{G}_\sigma, F_\sigma^{\mathfrak{b}}, F_\sigma^{\mathfrak{c}}), \qquad F_1^{\mathfrak{a}} = F^{\mathfrak{a}}, \tag{4.13}$$

where the operators $B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}$ are implicitly defined by

$$\sum_{\mathfrak{b},\mathfrak{c}} B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot{G}_\sigma, F_\sigma^{\mathfrak{b}}, F_\sigma^{\mathfrak{c}}) := \sum_{\ell'=0}^{\ell(\mathfrak{a})-1} \sum_{k'=0}^{k(\mathfrak{a})} (k'+1)\, \mathfrak{C}(\dot{G}_\sigma)\big(F_\sigma^{[\ell(\mathfrak{a})-1-\ell'],(k'+1)} \otimes F_\sigma^{[\ell'],(k(\mathfrak{a})-k')}\big) \tag{4.14}$$

and in particular $B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}} = 0$ unless

$$\begin{cases} \ell(\mathfrak{a}) = \ell(\mathfrak{b}) + \ell(\mathfrak{c}) + 1, \\ k(\mathfrak{a}) = k(\mathfrak{b}) + k(\mathfrak{c}) - 1, \\ -[\mathfrak{a}] + [\mathfrak{b}] + [\mathfrak{c}] = -\alpha + \beta - \delta. \end{cases} \tag{4.15}$$

Lemma 4.9. *For all* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{A}$*, the following bound*

$$[\![\sigma]\!]^{-[\mathfrak{a}]} \|B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot{G}_\sigma, F_\sigma^{\mathfrak{b}}, F_\sigma^{\mathfrak{c}})\|_\sigma \lesssim [\![\sigma]\!]^{-[\mathfrak{b}]-[\mathfrak{c}]-1} \|F_\sigma^{\mathfrak{b}}\|_\sigma \|F_\sigma^{\mathfrak{c}}\|_\sigma$$

holds uniformly in $\sigma \in [1/2, 1)$*, provided*

$$2s - \alpha + \beta - \delta \ge 0. \tag{4.16}$$

Proof. By Def. 4.6 and 4.5, together with (4.14), (4.12) and the identity $L_\sigma K_\sigma = 1$, we obtain

$$\begin{aligned}\|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)\|_\sigma &= \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K^{\mathfrak{a}}_\sigma B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)]\cdot \tilde w^{\mathfrak{a}}_\sigma\|_\sigma\\ &= \|\mathfrak{o}^{\mathfrak{a}}\cdot[B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(L^3_\sigma\dot G_\sigma, \tilde K^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma, \tilde K^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma)]\cdot \tilde w^{\mathfrak{a}}_\sigma\|_\sigma\\ &\leqslant \|\mathfrak{o}^{\mathfrak{a}}\cdot[B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(|L^3_\sigma\dot G_\sigma|, |\tilde K^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma|, |\tilde K^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma|)]\cdot \tilde w^{\mathfrak{a}}_\sigma\|_\sigma.\end{aligned}$$

Hence, by (4.4), (4.14), (4.12), Def. 4.6 and Lemma 1.17, we arrive at

$$\begin{aligned}\|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)\|_\sigma &\lesssim \|[B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(|L^3_\sigma\dot G_\sigma|\cdot w^{\flat}_\sigma, \mathfrak{o}^{\mathfrak{b}}\cdot|\tilde K^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma|\cdot\tilde w^{\mathfrak{b}}_\sigma, \mathfrak{o}^{\mathfrak{c}}\cdot|\tilde K^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma|\cdot\tilde w^{\mathfrak{c}}_\sigma)]\|_\sigma\\ &\lesssim \|L^3_\sigma\dot G_\sigma\|_{\mathrm{TV}(w^{\flat}_\sigma)}\|\mathfrak{o}^{\mathfrak{b}}\cdot|\tilde K^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma|\cdot\tilde w^{\mathfrak{b}}_\sigma\|\,\|\mathfrak{o}^{\mathfrak{c}}\cdot|\tilde K^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma|\cdot\tilde w^{\mathfrak{c}}_\sigma\|\\ &\lesssim \|L^3_\sigma\dot G_\sigma\|_{\mathrm{TV}(w^{\flat}_\sigma)}\|F^{\mathfrak{b}}_\sigma\|_\sigma\,\|F^{\mathfrak{c}}_\sigma\|_\sigma\\ &\lesssim [\![\sigma]\!]^{2s-1}\|F^{\mathfrak{b}}_\sigma\|_\sigma\,\|F^{\mathfrak{c}}_\sigma\|_\sigma.\end{aligned}$$

Moreover, combining (4.15) and (4.16) yields

$$-[\mathfrak{a}]+2s-1 = -[\mathfrak{b}]-[\mathfrak{c}]-\alpha+\beta-\delta+2s-1 \geqslant -[\mathfrak{b}]-[\mathfrak{c}]-1.$$

This finishes the proof. □

4.3 Norms for cumulants

In order to prove probabilistic bounds for the moments of the norm $\|F^{\mathfrak{A}}\|$, we start by establishing bounds for the cumulants of the family $(F^{\mathfrak{a}}_\sigma)_{\mathfrak{a}\in\mathfrak{A},\sigma\in[1/2,1]}$ of the effective force kernels. To this end, let us first introduce a useful notation. For

$$\boldsymbol{a}\in\boldsymbol{A} := \{(\mathfrak{a}_1,\dots,\mathfrak{a}_n)\,|\,\mathfrak{a}_k\in\mathfrak{A}, L(\boldsymbol{a})\leqslant 2\bar\ell\},$$

we write

$$n(\boldsymbol{a}) := n,\qquad L(\boldsymbol{a}) := \ell(\mathfrak{a}_1)+\cdots+\ell(\mathfrak{a}_n),\qquad K(\boldsymbol{a}) := k(\mathfrak{a}_1)+\cdots+k(\mathfrak{a}_n).$$

We denote by $\mathfrak{K}_n(X_1,\dots,X_n)$ the n-th order joint cumulant of the random variable $X_1,\dots,X_n$. For $\boldsymbol{a}=(\mathfrak{a}_1,\dots,\mathfrak{a}_n)\in\boldsymbol{A}$ and $\sigma\in[1/2,1]$, we use the following shorthand notation

$$\mathcal{F}^{\boldsymbol{a}}_\sigma(z_1,\mathrm{d}Z_1,\dots,z_n,\mathrm{d}Z_n) := \mathfrak{K}_n(F^{\mathfrak{a}_1}_\sigma(z_1,\mathrm{d}Z_1),\cdots,F^{\mathfrak{a}_n}_\sigma(z_n,\mathrm{d}Z_n)),$$

where

$$Z_i=(z_{i,1},\dots,z_{i,k(\mathfrak{a}_i)})\in\Lambda^{k(\mathfrak{a}_i)},\qquad i\in\{1,\dots n\},\quad n=n(\boldsymbol{a}),\tag{4.17}$$

are the input variables of the i-th effective force kernel involved in the joint cumulant. Since $F^{\mathfrak{a}_i}_\sigma$ is a kernel of an operator $\mathcal{E}^{\otimes k(\mathfrak{a}_i)}\longrightarrow\hat{\mathcal{E}}$, it is natural to view $\mathcal{F}^{\boldsymbol{a}}_\sigma$ as a kernel of an operator

$$\mathcal{E}^{\otimes K(\boldsymbol{a})}\longrightarrow\hat{\mathcal{E}}^{\otimes n(\boldsymbol{a})}.\tag{4.18}$$

We define the global homogeneity of the kernel $\mathcal{F}^{\boldsymbol{a}}_\sigma$ as

$$[\boldsymbol{a}] := -\varrho+n(\boldsymbol{a})(\theta+\alpha)+[\mathfrak{a}_1]+\cdots+[\mathfrak{a}_n],$$

for suitable parameters ϱ and θ whose values will be fixed in Sec. 4.4. By (4.2), we have

$$[\boldsymbol{a}] = -\varrho+\theta n(\boldsymbol{a})+\delta L(\boldsymbol{a})+\beta K(\boldsymbol{a}).\tag{4.19}$$

We say that a cumulant $\mathcal{F}^{\boldsymbol{a}}$ is *relevant* if $[\boldsymbol{a}]<0$, *marginal* if $[\boldsymbol{a}]=0$ and *irrelevant* if $[\boldsymbol{a}]>0$.

Definition 4.10. *For $\boldsymbol{a}\in\boldsymbol{A}$ and $\sigma,\mu\in(0,1)$ we define*

$$w^{\boldsymbol{a}}_\mu := \bigotimes_{i=1}^{n(\boldsymbol{a})} w^{\mathfrak{a}_i}_\mu,\qquad K^{\boldsymbol{a}}_\mu := \bigotimes_{i=1}^{n(\boldsymbol{a})} K^{1,k(\mathfrak{a}_i)}_\mu,\qquad L^{\boldsymbol{a}}_\sigma := \bigotimes_{i=1}^{n(\boldsymbol{a})} L^{1,k(\mathfrak{a}_i)}_\sigma,\qquad K^{\boldsymbol{a}}_{\eta,\sigma} := L^{\boldsymbol{a}}_\sigma K^{\boldsymbol{a}}_\eta,$$

where $w^{\mathfrak{a}_i}_\mu$ is introduced in Def. 4.3, and $K^{1,k}_\mu$ and $L^{1,k}_\sigma$ are introduced in Def. 4.5

The space of cumulants is endowed with the norm $|||\bullet|||$ defined by taking L^∞ norm on the first output variable and the total variation norm in all other variables, with the output variables restricted to the first period in space. The restriction of the integration of the output variables to the first period is natural since cumulants are kernels of operators acting between spaces of periodic functions, cf. (4.18).

Definition 4.11. *Let $\boldsymbol{a}\in\boldsymbol{A}$, $n=n(\boldsymbol{a})$ and $k=K(\boldsymbol{a})$. The norm of a kernel $\mathcal{F}^{\boldsymbol{a}}$ associated with an operator $\mathcal{E}^{\otimes k}\longrightarrow\mathcal{E}^{\otimes n}$ is defined by*

$$|||\mathcal{F}^{\boldsymbol{a}}|||:=\sup_{z_1\in\Lambda}\int_{\Lambda_M^{n-1}}\Big(\int_{\Lambda^k}|\mathcal{F}^{\boldsymbol{a}}(z_1,\mathrm{d}Z_1,\dots,z_n,\mathrm{d}Z_n)|\Big)\,\mathrm{d}z_2\cdots\mathrm{d}z_n, \tag{4.20}$$

with the notation as in (4.17). Given a weight $w\in C(\Lambda_0^{n+k})$, we write

$$(F\cdot w)(z_1,\mathrm{d}Z_1,\dots,z_n,\mathrm{d}Z_n):=\mathcal{F}(z_1,\mathrm{d}Z_1,\dots,z_n,\mathrm{d}Z_n)\,w(z_1,Z_1,\cdots,z_n,Z_n).$$

The norm of $\mathcal{F}^{\boldsymbol{a}}$ at scale $\sigma\in(1/2,1)$ is defined by

$$|||\mathcal{F}^{\boldsymbol{a}}|||_\sigma=|||[K_\sigma^{\boldsymbol{a}}\mathcal{F}^{\boldsymbol{a}}]\cdot w_\sigma^{\boldsymbol{a}}|||, \tag{4.21}$$

where $\tilde K_\sigma^{\boldsymbol{a}}\mathcal{F}^{\boldsymbol{a}}$ denotes the convolution of $\mathcal{F}^{\boldsymbol{a}}$, viewed as an element of $\mathcal{S}'(\Lambda^{n(\boldsymbol{a})+K(\boldsymbol{a})})$, with the kernel of $\tilde K_\sigma^{\boldsymbol{a}}$. We further introduce a norm for the family of cumulants of effective force kernels

$$\mathcal{F}^{\boldsymbol{A}}:=(\mathcal{F}_\sigma^{\boldsymbol{a}})_{\boldsymbol{a}\in\boldsymbol{A},\sigma\in(1/2,1)}$$

defined by

$$|||\mathcal{F}^{\boldsymbol{A}}|||:=\sup_{\boldsymbol{a}\in\boldsymbol{A}}\Big[\sup_{\sigma\in(1/2,1)}[\![\sigma]\!]^{-[\boldsymbol{a}]}\,|||\mathcal{F}_\sigma^{\boldsymbol{a}}|||_\sigma\Big]^{1/n(\boldsymbol{a})}. \tag{4.22}$$

Remark 4.12. Analogously to Remark 4.8, by $K_\mu^{\boldsymbol{a}}=K_{\mu,\sigma}^{\boldsymbol{a}}K_\sigma^{\boldsymbol{a}}$, Young's inequality, (4.6) and Lemma 1.17, we obtain

$$\begin{aligned}|||\mathcal{F}^{\boldsymbol{a}}|||_\mu=|||[K_\mu^{\boldsymbol{a}}\mathcal{F}^{\boldsymbol{a}}]\cdot w_\mu^{\boldsymbol{a}}||| &= |||[K_{\mu,\sigma}^{\boldsymbol{a}}K_\sigma^{\boldsymbol{a}}\mathcal{F}^{\boldsymbol{a}}]\cdot w_\mu^{\boldsymbol{a}}|||\\ &\lesssim \|K_{\mu,\sigma}\|_{\mathrm{TV}(w_\mu^{\flat})}\|K_{\mu,\sigma}^2\|_{\mathrm{TV}(w_\mu^{\flat})}^{K(\boldsymbol{a})}|||[K_\sigma^{\boldsymbol{a}}\mathcal{F}^{\boldsymbol{a}}]\cdot w_\mu^{\boldsymbol{a}}|||\\ &\lesssim |||[K_\sigma^{\boldsymbol{a}}\mathcal{F}^{\boldsymbol{a}}]\cdot w_\sigma^{\boldsymbol{a}}|||\\ &= |||\mathcal{F}^{\boldsymbol{a}}|||_\sigma,\end{aligned}$$

uniformly in $\mu\in[1/2,1)$ and $\sigma\in[\mu,1)$.

In Sec. 4.8, we will pass from estimates on the norm $|||\mathcal{F}^{\boldsymbol{A}}|||$ for the family of cumulants to estimates on the norm $\|F^{\mathfrak{A}}\|$ for the corresponding family of kernels, using a Kolmogorov-type argument. Following [Duc25a, Duc22], we introduce a flow equation for cumulants, which will allow us to control the norm $|||\mathcal{F}^{\boldsymbol{A}}|||$.

Lemma 4.13. *The cumulants satisfy the following flow equation:*

$$\partial_\sigma\mathcal{F}_\sigma^{\boldsymbol{a}}=\sum_{\boldsymbol{b}}\mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot G_\sigma,\mathcal{F}_\sigma^{\boldsymbol{b}})+\sum_{\boldsymbol{b},\boldsymbol{c}}\mathcal{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot G_\sigma,\mathcal{F}_\sigma^{\boldsymbol{b}},\mathcal{F}_\sigma^{\boldsymbol{c}}), \tag{4.23}$$

where the multilinear operators $\mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a}}$ and $\mathcal{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}$ are defined in Appendix B.1. We have $\mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a}}=0$ unless

$$\begin{cases} n(\boldsymbol{a})=n(\boldsymbol{b})-1,\\ L(\boldsymbol{a})=L(\boldsymbol{b})+1,\\ K(\boldsymbol{a})=K(\boldsymbol{b})-1,\\ [\boldsymbol{a}]=[\boldsymbol{b}]-\theta+\delta-\beta,\end{cases} \tag{4.24}$$

and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}} = 0$ *unless*

$$\begin{cases} n(\boldsymbol{a}) = n(\boldsymbol{b}) + n(\boldsymbol{c}) - 1, \\ L(\boldsymbol{a}) = L(\boldsymbol{b}) + L(\boldsymbol{c}) + 1, \\ K(\boldsymbol{a}) = K(\boldsymbol{b}) + K(\boldsymbol{c}) - 1, \\ [\boldsymbol{a}] = [\boldsymbol{b}] + [\boldsymbol{c}] + \varrho - \theta + \delta - \beta. \end{cases} \tag{4.25}$$

Moreover, the operators $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ *and* $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$ *satisfy the following bounds:*

$$\begin{aligned} [\![\sigma]\!]^{-[\boldsymbol{a}]} |\!|\!| \mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) |\!|\!|_\sigma &\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} |\!|\!| \mathcal{F}^{\boldsymbol{a}} |\!|\!|_\sigma, \\ [\![\sigma]\!]^{-[\boldsymbol{a}]} |\!|\!| \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) |\!|\!|_\sigma &\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1} |\!|\!| \mathcal{F}^{\boldsymbol{b}}_\sigma |\!|\!|_\sigma |\!|\!| \mathcal{F}^{\boldsymbol{c}}_\sigma |\!|\!|_\sigma, \end{aligned}$$

provided the following compatibility conditions hold:

$$\theta + \beta - \delta - d \geq 0, \qquad -\varrho + \theta + \beta - \delta + 2s \geq 0. \tag{4.26}$$

Proof. The derivation of the flow equation is a direct consequence of the definition of cumulants, see [Duc25a, Duc22]. The explicit form of the operators is not essential for the subsequent discussion and is provided in Appendix B.1, where the stated bounds are also proved (see Lemma B.1 and Lemma B.2). □

This general structure of the flow equation (4.23) allows us to propagate estimates for the cumulants of the form

$$\sup_{\sigma \in (1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]} |\!|\!| \mathcal{F}^{\boldsymbol{a}}_\sigma |\!|\!|_\sigma < \infty.$$

However, depending on the sign of $[\boldsymbol{a}]$, we shall handle differently the cumulants: in particular, for $[\boldsymbol{a}] > 0$, namely for irrelevant cumulants, the flow equation can be solved *backward* starting from the final condition at $\sigma = 1$. On the other hand, this approach does not work for cumulants for which $[\boldsymbol{a}] < 0$ as in this case the flow equation cannot be integrated close to $\sigma = 1$. As we shall see in Sec. 4.6, we will solve the flow equation for this class of cumulants, called *relevant cumulants*, by integrating it *forward*.

Remark 4.14. Before proceeding with the analysis of the flow equations for cumulants, we record a few remarks about symmetries. First, we observe that the SPDE under consideration, namely (1.4), is invariant under the transformation

$$\phi \mapsto -\phi, \qquad \xi \mapsto -\xi,$$

which also preserves the law of the noise $\xi = \xi^{(\varepsilon,M)}$. By the discussion in Sec. 4.4, this invariance implies that $\mathcal{F}^{\boldsymbol{a}}_\sigma = 0$ for all $\sigma \in [0,1]$ and $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = 0$ and

$$n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

To extend this property to higher levels $L(\boldsymbol{a}) > 0$, we exploit the fact that it is preserved by the flow equation (4.23) thanks to the compatibility conditions (4.24) and (4.25). Indeed, fix an arbitrary $\ell \in \mathbb{N}_+$ and assume inductively that

$$\mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \forall \sigma \in [0,1], \boldsymbol{a} \in \boldsymbol{A} \quad \text{such that} \quad L(\boldsymbol{a}) < \ell \quad \text{and} \quad n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

Then from the flow equation (4.23) and the conditions (4.24) and (4.25), it follows that

$$\partial_\sigma \mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \forall \sigma \in (0,1], \boldsymbol{a} \in \boldsymbol{A} \quad \text{such that} \quad L(\boldsymbol{a}) = \ell \quad \text{and} \quad n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

Since by construction $\mathcal{F}^{\boldsymbol{a}}_{\sigma=1} = 0$ for all $\boldsymbol{a} \in \boldsymbol{A}$ such that $n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1$, we conclude that

$$\mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \forall \sigma \in [0,1], \boldsymbol{a} \in \boldsymbol{A} \quad \text{such that} \quad n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

A further symmetry is given by spatial reflection, that is, the transformation

$$\phi(t,x) \mapsto \phi(t,-x), \qquad \xi(t,x) \mapsto \xi(t,-x),$$

which also leaves the law of ξ invariant. By an argument analogous to the one above, we infer that for any $\boldsymbol{a} \in \boldsymbol{A}$, the cumulant $\mathcal{F}_\mu^{\boldsymbol{a}}$ is symmetric under spatial reflections.

4.4 Bounds on parameters

We shall now fix the parameters

$$\beta, \theta, \varrho, \alpha, \delta, \flat$$

introduced in the analysis of the kernels. We have to choose these parameters so that (4.26) is satisfied. Another constraint comes from the requirement that $|||\mathcal{F}_\sigma^{\boldsymbol{a}}|||_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{a}]}$ for $\boldsymbol{a} \in \boldsymbol{A}$ with $L(\boldsymbol{a}) = 0$. Note that for $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = 0$, we have $\partial_\sigma \mathcal{F}_\sigma^{\boldsymbol{a}} = 0$ and $\mathcal{F}_\sigma^{\boldsymbol{a}} = \mathcal{F}_1^{\boldsymbol{a}}$. Using the equality

$$F_\sigma^{[0]}(\varphi) = F_1^{[0]}(\varphi) = -\lambda\varphi^3 + \bar{r}\varphi + \xi^{(\varepsilon,M)}$$

and the fact that the noise $\xi^{(\varepsilon,M)}$ is Gaussian, one shows that for $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = 0$ the cumulant $\mathcal{F}_\sigma^{\boldsymbol{a}}$ is nonzero only if:

a) it is the covariance of the noise, that is $n(\boldsymbol{a}) = 2$, $k(\boldsymbol{a}) = 0$ and

$$\mathcal{F}_\sigma^{\boldsymbol{a}}(z, z') = \mathfrak{K}_2(F^{[0],(0)}(z), F^{[0],(0)}(z')) = \mathbb{E}[\xi^{(\varepsilon,M)}(z)\,\xi^{(\varepsilon,M)}(z')] = \delta^M(z - z'),$$

where δ^M is the periodisation in space of the Dirac delta with period M, or

b) it is the expected value of the (deterministic) kernel $F_1^{[0],(3)}$, that is $n(\boldsymbol{a}) = 1$, $k(\boldsymbol{a}) = 3$ and

$$\mathcal{F}_\sigma^{\boldsymbol{a}}(z, z_1, z_2, z_3) = -\lambda\,\delta(z - z_1)\delta(z - z_2)\delta(z - z_3),$$

or

c) it is the expected value of the (deterministic) kernel $F_1^{[0],(1)}$, that is $n(\boldsymbol{a}) = 1$, $k(\boldsymbol{a}) = 1$ and

$$\mathcal{F}_\sigma^{\boldsymbol{a}}(z, z_1) = \bar{r}\,\delta(z - z_1).$$

In the case a) using the fact that $w_\sigma^{\boldsymbol{a}} = 1$, we conclude that

$$\|\mathcal{F}_1^{\boldsymbol{a}}\|_\sigma = \int_{\Lambda_M} \delta^M(\mathrm{d}z) = 1. \tag{4.27}$$

As a consequence, we have to require that

$$[\boldsymbol{a}] = -\varrho + 2\theta \leqslant 0. \tag{4.28}$$

In the case b) we observe that

$$(K_\sigma^{\boldsymbol{a}}\mathcal{F}_\sigma^{\boldsymbol{a}})(z; z_1, z_2, z_3) = -\lambda \prod_{i=1}^{3} K_\sigma(z - z_i),$$

and thus $|||\mathcal{F}_\sigma^{\boldsymbol{a}}|||_\sigma \lesssim 1$ uniformly in $\sigma > 0$ by Lemma 1.17. Consequently, we have to require that

$$[\boldsymbol{a}] = -\varrho + \theta + 3\beta \leqslant 0. \tag{4.29}$$

In the case c) we have

$$(K_\sigma^{\boldsymbol{a}}\mathcal{F}_\sigma^{\boldsymbol{a}})(z; z_1) = \bar{r}K_\sigma(z - z_1),$$

and thus $|||\mathcal{F}_\sigma^{\boldsymbol{a}}|||_\sigma \lesssim 1$ uniformly in $\sigma > 0$. Consequently, we have to require that

$$[\boldsymbol{a}] = -\varrho + \theta + \beta \leqslant 0.$$

Note that since $\beta > 0$ the last condition is implied by the bound (4.29).

Let us collect in a table the various conditions which influence the choice of parameters. Some of them we already encountered, while other will appear later on (the constraints $[K]$, $[\bar{F}]$ below). We prefer to collect here all our constraints and fix the values of the parameters to proceed later in a straightforward way to the discussion of various conditions.

$$\begin{array}{lll} [B] & \text{Flow kernels (4.16)} & 2s-\alpha+\beta-\delta\geq 0 \\ [\mathbb{A}] & \text{Flow cumulants } \mathcal{A} \text{ (4.26)} & \theta+\beta-\delta-d\geq 0 \\ [\mathbb{B}] & \text{Flow cumulants } \mathcal{B} \text{ (4.26)} & 2s-\varrho+\theta+\beta-\delta\geq 0 \\ [\Xi] & \text{Initial condition } F^{[0],(0)} \text{ (4.28)} & -\varrho+2\theta\leq 0 \\ [\Phi^3] & \text{Initial condition } F^{[0],(3)} \text{ (4.29)} & -\varrho+\theta+3\beta\leq 0 \\ [K] & \text{Kolmogorov (4.56)} & \alpha-\frac{\varrho}{2}+\theta-\frac{d+2s}{2}-\kappa>0 \\ [\bar{F}] & \text{Kolmogorov } \bar{F} \text{ (4.59)} & \alpha-\varrho+\theta-\kappa\geq 0 \end{array} \tag{4.30}$$

The parameter $\kappa>0$ quantifies the loss of regularity in the Kolmogorov-type argument used to estimate the pathwise behaviour of the random kernels and will be chosen sufficiently small.

From the constraints $[\Phi^3]$ and $[\mathbb{B}]$ we have

$$\beta\leq s-\delta/2,$$

while from $[\Xi]$, $[\mathbb{B}]$ and this last inequality we deduce that

$$\theta\leq\beta-\delta+2s\leq 3s-3\delta/2.$$

Using now $[\mathbb{B}]$ again we have

$$\varrho\leq\theta+\beta-\delta+2s\leq 6s-3\delta.$$

And $[\mathbb{A}]$ now gives

$$0\leq\theta+\beta-\delta-d\leq 4s-d-3\delta=3(\delta_\star-\delta), \tag{4.31}$$

where the strict positivity of

$$\delta_\star:=\frac{4s-d}{3}>0,$$

defines the subcritical regime of this model. We now fix and choose the other parameters to saturate most of the inequalities we just found giving:,

$$\boxed{\beta=s-\delta/2-\kappa/2,\quad \theta=3\beta,\quad \varrho=2\theta,\quad \alpha=3\beta+\kappa,\quad \delta=\delta_\star/2.} \tag{4.32}$$

By substituting these values into the inequalities (4.30) above, we check the following.

$$\begin{array}{llll} [B] & 0\leq 2s-\alpha+\beta-\delta & =2s-2\beta-\kappa-\delta & =0 \\ [\mathbb{A}] & 0\leq\theta+\beta-\delta-d & =4\beta-\delta-d & =\frac{3}{2}\delta_\star-2\kappa \\ [\mathbb{B}] & 0\leq-\varrho+\theta+\beta-\delta+2s & =-2\beta-\delta+2s & =\kappa \\ [\Xi] & 0\geq-\varrho+2\theta & =0 & =0 \\ [\Phi^3] & 0\geq-\varrho+\theta+3\beta & =0 & =0 \\ [K] & 0<\alpha-\frac{\varrho}{2}+\theta-\frac{d+2s}{2}-\kappa & =3\beta-\frac{d+2s}{2} & =\frac{3}{4}\delta_\star-\frac{3}{2}\kappa \\ [\bar{F}] & 0\leq\alpha-\varrho+\theta-\kappa & =0 & =0 \end{array} \tag{4.33}$$

We used that

$$\frac{d+2s}{2}=\frac{6s-3\delta_\star}{2}=3\beta-\frac{3}{2}(\delta_\star-\delta)+3\kappa. \tag{4.34}$$

All the inequalities are satisfied provided

$$\kappa \in (0, \delta_\star/2). \tag{4.35}$$

In addition, in Lemma 4.17 below we use the fact that the parameter of the weight $\mathfrak{o}$ introduced in Def. 4.3 satisfies

$$\kappa_{\mathfrak{o}} \leq \delta \wedge \mathfrak{b}/(1+\bar{\ell}). \tag{4.36}$$

Finally, we explain how the value of the parameter $\mathfrak{b}$, introduced in Def. 4.3, is fixed. The role of this parameter is to compensate for the loss of spatial weight arising in the localisation procedure for the relevant cumulants discussed below, where the Taylor expansion allows to gain in homogeneity and establish that only a local renormalisation is needed. In Lemma 4.15 and Lemma 4.17, we shall need $\mathfrak{b} > 2s - \delta$. On the other hand, as argued in Remark 1.18, we must also impose $\mathfrak{b} < 2s$. Thus, we fix $\mathfrak{b}$ such that

$$2s - \delta < \mathfrak{b} < 2s. \tag{4.37}$$

In Sec. 4.9, we shall introduce additional constraints on the parameters κ and $\kappa_{\mathfrak{o}}$, and we will fix all remaining parameters

$$\gamma, \vartheta, \bar{\kappa}, \bar{\ell}, \bar{k}, a, \nu, \kappa_{\mathfrak{o}}, \kappa,$$

which play a role in our analysis. We postpone this discussion because, apart from κ and $\kappa_{\mathfrak{o}}$, which must satisfy conditions (4.35) and (4.36), these parameters are only relevant for the results established in Sec. 4.9. We emphasise once again that all these parameters are to be regarded as fixed once and for all, and their choice depends solely on the exponent $s \in (3/4, 1)$ of the fractional Laplacian.

4.5 Classification of cumulants

Given the bounds on the parameters from the previous section, we can now examine the class of cumulants that are relevant or marginal, *i.e.*, those satisfying $[\boldsymbol{a}] \leq 0$. Recalling that

$$\varrho = 2\theta, \quad \text{and} \quad \theta = \varrho - 3\beta,$$

observe that the condition $[\boldsymbol{a}] \leq 0$ can be written as

$$[\boldsymbol{a}] = \theta(n(\boldsymbol{a}) - 2) + \beta K(\boldsymbol{a}) + \delta L(\boldsymbol{a}) = \beta(3n(\boldsymbol{a}) - 6 + K(\boldsymbol{a})) + \delta L(\boldsymbol{a}) \leq 0.$$

Then

a) if $n(\boldsymbol{a}) > 2$, there are no relevant/marginal cumulants;

b) if $n(\boldsymbol{a}) = 2$, the only relevant/marginal cumulant is the one with $L(\boldsymbol{a}) = K(\boldsymbol{a}) = 0$, that is the covariance of the noise: $\mathcal{F}^{\boldsymbol{a}} = \mathfrak{K}_2(F_\sigma^{[0](0)}, F_\sigma^{[0](0)})$.

c) if $n(\boldsymbol{a}) = 1$, the only relevant/marginal cumulants are (at most) those with $K(\boldsymbol{a}) \leq 3$.

Summarising, the only relevant/marginal cumulants are

$$\mathfrak{K}_2(F_\sigma^{[0](0)}, F_\sigma^{[0](0)}), \qquad \mathfrak{K}_1(F_\sigma^{[\ell](k)}) = \mathbb{E} F_\sigma^{[\ell](k)}, \qquad k = 0, 1, 2, 3.$$

We can further restrict the set of cumulants to be analysed. Indeed, the flow equation for the cumulants with $L(\boldsymbol{a}) = 0$ is trivial and there is no evolution, so they coincide with their initial values. This applies to $\mathfrak{K}_2(F_\sigma^{[0](0)}, F_\sigma^{[0](0)})$ and $\mathfrak{K}_1(F_\sigma^{[0](3)})$. Moreover $\mathfrak{K}_1(F_\sigma^{[\ell](3)})$ for $\ell \geq 1$ is irrelevant. As for the others, by Remark 4.14, we know that the cumulants $\mathfrak{K}_1(F_\sigma^{[\ell](0)})$ and $\mathfrak{K}_1(F_\sigma^{[\ell](2)})$ vanish due to symmetry arguments. Thus the only remaining cumulants that we have to consider in detail are

$$\mathfrak{K}_1(F_\sigma^{[\ell](1)}) = \mathbb{E} F_\sigma^{[\ell](1)}, \qquad \ell \in \{1, \dots, \hat{\ell}\}, \tag{4.38}$$

where by definition $\hat{\ell} \in \mathbb{N}_+$ is the smallest natural number such that $\delta(\hat{\ell}+1) - 2\beta > 0$. Since the paramter $\hat{\ell}$ appears only in this and following section, it is not included in the list (1.22). Note that there are no marginal cumulants $\mathcal{F}^{\boldsymbol{a}}$ with $L(\boldsymbol{a}) > 0$.

4.6 Inductive procedure

The aim of this section is to derive bounds for the cumulants $\mathcal{F}^{\boldsymbol{a}}_\sigma$ using an induction on $L(\boldsymbol{a})$, based on the flow equation (4.23). More precisely, Lemma 4.13 shows that the flow equation defines a triangular system with respect to $L(\boldsymbol{a})$. For irrelevant cumulants, we have $\mathcal{F}^{\boldsymbol{a}}_{\sigma=1} = 0$, and the desired bounds follow by directly integrating the flow equation from $\sigma = 1$. For relevant cumulants $\mathcal{F}^{\boldsymbol{a}}_\sigma$, which for $L(\boldsymbol{a}) > 0$ coincide with (4.38), the bound for $\partial_\sigma \mathcal{F}^{\boldsymbol{a}}_\sigma$ is not integrable at $\sigma = 1$. To overcome this, we decompose relevant cumulants into a local relevant part and a nonlocal irrelevant part, using the procedure detailed in Appendix B.2. The irrelevant part can again be integrated from $\sigma = 1$ with zero boundary condition. For the relevant part, we instead impose the boundary condition at $\sigma = 1/2$, which acts as the renormalisation condition fixing the mass counterterm $r_{(\varepsilon,M)}$. This allows the integration of the flow equation from $\sigma = 1/2$, thereby avoiding the non-integrable singularity at $\sigma = 1$.

Lemma 4.15. *For any $\bar{r} \in \mathbb{R}$ there exist (nonunique) choice of constants $\big(r^{[\ell]}_{\varepsilon,M}\big)_{\ell=1,\dots,\hat{\ell}}$ such that the solution*

$$F^{\mathfrak{A}} := (F^{\mathfrak{a}}_\sigma)_{\mathfrak{a}\in\mathfrak{A},\sigma\in(1/2,1)},$$

of the approximate flow equation with the final condition

$$\begin{aligned} F^{[0]}_1(\phi) &= -\lambda\phi^3 + \bar{r}\,\phi + \xi^{(\varepsilon,M)}, \\ F^{[\ell]}_1(\phi) &= r^{[\ell]}_{\varepsilon,M}\phi, && \ell \in \{1,\dots,\hat{\ell}\}, \\ F^{[\ell]}_1(\phi) &= 0, && \ell \in \{\hat{\ell}+1,\dots,\bar{\ell}\}, \end{aligned} \tag{4.39}$$

satisfies

$$\|\!|\mathcal{F}^{\boldsymbol{A}}|\!\| = \sup_{\boldsymbol{a}\in\boldsymbol{A}} \left[\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\!|\mathcal{F}^{\boldsymbol{a}}_\sigma|\!\|_\sigma \right]^{1/n(\boldsymbol{a})} < \infty.$$

Remark 4.16. Note that

$$F(\phi) = \sum_{\ell=1}^{\bar{\ell}} F^{[\ell]}_1(\phi), \qquad r_{\varepsilon,M} = \sum_{\ell=1}^{\hat{\ell}} r^{[\ell]}_{\varepsilon,M},$$

where $F = F^{(\varepsilon,M)}$ is the force (2.1) and $r_{\varepsilon,M}$ is the mass renormalisation, which appears in the expression (1.2) for the action. The parameter $\hat{\ell} \in \mathbb{N}_+$ was fixed at the end of Sec. 4.5, whereas the parameter $\bar{\ell} \geq \hat{\ell}$ will be fixed in Sec. 4.9.

Proof. For $\ell \in \{0,\dots,2\bar{\ell}\}$, define

$$M_\ell := 1 + \sup_{\boldsymbol{a}|L(\boldsymbol{a})\leq\ell} \left[\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\!|\mathcal{F}^{\boldsymbol{a}}_\sigma|\!\|_\sigma \right]^{1/n(\boldsymbol{a})}.$$

We will prove by induction on ℓ that

$$M_\ell < \infty, \qquad \text{and} \qquad \mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \text{for all } \boldsymbol{a} \text{ such that } L(\boldsymbol{a}) \leq \ell \text{ and } n(\boldsymbol{a}) > 2^{\ell+1}, \tag{4.40}$$

for any $\ell \in \{0,\dots,2\bar{\ell}\}$.

Let us remark that the second part of the above statement implies that there are only finitely many nonzero cumulants $\mathcal{F}^{\boldsymbol{a}}_\sigma$ such that $L(\boldsymbol{a}) \leq \ell$. We first note that the case $\ell = 0$ was already discussed in Sec. 4.4, where it was proved that $M_0 < \infty$.

Let us now consider the induction step. Assume that the conditions expressed by (4.40) hold true at the order $\ell-1$, for a fixed $\ell\in\mathbb{N}_+$. We shall prove that then the same is true at the order ℓ. The proof is based on the flow equation (4.23). We first note that, by Lemma 4.13, we have

$$\begin{aligned}\|\!|\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma)|\!\|_\sigma &\lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1}M_{\ell-1}^{n(b)},\\ \|\!|\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)|\!\|_\sigma &\lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1}M_{\ell-1}^{n(\boldsymbol{b})}M_{\ell-1}^{n(\boldsymbol{c})},\end{aligned}$$

for all $\boldsymbol{a}\in\boldsymbol{A}$ such that $L(\boldsymbol{a})=\ell$. Recall that $n(\boldsymbol{a})=n(\boldsymbol{b})-1$ and $L(\boldsymbol{b})=L(\boldsymbol{a})-1=\ell-1$ in the first line and $n(\boldsymbol{a})=n(\boldsymbol{b})+n(\boldsymbol{c})-1$ and $L(\boldsymbol{b})+L(\boldsymbol{c})=L(\boldsymbol{a})-1=\ell-1$ in the second line above. Hence, using the induction hypothesis we conclude that

$$\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma)=0,\qquad \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)=0,$$

if $n(\boldsymbol{a})>2^{\ell+1}$, and

$$\|\!|\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma)|\!\|_\sigma\lesssim[\![\sigma]\!]^{[\boldsymbol{a}]-1},\qquad \|\!|\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)|\!\|_\sigma\lesssim[\![\sigma]\!]^{[\boldsymbol{a}]-1}. \tag{4.41}$$

In the rest of the proof of the induction step we treat separately two cases: $\mathcal{F}^{\boldsymbol{a}}_\sigma$ is irrelevant, *i.e.* $[\boldsymbol{a}]>0$, or $\mathcal{F}^{\boldsymbol{a}}_\sigma$ is relevant, *i.e.*, $[\boldsymbol{a}]<0$. Note that, according to the analysis in Sec. 4.5, there are no marginal cumulants at level $\ell>0$.

If $\mathcal{F}^{\boldsymbol{a}}$ is irrelevant, *i.e.* $[\boldsymbol{a}]>0$, then we can bound it by integrating the flow equation (4.23) *backward* from the final condition at $\sigma=1$. We stress that for any irrelevant cumulant this final condition is vanishing, namely $\mathcal{F}^{\boldsymbol{a}}_1=0$. To see this fact, note that $\mathcal{F}^{\boldsymbol{a}}_1$ is a cumulant of kernels $F_1^{[\ell]}$ of the force (4.39). Since for $\ell>0$ these kernels are

$$F_1^{[\ell]}(z,z_1)=r^{[\ell]}_{\varepsilon,M}\delta(z,z_1)$$

which are deterministic, for $L(\boldsymbol{a})>0$ every non-vanishing cumulant $\mathcal{F}^{\boldsymbol{a}}_1$ coincides with the expected value of $F_1^{[\ell]}$, which is relevant. As a consequence, for $[\boldsymbol{a}]>0$ and $L(\boldsymbol{a})>0$, $\mathcal{F}^{\boldsymbol{a}}_1=0$. Using (4.23), Remark 4.12 and (4.41) we show easily that

$$\|\!|\mathcal{F}^{\boldsymbol{a}}_\mu|\!\|_\mu\lesssim\int_\mu^1[\![\sigma]\!]^{[\boldsymbol{a}]-1}\,\mathrm{d}\sigma\lesssim[\![\mu]\!]^{[\boldsymbol{a}]}.$$

On the other hand, if $\mathcal{F}^{\boldsymbol{a}}$ is relevant, *i.e.* $[\boldsymbol{a}]<0$, then on account of the discussion of Sec. 4.5 $\mathcal{F}^{\boldsymbol{a}}$ is of the form $\bar F^{[\ell],(1)}_\mu=\mathfrak{K}_1\big(F^{[\ell](1)}_\sigma\big)$ with $\ell\in\{1,\dots,\hat\ell\}$. The treatment of the relevant cumulant proceeds via a localisation procedure, which shows that only the local part of the cumulant requires renormalisation. However, in our fractional setting, where the kernels exhibit only limited polynomial decay, this must be done with care. Specifically, we first introduce a preliminary truncation before performing a Taylor expansion to localise the fields. Consequently, to bound $\bar F^{[\ell],(1)}_\mu$, we shall use the following decomposition

$$\begin{aligned}\bar F^{[\ell],(1)}_\mu \;=\;& \bar F^{[\ell],(1)}_1-\int_\mu^1\big[(1-K^{1,1}_\sigma)\dot{\bar F}^{[\ell],(1)}_\sigma\big]\,\mathrm{d}\sigma-\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\cdot(1-h_\mu)\,\mathrm{d}\sigma\\ &+\boldsymbol{L}\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\cdot(1-h_\mu)\,\mathrm{d}\sigma-\boldsymbol{L}\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\,\mathrm{d}\sigma\\ &-\mathbf{R}\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\cdot h_\mu\,\mathrm{d}\sigma.\end{aligned}\tag{4.42}$$

Here

$$\dot{\bar F}^{[\ell],(1)}_\sigma:=\partial_\sigma\bar F^{[\ell],(1)}_1,$$

the operators $\boldsymbol{L}$ and $\boldsymbol{R}$ are suitable localisation and remainder operators defined in Appendix B.2 and the polynomial weight h_μ was introduced in Def. 4.3. Recall that, by Def. 4.5, in the present case $K^{\boldsymbol{a}}_\sigma=K^{1,1}_\sigma=1\otimes K_\sigma$. We impose the following renormalisation condition

$$\bar F^{[\ell],(1)}_1=-\boldsymbol{L}\int_{1/2}^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\,\mathrm{d}\sigma.$$

Since $\bar{F}_1^{[\ell],(1)}(z,z_1)=\delta(z-z_1)\,r_{\varepsilon,M}^{[\ell]}$ the above condition fixes uniquely the counter-term $r_{\varepsilon,M}^{[\ell]}$. Note that the above integral is finite for all $\varepsilon\in(0,1)$ since on account of (4.23), the estimates (4.41) and Remark B.3 we have

$$\|\!|K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\| = \|K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\| \lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1}\wedge\varepsilon^{-\mathfrak{b}}[\![\sigma]\!]^{[\boldsymbol{a}]+\mathfrak{b}-1}. \tag{4.43}$$

Actually, the above bound implies that

$$|r_1^{\ell,\varepsilon}|\lesssim\|K_{1/2}^{1,1}\bar{F}_1^{[\ell],(1)}\|\lesssim\int_0^\varepsilon\|K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\|\,\mathrm{d}[\![\sigma]\!]\lesssim\int_0^\varepsilon\varepsilon^{-\mathfrak{b}}[\![\sigma]\!]^{[\boldsymbol{a}]+\mathfrak{b}-1}\mathrm{d}[\![\sigma]\!]+\int_\varepsilon^{1/2}[\![\sigma]\!]^{[\boldsymbol{a}]-1}\mathrm{d}[\![\sigma]\!]\lesssim\varepsilon^{[\boldsymbol{a}]},$$

where we used the fact that

$$[\boldsymbol{a}]+\mathfrak{b}\geqslant\mathfrak{b}-\varrho+\theta+\delta+\beta=\mathfrak{b}-2\beta+\delta=\mathfrak{b}-2s+2\delta+\kappa>0 \tag{4.44}$$

by (4.19), (4.32) and (4.37). Taking into account the renormalisation condition, we obtain

$$\begin{aligned}\|\!|\bar{F}_\mu^{[\ell],(1)}|\!\|_\mu \;\leq\; &\int_\mu^1\|\!|(1-K_\sigma^{1,1})\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\|_\mu\mathrm{d}\sigma+\int_\mu^1\|\!|\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)|\!\|_\mu\mathrm{d}\sigma\\
&+\int_\mu^1\|\!|\boldsymbol{L}\big(\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big)|\!\|_\mu\mathrm{d}\sigma+\int_{1/2}^\mu\|\!|\boldsymbol{L}\big(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big)|\!\|_\mu\mathrm{d}\sigma\\
&+\int_\mu^1\|\!|\mathbf{R}\big(\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)|\!\|_\mu\mathrm{d}\sigma.\end{aligned}$$

Finally, applying Lemmas A.13, A.18, B.5 and B.6, we arrive at

$$\begin{aligned}\|\!|\bar{F}_\mu^{[\ell],(1)}|\!\|_\mu \;\lesssim\; &[\![\mu]\!]^{-2s}\int_\mu^1[\![\sigma]\!]^{[\boldsymbol{a}]+2s-1}\mathrm{d}\sigma+[\![\mu]\!]^{-\mathfrak{b}}\int_\mu^1[\![\sigma]\!]^{[\boldsymbol{a}]+\mathfrak{b}-1}\mathrm{d}\sigma\\
&+[\![\mu]\!]^{-\mathfrak{b}}\int_\mu^1[\![\sigma]\!]^{[\boldsymbol{a}]+\mathfrak{b}-1}\mathrm{d}\sigma+\int_{1/2}^\mu[\![\sigma]\!]^{[\boldsymbol{a}]-1}\mathrm{d}\sigma\\
&+[\![\mu]\!]^{-2}\int_\mu^1[\![\sigma]\!]^{[\boldsymbol{a}]+2}\mathrm{d}\sigma\\
\lesssim\; &[\![\mu]\!]^{[\boldsymbol{a}]},\end{aligned}$$

where we used again (4.44). This concludes the proof of the induction step. □

4.7 Local estimates for the flow of kernels

In this section, we prove an auxiliary result showing that the norms of the effective force kernels $F_\mu^{\mathfrak{a}}$ can be controlled in terms of the norms of the localised kernels $F_\mu^{\mathfrak{a}}\cdot\nu_\mu^{\mathfrak{a}}$ restricted to a neighbourhood of the diagonal, where $\nu_\mu^{\mathfrak{a}}$ is the weight introduced in Def. 4.3. We use the notation

$$\|X\|_{L_{\mathbb{P}}^N}:=(\mathbb{E}|X|^N)^{1/N}.$$

Lemma 4.17. *For all $\ell\in\mathbb{N}_0, N\in\mathbb{N}_+$ it holds,*

$$\sup_{\mathfrak{a}|\ell(\mathfrak{a})=\ell}\sup_{\mu\in(1/2,1)}\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa}\|F_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L_{\mathbb{P}}^N}\lesssim_\ell 1+\sup_{\mathfrak{a}|\ell(\mathfrak{a})\leq\ell,[\mathfrak{a}]<0}\sup_{\mu\in(1/2,1)}\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa}\,\|F_\mu^{\mathfrak{a}}\cdot\nu_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L_{\mathbb{P}}^{2^\ell N}}^{2^\ell}, \tag{4.45}$$

and

$$\sup_{\mathfrak{a}|\ell(\mathfrak{a})=\ell}\sup_{\mu\in(1/2,1)}\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa+1}\|\partial_\mu F_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L_{\mathbb{P}}^N}\lesssim_\ell 1+\sup_{\mathfrak{a}|\ell(\mathfrak{a})\leq\ell,[\mathfrak{a}]<0}\sup_{\mu\in(1/2,1)}\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa}\,\|F_\mu^{\mathfrak{a}}\cdot\nu_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L_{\mathbb{P}}^{2^\ell N}}^{2^\ell}. \tag{4.46}$$

Proof. We shall prove the lemma by induction on $\ell\in\mathbb{N}_0$. First, we discuss the case $\ell=0$ for which we have $\partial_\mu F_\mu^{\mathfrak{a}}=0$ and thus $F_\mu^{\mathfrak{a}}=F_1^{\mathfrak{a}}$ for all $\mu\in(1/2,1)$. Observe that the kernels $F_1^{\mathfrak{a}}$ are local, that is supported on the diagonal, as can be seen from (2.1). Hence,

$$F_\mu^{\mathfrak{a}}\cdot\nu_\mu^{\mathfrak{a}}=F_1^{\mathfrak{a}}\cdot\nu_\mu^{\mathfrak{a}}=F_1^{\mathfrak{a}}=F_\mu^{\mathfrak{a}},\qquad\ell(\mathfrak{a})=0,$$

for all $\mu\in(1/2,1)$ since $v_\mu^{\mathfrak{a}}=1$ on the diagonal. Moreover, for the irrelevant kernels, that is the kernels $F_\mu^{\mathfrak{a}}$ such that $[\mathfrak{a}]>0$, we have $F_\mu^{\mathfrak{a}}=0$ for all $\mu\in(1/2,1)$ since $F_1^{\mathfrak{a}}=0$ for $\mathfrak{a}\in\mathfrak{A}$ such that $\ell(\mathfrak{a})=0$ and $[\mathfrak{a}]>0$. This proves the bounds (4.45) and (4.46) for $\ell=0$.

Let $\ell_0\in\mathbb{N}_0$ and suppose that the bounds (4.45) and (4.46) are true for all $\ell\leq\ell_0$. We shall prove the bounds for $\ell=\ell_0+1$. It follows from (4.13) and Lemma 4.9 that

$$[\![\mu]\!]^{-[\mathfrak{a}]-\kappa+1}\|\partial_\mu F_\mu^{\mathfrak{a}}\|_\mu\lesssim\sum_{\mathfrak{b},\mathfrak{c}}[\![\mu]\!]^{-[\mathfrak{a}]-2\kappa+1}\|B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot G_\mu,F_\mu^{\mathfrak{b}},F_\mu^{\mathfrak{c}})\|_\mu\lesssim\sum_{\mathfrak{b},\mathfrak{c}}[\![\mu]\!]^{-[\mathfrak{b}]-\kappa}\|F_\mu^{\mathfrak{b}}\|_\mu[\![\mu]\!]^{-[\mathfrak{c}]-\kappa}\|F_\mu^{\mathfrak{c}}\|_\mu.$$

Consequently, by Hölder's inequality

$$\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa+1}\|\partial_\mu F_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L^N_{\mathbb{P}}}\lesssim\sum_{\mathfrak{b},\mathfrak{c}}\big\|[\![\mu]\!]^{-[\mathfrak{b}]-\kappa}\|F_\mu^{\mathfrak{b}}\|_\mu\big\|_{L^{2N}_{\mathbb{P}}}\big\|[\![\mu]\!]^{-[\mathfrak{c}]-\kappa}\|F_\mu^{\mathfrak{c}}\|_\mu\big\|_{L^{2N}_{\mathbb{P}}}.$$

Since $\ell(\mathfrak{b})\vee\ell(\mathfrak{c})\leq\ell(\mathfrak{a})-1\leq\ell_0$ and $\ell=\ell(\mathfrak{a})=\ell_0+1$ the above bound together with the induction hypothesis imply the bound (4.46). The dependence on ℓ in the bound (4.46) comes from the estimate on the (finite) number of terms in the sum over $\mathfrak{b},\mathfrak{c}$.

Let $F_\mu^{\mathfrak{a}}$ be an irrelevant kernel, for which $[\mathfrak{a}]>0$. Then $F_1^{\mathfrak{a}}=0$, $F_\mu^{\mathfrak{a}}=-\int_\mu^1\partial_\eta F_\eta^{\mathfrak{a}}\,\mathrm{d}\eta$ and

$$\|F_\mu^{\mathfrak{a}}\|_\mu\lesssim\int_\mu^1\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu\,\mathrm{d}\eta\lesssim\int_\mu^1\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta\,\mathrm{d}\eta$$

where we replaced the norm $\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu$ with $\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta$ for $\eta>\mu$ using (4.10). Hence, by the Minkowski inequality

$$\Big\|\|F_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L^N_{\mathbb{P}}}\lesssim\int_\mu^1\Big\|\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta\Big\|_{L^N_{\mathbb{P}}}\mathrm{d}\eta.$$

Consequently, for the irrelevant kernels the bound (4.45) follows from the bound (4.46).

Let us now consider relevant kernels $F^{\mathfrak{a}}$, for which $[\mathfrak{a}]<0$. If $k(\mathfrak{a})=0$, then $v_\mu^{\mathfrak{a}}F_\mu^{\mathfrak{a}}=F_\mu^{\mathfrak{a}}$ and the bound (4.45) is trivial. For $k(\mathfrak{a})>0$ we shall use the following decomposition

$$\begin{aligned}F_\mu^{\mathfrak{a}}\cdot(1-v_\mu^{\mathfrak{a}}) &= F_1^{\mathfrak{a}}\cdot(1-v_\mu^{\mathfrak{a}})-\int_\mu^1\partial_\eta F_\eta^{\mathfrak{a}}\cdot(1-v_\mu^{\mathfrak{a}})\,\mathrm{d}\eta\\ &= -\int_\mu^1[(1-\tilde K_\eta^{\mathfrak{a}})\partial_\eta F_\eta^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\,\mathrm{d}\eta-\int_\mu^1[\tilde K_\eta^{\mathfrak{a}}\partial_\eta F_\eta^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\,\mathrm{d}\eta.\end{aligned}\tag{4.47}$$

Note that $F_1^{\mathfrak{a}}\cdot(1-v_\mu^{\mathfrak{a}})=0$, which follows from the fact that the kernels $F_1^{\mathfrak{a}}$ are local, that is supported on the diagonal, and $v_\mu^{\mathfrak{a}}=1$ on the diagonal. Then

$$\|F_\mu^{\mathfrak{a}}-F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu\leq\int_\mu^1\|[(1-\tilde K_\eta^{\mathfrak{a}})\partial_\eta F_\eta^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\|_\mu\,\mathrm{d}\eta+\int_\mu^1\|[\tilde K_\eta^{\mathfrak{a}}\partial_\eta F_\eta^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\|_\mu\,\mathrm{d}\eta.$$

We observe now that on account of Lemma A.14 and Lemma A.17, it holds that

$$\|F_\mu^{\mathfrak{a}}-F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu\leq\int_\mu^1\left(\frac{[\![\eta]\!]^{2s}}{[\![\mu]\!]^{2s}}+\frac{[\![\eta]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}}{[\![\mu]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}}\right)\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu\,\mathrm{d}\eta\lesssim\int_\mu^1\frac{[\![\eta]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}}{[\![\mu]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}}\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu\,\mathrm{d}\eta.$$

Consequently, by Minkowski's inequality

$$\Big\|\|F_\mu^{\mathfrak{a}}-F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu\Big\|_{L^N_{\mathbb{P}}}\lesssim\int_\mu^1\frac{[\![\eta]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}}{[\![\mu]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}}\Big\|\|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu\Big\|_{L^N_{\mathbb{P}}}\mathrm{d}\eta.$$

Note that by (4.2), (4.32), (4.37) and (4.36),

$$[\mathfrak{a}]+\mathfrak{b}-\ell(\mathfrak{a})\kappa_0\geq\mathfrak{b}-\alpha+(\delta-\kappa_0)\ell(\mathfrak{a})+\beta=\mathfrak{b}+(\delta-\kappa_0)\ell(\mathfrak{a})-2\beta-\kappa=(\delta-\kappa_0)\ell(\mathfrak{a})+\delta+\mathfrak{b}-2s>0,$$

for $k(\mathfrak{a})>0$ and $\ell(\mathfrak{a})>0$. Thus, the previous estimate and the bound (4.46) imply the bound (4.45) for the relevant kernels. This concludes the proof. □

4.8 From cumulants to random kernels

The bound (2.29) stated in Theorem 2.7 is proved by combining the cumulant estimates from Lemma 4.15 with the auxiliary bounds established in Lemma 4.17, via the following lemma, which shows how estimates on the cumulants translate into estimates on the effective force kernels. The remaining part of Theorem 2.7, namely, the proof that the estimates (2.17) hold with $C_F = 1 + \|F^{\mathfrak{A}}\|^2$, is presented in Sec. 4.9.

Our analysis relies on the decomposition of the effective force kernels into mean and fluctuation parts:

$$F_\mu^{\mathfrak{a}} = \bar{F}_\mu^{\mathfrak{a}} + \tilde{F}_\mu^{\mathfrak{a}}, \qquad \bar{F}_\mu^{\mathfrak{a}} := \mathbb{E} F_\mu^{\mathfrak{a}}, \qquad \tilde{F}_\mu^{\mathfrak{a}} := F_\mu^{\mathfrak{a}} - \mathbb{E} F_\mu^{\mathfrak{a}}.$$

Lemma 4.18. *Let $(F_\mu^\varepsilon)_{\mu\in(1/2,1)}$ be the solution of the approximate flow equation with initial condition (4.39) with $r_{\varepsilon,M}$ as in Lemma 4.15. For every $N \in \mathbb{N}_+$, we have*

$$\mathbb{E}\|F^{\mathfrak{A}}\|^N = \Big\| \|F^{\mathfrak{A}}\| \Big\|_{L^N_{\mathbb{P}}}^N < \infty, \tag{4.48}$$

where $\|F^{\mathfrak{A}}\|$ was defined in (4.9).

Proof. If the kernel $F_\mu^{\mathfrak{a}}$ is irrelevant, that is $[\mathfrak{a}] > 0$, then $F_1^{\mathfrak{a}} = 0$, $F_\mu^{\mathfrak{a}} = -\int_\mu^1 \partial_\eta F_\eta^{\mathfrak{a}} \, \mathrm{d}\eta$ and

$$[\![\mu]\!]^{-[\mathfrak{a}]} \|F_\mu^{\mathfrak{a}}\|_\mu \lesssim [\![\mu]\!]^{-[\mathfrak{a}]} \int_\mu^1 \|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu \, \mathrm{d}\eta \lesssim \int_\mu^1 [\![\eta]\!]^{-[\mathfrak{a}]} \|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta \, \mathrm{d}\eta \lesssim \int_{1/2}^1 [\![\eta]\!]^{-[\mathfrak{a}]} \|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta \, \mathrm{d}\eta.$$

On the other hand, if the kernel $F_\mu^{\mathfrak{a}}$ is relevant, that is $[\mathfrak{a}] < 0$, then $F_\mu^{\mathfrak{a}} = F_1^{\mathfrak{a}}$ and $[\![\mu]\!]^{-[\mathfrak{a}]} = 0$ at $\mu = 1$. Consequently, we have $[\![\mu]\!]^{-[\mathfrak{a}]} F_\mu^{\mathfrak{a}} = 0$ at $\mu = 1$. From

$$\partial_\mu ([\![\mu]\!]^{-[\mathfrak{a}]} F_\mu^{\mathfrak{a}}) = [\![\mu]\!]^{-[\mathfrak{a}]} \partial_\mu F_\mu^{\mathfrak{a}} - [\mathfrak{a}] \, [\![\mu]\!]^{-[\mathfrak{a}]-1} F_\mu^{\mathfrak{a}}, \tag{4.49}$$

we deduce that

$$[\![\mu]\!]^{-[\mathfrak{a}]} \|F_\mu^{\mathfrak{a}}\|_\mu \lesssim \int_\mu^1 ([\![\eta]\!]^{-[\mathfrak{a}]} \|\partial_\eta F_\eta^{\mathfrak{a}}\|_\mu + [\![\eta]\!]^{-[\mathfrak{a}]-1} \|F_\eta^{\mathfrak{a}}\|_\mu) \, \mathrm{d}\eta \lesssim \int_{1/2}^1 ([\![\eta]\!]^{-[\mathfrak{a}]} \|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta + [\![\eta]\!]^{-[\mathfrak{a}]-1} \|F_\eta^{\mathfrak{a}}\|_\eta) \, \mathrm{d}\eta.$$

Next, Minkowski inequality gives

$$\Big\| \sup_{\mu\in(1/2,1)} [\![\mu]\!]^{-[\mathfrak{a}]} \|F_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^N_{\mathbb{P}}} \lesssim \int_{1/2}^1 (\| [\![\eta]\!]^{-[\mathfrak{a}]} \|\partial_\eta F_\eta^{\mathfrak{a}}\|_\eta \|_{L^N_{\mathbb{P}}} + \| [\![\eta]\!]^{-[\mathfrak{a}]-1} \|F_\eta^{\mathfrak{a}}\|_\eta \|_{L^N_{\mathbb{P}}}) \, \mathrm{d}\eta,$$

for both relevant and irrelevant kernels. Since $\int_{1/2}^1 [\![\eta]\!]^{\kappa-1} \, \mathrm{d}\eta < \infty$, we arrive at

$$\Big\| \sup_{\mu\in(1/2,1)} [\![\mu]\!]^{-[\mathfrak{a}]} \|F_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^N_{\mathbb{P}}} \lesssim \sup_{\mu\in(1/2,1)} \| [\![\mu]\!]^{-[\mathfrak{a}]+1-\kappa} \|\partial_\mu F_\mu^{\mathfrak{a}}\|_\mu \|_{L^N_{\mathbb{P}}} + \sup_{\mu\in(1/2,1)} \| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|F_\mu^{\mathfrak{a}}\|_\mu \|_{L^N_{\mathbb{P}}}.$$

Using Lemma 4.17 and the decomposition $F_\mu^{\mathfrak{a}} = \bar{F}_\mu^{\mathfrak{a}} + \tilde{F}_\mu^{\mathfrak{a}}$ we obtain

$$\begin{aligned}
\Big\| \sup_{\mu\in(1/2,1)} [\![\mu]\!]^{-[\mathfrak{a}]} \|F_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^N_{\mathbb{P}}} &\lesssim 1 + \sup_{\mathfrak{a}|\ell(\mathfrak{a})\le\ell, [\mathfrak{a}]<0} \sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\nu_\mu^{\mathfrak{a}} F_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^{2^\ell N}_{\mathbb{P}}}^{2^\ell} \\
&\lesssim 1 + \sup_{\mathfrak{a}|\ell(\mathfrak{a})\le\ell, [\mathfrak{a}]<0} \sup_{\mu\in(1/2,1)} \Big(\Big\| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\nu_\mu^{\mathfrak{a}} \tilde{F}_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^{2^\ell N}_{\mathbb{P}}}^{2^\ell} + \big([\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\nu_\mu^{\mathfrak{a}} \bar{F}_\mu^{\mathfrak{a}}\|_\mu \big)^{2^\ell} \Big).
\end{aligned}$$

Applying Lemmas 4.19 and 4.21 and recalling that $\|\!|\mathcal{L}^{\boldsymbol{A}}|\!\| < \infty$, on account of Lemma 4.15, we conclude that

$$\Big\| \sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\mathfrak{a}]} \|F_\sigma^{\mathfrak{a}}\|_\sigma \Big\|_{L^N_{\mathbb{P}}} < \infty.$$

Finally, to establish the slightly stronger bound

$$\Big\| \sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{d/2+s+2\kappa} \, \|F_\sigma^{[0],(0)}\|_\sigma \Big\|_{L^N_{\mathbb{P}}} < \infty$$

for $F_\sigma^{[0],(0)} = \xi^{(\varepsilon,M)}$, we proceed as before, replacing $[\mathfrak{a}]$ by $-d/2-s-2\kappa$ and invoke the second bound stated in Lemma 4.19. □

Lemma 4.19. *For all $\mathfrak{a}\in\mathfrak{A}$ and $N\in\mathbb{N}_+$, we have*

$$\sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\tilde F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} \lesssim_N |\!|\!| \mathcal{F}_x \boldsymbol{A} |\!|\!|^{1/2},$$

$$\sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{d/2+s+\kappa} \|F_\mu^{[0],(0)}\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} \lesssim_N 1.$$

Proof. Let $k = k(\mathfrak{a})$. By Lemma A.15, there exists is a weight $\tilde v_\mu^{\mathfrak{a}}$ with support properties analogous to those of $v_\mu^{\mathfrak{a}}$ such that

$$\|\tilde F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu = \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}(\tilde F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}})]\cdot \tilde w_\mu^{\mathfrak{a}}\|_\mu \lesssim \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}\tilde F_\mu^{\mathfrak{a}}]\cdot \tilde v_\mu^{\mathfrak{a}}\tilde w_\mu^{\mathfrak{a}}\|.$$

Moreover, it holds that

$$\sup_{z\in\Lambda}\int_{\Lambda^k} \tilde w_\mu^{\mathfrak{a}}(z,Z^{\mathfrak{a}})\tilde v_\mu^{\mathfrak{a}}(z,Z^{\mathfrak{a}})\,\mathrm{d}Z^{\mathfrak{a}} \lesssim [\![\mu]\!]^{2sk}(\varepsilon\vee[\![\mu]\!])^{dk}, \qquad Z^{\mathfrak{a}} = (z_1,\dots,z_k)\in\Lambda^k.$$

Thus, we have

$$\|\tilde F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu \lesssim \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}\tilde F_\mu^{\mathfrak{a}}]\cdot\tilde v_\mu^{\mathfrak{a}}\tilde w_\mu^{\mathfrak{a}}\| \lesssim [\![\mu]\!]^{2sk}(\varepsilon\vee[\![\mu]\!])^{dk}\,\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}\tilde F_\mu^{\mathfrak{a}}]\|_{L^\infty},$$

where, here and below, $\|\bullet\|_{L^\infty}$ denotes the supremum norm in all the variables. Hence,

$$\Big\| \|\tilde F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} \lesssim [\![\mu]\!]^{2sk}(\varepsilon\vee[\![\mu]\!])^{dk} \Big\| \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}\tilde F_\mu^{\mathfrak{a}}]\|_{L^\infty} \Big\|_{L^{2N}_{\mathbb{P}}}. \tag{4.50}$$

To analyse the last term, we decompose the operator K_μ as

$$K_\mu = \check K_\mu \hat K_\mu,$$

where

$$\hat K_\mu := (1+[\![\mu]\!]^{2s}\partial_t)^{\hat\kappa-1}(1-[\![\mu]\!]^2\Delta)^{\hat\kappa-2}, \qquad \check K_\mu := (1+[\![\mu]\!]^{2s}\partial_t)^{-\hat\kappa}(1-[\![\mu]\!]^2\Delta)^{-\hat\kappa}, \tag{4.51}$$

and $\hat\kappa>0$ is a small constant to be fixed later. By Lemma A.5,

$$\|\check K_\mu\|_{L^{2N/(2N-1)}(1/\mathfrak{o}^{\mathfrak{a}})} \lesssim [\![\mu]\!]^{-(d+2s)/2N}, \tag{4.52}$$

provided $4N\hat\kappa>3$. In what follows, we choose $\hat\kappa\in(0,1/(2+2\bar k))$ and assume that $N>\bar\kappa^{-1}>2+2\bar k$. Recall that

$$\tilde K_\mu^{\mathfrak{a}} = \check K_\mu^{\otimes(1+k)}\hat K_\mu^{\otimes(1+k)}K_\mu^{1,k}$$

by Def. 1.13 and 4.5. Hence, by weighted Young's inequality, Fubini's theorem and (4.52), it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}\tilde F_\mu^{\mathfrak{a}}]\|_{L^\infty}^{2N}\Big] \lesssim \\
&\quad\lesssim [\![\mu]\!]^{-(d+2s)(1+k)}\int_{\Lambda^{1+k}}\mathbb{E}\Big[\big(\mathfrak{o}^{\mathfrak{a}}(z)\,(\hat K_\mu^{\otimes(1+k)}K_\mu^{1,k}\tilde F_\mu^{\mathfrak{a}})(z;Z^{\mathfrak{a}})\big)^{2N}\Big]\mathrm{d}z\,\mathrm{d}Z^{\mathfrak{a}} \\
&\quad\lesssim [\![\mu]\!]^{-(d+2s)(1+k)}\int_{\Lambda^{1+k}}\mathbb{E}\Big[\big((\hat K_\mu^{\otimes(1+k)}K_\mu^{1,k}\tilde F_\mu^{\mathfrak{a}})(z;Z^{\mathfrak{a}})\,w_\mu^{\mathfrak{a}}(z,Z^{\mathfrak{a}})\big)^{2N}\Big]\Big(\frac{\mathfrak{o}^{\mathfrak{a}}(z)}{w_\mu^{\mathfrak{a}}(z,Z^{\mathfrak{a}})}\Big)^{2N}\mathrm{d}z\,\mathrm{d}Z^{\mathfrak{a}} \\
&\quad\lesssim [\![\mu]\!]^{-(d+2s)(1+k)}\Big\|\mathbb{E}\Big[\big([\hat K_\mu^{\otimes(1+k)}K_\mu^{1,k}\tilde F_\mu^{\mathfrak{a}}]\cdot w_\mu^{\mathfrak{a}}\big)^{2N}\Big]\Big\|_{L^\infty},
\end{aligned}$$

where we used that,

$$\int_{\Lambda^{1+k}} \left(\frac{\mathfrak{o}^{\mathfrak{a}}(z)}{w_\mu^{\mathfrak{a}}(z, Z^{\mathfrak{a}})} \right)^{2N} \mathrm{d}z \mathrm{d}Z^{\mathfrak{a}} \lesssim 1,$$

for $N \in \mathbb{N}_+$ large enough. Thus,

$$\left\| \| \mathfrak{o}^{\mathfrak{a}} \cdot [\hat{K}_\mu^{\mathfrak{a}} \tilde{F}_\mu^{\mathfrak{a}}] \|_{L^\infty} \right\|_{L^{2N}_{\mathbb{P}}} \lesssim [\![\mu]\!]^{-\frac{(d+2s)(1+k)}{2N}} \left\| \mathbb{E}\Big[\Big([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} \tilde{F}_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}} \Big)^{2N} \Big] \right\|_{L^\infty}^{1/2N}. \tag{4.53}$$

We need to control the $L^{2N}_{\mathbb{P}}$ norm of the centred random variable

$$(\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} \tilde{F}_\mu^{\mathfrak{a}})(z; Z^{\mathfrak{a}})\, w_\mu^{\mathfrak{a}}(z, Z^{\mathfrak{a}}).$$

To this end, we observe that, by induction it is easy to show that $\tilde{F}_\mu^{\mathfrak{a}}$ is a polynomial of the Gaussian random field $\xi^{(\varepsilon,M)}$. Consequently, by Nelson's hypercontractivity estimate, all higher moments can be controlled by the second one. In particular, the $2N$-th moment can be bounded in terms of the second-order cumulant

$$\mathbb{E}(\tilde{F}_\mu^{\mathfrak{a}} \otimes \tilde{F}_\mu^{\mathfrak{a}}) = \mathfrak{K}_2(F_\mu^{\mathfrak{a}}, F_\mu^{\mathfrak{a}}) = \mathcal{F}_\mu^{(\mathfrak{a}\mathfrak{a})}.$$

Using the notation

$$w_\mu^{(\mathfrak{a}\mathfrak{a})} := w_\mu^{\mathfrak{a}} \otimes w_\mu^{\mathfrak{a}}, \qquad K_\mu^{(\mathfrak{a}\mathfrak{a})} := K_\mu^{1,k} \otimes K_\mu^{1,k},$$

we have

$$\begin{aligned} &\left\| \mathbb{E}\Big[\Big([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} \tilde{F}_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}} \Big)^{2N} \Big] \right\|_{L^\infty} \\ &\lesssim_N \left\| \mathbb{E}\Big[\big([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} \tilde{F}_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}} \big)^2 \Big] \right\|_{L^\infty}^N \\ &\lesssim \left\| \mathfrak{K}_2\big([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} F_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}}, [\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} F_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}} \big) \right\|_{L^\infty}^N. \end{aligned} \tag{4.54}$$

Since

$$\| \mathcal{F}_\mu^{(\mathfrak{a}\mathfrak{a})} \|_\mu = \| [K_\mu^{(\mathfrak{a}\mathfrak{a})} \mathcal{F}_\mu^{(\mathfrak{a}\mathfrak{a})}] \cdot w_\mu^{(\mathfrak{a}\mathfrak{a})} \|,$$

by Lemma 4.20 below, we obtain

$$\left\| \mathfrak{K}_2\big([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} F_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}}, [\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} F_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}} \big) \right\|_{L^\infty} \lesssim [\![\mu]\!]^{-2s(1+2k)} (\varepsilon \vee [\![\mu]\!])^{-d(1+2k)} \| \mathcal{F}_\mu^{(\mathfrak{a}\mathfrak{a})} \|_\mu.$$

Hence, by (4.54),

$$\left\| \mathbb{E}\Big([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} \tilde{F}_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}} \Big)^{2N} \right\|_{L^\infty}^{\frac{1}{2N}} \lesssim [\![\mu]\!]^{-s(1+2k)} (\varepsilon \vee [\![\mu]\!])^{-d(1+2k)/2} \| \mathcal{F}_\mu^{(\mathfrak{a}\mathfrak{a})} \|_\mu^{1/2}.$$

Overall, using (4.50), (4.53) together with the definition (4.22) of $\| \mathcal{F}^{\boldsymbol{A}} \|$ and the fact that

$$[(\mathfrak{a}\mathfrak{a})] = -\varrho + 2(\theta + \alpha) + 2[\mathfrak{a}],$$

we arrive at

$$\sup_\mu \left\| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \| v_\mu^{\mathfrak{a}} \tilde{F}_\mu^{\mathfrak{a}} \|_\mu \right\|_{L^{2N}_{\mathbb{P}}} \lesssim_N \| \mathcal{F}^{\boldsymbol{A}} \|^{1/2} \sup_\mu [\![\mu]\!]^{-\frac{\varrho}{2} + \theta + \alpha - \frac{d+2s}{2} - \frac{(d+2s)(1+k)}{2N} - \kappa}. \tag{4.55}$$

In order to conclude, we need the right-hand side above to be finite, that is, we require

$$\theta + \alpha - \frac{\varrho}{2} - \frac{d+2s}{2} - \frac{(d+2s)(1+\bar{k})}{2N} - \kappa \geq 0. \tag{4.56}$$

Given our choices of parameters (4.32) this bounds has the form

$$\frac{3}{4}\delta_\star - \frac{(d+2s)(1+\bar{k})}{2N} - \frac{3}{2}\kappa \geq 0. \tag{4.57}$$

Since $\kappa \in (0, \delta_\star/2)$ by (4.35), it suffices to take $N \in \mathbb{N}_+$ large enough to satisfy this inequality and conclude our bound. The bound for smaller values of $N \in \mathbb{N}_+$ follows then immediately by Jensen's inequality.

The second of the stated bounds, namely the estimate for the noise $F^{[0],(0)} = \xi^{(\varepsilon,M)}$, is proved by the same argument as for (4.55), except that we now use the precise estimate

$$\|\mathcal{F}_\sigma^{(\mathfrak{a},\mathfrak{a})}\|_\sigma = 1$$

for the covariance of the noise

$$\mathcal{F}_\sigma^{(\mathfrak{a},\mathfrak{a})} = \mathfrak{K}_2\big(\xi^{(\varepsilon,M)}, \xi^{(\varepsilon,M)}\big), \qquad \mathfrak{a} = (0,0),$$

which follows from (4.27), instead of the weaker bound

$$\|\mathcal{F}_\sigma^{(\mathfrak{a},\mathfrak{a})}\|_\sigma \lesssim \|\!|\mathcal{F}^{\boldsymbol{A}}|\!\| \, [\![\sigma]\!]^{[(\mathfrak{a}\mathfrak{a})]}.$$

Specifically, in place of (4.55) we obtain

$$\sup_\mu \Big\| [\![\mu]\!]^{d/2+s+\kappa} \|F^{[0],(0)}\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} = \sup_\mu \Big\| [\![\mu]\!]^{d/2+s+\kappa} \|\mathfrak{o} K_\mu F^{[0],(0)}\| \Big\|_{L^{2N}_{\mathbb{P}}} \lesssim \sup_\mu [\![\mu]\!]^{-\frac{(d+2s)}{2N}+\kappa} \lesssim 1,$$

for $N \in \mathbb{N}_+$ large enough, depending on $\kappa > 0$. □

Lemma 4.20. *Let $F \in C(\Lambda^{1+k})$ be a random field whose law is invariant under spacetime translations and such that*

$$\Lambda \ni z \mapsto F(z, z+z_1, \dots, z+z_k) \in \mathbb{R} \tag{4.58}$$

is a.s. periodic in space with period M for all $z_1, \dots, z_k \in \Lambda$. Define

$$\mathcal{F}(z, z_1, \dots, z_{2k}) = \mathfrak{K}_2(F(0, z_1, \dots, z_k), F(z, z_{k+1} \dots, z_{2k})), \qquad z, z_1, \dots, z_{2k} \in \Lambda.$$

For all $\omega \geq 0$ and $\hat\kappa \in [0, 1/(2k+1))$, we have

$$\begin{aligned} \|\mathfrak{K}_2\big((\hat K_\mu^{\otimes(1+k)} F), (\hat K_\mu^{\otimes(1+k)} F)\big) \cdot \big(w_\mu^{(1+k),\omega} \otimes w_\mu^{(1+k),\omega}\big)\|_{L^\infty} &\lesssim [\![\mu]\!]^{-2s(1+2k)} (\varepsilon \vee [\![\mu]\!])^{-d(1+2k)} \\ \times \int_{\Lambda_M \times \Lambda^{2k}} w_\mu^{(1+k),\omega}(0, z_1, \dots, z_k) w_\mu^{(1+k),\omega}(z, z_{k+1}, \dots, z_{2k}) \, |\mathcal{F}(z, z_1, \dots, z_{2k})| \, \mathrm{d}z \mathrm{d}z_1 \dots \mathrm{d}z_{2k}, \end{aligned}$$

uniformly over F and $\mu \in [1/2, 1)$, where

$$\hat K_\mu := (1 + [\![\mu]\!]^{2s} \partial_t)^{\hat\kappa - 1} (1 - [\![\mu]\!]^2 \Delta)^{\hat\kappa - 2}.$$

Proof. We claim that

$$\mathfrak{K}_2\big((\hat K_\mu^{\otimes(1+k)} F)(0, z_1, \dots, z_k), (\hat K_\mu^{\otimes(1+k)} F)(z, z_{k+1,}, \dots, z_{2k})\big) = (\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k})$$

for all $z, z_1, \dots, z_{2k} \in \Lambda$, where $*$ stands for the convolution on Λ^{1+2k} and

$$\mathcal{K}_\mu(z, z_1, \dots, z_{2k}) := \int_\Lambda \hat K_\mu(z'') \, \hat K_\mu(z + z'') \, \hat K_\mu(z_1 + z'') \dots \hat K_\mu(z_{2k} + z'') \, \mathrm{d}z''.$$

Indeed, we have

$$\begin{aligned} &(\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k}) \\ &= \int_{\Lambda^{2+2k}} \hat K_\mu(z'') \, \hat K_\mu(z' + z'') \, \hat K_\mu(z_1' + z'') \dots \hat K_\mu(z_{2k}' + z'') \\ &\qquad \times \mathcal{F}(z - z', z_1 - z_1', \dots, z_{2k} - z_{2k}') \, \mathrm{d}z' \mathrm{d}z'' \mathrm{d}z_1' \dots \mathrm{d}z_{2k}' \\ &= \int_{\Lambda^{2+2k}} \hat K_\mu(z'') \, \hat K_\mu(z') \, \hat K_\mu(z_1') \dots \hat K_\mu(z_{2k}') \\ &\qquad \times \mathfrak{K}_2(F(0 - z'', z_1 - z_1', \dots, z_k - z_k'), F(z - z', z_{k+1} - z_{k+1}', \dots, z_{2k} - z_{2k}')) \, \mathrm{d}z' \mathrm{d}z'' \mathrm{d}z_1' \dots \mathrm{d}z_{2k}' \\ &= \mathfrak{K}_2\big((\hat K_\mu^{\otimes(1+k)} F)(0, z_1, \dots, z_k), (\hat K_\mu^{\otimes(1+k)} F)(z, z_{k+1,}, \dots, z_{2k})\big), \end{aligned}$$

where the second equality above follows from the translation invariance of the law of F.

It remains to show that

$$\begin{aligned}&|(\mathcal{K}_\mu * \mathcal{F})(z,z_1,\dots,z_{2k})|\, w_\mu^{(1+k),\omega}(0,z_1,\dots,z_k) w_\mu^{(1+k),\omega}(z,z_{k+1},\dots,z_{2k})\\ &\lesssim [\![\mu]\!]^{-2s(1+2k)}(\varepsilon\vee[\![\mu]\!])^{-d(1+2k)}\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1,\dots,z_k) w_\mu^{(1+k),\omega}(z,z_{k+1},\dots,z_{2k})\\ &\qquad\times|\mathcal{F}(z,z_1,\dots,z_{2k})|\,\mathrm{d}z\mathrm{d}z_1\dots\mathrm{d}z_{2k}.\end{aligned}$$

We have

$$\begin{aligned}|(\mathcal{K}_\mu * \mathcal{F})(z,z_1,\dots,z_{2k})| \lesssim \sum_{n\in\mathbb{Z}^d}\int_{\Lambda_M\times\Lambda^{2k}} &|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')|\\ &\times|\mathcal{F}(z-z'+Mn,z_1-z_1',\dots,z_{2k}-z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'.\end{aligned}$$

Thus, by (4.6) we obtain

$$\begin{aligned}&|(\mathcal{K}_\mu * \mathcal{F})(z,z_1,\dots,z_{2k})|\, w_\mu^{(1+k),\omega}(0,z_1,\dots,z_k) w_\mu^{(1+k),\omega}(z,z_{k+1},\dots,z_{2k})\\ &\lesssim \sum_{n\in\mathbb{Z}^d}\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{\omega}(z'+Mn)\, w_\mu^{\omega}(z_1')\cdots w_\mu^{\omega}(z_{2k}')\,|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')|\\ &\qquad\times w_\mu^{(1+k),\omega}(0,z_1-z_1',\dots,z_k-z_k')\, w_\mu^{(1+k),\omega}(z-z'+Mn,z_{k+1}-z_{k+1}',\dots,z_{2k}-z_{2k}')\\ &\qquad\times|\mathcal{F}(z-z'+Mn,z_1-z_1',\dots,z_{2k}-z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}',\end{aligned}$$

where for $z=(t,x)\in\Lambda$ and $n\in\mathbb{Z}^d$ we write $z+Mn=(t,x+Mn)\in\Lambda$. Using Def. 4.3 we arrive at

$$\begin{aligned}&\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1-z_1',\dots,z_k-z_k') w_\mu^{(1+k),\omega}(z-z'+Mn,z_{k+1}-z_{k+1}',\dots,z_{2k}-z_{2k}')\\ &\qquad\times|\mathcal{F}(z-z'+Mn,z_1-z_1',\dots,z_{2k}-z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'\\ &=\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1',\dots,z_k') w_\mu^{(1+k),\omega}(0,z_{k+1}',\dots,z_{2k}')\\ &\qquad\times|\mathcal{F}(z-z'+Mn,z_1',\dots z_k',z-z'+Mn+z_{k+1}',\dots,z-z'+Mn+z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'.\end{aligned}$$

Note that by (4.58), the above expression can be rewritten as

$$\begin{aligned}&\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1',\dots,z_k') w_\mu^{(1+k),\omega}(0,z_{k+1}',\dots,z_{2k}')\\ &\qquad\times|\mathcal{F}(z',z_1',\dots z_k',z'+z_{k+1}',\dots,z'+z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'\\ &=\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1',\dots,z_k') w_\mu^{(1+k),\omega}(z',z_{k+1}',\dots,z_{2k}')\\ &\qquad\times|\mathcal{F}(z',z_1',\dots z_k',z_{k+1}',\dots,z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'.\end{aligned}$$

Consequently,

$$\begin{aligned}&|(\mathcal{K}_\mu * \mathcal{F})(z,z_1,\dots,z_{2k})|\, w_\mu^{(1+k),\omega}(0,z_1,\dots,z_k) w_\mu^{(1+k),\omega}(z,z_{k+1},\dots,z_{2k})\\ &\lesssim \sup_{z',z_1',\dots,z_{2k}'\in\Lambda}\sum_{n\in\mathbb{Z}^d} w_\mu^{\omega}(z'+Mn)\, w_\mu^{\omega}(z_1')\cdots w_\mu^{\omega}(z_{2k}')\,|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')|\\ &\times\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1',\dots,z_k') w_\mu^{(1+k),\omega}(z',z_{k+1}',\dots,z_{2k}')\,|\mathcal{F}(z',z_1',\dots,z_{2k}')|\,\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'.\end{aligned}$$

The statement follows now from the estimate

$$\sup_{z',z_1',\dots,z_{2k}'\in\Lambda}\sum_{n\in\mathbb{Z}^d} w_\mu^{\omega}(z'+Mn)\, w_\mu^{\omega}(z_1')\cdots w_\mu^{\omega}(z_{2k}')\,|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')|\lesssim [\![\mu]\!]^{-2s(1+2k)}(\varepsilon\vee[\![\mu]\!])^{-d(1+2k)},$$

which is a consequence of Lemma A.6. □

Lemma 4.21. *We have*

$$[\![\mu]\!]^{-[\mathfrak{a}]-\kappa}\,\|\bar{F}_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu \lesssim |||\mathcal{F}^A|||,$$

uniformly in $\mu \in [1/2, 1)$.

Proof. By Lemma A.15, we have

$$\|\bar{F}^{\mathfrak{a}}_{\mu} \cdot v^{\mathfrak{a}}_{\mu}\|_{\mu} \lesssim \|\bar{F}^{\mathfrak{a}}_{\mu}\|_{\mu} = \|[\tilde{K}^{\mathfrak{a}}_{\mu}\bar{F}^{\mathfrak{a}}_{\mu}] \cdot \tilde{w}^{\mathfrak{a}}_{\mu}\|.$$

Moreover, by Def. 4.3,

$$\mathfrak{o}^{\mathfrak{a}}\tilde{w}^{\mathfrak{a}}_{\mu} \leq w^{\mathfrak{a}}_{\mu}.$$

Thus, by Lemma 1.17, (4.2) and (4.19) we obtain

$$\begin{aligned}
[\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\bar{F}^{\mathfrak{a}}_{\mu} \cdot v^{\mathfrak{a}}_{\mu}\|_{\mu} &\lesssim [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|[\tilde{K}^{\mathfrak{a}}_{\mu}\bar{F}^{\mathfrak{a}}_{\mu}] \cdot w^{\mathfrak{a}}_{\mu}\| \\
&= [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|[K^{\otimes(1+k(\mathfrak{a}))}_{\mu} K^{1,k(\mathfrak{a})}_{\mu}\bar{F}^{\mathfrak{a}}_{\mu}] \cdot w^{\mathfrak{a}}_{\mu}\| \\
&\lesssim [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|K_{\mu}\|^{1+k(\mathfrak{a})}_{\mathrm{TV}(w^{\flat}_{\mu})} \|[K^{1,k(\mathfrak{a})}_{\mu}\bar{F}^{\mathfrak{a}}_{\mu}] \cdot w^{\mathfrak{a}}_{\mu}\| \\
&\lesssim |||\mathcal{F}_{\cdot}A||| \, [\![\mu]\!]^{[\boldsymbol{a}]-[\mathfrak{a}]-\kappa} \\
&\lesssim |||\mathcal{F}_{\cdot}A||| \, [\![\mu]\!]^{\alpha-\varrho+\theta-\kappa}.
\end{aligned}$$

To conclude we use the inequality

$$\alpha - \varrho + \theta - \kappa \geq 0, \tag{4.59}$$

which coincides with the constraint $[\bar{F}]$ stated in (4.30). □

4.9 Post-processing

To complete the proof of Theorem 2.7, it remains to exhibit the bounds (2.17) with $C_F = 1 + \|F^{\mathfrak{A}}\|^2$, based on the analysis of the flow equation carried out so far. This is the content of Lemmas 4.22, 4.25 and 4.26 below.

Before continuing with the specific computations leading to (2.17) it will be useful to discuss how to fix the values of the parameters

$$\gamma, \vartheta, \bar{\kappa}, \bar{\ell}, \bar{k}, a, \nu, \kappa_{\mathfrak{o}}, \kappa.$$

The validity of the Lemmas 4.22–4.27 below, which together yield (2.17), depends on a series of conditions on these parameters, namely (4.67), (4.71), (4.74), (4.75), (4.77) and (4.78) as well as the constraints (2.16) in Theorem 2.5. The parameters $\kappa, \kappa_{\mathfrak{o}}$ are further constrained by conditions (4.35) and (4.36).

For the reader's convenience, these conditions are summarised in the following table:

$$\begin{aligned}
&[\mathbb{A}] \quad \vartheta \leq 3(\gamma - \beta) + \delta - a\kappa_{\mathfrak{o}}(1 + \bar{\ell}) - \kappa \\
&[\mathbb{B}] \quad \vartheta \leq 2\gamma \wedge \left(3\gamma - \frac{d + 2s}{2} - 2\kappa - a\kappa_{\mathfrak{o}}\right) \\
&[\mathbb{C}] \quad \vartheta \leq \delta\bar{\ell} - \alpha - a\kappa_{\mathfrak{o}}(1 + 2\bar{\ell}) \\
&[\mathbb{D}] \quad \bar{\kappa} \geq \bar{k}\nu + (1 + \bar{\ell})\kappa_{\mathfrak{o}} \\
&[\mathbb{E}] \quad \vartheta \leq \gamma \wedge (2s - \gamma) \\
&[\mathbb{F}] \quad \vartheta \geq \frac{4(s + \gamma)\,\bar{\kappa}}{1 - \bar{\kappa}} \\
&[\mathbb{G}] \quad \kappa \in (0, \delta_{\star}/2) \\
&[\mathbb{H}] \quad \kappa_{\mathfrak{o}} \leq \delta \wedge \flat/(1 + \bar{\ell})
\end{aligned}$$

In order to satisfy all these constraints, together with the basic bounds $\vartheta>0$ and $\delta>0$, we proceed as follows. First, we set

$$\boxed{\gamma=\beta-\vartheta,\qquad \vartheta=\frac{1}{16}\delta_\star.} \tag{4.60}$$

Next, we fix $\bar\kappa>0$ small enough so that

$$\frac{4(s+\beta)\,\bar\kappa}{1-\bar\kappa}\le\frac{1}{16}\delta_\star,$$

which allows to satisfy [𝔽]. The constraint [𝔸] is satisfied provided

$$a\kappa_0(1+\bar\ell)+\kappa\le\frac{1}{4}\delta_\star. \tag{4.61}$$

Since from (4.32) we have

$$3\gamma-\frac{d+2s}{2}-2\kappa-a\kappa_0=3(\gamma-\beta)+3\beta-\frac{d+2s}{2}-2\kappa-a\kappa_0=3(\gamma-\beta)+\frac{3}{4}\delta_\star-\frac{7}{2}\kappa-a\kappa_0,$$

the condition [𝔹] is satisfied provided

$$4\vartheta\le\frac{3}{4}\delta_\star-\frac{7}{2}\kappa-a\kappa_0,$$

and

$$3\vartheta\le 2\beta=2s-\delta-\kappa=2s-\frac{\delta_\star}{2}-\kappa,$$

that is, whenever

$$\frac{7}{2}\kappa+a\kappa_0\le\frac{\delta_\star}{2},\qquad \kappa\le 2s-\frac{11}{16}\delta_\star. \tag{4.62}$$

Note that

$$2s-\frac{11}{16}\delta_\star\ge 2s-\delta_\star\ge 2s-\frac{4s-d}{3}=\frac{2s+d}{3}>0,$$

so the second condition in (4.62) can be indeed satisfied. To meet condition [ℂ], we require

$$2a\kappa_0\le\frac{\delta_\star}{4}, \tag{4.63}$$

and choose $\bar\ell$ large enough so that

$$\alpha+\frac{1}{16}\delta_\star\le\frac{\delta_\star}{4}\bar\ell-\frac{\delta_\star}{8}\le\left(\frac{\delta_\star}{2}-2a\kappa_0\right)\bar\ell-a\kappa_0.$$

Given $\bar\ell$ as above, we fix $\bar k$ large enough as discussed in Sec. 4.1. Then we fix a sufficiently large so that

$$a\ge\bar k(2s-\delta)/\bar\kappa\ge 2\bar k\beta/\bar\kappa\ge 2\bar k\gamma/\bar\kappa,$$

Consequently, relation (1.6) compels us to set $\nu=\gamma/a$. To ensure condition [𝔻] we introduce a constraint

$$(1+\bar\ell)\kappa_0\le\bar\kappa/2. \tag{4.64}$$

Condition [𝔼] is met if

$$\kappa\le\frac{8s+d}{6}, \tag{4.65}$$

since then

$$\vartheta=2\vartheta-\vartheta=\frac{1}{16}\delta_\star-\vartheta\le\left(\frac{s}{2}-\frac{\delta_\star}{16}-\frac{\kappa}{4}\right)-\vartheta=\beta-\vartheta=\gamma$$

and

$$\vartheta=\beta-\gamma=s-\frac{\delta_\star}{4}-\frac{\kappa}{2}-\gamma\le 2s-\gamma.$$

To conclude, we fix κ_0 and κ small enough to satisfy the remaining constraints, namely [G], [H] as well as (4.61), (4.62), (4.63), (4.64), (4.65).

Let us now detail the analysis. As a first step, we extract the coercive contribution from the effective force by defining

$$\begin{aligned} Q_\sigma(\psi) &:= \mathcal{J}_\sigma F_\sigma(\psi_\sigma) - (-\lambda\,\psi_\sigma^3) \\ &= \mathcal{J}_\sigma F_\sigma(\psi_\sigma) - \mathcal{J}_\sigma(-\lambda\,\psi_\sigma^3) - (1-\mathcal{J}_\sigma)(\lambda\,\psi_\sigma^3) \\ &= \mathcal{J}_\sigma F_\sigma^{[>0]}(\psi_\sigma) + \mathcal{J}_\sigma F^{[0],(1)}\psi_\sigma + \mathcal{J}_\sigma F^{[0],(0)} - (1-\mathcal{J}_\sigma)(\lambda\,\psi_\sigma^3), \end{aligned} \tag{4.66}$$

where ψ is a generic field,

$$\psi_\sigma := \mathcal{J}_\sigma \psi$$

and

$$F^{[0],0} = \xi^{(\varepsilon,M)}, \qquad F^{[0],1}(\psi) = \bar{r}\,\psi, \qquad F_\sigma^{[>0]}(\psi) := \sum_{\mathfrak{a}\,|\,\ell(\mathfrak{a})>0} F_\sigma^{\mathfrak{a}}(\psi^{\otimes k(\mathfrak{a})}).$$

Recall that the norms $|||\bullet||| = |||\bullet|||_{\bar\mu}$ and $|||\bullet|||_\# = |||\bullet|||_{\#,\bar\mu}$ depending on the terminal scale $\bar\mu \in [1/2,1)$ were introduced in Def. 2.2.

Lemma 4.22. *Assume* (4.71), (4.74), (4.78) *and*

$$\bar{k} \geq 5, \qquad \bar\kappa \geq \bar{k}\nu + (1+\bar\ell)\kappa_0. \tag{4.67}$$

Then the bounds

$$\|\zeta_\mu^{\bar\kappa} Q_\sigma(\psi_\sigma)\| \;\lesssim\; [\![\sigma]\!]^{-3\gamma+\vartheta}\Big[\|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar{k}} + (1+|||\psi|||)^2\,|||\mathcal{L}\psi|||_\#\Big] \tag{4.68}$$

and

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma(\psi_\sigma)\| \;\lesssim\; [\![\sigma]\!]^{-3\gamma}\,\|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar{k}}, \tag{4.69}$$

hold uniformly in $\bar\mu,\mu \in [1/2,1)$, $\sigma \in [\mu\vee\bar\mu, 1)$ and $\psi \in \mathcal{S}'(\Lambda)$.

Proof. First, observe that by the triangular inequality applied to the decomposition (4.66), we have

$$\begin{aligned} \|\zeta_\mu^{\bar\kappa} Q_\sigma(\psi_\sigma)\| &\leq \|\zeta_\mu^{\bar\kappa}\mathcal{J}_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| + \|\zeta_\mu^{\bar\kappa}\mathcal{J}_\sigma(F^{[0],(1)}\psi_\sigma + F^{[0],(0)})\| + \|\zeta_\mu^{\bar\kappa}(1-\mathcal{J}_\sigma)(\lambda\,\psi_\sigma^3)\| \\ &\lesssim \|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| + \|\zeta_\mu^{\bar\kappa} K_\sigma(F^{[0],(1)}\psi_\sigma + F^{[0],(0)})\| + \|\zeta_\mu^{5\nu}(1-\mathcal{J}_\sigma)(\lambda\,\psi_\sigma^3)\|. \end{aligned}$$

To obtain the second estimate, we used that $\bar\kappa \geq 5\nu$, which is a consequence of (4.67), together with the identity $\mathcal{J}_\sigma = L_\sigma \mathcal{J}_\sigma K_\sigma$ and the bound $\|L_\sigma\mathcal{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1$, the latter being a consequence of Lemma 1.17. The bound (4.68) follows now from Lemmas 4.23, 4.24 and 4.27. Since $\bar\kappa \geq 3\nu$ and $\|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1$ by Lemma 1.17, we have

$$\|\zeta_\mu^{\bar\kappa} K_\sigma\,\psi_\sigma^3\| \lesssim \|\zeta_\mu^{3\nu} K_\sigma\,\psi_\sigma^3\| \lesssim \|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\,\|\zeta_\mu^{3\nu}\psi_\sigma^3\| \lesssim \|\zeta_\mu^{3\nu}\psi_\sigma^3\| \lesssim [\![\sigma]\!]^{-3\gamma}\,|||\psi|||^3.$$

The bound (4.69) follows now from the estimate

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma(\psi_\sigma)\| \lesssim \|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| + \|\zeta_\mu^{\bar\kappa} K_\sigma(F^{[0],(1)}\psi_\sigma + F^{[0],(0)})\| + \|\zeta_\mu^{\bar\kappa} K_\sigma\,\psi_\sigma^3\|$$

and Lemmas 4.23 and 4.24. □

Lemma 4.23. *Assume* (4.67). *Then for every $\mathfrak{a} \in \mathfrak{A}$ such that $\ell(\mathfrak{a}) > 0$, the bound*

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{\mathfrak{a}}\big(\psi_\sigma^{\otimes k(\mathfrak{a})}\big)\| \lesssim [\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-a\kappa_0(1+\bar\ell)}\,\|F^{\mathfrak{A}}\|\,|||\psi|||^{k(\mathfrak{a})} \tag{4.70}$$

holds uniformly in $\bar\mu,\mu \in [1/2,1)$, $\sigma \in [\mu\vee\bar\mu,1)$ and $\psi \in \mathcal{S}'(\Lambda)$. Moreover, provided

$$\vartheta \leq 3(\gamma-\beta) + \delta - a\kappa_0(1+\bar\ell) - \kappa, \tag{4.71}$$

the following bound

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\, \|F^{\mathfrak{A}}\|\, (1+|||\psi|||)^{\bar k}.$$

holds uniformly in $\bar\mu,\mu\in[1/2,1)$, $\sigma\in[\mu\vee\bar\mu,1)$ *and* $\psi\in\mathcal{S}'(\Lambda)$.

Proof. We first observe that by $\tilde{\mathcal{J}}_\sigma\mathcal{J}_\sigma=\mathcal{J}_\sigma$ and $K_\sigma L_\sigma=1$, it holds that

$$\psi_\sigma=\tilde{\mathcal{J}}_\sigma\psi_\sigma=K_\sigma^2L_\sigma^2\tilde{\mathcal{J}}_\sigma\psi_\sigma.$$

Consequently, it follows from $\tilde K_\mu^{\mathfrak{a}}=K_\mu\otimes(K_\mu^2)^{\otimes k(\mathfrak{a})}$ and Def. 4.6 of the kernel norm that

$$\|\zeta_\mu^{\bar\kappa}K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma^{\otimes k(\mathfrak{a})})\|\lesssim\|\mathfrak{w}_{\mu,\sigma}\|\,\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\sigma^{\mathfrak{a}}F_\sigma^{\mathfrak{a}}]\cdot\tilde w_\sigma^{\mathfrak{a}}\|\,\|\rho_\mu L_\sigma^2\tilde{\mathcal{J}}_\sigma\psi_\sigma\|^{k(\mathfrak{a})},$$

where

$$\mathfrak{w}_{\mu,\sigma}(z,z_1,\dots,z_{k(\mathfrak{a})}):=\frac{\zeta_\mu^{\bar\kappa}(z)}{\mathfrak{o}^{\mathfrak{a}}(z)\,\tilde w_\sigma^{\mathfrak{a}}(z,z_1,\dots,z_{k(\mathfrak{a})})\,\rho_\mu(z_1)\dots\rho_\mu(z_{k(\mathfrak{a})})},\qquad \|\mathfrak{w}_{\mu,\sigma}\|=\|\mathfrak{w}_{\mu,\sigma}\|_{L^\infty(\Lambda^{1+k(\mathfrak{a})})}.$$

Using that the weight $\mathfrak{o}$ is at scale 1 (cf. Def. 4.3) and that the weight ζ_μ is at scale $[\![\mu]\!]^{-a}$ (cf. Def. 1.5), we deduce

$$\mathfrak{o}(z)^{-1}=\langle z\rangle_s^{\kappa_\mathfrak{o}}\lesssim[\![\mu]\!]^{-a\kappa_\mathfrak{o}}\langle[\![\mu]\!]^a.z\rangle_s^{\kappa_\mathfrak{o}}=[\![\mu]\!]^{-a\kappa_\mathfrak{o}}\zeta_\mu^{-\kappa_\mathfrak{o}}(z).\tag{4.72}$$

Noting that the Steiner diameter of a collection of points is always at least as large as the distance between any two points, we exploit the weight $(\tilde w_\sigma^a)^{-1}$ to propagate the weights ρ_μ^{-1} to the output variable. Using furthermore that $\rho_\mu=\zeta_\mu^\nu$ and $\mathfrak{o}^{\mathfrak{a}}=\mathfrak{o}^{1+\ell(\mathfrak{a})}$ as well as $\ell(\mathfrak{a})\le\bar\ell$, $k(\mathfrak{a})\le\bar k$ and (4.67), we obtain

$$\begin{aligned}\|\mathfrak{w}_{\mu,\sigma}\| &\lesssim \|\zeta_\mu^{\bar\kappa}\big(\mathfrak{o}^{\mathfrak{a}}\rho_\mu^{k(\mathfrak{a})}\big)^{-1}\|_{L^\infty(\Lambda)}\\ &\lesssim [\![\mu]\!]^{-a\kappa_\mathfrak{o}(1+\ell(\mathfrak{a}))}\,\|\zeta_\mu^{\bar\kappa-\kappa_\mathfrak{o}(1+\ell(\mathfrak{a}))-\nu k(\mathfrak{a})}\|_{L^\infty(\Lambda)}\\ &\lesssim [\![\mu]\!]^{-a\kappa_\mathfrak{o}(1+\bar\ell)}\end{aligned}\tag{4.73}$$

We also observe that, by Lemmas 1.17 and 2.3,

$$\|\rho_\mu L_\sigma^2\tilde{\mathcal{J}}_\sigma\psi_\sigma\|\lesssim\|L_\sigma^2\tilde{\mathcal{J}}_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\|\rho_\mu\psi_\sigma\|_{L^\infty}\lesssim[\![\sigma]\!]^{-\gamma}\,|||\psi|||.$$

Moreover, by Def. 4.6,

$$\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\sigma^{\mathfrak{a}}F_\sigma^{\mathfrak{a}}]\cdot\tilde w_\sigma^{\mathfrak{a}}\|=\|F_\sigma^{\mathfrak{a}}\|_\sigma\lesssim[\![\sigma]\!]^{[\mathfrak{a}]}\|F^{\mathfrak{A}}\|.$$

Combining the above estimates and using (4.2), we conclude that

$$\|\zeta_\mu^{\bar\kappa}K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma^{\otimes k(\mathfrak{a})})\|\lesssim[\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-a\kappa_\mathfrak{o}(1+\bar\ell)}\,\|F^{\mathfrak{A}}\|\,|||\psi|||^{k(\mathfrak{a})},$$

which proves (4.70).

To prove the second part, we observe that

$$\begin{aligned}\|\zeta_\mu^{\bar\kappa}K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| &\le \sum_{\mathfrak{a}\,|\,\ell(\mathfrak{a})>0,k(\mathfrak{a})\le\bar k}\|\zeta_\mu^{\bar\kappa}K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma)\|\\ &\lesssim \sum_{\mathfrak{a}\,|\,\ell(\mathfrak{a})>0,k(\mathfrak{a})\le\bar k}[\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-a\kappa_\mathfrak{o}(1+\bar\ell)}\,\|F^{\mathfrak{A}}\|\,|||\psi|||^{k(\mathfrak{a})}\\ &\lesssim [\![\sigma]\!]^{-\alpha+\delta-a\kappa_\mathfrak{o}(1+\bar\ell)}\,\|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar k},\end{aligned}$$

where we used that $\beta\ge\gamma$, $1\le\ell(\mathfrak{a})\le\bar\ell$ and $k(\mathfrak{a})\le\bar k$. Since, by $\alpha=3\beta+\kappa$ and (4.71), we have

$$3\gamma-\alpha+\delta-a\kappa_\mathfrak{o}(1+\bar\ell)=3(\gamma-\beta)+\delta-a\kappa_\mathfrak{o}(1+\bar\ell)-\kappa\le\vartheta,$$

the proof is complete. □

Lemma 4.24. *Assume* (4.67) *and*

$$\vartheta \le 2\gamma \wedge \Big(3\gamma - \frac{d+2s}{2} - 2\kappa - a\kappa_{\mathfrak{o}}\Big). \tag{4.74}$$

Then the bound

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[0],(0)}\| \vee \|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[0],(1)}(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\, \|F^{\mathfrak{A}}\|\,(1+|||\psi|||),$$

holds uniformly in $\bar\mu,\mu\in[1/2,1), \sigma\in[\mu\vee\bar\mu,1)$ *and* $\psi\in\mathcal{S}'(\Lambda)$.

Proof. On account of (4.13) and (4.14), $\partial_\sigma F_\sigma^{[0]}=0$. Thus,

$$F_\sigma^{[0],(0)}(z) = F_1^{[0],(0)}(z) = \xi^{(\varepsilon,M)}(z), \qquad F_\sigma^{[0],(1)}(z,z_1) = F_1^{[0],(1)}(z,z_1) = \bar r\,\delta(z,z_1).$$

Recall the definition (4.9) of $\|F^{\mathfrak{A}}\|$. By Lemmas 1.17 and 2.3, $\bar\kappa\ge\nu$ and $\vartheta\le 2\gamma$, we obtain

$$\begin{aligned}\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[0],(1)}(\psi_\sigma)\| &= |\bar r|\,\|\zeta_\mu^{\bar\kappa} K_\sigma\psi_\sigma\| \lesssim \|F^{\mathfrak{A}}\|\,\|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\,\|\zeta_\mu^{\bar\kappa}\psi_\sigma\| \\ &\lesssim [\![\sigma]\!]^{-\gamma}\,\|F^{\mathfrak{A}}\|\,|||\psi||| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\,\|F^{\mathfrak{A}}\|\,|||\psi|||.\end{aligned}$$

Moreover, by (4.74), we have

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[0],(0)}\| \lesssim \|\zeta_\mu^{\bar\kappa}\mathfrak{o}^{-1}\|\,\|\mathfrak{o} K_\sigma F^{[0],(0)}\| \lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}}[\![\sigma]\!]^{-(d+2s)/2-2\kappa}\,\|F^{\mathfrak{A}}\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\,\|F^{\mathfrak{A}}\|,$$

where the bound $\|\zeta_\mu^{\bar\kappa}\mathfrak{o}^{-1}\|\lesssim[\![\mu]\!]^{-a\kappa_{\mathfrak{o}}}$ follows from (4.72), since $\bar\kappa\ge\kappa_{\mathfrak{o}}$ by (4.67). □

Lemma 4.25. *Assume* (4.67) *and*

$$\vartheta \le \delta - a\kappa_{\mathfrak{o}}(1+\bar\ell). \tag{4.75}$$

Then the bound

$$\|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot G_\sigma\hat\psi))\| \lesssim [\![\sigma]\!]^{\vartheta-1}\,\|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar k}\,\|\zeta_\mu^{1-\bar\kappa}\tilde{\mathcal{J}}_\sigma^2\hat\psi\|,$$

holds uniformly in $\bar\mu\in[1/2,1)$, $\mu\in[1/2,1)$, $\sigma\in[\mu\vee\bar\mu,1)$ *and* $\psi,\hat\psi\in\mathcal{S}'(\Lambda)$.

Proof. We have

$$\|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot G_\sigma\hat\psi))\| \lesssim \sum_{\mathfrak{a}\,|\,k(\mathfrak{a})>0} \|\zeta_\mu K_\sigma F_\sigma^{\mathfrak{a}}\big(\psi_\sigma^{\otimes(k(\mathfrak{a})-1)}\otimes\dot G_\sigma\hat\psi\big)\|.$$

Observe that since $\dot G_\sigma := \mathcal{L}_\varepsilon^{-1}\dot{\mathcal{J}}_\sigma$, $\tilde{\mathcal{J}}_\sigma^2\dot{\mathcal{J}}_\sigma=\dot{\mathcal{J}}_\sigma$, $\tilde{\mathcal{J}}_\sigma\mathcal{J}_\sigma=\mathcal{J}_\sigma$ and $K_\sigma L_\sigma=1$,

$$\psi_\sigma = K_\sigma^2 L_\sigma^2\tilde{\mathcal{J}}_\sigma\psi_\sigma, \qquad \dot G_\sigma\hat\psi = \tilde{\mathcal{J}}_\sigma^2\dot G_\sigma\hat\psi = K_\sigma^2L_\sigma^2\dot G_\sigma\tilde{\hat\psi}, \qquad \tilde{\hat\psi}=\tilde{\mathcal{J}}_\sigma^2\hat\psi.$$

Using Def. 4.6 of the kernel norm, we obtain

$$\|\zeta_\mu K_\sigma F_\sigma^{\mathfrak{a}}\big(\psi_\sigma^{\otimes(k(\mathfrak{a})-1)}\otimes\dot G_\sigma\tilde{\hat\psi}\big)\| \lesssim \|\mathfrak{w}_{\mu,\sigma}\|\,\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\sigma^{\mathfrak{a}}F_\sigma^{\mathfrak{a}}]\cdot\tilde w_\sigma^{\mathfrak{a}}\|\,\|\rho_\mu L_\sigma^2\tilde{\mathcal{J}}_\sigma\psi_\sigma\|^{k(\mathfrak{a})-1}\,\|\zeta_\mu^{1-\bar\kappa}L_\sigma^2\dot G_\sigma\tilde{\hat\psi}\|,$$

where

$$\mathfrak{w}_{\mu,\sigma}(z,z_1,\dots,z_{k(\mathfrak{a})}) := \frac{\zeta_\mu(z)}{\mathfrak{o}^{\mathfrak{a}}(z)\,\tilde w_\sigma^{\mathfrak{a}}(z,z_1,\dots,z_{k(\mathfrak{a})})\,\zeta_\mu^{1-\bar\kappa}(z_1)\rho_\mu(z_2)\dots\rho_\mu(z_{k(\mathfrak{a})})}, \qquad \|\mathfrak{w}_{\mu,\sigma}\| = \|\mathfrak{w}_{\mu,\sigma}\|_{L^\infty(\Lambda^{1+k(\mathfrak{a})})}.$$

Using furthermore that $\rho_\mu=\zeta_\mu^\nu$ and $\mathfrak{o}^{\mathfrak{a}}=\mathfrak{o}^{1+\ell(\mathfrak{a})}$ as well as $\ell(\mathfrak{a})\le\bar\ell$, $k(\mathfrak{a})\le\bar k$ and (4.67), we obtain

$$\begin{aligned}\|\mathfrak{w}_{\mu,\sigma}\|_{L^\infty} &\lesssim \|\zeta_\mu\big(\mathfrak{o}^{\mathfrak{a}}\,\zeta_\mu^{1-\bar\kappa}\rho_\mu^{(k(\mathfrak{a})-1)}\big)^{-1}\|_{L^\infty(\Lambda)} \\ &\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))}\,\|\zeta_\mu^{\bar\kappa-\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))-\nu(k(\mathfrak{a})-1)}\|_{L^\infty(\Lambda)} \\ &\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\bar\ell)}.\end{aligned} \tag{4.76}$$

We also observe that, by Lemmas 1.17 and 2.3,

$$\|\rho_\mu L_\sigma^2 \tilde{J}_\sigma \psi_\sigma\| \lesssim \|L_\sigma^2 \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \|\rho_\mu \psi_\sigma\|_{L^\infty} \lesssim [\![\sigma]\!]^{-\gamma} |||\psi|||$$

and

$$\|\zeta_\mu^{1-\bar\kappa} L_\sigma^2 \dot{G}_\sigma \tilde\psi\| \lesssim \|L_\sigma^2 \dot{G}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \|\zeta_\mu^{1-\bar\kappa} \tilde\psi\| \lesssim [\![\sigma]\!]^{2s-1} \|\zeta_\mu^{1-\bar\kappa} \tilde\psi\|.$$

Moreover, by Def. 4.6,

$$\|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K_\sigma^{\mathfrak{a}} F_\sigma^{\mathfrak{a}}] \cdot \tilde w_\sigma^{\mathfrak{a}}\| = \|F_\sigma^{\mathfrak{a}}\|_\sigma \lesssim [\![\sigma]\!]^{[\mathfrak{a}]} \|F^{\mathfrak{A}}\|.$$

Combining the above estimates and using (4.2), we conclude that

$$\begin{aligned} \|\zeta_\mu K_\sigma F_\sigma^{\mathfrak{a}} (\psi_\sigma^{\otimes(k(\mathfrak{a})-1)} \otimes \dot{G}_\sigma \tilde\psi)\| &\lesssim [\![\sigma]\!]^{2s-\alpha+\beta+(\beta-\gamma)(k(\mathfrak{a})-1)-a\kappa_0(1+\bar\ell)-1} \|F^{\mathfrak{A}}\| (1+|||\psi|||)^{\bar k} \|\zeta_\mu^{1-\bar\kappa}\tilde\psi\| \\ &\lesssim [\![\sigma]\!]^{\delta-a\kappa_0(1+\bar\ell)-1} \|F^{\mathfrak{A}}\| (1+|||\psi|||)^{\bar k} \|\zeta_\mu^{1-\bar\kappa}\tilde\psi\|, \end{aligned}$$

where in the last step we used $2s-\alpha+\beta=\delta$ and that $\beta \geq \gamma$. This finishes the proof. □

Lemma 4.26. *Assume that*

$$\vartheta \leq \delta\bar\ell - \alpha - a\kappa_0(1+2\bar\ell). \tag{4.77}$$

Then

$$H_\sigma(\psi) := \partial_\sigma F_\sigma(\psi) + \mathrm{D}F_\sigma(\psi)(\dot G_\sigma F_\sigma(\psi)), \qquad \psi \in \mathcal{S}'(\Lambda),$$

satisfies the bound

$$\|\zeta_\mu K_\sigma H_\sigma(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{\vartheta-1} \|F^{\mathfrak{A}}\|^2 (1+|||\psi|||)^{2\bar k}.$$

uniformly in $\bar\mu, \mu \in [1/2, 1)$, $\sigma \in [\mu \vee \bar\mu, 1)$ and $\psi \in \mathcal{S}'(\Lambda)$.

Proof. We observe that on account of the perturbative flow equation, it holds

$$H_\sigma(\psi_\sigma) = \sum_{\mathfrak{a} \notin \mathfrak{A}(\bar\ell)} \sum_{\mathfrak{b},\mathfrak{c} \in \mathfrak{A}(\bar\ell)} B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot G_\sigma, F_\sigma^{\mathfrak{b}}, F_\sigma^{\mathfrak{c}})(\psi_\sigma^{\otimes k(\mathfrak{a})}),$$

where $\mathfrak{A}(\bar\ell) := \{\mathfrak{a} \in \mathfrak{A} | \ell(\mathfrak{a}) \leq \bar\ell\}$. Then, working as in the proof of Lemma 4.23 and using Lemma 4.9 as well as $k(\mathfrak{a}) = k(\mathfrak{b}) + k(\mathfrak{c}) - 1 \leq 2\bar k$, we obtain

$$\begin{aligned} \|\zeta_\mu K_\sigma H_\sigma(\psi_\sigma)\| &\leq \sum_{\mathfrak{a} \notin \mathfrak{A}(\bar\ell)} \sum_{\mathfrak{b},\mathfrak{c} \in \mathfrak{A}(\bar\ell)} \|\zeta_\mu K_\sigma B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot G_\sigma, F_\sigma^{\mathfrak{b}}, F_\sigma^{\mathfrak{c}})(\psi_\sigma^{\otimes k(\mathfrak{a})})\| \\ &\lesssim \sum_{\mathfrak{a} \notin \mathfrak{A}(\bar\ell)} \sum_{\mathfrak{b},\mathfrak{c} \in \mathfrak{A}(\bar\ell)} [\![\mu]\!]^{-a\kappa_0(1+\ell(\mathfrak{a}))} [\![\sigma]\!]^{-\gamma k(\mathfrak{a})} \|B_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}}(\dot G_\sigma, F_\sigma^{\mathfrak{b}}, F_\sigma^{\mathfrak{c}})\|_\sigma |||\psi|||^{k(\mathfrak{a})} \\ &\lesssim \sum_{\mathfrak{a} \notin \mathfrak{A}(\bar\ell)} \sum_{\mathfrak{b},\mathfrak{c} \in \mathfrak{A}(\bar\ell)} [\![\mu]\!]^{-a\kappa_0(1+\ell(\mathfrak{a}))} [\![\sigma]\!]^{[\mathfrak{a}]-[\mathfrak{b}]-[\mathfrak{c}]-\gamma k(\mathfrak{a})} \|F_\sigma^{\mathfrak{b}}\|_\sigma \|F_\sigma^{\mathfrak{c}}\|_\sigma |||\psi|||^{k(\mathfrak{a})} \\ &\lesssim \sum_{\mathfrak{a} \notin \mathfrak{A}(\bar\ell)} \sum_{\mathfrak{b},\mathfrak{c} \in \mathfrak{A}(\bar\ell)} [\![\sigma]\!]^{[\mathfrak{a}]-\gamma k(\mathfrak{a})-a\kappa_0(1+\ell(\mathfrak{a}))-1} \|F^{\mathfrak{A}}\|^2 (1+|||\psi|||)^{2\bar k}. \end{aligned}$$

The factor $[\![\mu]\!]^{-a\kappa_0(1+\ell(\mathfrak{a}))}$ appearing in the second line arises due to the estimate (4.72) by a bound similar to (4.73). Now since $\beta \geq \gamma$ and $\ell(\mathfrak{a}) \in [\bar\ell, 2\bar\ell]$, we have

$$[\mathfrak{a}] - \gamma k(\mathfrak{a}) - a\kappa_0(1+\ell(\mathfrak{a})) = -\alpha + (\beta-\gamma)k(\mathfrak{a}) + \delta\ell(\mathfrak{a}) - a\kappa_0(1+\ell(\mathfrak{a})) \geq -\alpha + \delta\bar\ell - a\kappa_0(1+2\bar\ell) > 0,$$

where we exploited (4.2) and (4.77). This proves the claim. □

Lemma 4.27. *For all*

$$\vartheta \leq 2s - 2\gamma \tag{4.78}$$

the following bound

$$\|\zeta_\mu^{5\nu}(1-\mathcal{J}_\sigma)(\psi_\sigma^3)\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\,|||\psi|||^2\,|||\mathcal{L}\psi_\bullet|||_\#.$$

holds uniformly in $\bar\mu,\mu\in[1/2,1)$, $\sigma\in[\mu\vee\bar\mu,1)$ *and* $\psi\in\mathcal{S}'(\Lambda)$.

Proof. Let $\mathcal{J}_{>\eta}:=1-\mathcal{J}_\eta$. Then

$$(1-\mathcal{J}_\sigma)(\psi_\sigma^3)=(1-\mathcal{J}_\sigma)\big[(\mathcal{J}_\eta\psi_\sigma)^3+(\mathcal{J}_{>\eta}\psi_\sigma)^3+3(\mathcal{J}_\eta\psi_\sigma)^2(\mathcal{J}_{>\eta}\psi_\sigma)+3(\mathcal{J}_\eta\psi_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^2\big].$$

Choosing $\eta=\frac{\sigma}{6-5\sigma}$, which implies $6\eta(1-\eta)^{-1}=\sigma(1-\sigma)^{-1}$, the first contribution $(1-\mathcal{J}_\sigma)(\mathcal{J}_\eta\psi_\sigma)^3$ vanishes by the Fourier space support property of the product. As for the other contributions, we have, for example,

$$\|\zeta_\mu^{5\nu}(1-\mathcal{J}_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^3\|=\|\rho_\mu^5(1-\mathcal{J}_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^3\|\lesssim\|\rho_\mu^5(\mathcal{J}_{>\eta}\psi_\sigma)^3\|\lesssim\|\rho_\mu\,\psi_\sigma\|^2\,\|\rho_\mu^3(\mathcal{J}_{>\eta}\psi_\sigma)\|.$$

Since $\sigma\geqslant\mu$ and $[\![\eta]\!]\approx[\![\sigma]\!]$, by Lemmas 2.3 and A.22, we have

$$\|\zeta_\mu^{5\nu}(1-\mathcal{J}_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^3\|\lesssim[\![\eta]\!]^{2s}[\![\sigma]\!]^{-5\gamma}\,|||\psi|||^2\,|||\mathcal{L}\psi_\bullet|||_\#\lesssim[\![\sigma]\!]^{2s-5\gamma}\,|||\psi|||^2\,|||\mathcal{L}\psi_\bullet|||_\#.$$

The contributions coming from the terms $(\mathcal{J}_\eta\psi_\sigma)^2(\mathcal{J}_{>\eta}\psi_\sigma)$ and $(\mathcal{J}_\eta\psi_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^2$ satisfy the same bound. Since $\vartheta\leqslant 2s-2\gamma$, this finishes the proof. □

Appendix A Auxiliary estimates

We collect in this appendix various technical estimates of general character.

A.1 Kernel estimates

Definition A.1. *For* $A\in\mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ *and* $(k_0,\tilde k)\in\mathbb{R}\times\mathbb{R}^d$, *we define*

$$|\bar A|:=\sum_{i=1}^d(|A^{i+}|+|A^{i-}|),\qquad |A|:=2s|A_0|+|\bar A|,\qquad \partial^A:=\partial_0^{A_0}\prod_{i=1}^d(\partial^{i+})^{A_{i+}}(\partial^{i-})^{A_{i-}}$$

and

$$d_\varepsilon^{i,\pm}(\tilde k):=\pm\big(e^{\pm i\varepsilon\tilde k e_i}-1\big)/\varepsilon,\qquad d_\varepsilon^A(k_0,\tilde k):=(ik_0)^{A_0}\prod_{i=1}^d(d_\varepsilon^{i,+}(\tilde k))^{A_{i,+}}(d_\varepsilon^{i,-}(\tilde k))^{A_{i,-}}.$$

Lemma A.2. *Let* $\check j_{\sigma,\ell}$ *be the kernel of* $\tilde{\mathcal{J}}_{\sigma,\ell}$. *For all* $n,\ell\in\mathbb{N}_+$ *and* $A\in\mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ *it holds*

$$|\partial^A\check j_{\sigma,\ell}(t,x)|\lesssim[\![\sigma]\!]^{-d-2s-|A|}(1+|t,x|_s/[\![\sigma]\!])^{-n},$$
$$|\partial^A\partial_\sigma\check j_{\sigma,\ell}(t,x)|\lesssim[\![\sigma]\!]^{-d-2s-|A|-1}(1+|t,x|_s/[\![\sigma]\!])^{-n},$$

uniformly in $(t,x)\in\Lambda$ *and* $\sigma\in[1/2,1)$.

Proof. Recall that

$$\partial^A\check j_{\sigma,\ell}(t,x)=\int_{\Lambda_\varepsilon^*}j_{\sigma,\ell}(|k_0'|^{1/2s})\,j_{\sigma,\ell}(q_\varepsilon(\tilde k'))d_\varepsilon^A(k_0',\tilde k')\,e^{i(k_0't+\tilde k'x)}\frac{\mathrm{d}k_0'\,\mathrm{d}\tilde k'}{(2\pi)^{d+1}}.$$

We start with the proof of the bound for $\partial_\sigma\check j_{\sigma,\ell}$. Let $h:\mathbb{R}\longrightarrow\mathbb{R}_+$ be defined by $h(\omega):=-\omega\partial_\omega j(\omega)$ for $\omega\in\mathbb{R}$. It holds

$$\partial_\sigma[j_{\sigma,\ell}(|k_0'|^{1/2s})\,j_{\sigma,\ell}(q_\varepsilon(\tilde k'))]=([\![\sigma]\!]\sigma)^{-1}[h(\tau|k_0'|^{1/2s})\,j(\tau q_\varepsilon(\tilde k'))+j(\tau|k_0'|^{1/2s})\,h(\tau q_\varepsilon(\tilde k'))]\big|_{\tau=2^{-\ell}\sigma^{-1}[\![\sigma]\!]}.$$

By the change of variables $k_0 = \tau^{2s} k_0'$ and $\tilde{k} = \tau \tilde{k}'$ with $\tau \equiv 2^{-\ell}\sigma^{-1}[\![\sigma]\!]$, we get

$$(\partial^A \partial_\sigma \check{j}_{\sigma,\ell})(t,x) = \\ = \frac{\tau^{-d-2s-|A|-1}}{2^\ell \sigma^2} \int_{\Lambda^*_{\varepsilon/\tau}} (h(|k_0|^{1/2s})\, j(q_{\varepsilon/\tau}(\tilde{k})) + j(|k_0|^{1/2s})\, h(q_{\varepsilon/\tau}(\tilde{k})))\, d^A_{\varepsilon/\tau}(k_0,\tilde{k}) e^{i(k_0 t/\tau^{2s} + \tilde{k}x/\tau)} \frac{\mathrm{d}k_0 \mathrm{d}\tilde{k}}{(2\pi)^{d+1}}.$$

Recall that $q_\varepsilon(k) := [\sum_{i=1}^d (\sin(\varepsilon k_i)/\varepsilon)^2]^{1/2}$. Using the bound

$$\left|\partial_{k_0}^{a_0} \partial_{\tilde{k}}^{\tilde{a}} ([h(|k_0|^{1/2s})\, j(q_{\varepsilon/\tau}(\tilde{k})) + j(|k_0|^{1/2s})\, h(q_{\varepsilon/\tau}(\tilde{k}))]\, d^A_{\varepsilon/\tau}(k_0,\tilde{k})\right| \lesssim 1_{[0,2]}(|k_0|^{1/2s})\, 1_{[0,2]}(q_{\varepsilon/\tau}(\tilde{k})),$$

for all $(a_0,\tilde{a}) \in \mathbb{N}_0^{1+d}$ uniform in $(k_0,\tilde{k}) \in \Lambda^*_{\varepsilon/\tau}$, we show that, for all $p, q \in \mathbb{N}_+$,

$$\begin{aligned}(\tau^{-2s}|t|)^p (\tau^{-1}|x|)^q |\partial_\sigma \check{j}_{\sigma,\ell}(t,x)| &\lesssim \frac{\tau^{-d-2s-|A|-1}}{2^\ell\sigma^2} \int_{\Lambda^*_{\varepsilon/\tau}} 1_{[0,2]}(|k_0|^{1/2s}) 1_{[0,2]}(q_{\varepsilon/\tau}(\tilde{k})) \frac{\mathrm{d}k_0 \mathrm{d}\tilde{k}}{(2\pi)^{d+1}} \\ &\lesssim \tau^{-d-2s-|A|-1}.\end{aligned}$$

This proves the second of the stated bounds. The first follows by an analogous argument. □

Lemma A.3. *Let $k \in \mathbb{N}_0$ and $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ be such that $A_0 \in \{0,1\}$ and $|\bar{A}| \leq 2$. The following bounds*

$$\|\partial^A K_\mu\|_{\mathrm{TV}(w_\mu^2)} \lesssim [\![\mu]\!]^{-|A|}, \qquad \|K_{\mu,\eta}\|_{\mathrm{TV}(w_\mu^2)} \lesssim 1,$$

$$\|(L_\sigma - 1)K_\mu\|_{\mathrm{TV}(w_\mu^2)} \vee \|(\tilde{L}_\sigma^{1,k} - 1)\tilde{K}_\mu^{1,k}\|_{\mathrm{TV}((w_\mu^2)^{\otimes(1+k)})} \lesssim [\![\mu]\!]^{-2s} [\![\sigma]\!]^{2s},$$

hold uniformly in $\mu \in [0,1)$ and $\eta \in [\mu,1)$.

Proof. The lemma follows from Lemma A.4, Def. 1.16, 1.13 and 4.5 and the bound

$$w_\mu^2(t,x) \lesssim (1 + |t|^2/[\![\mu]\!]^{4s})(1 + |x|^2/[\![\mu]\!]^2),$$

which is a consequence of (1.23). □

Lemma A.4. *For $\tau \in [0,1]$ and $\eta \in [\tau,1]$ define*

$$\hat{L}_\tau := (1 + \tau^{2s}\partial_t), \qquad \bar{L}_\tau := (1 - \tau^2\Delta), \qquad \hat{K}_\tau := \hat{L}_\tau^{-1}, \qquad \bar{K}_\tau := \bar{L}_\tau^{-1}, \qquad \hat{K}_{\eta,\tau} := \hat{L}_\tau \hat{K}_\eta, \qquad \bar{K}_{\eta,\tau} := \bar{L}_\tau \bar{K}_\eta$$

and

$$\hat{w}_\tau(t) := 1 + |t|^2/\tau^{4s}, \qquad \bar{w}_\tau(x) := 1 + |x|^2/\tau^2.$$

Recall that we identify operators with their integral kernels. The following statements are true.

a) The bounds

$$\|\hat{K}_\tau\|_{\mathrm{TV}(\hat{w}_\tau)} \vee \tau^{2s} \|\partial_t \hat{K}_\tau\|_{\mathrm{TV}(\hat{w}_\tau)} \lesssim 1, \qquad \|\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \vee \tau \|\partial^{i,\pm} \bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \vee \tau^2 \|\Delta \bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \lesssim 1,$$

hold uniformly in $\tau \in (0,1]$.

b) The bounds

$$\|(1 - \hat{L}_\tau)\hat{K}_\eta\|_{\mathrm{TV}(\hat{w}_\eta)} \lesssim \tau^{2s}\eta^{-2s}, \qquad \|(1 - \bar{L}_\tau)\bar{K}_\eta\|_{\mathrm{TV}(\bar{w}_\eta)} \lesssim \tau^2 \eta^{-2} \leq \tau^{2s}\eta^{-2s},$$

hold uniformly in $\tau, \eta \in (0,1]$.

c) The bounds

$$\|\hat{K}_{\eta,\tau}\|_{\mathrm{TV}(\hat{w}_\eta)} \lesssim 1, \qquad \|\bar{K}_{\eta,\tau}\|_{\mathrm{TV}(\bar{w}_\eta)} \lesssim 1,$$

hold uniformly in $\tau\in(0,1]$ *and* $\eta\in[\tau,1]$.

Proof. The first of the bounds stated in Item a) is an immediate consequence of the fact that

$$\hat{K}_\tau(t)=\tau^{-2s}\,\mathbb{1}_{t\geqslant 0}\exp(-t/\tau^{2s}).$$

Let us proceed to the proof of the second of the bounds stated in Item a). First, recall that the heat kernel

$$G^{(\varepsilon)}(t,\bullet)=e^{-(m^2-\Delta_\varepsilon)t}$$

satisfies the following standard estimate

$$(\partial^A G^{(\varepsilon)})(t,x)\lesssim \mathbb{1}_{t\geqslant 0}\,t^{-d/2-|A|/2}\,e^{-m^2t-c|x|^2/t},\qquad A\in\mathbb{N}_0^{\{1\pm,\ldots,d\pm\}}.$$

Note that

$$\bar{K}_\tau=\int_0^\infty e^{-(1-\tau^2\Delta)\lambda}\,\mathrm{d}\lambda.$$

Consequently, we have

$$\begin{aligned}
|(\partial^A\bar{K}_\tau)(x)| &\lesssim \tau^{-d-|A|/2}\int_0^\infty \lambda^{-d/2-|A|/2}\,e^{-\lambda-c(|x|/\tau)^2/\lambda}\,\mathrm{d}\lambda\\
&\leqslant \tau^{-d-|A|/2}e^{-c^{1/2}(|x|/\tau)}\int_0^\infty \lambda^{-d/2-|A|/2}\,e^{-\lambda/2-c(|x|/\tau)^2/\lambda/2}\,\mathrm{d}\lambda\\
&= \tau^{-d-|A|/2}(|x|/\tau)^{2-d-|A|}e^{-c^{1/2}(|x|/\tau)}\int_0^\infty \lambda^{-d/2-|A|/2}\,e^{-\lambda(|x|/\tau)^2-c/\lambda}\,\mathrm{d}\lambda\\
&\leqslant \tau^{-d-|A|/2}(|x|/\tau)^{2-d-|A|}e^{-c^{1/2}(|x|/\tau)}\int_0^\infty \lambda^{-d/2-|A|/2}\,e^{-c/\lambda}\,\mathrm{d}\lambda\\
&\lesssim \tau^{-d-|A|/2}(|x|/\tau)^{2-d-|A|}(1+|x|^2/\tau^2)^{-2},
\end{aligned}$$

uniformly in $x\in R_\varepsilon^d\setminus\{0\}$ and $\tau\in(0,1]$. We also have $|\partial^A\bar{K}_\tau(0)|\lesssim\varepsilon^{2-d-|A|}$. As a result, since $d=3$, for all $A\in\mathbb{N}_0^{\{1\pm,\ldots,d\pm\}}$ such that $|A|\leqslant 1$, we obtain

$$\sum_{x\in\mathbb{R}_\varepsilon^d}\varepsilon^d(1+|x|^2/\tau^2)\,|\partial^A\bar{K}_\tau(x)|\lesssim\varepsilon^{2-|A|}+\tau^{-|A|}\sum_{x\in\mathbb{R}^d_{\varepsilon/\tau}\setminus\{0\}}(\varepsilon/\tau)^d(|x|^2+|x|^4)^{-1}\lesssim\tau^{-|A|},$$

which implies $\|\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)}\vee\tau\,\|\partial^{i,\pm}\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)}\lesssim 1$. The bound $\tau^2\,\|\Delta\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)}\lesssim 1$ follows now from the identity $\tau^2\Delta\bar{K}_\tau=\bar{K}_\tau-1$. This finishes the proof of Item a).

To prove Item b) observe that

$$(1-\hat{L}_\tau)\hat{K}_\eta=\hat{K}_\eta-\hat{K}_{\eta,\tau}=\tau^{2s}\eta^{-2s}(\hat{K}_\eta-1)$$

and

$$(1-\bar{L}_\tau)\bar{K}_\eta=\bar{K}_\eta-\bar{K}_{\eta,\tau}=\tau^2\eta^{-2}(\bar{K}_\eta-1).$$

Consequently, by Item a) we have

$$\|(1-\hat{L}_\tau)\hat{K}_\eta\|_{\mathrm{TV}(\hat{w}_\eta)}\lesssim\tau^{2s}\eta^{-2s}(\|1\|_{\mathrm{TV}(\hat{w}_\eta)}+\|\hat{K}_\eta\|_{\mathrm{TV}(\hat{w}_\eta)})\lesssim\tau^{2s}\eta^{-2s}$$

and

$$\|(1-\bar{L}_\tau)\bar{K}_\eta\|_{\mathrm{TV}(\bar{w}_\tau)}\lesssim\tau^{2}\eta^{-2}(\|1\|_{\mathrm{TV}(\bar{w}_\tau)}+\|\bar{K}_\eta\|_{\mathrm{TV}(\bar{w}_\tau)})\lesssim\tau^{2}\eta^{-2}.$$

This proves Item b). Item c) follows from Items a) and b). □

Lemma A.5. *Let* $\check{K}_\mu:=(1+[\![\mu]\!]^{2s}\partial_t)^{-\hat{\kappa}}(1-[\![\mu]\!]^2\Delta)^{-\hat{\kappa}}$. *For* $N\in\mathbb{N}_+$, $\hat{\kappa}>3/(4N)$ *the following bound*

$$\|\check{K}_\mu\|_{L^{2N/(2N-1)}(\zeta^{-1})}\lesssim[\![\mu]\!]^{-(d+2s)/2N},$$

holds uniformly in $\mu\in(0,1)$.

Proof. We have $\check{K}_\mu = \check{K}_\mu \otimes \bar{K}_\mu$, where $\check{K}_\mu$ and $\bar{K}_\mu$ are the kernels of the operators $(1+[\![\mu]\!]^{2s}\partial_t)^{-\hat{\kappa}}$ and $(1-[\![\mu]\!]^2\Delta)^{-\hat{\kappa}}$, respectively. Note that

$$|\check{K}_{\mu=0}(t)| \lesssim_\omega |t|^{\hat{\kappa}-1}(1+|t|)^{-\omega},$$

uniformly in $t\in\mathbb{R}$ for all $\omega\in\mathbb{N}_+$. Hence,

$$\|(1+|\bullet|)^\omega \check{K}_\mu(\bullet)\|_{L^{2N/(2N-1)}} \lesssim [\![\mu]\!]^{-2s/2N}\|(1+|\bullet|)^\omega \check{K}_0(\bullet)\|_{L^{2N/(2N-1)}} \lesssim_\omega [\![\mu]\!]^{-2s/2N},$$

uniformly in $\mu\in(0,1)$ for all $\omega\in\mathbb{N}_+$ provided $2N\hat{\kappa}>1$. Observe that

$$(1-[\![\mu]\!]^2\Delta)^{-\hat{\kappa}} = C_{\hat{\kappa}}\int_0^\infty e^{-(1-[\![\mu]\!]^2\Delta)\lambda}\lambda^{\hat{\kappa}-1}\mathrm{d}\lambda,$$

where $C_{\hat{\kappa}}>0$ is some constant. Consequently, by the standard estimate for the heat kernel (1.21), the kernel $\bar{K}_\mu$ satisfies the bound

$$\begin{aligned}
|\bar{K}_\mu(x)| &\lesssim [\![\mu]\!]^{-d}\int_0^\infty \lambda^{-d/2}\, e^{-\lambda - c(|x|/[\![\mu]\!])^2/\lambda}\lambda^{\hat{\kappa}-1}\,\mathrm{d}\lambda \\
&\le [\![\mu]\!]^{-d} e^{-c^{1/2}(|x|/[\![\mu]\!])}\int_0^\infty \lambda^{-d/2}\, e^{-\lambda/2 - c(|x|/[\![\mu]\!])^2/\lambda/2}\lambda^{\hat{\kappa}-1}\,\mathrm{d}\lambda \\
&= [\![\mu]\!]^{-d}(|x|/[\![\mu]\!])^{2\hat{\kappa}-d} e^{-c^{1/2}(|x|/[\![\mu]\!])}\int_0^\infty \lambda^{-d/2}\, e^{-\lambda/(|x|/[\![\mu]\!])^2 - c/\lambda}\lambda^{\hat{\kappa}-1}\,\mathrm{d}\lambda \\
&\le [\![\mu]\!]^{-d}(|x|/[\![\mu]\!])^{2\hat{\kappa}-d} e^{-c^{1/2}(|x|/[\![\mu]\!])}\int_0^\infty \lambda^{-d/2}\, e^{-c/\lambda}\lambda^{\hat{\kappa}-1}\,\mathrm{d}\lambda \\
&\lesssim_\omega [\![\mu]\!]^{-d}(|x|/[\![\mu]\!])^{2\hat{\kappa}-d}(1+|x|/[\![\mu]\!])^{-\omega},
\end{aligned}$$

where we used

$$e^{-\lambda - c(|x|/[\![\mu]\!])^2/\lambda} \le e^{-\lambda/2 - c(|x|/[\![\mu]\!])^2/\lambda/2}\, e^{-c^{1/2}(|x|/[\![\mu]\!])}.$$

We also have $|\bar{K}_\tau(0)| \lesssim \varepsilon^{2\hat{\kappa}-d}$. Hence, for all $\omega\in\mathbb{N}_+$

$$\|(1+|\bullet|)^\omega \bar{K}_\mu(\bullet)\|_{L^{2N/(2N-1)}} \lesssim_\omega [\![\mu]\!]^{-d/2N},$$

uniformly in $\mu\in(0,1)$, provided $4N\hat{\kappa}>d=3$. Using the bounds for $\check{K}_\mu$, $\bar{K}_\mu$ one easily deduces the desired bound for $\check{K}_\mu$. □

Lemma A.6. *Let $m\in\mathbb{N}_+$ and $\hat{K}_\mu := (1+[\![\mu]\!]^{2s}\partial_t)^{\hat{\kappa}-1}(1-[\![\mu]\!]^2\Delta)^{\hat{\kappa}-2}$ and*

$$\mathcal{K}_\mu(z_1,\dots,z_m) := \int_{\Lambda_\varepsilon} \hat{K}_\mu(z)\,\hat{K}_\mu(z_1+z)\cdots\hat{K}_\mu(z_m+z)\,\mathrm{d}z.$$

For $\hat{\kappa}\in[0,1/(m+1))$, the function $\mathcal{K}_\mu\in C(\Lambda_\varepsilon^m)$ is Hölder continuous and satisfies, for all $\omega\ge 0$, the bound

$$\|(w_\mu^\omega)^{\otimes m}\mathcal{K}_\mu\|_{L^\infty} \lesssim [\![\mu]\!]^{-2sm}(\varepsilon\vee[\![\mu]\!])^{-dm},$$

uniformly in $\mu\in(0,1)$.

Proof. We have

$$\hat{K}_\mu = \check{K}_\mu\otimes\bar{K}_\mu, \qquad \mathcal{K}_\mu = \check{\mathcal{K}}_\mu\otimes\bar{\mathcal{K}}_\mu,$$

where $\check{K}_\mu$, $\bar{K}_\mu$ are the kernels of the operators $(1+[\![\mu]\!]^{2s}\partial_t)^{1-\hat{\kappa}}$ and $(1-[\![\mu]\!]^2\Delta)^{2-\hat{\kappa}}$ and

$$\begin{aligned}
\check{\mathcal{K}}_\mu(t_1,\dots,t_m) &:= \int_{\mathbb{R}} \check{K}_\mu(t)\check{K}_\mu(t_1+t)\cdots\check{K}_\mu(t_m+t)\,\mathrm{d}t, \\
\bar{\mathcal{K}}_\mu(x_1,\dots,x_m) &:= \int_{\mathbb{R}^d_\varepsilon} \bar{K}_\mu(x)\bar{K}_\mu(x_1+x)\cdots\bar{K}_\mu(x_m+x)\,\mathrm{d}x.
\end{aligned}$$

Since $\hat{\kappa} < 1/2$ and $d = 3$ the symbol of the operator $(1 - [\![\mu]\!]^2\Delta)^{\hat{\kappa}-2}$ and its derivates are absolutely integrable. As a result, one easily shows that for all $\omega \in \mathbb{N}_+$,

$$|\bar{K}_\mu(x)| \lesssim_\omega (\varepsilon \vee [\![\mu]\!])^{-d} (1 + |x|/[\![\mu]\!])^{-\omega},$$

uniformly in $\mu \in (0,1)$. This implies that for all $\omega \in \mathbb{N}_+$,

$$\|((1 + |\bullet|/[\![\mu]\!])^\omega)^{\otimes m} \bar{\mathcal{K}}_\mu\|_\infty \lesssim (\varepsilon \vee [\![\mu]\!])^{-dm},$$

uniformly in $\mu \in (0,1)$. Note that

$$\check{K}_0(t) = \int \frac{\exp(\mathrm{i}pt)}{(1 + \mathrm{i}p)^{1-\hat{\kappa}}} \frac{dp}{2\pi},$$

and

$$\check{\mathcal{K}}_0(t_1, \dots, t_m) = \int_{\mathbb{R}^m} \frac{\exp(\mathrm{i}p_1 t_1 + \cdots + \mathrm{i}p_m t_m)}{(1 - \mathrm{i}(p_1 + \cdots + p_m))^{1-\hat{\kappa}} (1 + \mathrm{i}p_1)^{1-\hat{\kappa}} \cdots (1 + \mathrm{i}p_m)^{1-\hat{\kappa}}} \frac{\mathrm{d}p_1 \cdots \mathrm{d}p_m}{(2\pi)^m}.$$

Observe that for all $\alpha, \beta \in (0,1)$ such that $1 < \alpha + \beta$ there exists $C \in (0, \infty)$ such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}p}{(1 + (p+q)^2)^{\alpha/2} (1 + p^2)^{\beta/2}} \leq \frac{C}{(1 + q^2)^{\alpha/2 \wedge \beta/2}},$$

for all $q \in \mathbb{R}$. Applying the above observation recursively, one shows that

$$\|((1 + |\bullet|/[\![\mu]\!]^{2s})^\omega)^{\otimes m} \check{\mathcal{K}}_\mu\|_{L^\infty} = [\![\mu]\!]^{-2sm} \|((1 + |\bullet|)^\omega)^{\otimes m} \check{\mathcal{K}}_0\|_{L^\infty} \lesssim [\![\mu]\!]^{-2sm},$$

where the factor $[\![\mu]\!]^{-2sm}$ in the first step comes from the rescaling

$$\check{\mathcal{K}}_\mu(t_1, \dots, t_m) = [\![\mu]\!]^{-2sm} \check{\mathcal{K}}_0(t_1/[\![\mu]\!]^{2s}, \dots, t_m/[\![\mu]\!]^{2s}).$$

Using the bounds for $\check{\mathcal{K}}_\mu$, $\bar{\mathcal{K}}_\mu$ one easily deduces the bound stated in the lemma. □

Lemma A.7. *For every $\alpha \in [0, 2s)$ and $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$, it holds*

$$|\partial^A G_{1/2}(t,x)| \lesssim (1 + |t,x|_s)^{-d-\alpha} \tag{A.1}$$

and

$$|\partial^A \dot{G}_\sigma(t,x)| \lesssim [\![\sigma]\!]^{-d-1-|A|} (1 + (\varepsilon \vee |t,x|_s)/[\![\sigma]\!])^{-d} (1 + |t,x|_s/[\![\sigma]\!])^{-\alpha}, \tag{A.2}$$

uniformly in $(t,x) \in \Lambda$ and $\sigma \in [1/2, 1)$.

Remark A.8. The above lemma implies that for any $\alpha \in [0, 2s)$, $\beta \in [0, d]$ and $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$,

$$|\partial^A \dot{G}_\sigma(t,x)| \lesssim \varepsilon^{-\beta} [\![\sigma]\!]^{\beta-d-1-|A|} (1 + |t,x|_s/[\![\sigma]\!])^{-d-\alpha+\beta},$$

uniformly in $(t,x) \in \Lambda$ and $\sigma \in (1/2, 1)$.

Proof. We only prove (A.2) since the proof of (A.1) follows the same lines. Observe that

$$\partial^A \dot{G}_\sigma(t,x) = \int_{\Lambda_\varepsilon^*} \frac{\partial_\sigma(j_\sigma(|k_0'|^{1/2s}) j_\sigma(q_\varepsilon(k'))) \, d_\varepsilon^A(k_0', \tilde{k}')}{ik_0' + q_\varepsilon^{2s}(\tilde{k}')} \, e^{i(k_0' t + \tilde{k}' x)} \frac{\mathrm{d}k_0' \, \mathrm{d}\tilde{k}'}{(2\pi)^{d+1}}.$$

To prove a bound for the L^∞-norm of

$$(t,x) \mapsto t^{a_0} x^{\tilde{a}} \partial^A \dot{G}_\sigma(t,x),$$

it suffices to control the L^1-norm of

$$(k_0', k') \mapsto \partial_{k_0}^{a_0} \partial_{\tilde{k}'}^{\tilde{a}} \left(\frac{\partial_\sigma(j_\sigma(|k_0'|^{1/2s}) j_\sigma(q_\varepsilon(k'))) \, d_\varepsilon^A(k_0', \tilde{k}')}{ik_0' + q_\varepsilon^{2s}(\tilde{k}')} \right). \tag{A.3}$$

Since for noninteger parameters s the bound

$$|\partial_{\tilde{k}'}^{\tilde{a}} q_\varepsilon^{2s}(\tilde{k}')| \lesssim |\tilde{k}'|^{2s-|\tilde{a}|},$$

uniform in $\varepsilon\in(0,1)$, is optimal, the L^1-norm of the function (A.3) is bounded uniformly in $\varepsilon\in(0,1)$ only if $|\tilde{a}|<d+2s$. As a result, the above simple strategy can only be used to prove the lemma for $\alpha\in[0,\lfloor 2s\rfloor]$. To establish the claim for every $\alpha\in[0,2s)$ a more refined argument is needed. To this end, let $h\colon\mathbb{R}\longrightarrow\mathbb{R}_+$ be

$$h(\omega) := -\omega\partial_\omega j(\omega)$$

for $\omega\in\mathbb{R}$. Moreover, let the families of kernels $(\tilde{G}_\tau)_{\tau\in(0,\infty)}$, $(\check{G}_\tau)_{\tau\in(0,\infty)}$, $(\hat{G}_{\tau,\eta})_{\tau,\eta\in(0,\infty)}$ be defined by

$$\begin{aligned}
\tilde{G}_\tau(t,x) &:= \tau^{-1}\int_{\Lambda_\varepsilon^*} \frac{j(\tau|k_0'|^{1/2s})h(\tau q_\varepsilon(\tilde{k}')) + h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))}{ik_0' + m^{2s} + q_\varepsilon^{2s}(\tilde{k}')}\, e^{i(k_0't+\tilde{k}'x)}\frac{\mathrm{d}k_0'\mathrm{d}\tilde{k}'}{(2\pi)^{d+1}},\\
\hat{G}_{\tau,\eta}(t,x) &:= \tau^{-1}\eta^{-1}\int_{\Lambda_\varepsilon^*} \frac{q_\varepsilon^{2s}(\tilde{k}')\, h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))\, h(\tau\eta q_\varepsilon(\tilde{k}'))}{(k_0' - im^{2s})^2 + q_\varepsilon^{4s}(\tilde{k}')}\, e^{i(k_0't+\tilde{k}'x)}\frac{\mathrm{d}k_0'\mathrm{d}\tilde{k}'}{(2\pi)^{d+1}},\\
\check{G}_\tau(t,x) &:= \tau^{-1}\int_{\Lambda_\varepsilon^*}\left(\frac{j(\tau|k_0'|^{1/2s})h(\tau q_\varepsilon(\tilde{k}'))}{ik_0' + m^{2s} + q_\varepsilon^{2s}(\tilde{k}')} - \frac{(ik_0' + m^{2s})\, h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))}{(k_0' - im^{2s})^2 + q_\varepsilon^{4s}(\tilde{k}')}\right) e^{i(k_0't+\tilde{k}'x)}\frac{\mathrm{d}k_0'\mathrm{d}\tilde{k}'}{(2\pi)^{d+1}}.
\end{aligned}$$

We claim that

$$\dot{G}_\sigma(t,x) = \sigma^{-2}\tilde{G}_{[\![\sigma]\!]/\sigma}(t,x), \qquad \tilde{G}_\tau = \check{G}_\tau + \int_{1/2}^\infty \hat{G}_{\tau,\eta}\,\mathrm{d}\eta. \tag{A.4}$$

The first identity follows from

$$\partial_\sigma j_\sigma(\omega) = [\![\sigma]\!]^{-1}\sigma^{-1}h([\![\sigma]\!]\sigma^{-1}\omega).$$

To verify the second, we use the identities

$$\frac{h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))}{ik_0' + m^{2s} + q_\varepsilon^{2s}(\tilde{k}')} = \frac{q_\varepsilon^{2s}(\tilde{k}')\, h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))}{(k_0' - im^{2s})^2 + q_\varepsilon^{4s}(\tilde{k}')} - \frac{(ik_0' + m^{2s})\, h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))}{(k_0' - im^{2s})^2 + q_\varepsilon^{4s}(\tilde{k}')}$$

as well as

$$\int_{1/2}^\infty \eta^{-1}h(\tau\eta q_\varepsilon(\tilde{k}'))\,\mathrm{d}\eta = -\int_{1/2}^\infty \partial_\eta j(\tau\eta q_\varepsilon(\tilde{k}'))\,\mathrm{d}\eta = j(\tau q_\varepsilon(\tilde{k}')/2)$$

and

$$j(\tau q_\varepsilon(\tilde{k}'))j(\tau q_\varepsilon(\tilde{k}')/2) = j(\tau q_\varepsilon(\tilde{k}')).$$

Let us motivate the usefulness of the representation of $\dot{G}_\sigma$ given by (A.4). First, note that if $|\tilde{a}|<d+4s$, then the L^1-norm of the function

$$(k_0',\tilde{k}') \mapsto \partial_{k_0'}^{a_0}\partial_{\tilde{k}'}^{\tilde{a}}\left(\left(\frac{j(\tau|k_0'|^{1/2s})h(\tau q_\varepsilon(\tilde{k}'))}{ik_0' + m^{2s} + q_\varepsilon^{2s}(\tilde{k}')} - \frac{(ik_0' + m^{2s})h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))}{(k_0' - im^{2s})^2 + q_\varepsilon^{4s}(\tilde{k}')}\right) d_\varepsilon^A(k_0',\tilde{k}')\right),$$

is bounded uniformly in $\varepsilon\in(0,1)$. This stands in contrast to the function in (A.3), whose L^1-norm remains uniformly bounded in $\varepsilon\in(0,1)$ only if $|\tilde{a}|<d+2s$. As a result, it is possible to control the L^∞-norm of the function

$$(t,x) \mapsto t^{a_0}x^{\tilde{a}}\partial^A\check{G}_\tau(t,x)$$

for $|\tilde{a}|<d+4s$. Since $\lfloor 4s\rfloor \geq 3 > 2s > \alpha$ this indicates that $\partial^A\check{G}_\tau$ indeed exhibits the desired decay at infinity. On the other hand, the L^1-norm of the function

$$(k_0',\tilde{k}') \mapsto \partial_{k_0'}^{a_0}\partial_{\tilde{k}'}^{\tilde{a}}\left(\frac{q_\varepsilon^{2s}(\tilde{k}')h(\tau|k_0'|^{1/2s})j(\tau\, q_\varepsilon(\tilde{k}'))h(\tau\,\eta\, q_\varepsilon(\tilde{k}'))}{(k_0' - im^{2s})^2 + q_\varepsilon^{4s}(\tilde{k}')} d_\varepsilon^A(k_0',\tilde{k}')\right)$$

is bounded uniformly in $\varepsilon\in(0,1)$ for all $a_0,\tilde{a}$ and consequently $\partial^A\hat{G}_{\tau,\eta}$ has good decay properties. However, the bound for the L^∞-norm of the function

$$(t,x)\mapsto t^{a_0}x^{\tilde{a}}\,\partial^A\hat{G}_{\tau,\eta}(t,x)$$

depends on η. To complete the argument, we must therefore control the above norms uniformly in $\varepsilon\in(0,1)$, $\tau\in(0,1)$ and $\eta\in(1/2,\infty)$.

We claim that the following bounds imply (A.2),

$$\begin{aligned}
|\partial^A\tilde{G}_\tau(t,0)| &\lesssim \tau^{-1-|A|}\varepsilon^{-d}(1+|t|/\tau^{2s})^{-d-4},\\
|\partial^A\check{G}_\tau(t,x)| &\lesssim \tau^{-d-1-|A|}(1+|x|/\tau)^{-d-3}(1+|t|/\tau^{2s})^{-d-4},\\
|\partial^A\hat{G}_{\tau,\eta}(t,x)| &\lesssim \eta^{-2s-1}\tau^{-d-1-|A|}(1+|x|/\tau)^{-d}(1+|t|/\tau^{2s})^{-d-4},\\
|\partial^A\hat{G}_{\tau,\eta}(t,x)| &\lesssim \eta^{1-2s}\tau^{-d-1-|A|}(1+|x|/\tau)^{-d-2}(1+|t|/\tau^{2s})^{-d-4}.
\end{aligned}\tag{A.5}$$

To prove this claim, first use the last two bounds to conclude that for all $\alpha\in[0,2]$ it holds

$$|\partial^A\hat{G}_{\tau,\eta}(t,x)|\lesssim\eta^{\alpha-1-2s}\tau^{-d-1-|A|}\,(1+|x|/\tau)^{-d-\alpha}(1+|t|/\tau^{2s})^{-d-4}.$$

Consequently, the second identity in (A.4) and the above bound yield

$$|\partial^A\tilde{G}_\tau(t,x)|\lesssim_\alpha\tau^{-d-1-|A|}\,(1+|t,x|_s/\tau)^{-d}(1+|t,x|_s/\tau)^{-\alpha}$$

for $\alpha\in[0,2s)$. Combining the above estimate with the first estimate in (A.5) we obtain

$$|\partial^A\tilde{G}_\tau(t,x)|\lesssim_\alpha\tau^{-d-1-|A|}\,(1+(\varepsilon\vee|t,x|_s)/\tau)^{-d}(1+|t,x|_s/\tau)^{-\alpha}.$$

The bound (A.2) follows now from the first identity in (A.4).

It remains to prove (A.5). Starting from the bound for $\partial^A\hat{G}_{\tau,\eta}(t,x)$, we have

$$\begin{aligned}
&(\partial^A\hat{G}_{\tau,\eta})(t,x)=\\
&\quad=\tau^{-1}\int_{\Lambda_\varepsilon^*}\frac{q_\varepsilon^{2s}(\tilde{k}')h(\tau|k_0'|^{1/2s})j(\tau q_\varepsilon(\tilde{k}'))h(\tau\eta q_\varepsilon(\tilde{k}'))}{(k_0'-im^{2s})^2+q_\varepsilon^{4s}(\tilde{k}')}d_\varepsilon^A(k_0',\tilde{k}')\,e^{i(k_0't+\tilde{k}'x)}\frac{\mathrm{d}k_0'\,\mathrm{d}\tilde{k}'}{(2\pi)^{d+1}}\\
&\quad=\tau^{-d-1-|A|}\eta^{-1}\int_{\Lambda_{\varepsilon/\tau}^*}\frac{q_{\varepsilon/\tau}^{2s}(\tilde{k})h(|k_0|^{1/2s})j(q_{\varepsilon/\tau}(\tilde{k}))h(\eta q_{\varepsilon/\tau}(\tilde{k}))d_{\varepsilon/\tau}^A(k_0,\tilde{k})}{(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}e^{i(k_0t/\tau^{2s}+\tilde{k}x/\tau)}\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}},
\end{aligned}$$

where we set $k_0=\tau^{2s}k_0'$ and $\tilde{k}=\tau\tilde{k}'$. It follows that

$$\begin{aligned}
&(\tau^{-2s}|t|)^{a_0}(\tau^{-1}|x|)^{\tilde{a}}|(\partial^A\hat{G}_{\tau,\eta})(t,x)|\\
&\quad\lesssim\tau^{-d-1-|A|}\eta^{-1}\int_{\Lambda_{\varepsilon/\tau}^*}\left|\partial_{k_0}^{a_0}\partial_{\tilde{k}}^{\tilde{a}}\left[\frac{q_{\varepsilon/\tau}^{2s}(\tilde{k})h(|k_0|^{1/2s})j(q_{\varepsilon/\tau}(\tilde{k}))h(\eta q_{\varepsilon/\tau}(\tilde{k}))d_{\varepsilon/\tau}^A(k_0,\tilde{k})}{(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\right]\right|\frac{\mathrm{d}k_0\,\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}.
\end{aligned}\tag{A.6}$$

Observe the following:

- In (A.6), the factor $h(|k_0|^{1/2s})$ restricts the integration domain to $|k_0|^{1/2s}\in[1,2]$. In this region, the denominator is never vanishing. More precisely

$$|(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})|^2\geqslant 1.$$

- Similarly, the factor $h(\eta q_{\varepsilon/\tau}(\tilde{k}))$ restricts the integration domain to those $\tilde{k}$ such that $q_{\varepsilon/\tau}(\tilde{k})\in[\eta^{-1},2\eta^{-1}]$. Thus, on the integration domain

$$|\partial_{\tilde{k}}^{\tilde{a}}q_{\varepsilon/\tau}^{2s}(\tilde{k})|\lesssim q_{\varepsilon/\tau}^{2s-|\tilde{a}|}(\tilde{k})\approx\eta^{|\tilde{a}|-2s}.$$

On account of these comments, we have the following bound for the integrand

$$\left|\partial_{k_0}^{a_0}\partial_{\tilde{k}}^{\tilde{a}}\left(\frac{q_{\varepsilon/\tau}^{2s}(\tilde{k})h(|k_0|^{1/2s})j(q_{\varepsilon/\tau}(\tilde{k}))h(\eta q_{\varepsilon/\tau}(\tilde{k}))d_{\varepsilon/\tau}^{A}(k_0,\tilde{k})}{(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\right)\right|\lesssim \eta^{|\tilde{a}|-2s}1_{[0,2]}(|k_0|^{1/2s})1_{[1,2]}(\eta q_{\varepsilon/\tau}(\tilde{k})),$$

for all $(a_0,\tilde{a})\in\mathbb{N}_0^{1+d}$. This implies that

$$\begin{aligned}(\tau^{-2s}|t|)^p(\tau^{-1}|x|)^q|(\partial^A\hat{G}_{\tau,\eta})(t,x)| &\lesssim \tau^{-d-|A|-1}\eta^{q-2s-1}\int_{\Lambda_{\varepsilon/\tau}^*}1_{[0,2]}(|k_0|^{1/2s})1_{[1,2]}(\eta q_{\varepsilon/\tau}(\tilde{k}))\frac{\mathrm{d}k_0\,\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}\\ &\lesssim \tau^{-d-|A|-1}\eta^{q-2s-d-1}\int_{\Lambda_{\varepsilon/(\tau\eta)}^*}1_{[0,2]}(|k_0|^{1/2s})\,1_{[1,2]}(q_{\varepsilon/(\tau\eta)}(\tilde{k}))\frac{\mathrm{d}k_0\,\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}\\ &\lesssim \tau^{-d-|A|-1}\eta^{q-2s-d-1},\end{aligned}$$

for any $p,q\in\mathbb{N}_+$. The proof for $\partial^A\check{G}_\tau(t,x)$ is similar. Working as in the previous case, we have

$$\begin{aligned}&(\tau^{-2s}|t|)^p(\tau^{-1}|x|)^q|\partial^A\check{G}_\tau(t,x)|\lesssim\tau^{-d-1-|A|}\times\\ &\times\int_{\Lambda_{\varepsilon/\tau}^*}\left|\partial_{k_0}^{a_0}\partial_{\tilde{k}}^{\tilde{a}}\left[\left(\frac{j(|k_0|^{1/2s})h(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^{2s}+q_{\varepsilon/\tau}^{2s}(\tilde{k})}+\frac{(ik_0'+\tau^{2s}m^{2s})h(|k_0|^{1/2s})j(q_{\varepsilon/\tau}(\tilde{k}))}{(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\right)d_{\varepsilon/\tau}^{A}(k_0,\tilde{k})\right]\right|\frac{\mathrm{d}k_0\,\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}.\end{aligned}$$

As above, also here the denominators are non-vanishing on the integration domain, more precisely,

$$|ik_0+\tau^{2s}m^{2s}+q_{\varepsilon/\tau}^{2s}(\tilde{k})|\geq 1\qquad\text{and}\qquad|(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})|\geq 1.$$

In the first term, this is due to the factor $h(q_{\varepsilon/\tau}^{2s}(\tilde{k}))$, which restricts the integration domain to $q_{\varepsilon/\tau}^{2s}(\tilde{k})\in[1,2]$, while in the second one this is due to the factor $h(|k_0|^{1/2s})$, which restricts the integration domain to $|k_0|^{1/2s}\in[1,2]$. Since $|\partial_{\tilde{k}}^{\tilde{a}}q_{\varepsilon/\tau}^{4s}(\tilde{k})|\lesssim q_{\varepsilon/\tau}^{4s-|\tilde{a}|}(\tilde{k})$, we get

$$\begin{aligned}&\left|\partial_{k_0}^{a_0}\partial_{\tilde{k}}^{\tilde{a}}\left[\left(\frac{j(|k_0|^{1/2s})\,h(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^{2s}+q_{\varepsilon/\tau}^{2s}(\tilde{k})}+\frac{(ik_0'+\tau^{2s}m^{2s})\,h(|k_0|^{1/2s})\,j(q_{\varepsilon/\tau}(\tilde{k}))}{(k_0-i\tau^{2s}m^{2s})^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\right)d_{\varepsilon/\tau}^{A}(k_0,\tilde{k})\right]\right|\lesssim\\ &\qquad\lesssim 1_{[0,2]}(|k_0|^{1/2s})\,1_{[0,2]}(q_{\varepsilon/\tau}(\tilde{k}))\,(1\vee q_{\varepsilon/\tau}^{4s-|\tilde{a}|}(\tilde{k})).\end{aligned}$$

This implies that, for $q<d+4s$,

$$\begin{aligned}(\tau^{-2s}|t|)^p(\tau^{-1}|x|)^q|\partial^A\check{G}_\tau(t,x)| &\lesssim \tau^{-d-|A|-1}\int_{\Lambda_{\varepsilon/\tau}^*}1_{[0,2]}(|k_0|^{1/2s})1_{[0,2]}(q_{\varepsilon/\tau}(\tilde{k}))(1\vee q_{\varepsilon/\tau}^{4s-q}(\tilde{k}))\mathrm{d}k_0\mathrm{d}\tilde{k}\\ &\lesssim \tau^{-d-|A|-1}.\end{aligned}$$

Since $d+4s>d+3$ the bound for $\partial^A\check{G}_\tau$ follows. Finally, we discuss the first bound in (A.5). By changing the variables as above, we arrive at

$$\begin{aligned}&(\tau^{-2s}|t|)^p|\partial^A\tilde{G}_\tau(t,0)|\leq\\ &\qquad\leq\tau^{-1-d-|A|}\int_{\Lambda_{\varepsilon/\tau}^*}\left|\partial_{k_0}^{p}\left[\frac{j(|k_0|^{1/2s})h(q_{\varepsilon/\tau}(\tilde{k}))+h(|k_0|^{1/2s})j(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^{2s}+q_{\varepsilon/\tau}^{2s}(\tilde{k})}d_{\varepsilon/\tau}^{A}(k_0,\tilde{k})\right]\right|\mathrm{d}k_0\mathrm{d}\tilde{k}.\end{aligned}$$

We use the same argument as for $\partial^A\check{G}_\tau$: indeed both terms in the integrand have non-vanishing denominators due to the factors $h(q_{\varepsilon/\tau}(\tilde{k}))$ and $h(|k_0|^{1/2s})$, respectively. Analogously, we obtain

$$\left|\partial_{k_0}^{p}\left[\frac{j(|k_0|^{1/2s})h(q_{\varepsilon/\tau}(\tilde{k}))+h(|k_0|^{1/2s})j(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^{2s}+q_{\varepsilon/\tau}^{2s}(\tilde{k})}d_{\varepsilon/\tau}^{A}(k_0,\tilde{k})\right]\right|\lesssim 1_{[0,2]}(|k_0|^{1/2s}),$$

and thus

$$(\tau^{-2s}|t|)^p|\partial^A\tilde{G}_\tau(t,x)|\lesssim\tau^{-1-d-|A|}\int_{\Lambda_{\varepsilon/\tau}^*}1_{[0,2]}(|k_0|^{1/2s})\frac{\mathrm{d}k_0\,\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}\lesssim\tau^{-1-d-|A|}(\varepsilon\tau^{-1})^{-d}=\tau^{-1-|A|}\varepsilon^{-d}.$$

This concludes the proof. □

A.2 Properties of the weights

We collect here some important properties of the various weights introduced throughout the paper.

Lemma A.9. *For $\alpha \geq 0$, we have*

$$\|(-\Delta_\varepsilon)^s \zeta_\sigma^\alpha\| \lesssim [\![\sigma]\!]^{2sa}$$

uniformly in $\sigma \in (0,1)$.

Proof. Let $f \in C^2(\mathbb{R}^d)$. Recall that, by (1.14), we have

$$(-\Delta_\varepsilon)^s f(x) = \int_{\mathbb{R}^d_\varepsilon} H_s^{(\varepsilon)}(y)\,(f(x) - f(x-y))\,\mathrm{d}y, \qquad x \in \mathbb{R}^d_\varepsilon,$$

where the kernel $H_s^{(\varepsilon)} : \mathbb{R}^d_\varepsilon \longrightarrow \mathbb{R}$ is positive and such that $H_s^{(\varepsilon)}(0) = 0$, $H_s^{(\varepsilon)}(x) = H_s^{(\varepsilon)}(-x)$ and

$$\left|H_s^{(\varepsilon)}(x)\right| \lesssim |x|^{-d-2s}, \qquad x \in \mathbb{R}^d_\varepsilon.$$

Let $B(\delta)$ the ball in $\mathbb{R}^d_\varepsilon$ of radius $\delta > 0$ centred at the origin and $B^c(\delta) := \mathbb{R}^d_\varepsilon \setminus B(\delta)$. For any $\delta > 0$, we have

$$|(-\Delta_\varepsilon)^s f(x)| \leq \frac{1}{2}\int_{B(\delta)} H_s^{(\varepsilon)}(y)\,(2f(x) - f(x+y) - f(x-y))\,\mathrm{d}y + \int_{B^c(\delta)} H_s^{(\varepsilon)}(y)\,(f(x) - f(x-y))\,\mathrm{d}y.$$

To bound the first term observe that

$$|2f(x) - f(x+y) - f(x-y)| \lesssim \|\nabla\nabla f\|\,|y|^2,$$

by Taylor's theorem, where $\|\nabla\nabla f\|$ denotes the supremum norm of the Hessian of f in the continuum. As a result, we obtain

$$\left|\int_{B(\delta)} H_s^{(\varepsilon)}(y)\,(2f(x) - f(x+y) - f(x-y))\,\mathrm{d}y\right| \lesssim \|\nabla\nabla f\| \int_{B(\delta)} |y|^{2-d-2s}\,\mathrm{d}y \lesssim \|\nabla\nabla f\|\,\delta^{2-2s}.$$

Moreover, we have

$$\left|\int_{B^c(\delta)} H_s^{(\varepsilon)}(y)\,(f(x) - f(x-y))\,\mathrm{d}y\right| \lesssim \|f\| \int_{B^c(\delta)} |y|^{-d-2s}\,\mathrm{d}y \lesssim \|f\|\,\delta^{-2s}.$$

To conclude, we apply the above estimates with $f(x) = \zeta_\mu^\alpha(t,x)$ and choose $\delta = [\![\mu]\!]^{-as}$. □

Lemma A.10. *We have*

$$\|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})\|^2 \lesssim [\![\sigma]\!]^{2sa},$$

uniformly in $\sigma \in (0,1)$.

Proof. Recall that μ_s denotes the kernel defined by (1.15), which is local in time. Let $B(z,\delta)$ the ball of radius $\delta > 0$ centred at $z \in \Lambda$. By (1.17), we have

$$\begin{aligned}
[\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})(z)]^2 &= \zeta_\sigma^{2/3}(z)\int [\zeta_\sigma^{-1/3}(z') - \zeta_\sigma^{-1/3}(z)]^2 \mu_s(z,\mathrm{d}z') \\
&= \zeta_\sigma^{2/3}(z)\int_{B^c(z,\delta)} [\zeta_\sigma^{-1/3}(z') - \zeta_\sigma^{-1/3}(z)]^2 \mu_s(z,\mathrm{d}z') \\
&\quad + \zeta_\sigma^{2/3}(z)\int_{B(z,\delta)} [\zeta_\sigma^{-1/3}(z') - \zeta_\sigma^{-1/3}(z)]^2 \mu_s(z,\mathrm{d}z') \\
&= \mathbb{I} + \mathbb{II}.
\end{aligned}$$

On $B^c(z,\delta) := \Lambda_\varepsilon \setminus B(z,\delta)$ we have

$$\begin{aligned}\mathbb{I} = \int_{B^c(z,\delta)} [\zeta_\sigma^{1/3}(z)\zeta_\sigma^{-1/3}(z') - 1]^2 \mu_s(z, \mathrm{d}z') &\lesssim \int_{B^c(z,\delta)} [\zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z') + 1]\, \mu_s(z, \mathrm{d}z') \\ &\lesssim \int_{B^c(z,\delta)} \zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z')\, \mu_s(z, \mathrm{d}z') + \delta^{-2s}.\end{aligned}$$

Recall that

$$\zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z') \lesssim \zeta_\sigma^{-2/3}(z - z') = \langle [\![\sigma]\!]^a . (z - z')\rangle_s^{2/3} = (1 + [\![\sigma]\!]^{2a} |\bar{z} - \bar{z}'|^2)^{1/3},$$

uniformly in $z = (z^0, \bar{z}) \in \Lambda$ and $z' = (z^0, \bar{z}') \in \Lambda$. Since $2s > 2/3$,

$$\begin{aligned}\int_{B^c(z,\delta)} \zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z')\, \mu_s(z, \mathrm{d}z') &\lesssim \int_{B^c(z,\delta)} (1 + [\![\sigma]\!]^{2a} |\bar{z} - \bar{z}'|^2)^{1/3} \mu_s(z, \mathrm{d}z') \\ &\lesssim \delta^{-2s} + [\![\sigma]\!]^{2a/3} \delta^{-2s+2/3}.\end{aligned}$$

Let us bound the second term. Thanks to the smoothness of $\zeta_\sigma^{-1/3}$ in space, by Taylor's theorem

$$\mathbb{II} \lesssim \zeta_\sigma^{2/3}(z) \int_{B(z,\delta)} \|\nabla \zeta_\sigma^{-1/3}\|^2 \, |\bar{z}' - z|_s^2 \, \mu_s(z, \mathrm{d}z').$$

Since $\|\nabla \zeta_\sigma^{-1/3}\|^2 \lesssim [\![\sigma]\!]^{2a}$ and $\|\zeta_\sigma^{2/3}\| \lesssim 1$, we conclude

$$\mathbb{II} \lesssim [\![\sigma]\!]^{2a} \int_{B(z,\delta)} |\bar{z}' - \bar{z}|_s^2 \, \mu_s(z, \mathrm{d}z') \lesssim \delta^{2-2s} [\![\sigma]\!]^{2a} = [\![\sigma]\!]^{2sa}.$$

Choosing now $\delta = [\![\sigma]\!]^{-a}$, we get

$$\mathbb{I} \lesssim [\![\sigma]\!]^{2as} + [\![\sigma]\!]^{2a/3} [\![\sigma]\!]^{2sa - 2a/3} \lesssim [\![\sigma]\!]^{2as}, \qquad \mathbb{II} \lesssim \delta^{2-2s} [\![\sigma]\!]^{2a} = [\![\sigma]\!]^{2sa}.$$

This finishes the proof. □

Lemma A.11. *The bound*

$$\frac{(1 - h_\mu) w_\mu^{(2),\mathfrak{b}}}{w_\sigma^{(2),\mathfrak{b}}} \lesssim [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}}$$

holds uniformly in $\mu, \sigma \in (0,1)$.

Proof. We have

$$\begin{aligned}\frac{(1 - h_\mu(z,z_1)) w_\mu^{(2),\mathfrak{b}}(z,z_1)}{w_\sigma^{(2),\mathfrak{b}}(z,z_1)} &= \frac{w_\mu^{(2),\mathfrak{b}}(z,z_1)}{w_\sigma^{(2),\mathfrak{b}}(z,z_1)} \frac{[\![\mu]\!]^{-2} |z - z_1|_s^2}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\ &= \frac{w_\mu^{(2),\mathfrak{b}}(z,z_1)}{w_\sigma^{(2),\mathfrak{b}}(z,z_1)} \frac{([\![\mu]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}} ([\![\mu]\!]^{-1} |z - z_1|_s)^{2-\mathfrak{b}}}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\ &= [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}} \frac{(1 + [\![\mu]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}}}{(1 + [\![\sigma]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}}} \frac{([\![\sigma]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}} ([\![\mu]\!]^{-1} |z - z_1|_s)^{2-\mathfrak{b}}}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\ &\leqslant [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}} \frac{(1 + [\![\mu]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}} ([\![\mu]\!]^{-1} |z - z_1|_s)^{2-\mathfrak{b}}}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\ &\leqslant [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}} \frac{(1 + [\![\mu]\!]^{-1} |z - z_1|_s)^2}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \lesssim [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}},\end{aligned}$$

which proves the claim. □

Lemma A.12. *For all* $\omega > 0$*, the following bound*

$$\frac{\big(1 - v_\mu^{(1+m)}\big) w_\mu^{(1+m),\omega}}{w_\sigma^{(1+m),\omega}} \lesssim [\![\sigma]\!]^{\omega} [\![\mu]\!]^{-\omega}$$

holds uniformly in $\mu,\sigma\in(0,1)$.

Proof. By Def. 4.3, $1-v_\mu^{(m+1)}=0$ if $[\![\mu]\!]^{-1}\mathrm{St}(z,y_1,\dots,y_m)\leq 1$. As a consequence,

$$\begin{aligned}\frac{\big(1-v_\mu^{(1+m)}\big)w_\mu^{(1+m),\omega}}{w_\sigma^{(1+m),\omega}} &= \frac{(1+[\![\mu]\!]^{-1}\mathrm{St}(z,y_1,\dots,y_m))^\omega}{(1+[\![\sigma]\!]^{-1}\mathrm{St}(z,y_1,\dots,y_m))^\omega}\big(1-v_\mu^{(1+m)}\big)\\ &= [\![\mu]\!]^{-\omega}[\![\sigma]\!]^\omega\frac{([\![\mu]\!]+\mathrm{St}(z,y_1,\dots,y_m))^\omega}{([\![\sigma]\!]+\mathrm{St}(z,y_1,\dots,y_m))^\omega}\big(1-v_\mu^{(1+m)}\big),\end{aligned}$$

is non-vanishing only if $\mathrm{St}(z,y_1,\dots,y_m)>[\![\mu]\!]$. Hence

$$\frac{\big(1-v_\mu^{(1+m)}\big)w_\mu^{(1+m),\omega}}{w_\sigma^{(1+m),\omega}}\leq[\![\mu]\!]^{-\omega}[\![\sigma]\!]^\omega\frac{(2\mathrm{St}(z,y_1,\dots,y_m))^\omega}{(\mathrm{St}(z,y_1,\dots,y_m))^\omega}\big(1-v_\mu^{(m+1)}\big)\lesssim[\![\mu]\!]^{-\omega}[\![\sigma]\!]^\omega,$$

which proves the claim. □

A.3 Norm estimates

Lemma A.13. *The cumulant norms introduced in Def. 4.11 satisfy the bound*

$$\big|\!\big|\!\big|\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big|\!\big|\!\big|_\mu\lesssim[\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\,\big|\!\big|\!\big|\dot{\bar{F}}_\sigma^{[\ell],(1)}\big|\!\big|\!\big|_\sigma,$$

uniformly over $\sigma,\mu\in[1/2,1)$ *and* $\dot{\bar{F}}_\sigma^{[\ell],(1)}$.

Proof. By (4.6), Young's inequality and Lemma A.11 we obtain

$$\begin{aligned}\big|\!\big|\!\big|\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big|\!\big|\!\big|_\mu &= \big|\!\big|\!\big|\big[K_\mu^{1,1}\big(\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big)\big]\cdot w_\mu^{(2),\mathfrak{b}}\big|\!\big|\!\big|\\ &\lesssim \|K_\mu^{1,1}\|_{\mathrm{TV}((w_\mu^{\mathfrak{b}})^{\otimes 2})}\big|\!\big|\!\big|\big[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)w_\mu^{(2),\mathfrak{b}}\big|\!\big|\!\big|\\ &\lesssim \|w_\mu^{(2),\mathfrak{b}}(1-h_\mu)\big(w_\sigma^{(2),\mathfrak{b}}\big)^{-1}\|_{L^\infty}\big|\!\big|\!\big|\dot{\bar{F}}_\sigma^{[\ell],(1)}\big|\!\big|\!\big|_\sigma\\ &\lesssim [\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\big|\!\big|\!\big|\dot{\bar{F}}_\sigma^{[\ell],(1)}\big|\!\big|\!\big|_\sigma.\end{aligned}$$

This proves the statement. □

Lemma A.14. *The kernel norms introduced in Def. 4.6 satisfy the bound*

$$\|[\tilde{K}_\sigma^{\mathfrak{a}}\dot{F}_\sigma^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\|_\mu\lesssim[\![\mu]\!]^{\ell(\mathfrak{a})\kappa_0-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}\|\dot{F}_\sigma^{\mathfrak{a}}\|_\sigma,$$

uniformly over $\sigma,\mu\in[1/2,1)$ *and* $\dot{F}_\sigma^{\mathfrak{a}}$.

Proof. By (4.7), Young's inequality, Lemma A.12 and $\tilde{w}_\mu^{\mathfrak{a}}:=w_\mu^{(1+k(\mathfrak{a})),\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}$ we obtain

$$\begin{aligned}\|[\tilde{K}_\sigma^{\mathfrak{a}}\dot{F}_\sigma^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\|_\mu &= \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde{K}_\mu^{\mathfrak{a}}([\tilde{K}_\sigma^{\mathfrak{a}}\dot{F}_\sigma^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}}))]\cdot\tilde{w}_\mu^{\mathfrak{a}}\|\\ &\lesssim \|\tilde{K}_\mu^{\mathfrak{a}}\|_{\mathrm{TV}(\mathfrak{w}_\mu)}\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde{K}_\sigma^{\mathfrak{a}}\dot{F}_\sigma^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\tilde{w}_\mu^{\mathfrak{a}}\|\\ &\lesssim \|(1-v_\mu^{\mathfrak{a}})\tilde{w}_\mu^{\mathfrak{a}}(\tilde{w}_\sigma^{\mathfrak{a}})^{-1}\|_{L^\infty}\|\dot{F}_\sigma^{\mathfrak{a}}\|_\sigma\\ &\lesssim [\![\mu]\!]^{\ell(\mathfrak{a})\kappa_0-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}\|\dot{F}_\sigma^{\mathfrak{a}}\|_\sigma,\end{aligned}$$

where

$$\mathfrak{w}_\mu(z,z_1,\dots,z_{k(\mathfrak{a})})=w_\mu^2(z)w_\mu^{\mathfrak{b}}(z_1)\cdots w_\mu^{\mathfrak{b}}(z_{k(\mathfrak{a})}).$$

This proves the statement. □

Lemma A.15. *The kernel norms introduced in Def. 4.6 satisfy the bound*

$$\|F^{\mathfrak{a}} \cdot \nu_\mu^{\mathfrak{a}}\|_\mu \lesssim \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}_\mu^{\mathfrak{a}} F^{\mathfrak{a}}] \cdot \tilde{\nu}_\mu^{\mathfrak{a}} \tilde{w}_\mu^{\mathfrak{a}}\| \lesssim \|F^{\mathfrak{a}}\|_\mu,$$

uniformly over $\mu \in [1/2, 1)$ *and* $F^{\mathfrak{a}}$, *where* $\tilde{\nu}_\mu^{\mathfrak{a}}$ *is a smooth weight on* $\Lambda_0^{1+k(\mathfrak{a})}$ *such that* $\|\tilde{\nu}_\mu^{\mathfrak{a}}\| \lesssim 1$ *uniformly in* $\mu \in [1/2, 1)$ *and*

$$\operatorname{supp} \tilde{\nu}_\mu^{\mathfrak{a}} \subset \Big\{(z, z_1, \dots, z_{k(\mathfrak{a})}) \,|\, \forall i\, |z_0 - z_{i,0}| \le c\,[\![\mu]\!], \forall i\, |\bar{z} - \bar{z}_i| \le c\,(\varepsilon \vee [\![\mu]\!])\Big\}$$

for some $c > 0$ *independent of* μ *and* ε.

Remark A.16. Note that for $[\![\mu]\!] \ge \varepsilon$ the weight $\tilde{\nu}_\mu^{\mathfrak{a}}$ has comparable support property to the weight $\nu_\mu^{\mathfrak{a}}$.

Proof. Let $k = k(\mathfrak{a})$. Since $F^{\mathfrak{a}} = \tilde{L}_\mu^{\mathfrak{a}} \tilde{K}_\mu^{\mathfrak{a}} F^{\mathfrak{a}}$ by Def. 1.13 and 4.5, we have

$$\tilde{K}_\mu^{\mathfrak{a}}(F^{\mathfrak{a}} \cdot \nu_\mu^{\mathfrak{a}}) \;=\; \tilde{K}_\mu^{\mathfrak{a}}([\tilde{L}_\mu^{\mathfrak{a}} \tilde{K}_\mu^{\mathfrak{a}} F^{\mathfrak{a}}] \cdot \nu_\mu^{\mathfrak{a}}) = \tilde{K}_\mu^{\mathfrak{a}}([(L_\mu \otimes (L_\mu^2)^{\otimes k})(\tilde{K}_\mu^{\mathfrak{a}} F^{\mathfrak{a}})] \cdot \nu_\mu^{\mathfrak{a}}),$$

where $L_\mu = (1 + [\![\mu]\!]^{2s}\partial_t)(1 - [\![\mu]\!]^2\Delta)^2$. Integrating by parts each of the differential operators appearing in the tensor product $L_\mu \otimes (L_\mu^2)^{\otimes k}$, we obtain

$$\tilde{K}_\mu^{\mathfrak{a}}(F^{\mathfrak{a}} \cdot \nu_\mu^{\mathfrak{a}}) = \sum_{A,B \in I} c_{A,B} (D_\mu^A \tilde{K}_\mu^{\mathfrak{a}})([\tilde{K}_\mu^{\mathfrak{a}} F^{\mathfrak{a}}] \cdot D_\mu^B \nu_\mu^{\mathfrak{a}}).$$

Here I is a finite index set, $(c_{A,B})_{A,B \in I}$ is a family of real constants and $(D^A)_{A \in I}$ is a family of differential operators on Λ^{1+k}. Each D^A has the form

$$D_\mu^A = \big(D_{t,\mu}^{(0)} \otimes D_{x,\mu}^{(0)}\big) \otimes \cdots \otimes \big(D_{t,\mu}^{(k)} \otimes D_{x,\mu}^{(k)}\big),$$

with temporal operators

$$D_{t,\mu}^{(0)}, \dots, D_{t,\mu}^{(k)} \in \{1, (1 + [\![\mu]\!]^{2s}\partial_t)\}$$

and spatial operators

$$D_{x,\mu}^{(0)}, \dots D_{x,\mu}^{(k)} \in \boldsymbol{D}^{(2)} \cup (1 - [\![\mu]\!]^2\Delta)^2 \boldsymbol{D}^{(1)} \cup (1 - [\![\mu]\!]^2\Delta)^2 \boldsymbol{D}^{(0)},$$

where

$$\boldsymbol{D}^{(l)} = \big\{ [\![\mu]\!]^{|A|}\, T_x\, \partial^A \,\big|\, A \in \mathbb{N}_0^{\{1\pm, \dots, d\pm\}}, |A| \le l, x \in \mathbb{R}_\varepsilon^d \big\}.$$

The symbol ∂^A denotes the derivative operator on $\mathbb{R}_\varepsilon^d$ introduced in Def. A.1, and T_x stands for the translation by $x \in \mathbb{R}_\varepsilon^d$. The highest-order operator is either $(1 + [\![\mu]\!]^{2s}\partial_t)$ or $(1 - [\![\mu]\!]^2\Delta)^2$; all lower-order operators are included as well. Since we are on a lattice, applying the Leibniz rule requires considering spatial operators that shift the spatial variable by multiples of lattice spacings in certain directions.

With this decomposition, it is straightforward to see that

$$D_\mu^A \tilde{K}_\mu^{\mathfrak{a}} = \bigotimes_{i=0}^{k} \Big(D_{t,\mu}^{(i)} (1 + [\![\mu]\!]^{2s}\partial_t)^{-1} \otimes D_{x,\mu}^{(i)} (1 - [\![\mu]\!]^2\Delta)^{-2} \Big).$$

By Lemma A.4, we have

$$\|D_\mu^A \tilde{K}_\mu^{\mathfrak{a}}\|_{\mathrm{TV}(\mathfrak{w}_\mu)} \lesssim 1,$$

where

$$\mathfrak{w}_\mu(z, z_1, \dots, z_k) = w_\mu^2(z) w_\mu^\flat(z_1) \cdots w_\mu^\flat(z_k).$$

Therefore, by (4.7) and Young's inequality, we conclude

$$\begin{aligned} \|F^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu &\lesssim \sum_{A,B} \|\mathfrak{o}^{\mathfrak{a}}\cdot[(D_\mu^A\tilde K_\mu^{\mathfrak{a}})([\tilde K_\mu^{\mathfrak{a}}F^{\mathfrak{a}}]\cdot D_\mu^B v_\mu^{\mathfrak{a}})]\cdot\tilde w_\mu^{\mathfrak{a}}\| \\ &\lesssim \sum_{A,B} \|D_\mu^A\tilde K_\mu^{\mathfrak{a}}\|_{\mathrm{TV}(\mathfrak{w}_\mu)}\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}F^{\mathfrak{a}}]\cdot(D_\mu^B v_\mu^{\mathfrak{a}})\tilde w_\mu^{\mathfrak{a}}\| \\ &\lesssim \|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\mu^{\mathfrak{a}}F^{\mathfrak{a}}]\cdot\tilde v_\mu^{\mathfrak{a}}\tilde w_\mu^{\mathfrak{a}}\|, \end{aligned} \tag{A.7}$$

where in the last step we introduced the weight

$$\tilde v_\mu^{\mathfrak{a}} := \sup_{B\in I}|D_\mu^B v_\mu^{\mathfrak{a}}|.$$

This proves the first of the stated bounds, while the second one is trivial. □

Lemma A.17. *The kernel norms introduced in Def. 4.6 satisfy the bound*

$$\|[(1-\tilde K_\sigma^{\mathfrak{a}})\dot F_\sigma^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\|_\mu \lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}\|\dot F_\sigma^{\mathfrak{a}}\|_\sigma,$$

uniformly over $\mu\in[1/2,1)$, $\sigma\in[\mu,1)$ and $\dot F_\sigma^{\mathfrak{a}}$.

Proof. By Lemma A.15 we have

$$\|[(1-\tilde K_\sigma^{\mathfrak{a}})\dot F_\sigma^{\mathfrak{a}}]\cdot(1-v_\mu^{\mathfrak{a}})\|_\mu \lesssim \|(1-\tilde K_\sigma^{\mathfrak{a}})\dot F_\sigma^{\mathfrak{a}}\|_\mu = \|\mathfrak{o}^{\mathfrak{a}}\cdot[(\tilde L_\sigma^{\mathfrak{a}}-1)\tilde K_\mu^{\mathfrak{a}}\tilde K_\sigma^{\mathfrak{a}}\dot F_\sigma^{\mathfrak{a}}]\cdot\tilde w_\mu^{\mathfrak{a}}\|_\mu.$$

By (4.4), Young's inequality, Lemma A.3 and $\tilde w_\mu^{\mathfrak{a}}\le\tilde w_\sigma^{\mathfrak{a}}$ we obtain

$$\begin{aligned} \|\mathfrak{o}^{\mathfrak{a}}\cdot[(\tilde L_\sigma^{\mathfrak{a}}-1)\tilde K_\mu^{\mathfrak{a}}\tilde K_\sigma^{\mathfrak{a}}\dot F_\sigma^{\mathfrak{a}}]\cdot\tilde w_\mu^{\mathfrak{a}}\| &\lesssim \|(\tilde L_\sigma^{\mathfrak{a}}-1)\tilde K_\mu^{\mathfrak{a}}\|_{\mathrm{TV}(\mathfrak{w}_\mu)}\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde K_\sigma^{\mathfrak{a}}\dot F_\sigma^{\mathfrak{a}}]\cdot\tilde w_\mu^{\mathfrak{a}}\| \\ &\lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}\|\dot F_\sigma^{\mathfrak{a}}\|_\sigma, \end{aligned}$$

where $\mathfrak{w}_\mu(z,z_1,\dots,z_{k(\mathfrak{a})}) = w_\mu^2(z)w_\mu^{\mathfrak{b}}(z_1)\cdots w_\mu^{\mathfrak{b}}(z_{k(\mathfrak{a})})$. □

Lemma A.18. *The cumulant norms introduced in Def. 4.11 satisfy the bound*

$$|||(1-K_\sigma^{1,1})\dot{\bar F}_\sigma^{[\ell],(1)}|||_\mu \lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}|||\dot{\bar F}_\sigma^{[\ell],(1)}|||_\sigma,$$

uniformly over $\mu\in[1/2,1)$ and $\sigma\in[\mu,1)$.

Proof. By Young's inequality, Lemma A.3 and $w_\mu^{\mathfrak{a}}\le w_\sigma^{\mathfrak{a}}$ we obtain

$$\begin{aligned} |||(1-K_\sigma^{1,1})\dot{\bar F}_\sigma^{[\ell],(1)}|||_\mu &= |||\big[(1\otimes(L_\sigma-1)K_\mu)K_\sigma^{1,1}\dot{\bar F}_\sigma^{[\ell],(1)}\big]\cdot w_\mu^{(2),\mathfrak{b}}||| \\ &\lesssim \|(L_\sigma-1)K_\mu\|_{\mathrm{TV}(w_\mu^{\mathfrak{b}})}|||\big[K_\sigma^{1,1}\dot{\bar F}_\sigma^{[\ell],(1)}\big]\cdot w_\mu^{(2),\mathfrak{b}}||| \\ &\lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}|||\dot{\bar F}_\sigma^{[\ell],(1)}|||_\sigma, \end{aligned}$$

as claimed. □

A.4 Schauder estimates

In this section, we establish estimates that characterise the spacetime regularity of distributions ϕ in terms of the size of $\mathcal{L}\phi$.

Lemma A.19. *For all $A\in\mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ the following bounds*

$$\|\partial^A\bar\Delta_i\|_{\mathrm{TV}(\zeta^{-1})}\lesssim[\![\mu_i]\!]^{-|A|},\qquad \|(1-\Delta_\varepsilon)^{-1}\bar\Delta_i\|_{\mathrm{TV}(\zeta^{-1})}\lesssim[\![\mu_i]\!]^2,\qquad \|(-\Delta_\varepsilon)^s\bar\Delta_i\|_{\mathrm{TV}(\zeta^{-1})}\lesssim[\![\mu_i]\!]^{-2s}$$

hold uniformly in $i \geq -1$.

Proof. The estimates follows from the identity $[\![\mu_i]\!] = 2^{-i-2}$ and Def. 1.14 of the spatial Littlewood-Paley blocks. □

Lemma A.20. *For all* $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ *the following bounds*

$$\|\bar{\Delta}_i G\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\mu_i]\!]^{2s}, \qquad \|\partial^A \dot{G}_\eta\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\eta]\!]^{2s-1-|A|_s}, \qquad \|(G - G_\eta)\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\eta]\!]^{2s},$$

hold uniformly in $i \geq -1$ *and* $\eta \in (1/2, 1)$.

Proof. Recall that $\bar{\Delta}_i$ is the spatial Littlewood-Paley block, $\zeta(z) := (1 + |z|_s^2)^{-1/2}$ and note that

$$G = \int_{\mu_i}^1 \dot{G}_\mu \,\mathrm{d}\mu + \int_{1/2}^{\mu_i} \dot{G}_\mu \,\mathrm{d}\mu + G_{1/2}.$$

Taking into account Remark 1.8 a), we obtain

$$\begin{aligned} \|\bar{\Delta}_i G\|_{\mathrm{TV}(\zeta^{-1})} \;\lesssim\; & \|\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \int_{\mu_i}^1 \|\dot{G}_\mu\|_{\mathrm{TV}(\zeta^{-1})} \,\mathrm{d}\mu \\ & + \|(1-\Delta_\varepsilon)^{-1}\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \int_{1/2}^{\mu_i} \|(1-\Delta_\varepsilon)\dot{G}_\mu\|_{\mathrm{TV}(\zeta^{-1})} \,\mathrm{d}\mu \\ & + \|(1-\Delta_\varepsilon)^{-1}\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \|(1-\Delta_\varepsilon)G_{1/2}\|_{\mathrm{TV}(\zeta^{-1})}. \end{aligned}$$

Consequently, using Lemma A.7 we arrive at

$$\begin{aligned} \|\bar{\Delta}_i G\|_{\mathrm{TV}(\zeta^{-1})} \;&\lesssim\; \int_{\mu_i}^1 [\![\mu]\!]^{2s-1} \,\mathrm{d}\mu + [\![\mu_i]\!]^2 \int_{1/2}^{\mu_i} [\![\mu]\!]^{2s-3} \,\mathrm{d}\mu + [\![\mu_i]\!]^2 \times 1 \\ &\lesssim\; [\![\mu_i]\!]^{2s} + [\![\mu_i]\!]^2 [\![\mu_i]\!]^{2s-2} + [\![\mu_i]\!]^2 \lesssim [\![\mu_i]\!]^{2s}. \end{aligned}$$

This proves the first bound. The second bound follows directly from Lemma A.7. To show the third bound notice that

$$\|(G - G_\eta)\|_{\mathrm{TV}(\zeta^{-1})} \lesssim \int_\eta^1 \|\dot{G}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \,\mathrm{d}\sigma \lesssim \int_\eta^1 [\![\sigma]\!]^{2s-1} \mathrm{d}\sigma \lesssim [\![\eta]\!]^{2s}.$$

This completes the proof. □

Lemma A.21. *It holds*

$$\|\rho_\eta \bar{\Delta}_i \phi\| \lesssim [\![\eta]\!]^{-\gamma} [\![\bar{\mu}]\!]^{\gamma} [\![\mu_i \vee \bar{\mu}]\!]^{-\gamma} \Big[|||\phi||| + [\![\bar{\mu}]\!]^{2s-2\gamma} |||K_\bullet \mathcal{L}\phi|||_\sharp \Big], \tag{A.8}$$

uniformly in $\phi \in \mathcal{S}'(\Lambda)$, $i \geq -1$ *and* $\eta \in [\bar{\mu}, 1)$, *where the norms* $|||\bullet||| = |||\bullet|||_{\bar{\mu}}$ *and* $|||\bullet|||_\sharp = |||\bullet|||_{\sharp,\bar{\mu}}$ *are defined by* (2.11) *and* (2.12). *In particular, for* $\hat{\gamma} > \gamma$, *it holds*

$$\sum_i 2^{-\hat{\gamma} i} \|\rho_\eta \bar{\Delta}_i \phi\| \lesssim [\![\eta]\!]^{-\gamma} \Big[|||\phi||| + [\![\bar{\mu}]\!]^{2s-2\gamma} |||K_\bullet \mathcal{L}\phi|||_\sharp \Big],$$

uniformly in $\bar{\mu} \in [1/2, 1)$, $\eta \in [\bar{\mu}, 1)$ *and* $\phi \in \mathcal{S}'(\Lambda)$.

Proof. First observe that

$$\|\rho_\eta \bar{\Delta}_i \phi\| \lesssim \sup_j \|\rho_\eta \chi_j \bar{\Delta}_i \phi\| \lesssim \sup_j \|\rho_\eta \chi_j \zeta_{\mu_j \vee \bar{\mu}}^{-1}\| \, \|\zeta_{\mu_j \vee \bar{\mu}} \bar{\Delta}_i \phi\| \lesssim [\![\eta]\!]^{-\gamma} \sup_j [\![\mu_j \vee \bar{\mu}]\!]^{\gamma} \|\zeta_{\mu_j \vee \bar{\mu}} \bar{\Delta}_i \phi\|, \tag{A.9}$$

uniformly in $i \geq -1$ and $\eta \geq \bar{\mu}$. To obtain the last estimate we used the fact that

$$\sup_j \|\rho_\eta \chi_j \zeta_{\mu_j \vee \bar{\mu}}^{-1}\| \lesssim \sup_j \|\rho_\eta \chi_j \zeta_{\mu_j}^{-1}\| \lesssim [\![\eta]\!]^{-\gamma} [\![\mu_j \vee \bar{\mu}]\!]^{\gamma}. \tag{A.10}$$

Indeed note that if $\bar{\mu} \le \eta \le \mu_j$, we have

$$\|\rho_\eta \chi_j \zeta_{\mu_j}^{-1}\| \le (1 + [\![\eta]\!]^a [\![\mu_j]\!]^{-a})^{-\nu} \lesssim [\![\eta]\!]^{-\gamma} [\![\mu_j]\!]^\gamma = [\![\eta]\!]^{-\gamma} [\![\mu_j \vee \bar{\mu}]\!]^\gamma.$$

On the other hand, if $\eta \ge \mu_j \vee \bar{\mu}$, then $\|\rho_\eta \chi_j \zeta_{\mu_j}^{-1}\| \le 1$ and $1 \le [\![\eta]\!]^{-\gamma} [\![\mu_j \vee \bar{\mu}]\!]^\gamma$. Hence, the bound (A.10) follows. By the estimate (A.9) it suffices to establish the bound (A.8) with the weight ρ_η replaced by ζ_η. To prove the latter bound note that

$$\begin{aligned}
\sup_{\eta \ge \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi\| &= \sup_{\bar{\mu} \le \eta \le \mu_i} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi\| \vee \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi\| \\
&\le [\![\bar{\mu}]\!]^\gamma \|\zeta_{\mu_i \vee \bar{\mu}} \bar{\Delta}_i \phi\| \vee [\![\bar{\mu}]\!]^\gamma [\![\mu_i \vee \bar{\mu}]\!]^{-\gamma} \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi\| \\
&\le [\![\bar{\mu}]\!]^\gamma [\![\mu_i \vee \bar{\mu}]\!]^{-\gamma} \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi\|.
\end{aligned}$$

In order to bound the supremum in the last line above we shall use the following decomposition

$$\phi = \phi_\eta + (G - G_\eta) \mathcal{L} \phi, \qquad \eta \ge \mu_i \vee \bar{\mu}.$$

Note that

$$\sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi\| \lesssim \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi_\eta\| + \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i (G - G_\eta) \mathcal{L} \phi\|,$$

and

$$\sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \bar{\Delta}_i \phi_\eta\| \lesssim \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta \phi_\eta\| \lesssim \sup_{\eta \ge \mu_i \vee \bar{\mu}} [\![\eta]\!]^\gamma \|\zeta_\eta^{1/3} \phi_\eta\| \le |\!|\!| \phi |\!|\!|.$$

To complete the proof it is now enough to show that

$$\|\zeta_\eta \bar{\Delta}_i (G - G_\eta) \mathcal{L} \phi\| \lesssim [\![\eta]\!]^{2s - 3\gamma} |\!|\!| K_\bullet \mathcal{L} \phi |\!|\!|_\sharp, \tag{A.11}$$

uniformly in $\eta \ge \mu_i \vee \bar{\mu}$. To this end, recall that $L_\mu = (1 + [\![\mu]\!]^{2s} \partial_t) \bar{L}_\mu$, where $\bar{L}_\mu := (1 - [\![\mu]\!]^2 \Delta)^2$. Now

$$\begin{aligned}
\bar{\Delta}_i (G - G_\eta) \phi &= \bar{\Delta}_i (G - G_\eta) L_\eta K_\eta \mathcal{L} \phi \\
&= \bar{L}_\eta \bar{\Delta}_i (G - G_\eta)(1 + [\![\eta]\!]^{2s} \partial_t)(K_\eta \mathcal{L} \phi) \\
&= \bar{L}_\eta \bar{\Delta}_i (G - G_\eta) K_\eta \mathcal{L} \phi + [\![\eta]\!]^{2s} \bar{L}_\eta \bar{\Delta}_i \partial_t G (1 - \mathcal{J}_\eta) K_\eta \mathcal{L} \phi \\
&= \mathbb{I} + \mathbb{II}.
\end{aligned}$$

We have the following bound for the first term

$$\begin{aligned}
\|\zeta_\eta \mathbb{I}\| = \|\zeta_\eta \bar{L}_\eta \bar{\Delta}_i (G - G_\eta) K_\eta \mathcal{L} \phi\| &\lesssim \|\bar{L}_\eta \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \|(G - G_\eta)\|_{\mathrm{TV}(\zeta^{-1})} \|\zeta_\eta K_\eta \mathcal{L} \phi\| \\
&\lesssim [\![\eta]\!]^{2s} \|\zeta_\eta K_\eta \mathcal{L} \phi\|.
\end{aligned}$$

The above estimates follow from Remark 1.8 a), Lemma A.20 and the bound

$$\|\bar{L}_\eta \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1,$$

which is a consequence of Lemma A.19. In order to bound $\|\zeta_\eta \mathbb{II}\|_{L^\infty}$ we first note that

$$\partial_t G = 1 - (m^2 + (-\Delta_\varepsilon)^s) G$$

and write

$$\begin{aligned}
\mathbb{II} &= [\![\eta]\!]^{2s} \bar{L}_\eta \bar{\Delta}_i \partial_t G (1 - \mathcal{J}_\eta) K_\eta \mathcal{L} \phi \\
&= [\![\eta]\!]^{2s} \bar{L}_\eta \bar{\Delta}_i (1 - \mathcal{J}_\eta) K_\eta \mathcal{L} \phi - [\![\eta]\!]^{2s} (m^2 + (-\Delta_\varepsilon)^s) \bar{L}_\eta \bar{\Delta}_i G (1 - \mathcal{J}_\eta) K_\eta \mathcal{L} \phi \\
&= \mathbb{II}_1 + \mathbb{II}_2.
\end{aligned}$$

We have the following bound for $\|\zeta_\eta \mathbb{II}_1\|_{L^\infty}$:

$$\begin{aligned}
\|\zeta_\eta \mathbb{II}_1\| = [\![\eta]\!]^{2s} \|\zeta_\eta \bar{L}_\eta \bar{\Delta}_i (1 - \mathcal{J}_\eta) K_\eta \mathcal{L} \phi\| &\lesssim [\![\eta]\!]^{2s} \|\bar{L}_\eta \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \|(1 - \mathcal{J}_\eta)\|_{\mathrm{TV}(\zeta^{-1})} \|\zeta_\eta K_\eta \mathcal{L} \phi\| \\
&\lesssim [\![\eta]\!]^{2s} \|\zeta_\eta K_\eta \mathcal{L} \phi\|.
\end{aligned}$$

For $\|\zeta_\eta \mathrm{III}_2\|$ we have, exploiting Lemma A.20,

$$\begin{aligned}\|\zeta_\eta \mathrm{III}_2\| &\le [\![\eta]\!]^{2s} \|\zeta_\eta (m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta(\bar{\Delta}_{i-1}+\bar{\Delta}_i+\bar{\Delta}_{i+1})\Delta_i G(1-\mathcal{J}_\eta)K_\eta \mathcal{L}\phi\| \\ &\lesssim [\![\eta]\!]^{2s} \|(m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta(\bar{\Delta}_{i-1}+\bar{\Delta}_i+\bar{\Delta}_{i+1})\|_{\mathrm{TV}(\zeta^{-1})}\|\bar{\Delta}_i G\|_{\mathrm{TV}(\zeta^{-1})}\|(1-\mathcal{J}_\eta)\|_{\mathrm{TV}(\zeta^{-1})}\|\zeta_\eta K_\eta \mathcal{L}\phi\| \\ &\lesssim [\![\eta]\!]^{2s}[\![\mu_i]\!]^{-2s}[\![\mu_i]\!]^{2s}\|\zeta_\eta K_\eta \mathcal{L}\phi\| \\ &= [\![\eta]\!]^{2s}\|\zeta_\eta K_\eta \mathcal{L}\phi\|,\end{aligned}$$

where we used that

$$\|\bar{\zeta}^{-1}(m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta(\bar{\Delta}_{i-1}+\bar{\Delta}_i+\bar{\Delta}_{i+1})\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\mu_i]\!]^{-2s},$$

which follows from Lemma A.19. Summing up, we have

$$\|\zeta_\eta(G-G_\eta)\mathcal{L}\phi\| \lesssim [\![\eta]\!]^{2s}\|\zeta_\eta K_\eta \mathcal{L}\phi\| \lesssim [\![\eta]\!]^{2s-3\gamma}[\![\eta]\!]^{3\gamma}\|\zeta_\eta K_\eta \mathcal{L}\phi\| \lesssim [\![\eta]\!]^{2s-3\gamma}|||K_\bullet \mathcal{L}\phi|||_\sharp.$$

This proves the bound (A.11) and completes the proof of the lemma. □

Lemma A.22. *Let $\mathcal{J}_{>\eta} := 1-\mathcal{J}_\eta$. We have*

$$\|\rho_\mu^3 \mathcal{J}_{>\eta}\phi_\sigma\| \lesssim [\![\eta]\!]^{2s}[\![\sigma]\!]^{-3\gamma}|||\mathcal{L}\phi_\bullet|||_\sharp,$$

uniformly in $\bar{\mu},\mu,\eta\in[1/2,1)$, $\sigma\in[\mu\vee\bar{\mu},1)$ and $\phi\in\mathcal{S}'(\Lambda)$, where the norm $|||\bullet|||_\sharp = |||\bullet|||_{\sharp,\bar{\mu}}$ is defined by (2.12).

Proof. Using Remark 1.8 a) and Lemma A.20, we obtain

$$\|\rho_\mu^3(G-G_\eta)\mathcal{L}\phi_\sigma\| \lesssim \|G-G_\eta\|_{\mathrm{TV}(\zeta^{-1})}\|\rho_\mu^3\mathcal{L}\phi_\sigma\| \lesssim [\![\eta]\!]^{2s}\|\rho_\mu^3\mathcal{L}\phi_\sigma\|.$$

Next, we note that

$$\|\rho_\mu^3\mathcal{J}_{>\eta}\phi_\sigma\| = \|\rho_\mu^3(G-G_\eta)\mathcal{L}\phi_\sigma\| \lesssim [\![\eta]\!]^{2s}\|\rho_\mu^3\mathcal{L}\phi_\sigma\| \lesssim [\![\eta]\!]^{2s}\|\rho_\sigma^3\mathcal{L}\phi_\sigma\| \lesssim [\![\eta]\!]^{2s}[\![\sigma]\!]^{-3\gamma}|||\mathcal{L}\phi_\bullet|||_\sharp,$$

on account of Lemma 2.3 and the identity $\rho_\sigma = \zeta_\sigma^\nu$. This finishes the proof. □

Appendix B Flow equation estimates

B.1 Estimates for $\mathcal{A}$ and $\mathcal{B}$

Goal of this section is to prove the bounds for the operators $\mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a}}$ and $\mathcal{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}$ appearing in the flow equation for cumulants (4.23), thereby proving Lemma 4.13. We begin with the definition of the first operator:

$$\sum_{\boldsymbol{b}} \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}) := \sum_{i=1}^{n(\boldsymbol{a})}\sum_{\boldsymbol{b}} \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}),$$

where

$$\begin{aligned}\sum_{\boldsymbol{b}} \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}) := \sum_{\ell'=0}^{\ell(\mathfrak{a}_i)-1}\sum_{k'=0}^{k(\mathfrak{a}_i)}(k'+1) \\ \times \mathfrak{C}_i(\dot{G}_\sigma)\mathfrak{K}_{n(\boldsymbol{a})+1}\big(F_\sigma^{\mathfrak{a}_1},\dots,F_\sigma^{\mathfrak{a}_{i-1}},F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)},F_\sigma^{[\ell'],(k(\mathfrak{a}_i)-k')},F_\sigma^{\mathfrak{a}_{i+1}},\dots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big).\end{aligned} \tag{B.1}$$

The summation index $\boldsymbol{b}$ is constrained by the allowed values of (ℓ',k') appearing on the right-hand side. The operator $\mathfrak{C}_i(\dot{G}_\sigma)$ acts by applying $\dot{G}_\sigma$ to the output variable of the $(i+1)$-th kernel, and by inserting the resulting function into the last input variable of the i-th kernel. This operator generalises $\mathfrak{C}(\dot{G}_\sigma)$ defined in (4.11), which appears in the flow equation for the effective force kernels (4.13).

We now proceed with the definition of the second operator:

$$\sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) := \sum_{i=1}^{n(\boldsymbol{a})} \sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma),$$

where

$$\begin{aligned}\sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) := \sum_{I_1,I_2} \sum_{\ell'=0}^{\ell(\mathfrak{a}_i)-1} \sum_{k'=0}^{k(\mathfrak{a}_i)} (k'+1) \times \\ \times \mathfrak{C}_{|I_1|+1}(\dot{G}_\sigma)\Big(\mathfrak{K}_{|I_1|+1}\big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1}, F^{(\ell(\mathfrak{a}_i)-1-\ell',k'+1)}_\sigma\big)\mathfrak{K}_{|I_2|+1}\big(F^{(\ell',k(\mathfrak{a}_i)-k')}_\sigma, (F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\big)\Big).\end{aligned} \tag{B.2}$$

Here, for a fixed $i\in\{1,\dots,n(\boldsymbol{a})\}$, the sum $\sum_{I_1,I_2}$ runs over all the partitions of the set

$$I_1 \sqcup I_2 = \{1,\dots,i-1,i+1,\dots,n(\boldsymbol{a})\}.$$

The summation index $\boldsymbol{b}$ is now constrained by the allowed values of (I_1, I_2, ℓ', k') on the right-hand side of (B.2). For precise definitions of the operators $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$, see [Duc25a].

Lemma B.1. *For all $\boldsymbol{a}\in\boldsymbol{A}$ and $i\leqslant n(\boldsymbol{a})$, the following bound*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\,|\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)|\!|\!|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma$$

holds uniformly in $\mathcal{F}^{\boldsymbol{b}}_\sigma$ and $\sigma\in[1/2,1)$.

Proof. Recalling (B.1), our task is to analyse the expression

$$\mathfrak{C}_i(\dot{G}_\sigma)\,\mathfrak{K}_{n(\boldsymbol{a})+1}\big(F^{\mathfrak{a}_1}_\sigma,\dots,F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma, F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma,\dots,F^{\mathfrak{a}_{n(\boldsymbol{a})}}_\sigma\big).$$

Using $L_\sigma K_\sigma = 1$ and Def. 4.5, we obtain

$$\begin{aligned}&K^{n(\boldsymbol{a}),K(\boldsymbol{a})}_\sigma\Big(\mathfrak{C}_i(\dot{G}_\sigma)\,\mathfrak{K}_{n(\boldsymbol{a})+1}\big(F^{\mathfrak{a}_1}_\sigma,\dots,F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma, F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma,\dots,F^{\mathfrak{a}_{n(\boldsymbol{a})}}_\sigma\big)\Big)\\ =&\mathfrak{C}_i(L_\sigma\dot{G}_\sigma)\,K^{n(\boldsymbol{a})+1,K(\boldsymbol{a})+1}_\sigma\mathfrak{K}_{n(\boldsymbol{a})+1}\big(F^{\mathfrak{a}_1}_\sigma,\dots,F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma, F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma,\dots,F^{\mathfrak{a}_{n(\boldsymbol{a})}}_\sigma\big).\end{aligned}$$

As a consequence,

$$K^{\boldsymbol{a}}_\sigma\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) = \mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(L_\sigma\dot{G}_\sigma, K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma).$$

Hence, by Def. 4.11 of the cumulant norm,

$$\begin{aligned}|\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)|\!|\!|_\sigma &= |\!|\!|\big[K^{\boldsymbol{a}}_\sigma\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)\big]\cdot w^{\boldsymbol{a}}_\sigma|\!|\!|\\ &= |\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(L_\sigma\dot{G}_\sigma, K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma)\cdot w^{\boldsymbol{a}}_\sigma|\!|\!|\\ &\leqslant |\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(|L_\sigma\dot{G}_\sigma|, |K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma|)\cdot w^{\boldsymbol{a}}_\sigma|\!|\!|\\ &\lesssim |\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(|L_\sigma\dot{G}_\sigma|\cdot w^{\flat}_\sigma, |K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma|\cdot w^{\boldsymbol{b}}_\sigma)|\!|\!|,\end{aligned}$$

where the last estimate follows from (4.3) and $|L_\sigma\dot{G}_\sigma|\cdot w^{\flat}_\sigma$ denotes the function

$$z\mapsto |(L_\sigma\dot{G}_\sigma)(z)|\,w^{\flat}_\sigma(z). \tag{B.3}$$

As a result, using Def. 4.11 one proves along the lines of Lemma 5.29 in [Duc25a] that

$$|\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)|\!|\!|_\sigma \lesssim |||(L_\sigma\dot{G}_\sigma)\cdot w^{\flat}_\sigma|_M||_{L^\infty}\,|\!|\!|[K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma]\cdot w^{\boldsymbol{b}}_\sigma|\!|\!| = |||(L_\sigma\dot{G}_\sigma)\cdot w^{\flat}_\sigma|_M||_{L^\infty}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma,$$

where $|(L_\sigma\dot{G}_\sigma)\cdot w^{\flat}_\sigma|_M$ denotes the periodisation in space of the function (B.3) with period M. Thus, since by Lemma A.7

$$|||(L_\sigma\dot{G}_\sigma)\cdot w^{\flat}_\sigma|_M||_{L^\infty} \lesssim [\![\sigma]\!]^{-1-d},$$

we obtain

$$|\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)|\!|\!|_\sigma \lesssim [\![\sigma]\!]^{-1-d}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma.$$

It follows that

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\,|\!|\!|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma)|\!|\!|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{a}]-1-d}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma.$$

In the last step, we used (4.19) and (4.26), which imply

$$-[\boldsymbol{a}]-1-d=-[\boldsymbol{b}]-1+\theta+\beta-\delta-d\geqslant -[\boldsymbol{b}]-1.$$

This finishes the proof. □

Lemma B.2. *For all $\boldsymbol{a}\in\boldsymbol{A}$ and $i\leqslant n(\boldsymbol{a})$ the following bound*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\,|\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)|\!|\!|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma\,|\!|\!|\mathcal{F}^{\boldsymbol{c}}_\sigma|\!|\!|_\sigma$$

holds uniformly in $\mathcal{F}^{\boldsymbol{b}}_\sigma$, $\mathcal{F}^{\boldsymbol{c}}_\sigma$ and $\sigma\in[1/2,1)$.

Proof. The proof proceeds along the same lines as the previous one. Recalling (B.2), we now need to analyse the expression

$$\mathfrak{C}_{|I_1|+1}(\dot G_\sigma)\Big(\mathfrak{K}_{|I_1|+1}\big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1},F^{(\ell(\mathfrak{a}_i)-1-\ell',k'+1)}_\sigma\big)\,\mathfrak{K}_{|I_2|+1}\Big(F^{(\ell',k(\mathfrak{a}_i)-k')}_\sigma,(F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\Big)\Big).$$

Using $L_\sigma K_\sigma=1$ and Def. 4.5, we obtain

$$\begin{aligned}
&K^{\boldsymbol{a}}_\sigma\Big(\mathfrak{C}_{|I_1|+1}(\dot G_\sigma)\Big(\mathfrak{K}_{|I_1|+1}\big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1},F^{(\ell(\mathfrak{a}_i)-1-\ell',k'+1)}_\sigma\big)\,\mathfrak{K}_{|I_2|+1}\Big(F^{(\ell',k(\mathfrak{a}_i)-k')}_\sigma,(F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\Big)\Big)\Big)\\
&=\mathfrak{C}_{|I_1|+1}(L_\sigma\dot G_\sigma)\Big(\big[K^{|I_1|+1,k_1}_\sigma\mathfrak{K}_{|I_1|+1}\big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1},F^{(\ell(\mathfrak{a}_i)-1-\ell',k'+1)}_\sigma\big)\big]\\
&\qquad\big[K^{|I_2|+1,k_2}_\sigma\mathfrak{K}_{|I_1|+1}\big(F^{(\ell',k(\mathfrak{a}_i)-k')}_\sigma,(F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\big)\big]\Big),
\end{aligned}$$

where $k_1=\sum_{j=1}^{i-1}k(\mathfrak{a}_j)+k'+1$ and $k_2=\sum_{j=i+1}^{n(\boldsymbol{a})}k(\mathfrak{a}_j)+k(\mathfrak{a}_i)-k'$. As a consequence,

$$K^{\boldsymbol{a}}_\sigma\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)=\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(L_\sigma\dot G_\sigma,K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma,K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma).$$

Hence, by Def. 4.11 of the cumulant norm,

$$\begin{aligned}
|\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)|\!|\!|_\sigma &= |\!|\!|\big[K^{\boldsymbol{a}}_\sigma\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)\big]\cdot w^{\boldsymbol{a}}_\sigma|\!|\!|\\
&= |\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(L_\sigma\dot G_\sigma,K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma,K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma)\cdot w^{\boldsymbol{a}}_\sigma|\!|\!|\\
&\leqslant |\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(|L_\sigma\dot G_\sigma|,|K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma|,|K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma|)\cdot w^{\boldsymbol{a}}_\sigma|\!|\!|\\
&\lesssim |\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(|L_\sigma\dot G_\sigma|\cdot w^{\flat}_\sigma,|K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma|\cdot w^{\boldsymbol{b}}_\sigma,|K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma|\cdot w^{\boldsymbol{c}}_\sigma)|\!|\!|,
\end{aligned}$$

where the last estimate follows from (4.3). As a result, by Def. 4.11 and Lemma 1.17, we have

$$\begin{aligned}
|\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)|\!|\!|_\sigma &\lesssim ||L_\sigma\dot G_\sigma||_{\mathrm{TV}(w^{\flat}_\sigma)}\,|\!|\!|K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma\cdot w^{\boldsymbol{b}}_\sigma|\!|\!|\,|\!|\!|K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma\cdot w^{\boldsymbol{c}}_\sigma|\!|\!|\\
&= ||L_\sigma\dot G_\sigma||_{\mathrm{TV}(w^{\flat}_\sigma)}\,|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma\,|\!|\!|\mathcal{F}^{\boldsymbol{c}}_\sigma|\!|\!|_\sigma\\
&\lesssim [\![\sigma]\!]^{2s-1}|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma\,|\!|\!|\mathcal{F}^{\boldsymbol{c}}_\sigma|\!|\!|_\sigma.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
[\![\sigma]\!]^{-[\boldsymbol{a}]}\,|\!|\!|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma,\mathcal{F}^{\boldsymbol{b}}_\sigma,\mathcal{F}^{\boldsymbol{c}}_\sigma)|\!|\!|_\sigma &\lesssim [\![\sigma]\!]^{-[\boldsymbol{a}]}[\![\sigma]\!]^{2s-1}|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma\,|\!|\!|\mathcal{F}^{\boldsymbol{c}}_\sigma|\!|\!|_\sigma,\\
&= [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}|\!|\!|\mathcal{F}^{\boldsymbol{b}}_\sigma|\!|\!|_\sigma\,|\!|\!|\mathcal{F}^{\boldsymbol{c}}_\sigma|\!|\!|_\sigma.
\end{aligned}$$

In the last step we used (4.19) and (4.26), which imply

$$-[\boldsymbol{a}]+2s-1=-[\boldsymbol{b}]-[\boldsymbol{c}]-\varrho+\theta+\beta-\delta+2s-1\geqslant -[\boldsymbol{b}]-[\boldsymbol{c}]-1.$$

This finishes the proof. □

Remark B.3. Note that Lemma A.7 implies that

$$|||(L_\sigma\dot G_\sigma)|_M||_{L^\infty}\lesssim\varepsilon^{-\flat}[\![\sigma]\!]^{\flat-1-d},\qquad ||L_\sigma\dot G_\sigma||_{\mathrm{TV}}\lesssim\varepsilon^{-\flat}[\![\sigma]\!]^{\flat+2s-1},$$

where $|L_\sigma \dot G_\sigma|_M$ denotes the periodisation in space of the function $z \mapsto |(L_\sigma \dot G_\sigma)(z)|$ with period M. Using the above bounds one shows, along the lines of the proofs of Lemmas B.1 and B.2, that

$$\begin{aligned}\bigl\|\!\bigl| K_\sigma^{\boldsymbol a} \mathcal{A}_{\boldsymbol b}^{\boldsymbol a,(i)}(\dot G_\sigma, \mathcal{F}_\sigma^{\boldsymbol b})\bigr|\!\bigr\| &\lesssim \varepsilon^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}+2s-1} \bigl\|\!\bigl| \mathcal{F}_\sigma^{\boldsymbol b}\bigr|\!\bigr\|_\sigma,\\ \bigl\|\!\bigl| K_\sigma^{\boldsymbol a} \mathcal{B}_{\boldsymbol b,\boldsymbol c}^{\boldsymbol a,(i)}(\dot G_\sigma, \mathcal{F}_\sigma^{\boldsymbol b}, \mathcal{F}_\sigma^{\boldsymbol c})\bigr|\!\bigr\| &\lesssim \varepsilon^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}+2s-1} \bigl\|\!\bigl| \mathcal{F}_\sigma^{\boldsymbol b}\bigr|\!\bigr\|_\sigma \bigl\|\!\bigl| \mathcal{F}_\sigma^{\boldsymbol c}\bigr|\!\bigr\|_\sigma.\end{aligned}$$

These bounds are used to prove the estimate (4.43). Note that the norms appearing on the left-hand sides of the above bounds, as well as the norm in (4.43), do not involve weights.

B.2 Localisation

In this section, we introduce the Taylor expansion on the (semi)-discrete lattice $\Lambda = \Lambda_\varepsilon$. At first order, we have

$$\psi(z_1) = \psi(z) + \sum_{i \in \{0,1\pm,2\pm,\dots,d\pm\}} \int_0^1 [\mathrm{d}\rho_{z_1-z}(t)]^i (\partial^i \psi)(z + \rho_{z_1-z}(t)),$$

where ∂^0 denotes the time derivative,

$$\partial^{k\pm}\psi(z) := \pm\varepsilon^{-1}[\psi(z \pm e_k) - \psi(z)],$$

denote the discrete forward $(k+)$ and backward $(k-)$ derivatives in the k-th spatial direction and for $h \in \Lambda$ the function $\rho_h : [0,1] \longrightarrow \Lambda$ is a bounded variation path such that $\rho_h(0) = 0$ and $\rho_h(1) = h$. We use the notation $[\mathrm{d}\rho_{z_1-z}(t)]^0 := \mathrm{d}\rho^0_{z_1-z}(t)$ and $[\mathrm{d}\rho_{z_1-z}(t)]^{k\pm} := (\mathrm{d}\rho^k_{z_1-z}(t))_\pm$, where $(\mathrm{d}\rho^k_{z_1-z}(t))_\pm$ denote the positive and negative parts of the signed measure $\mathrm{d}\rho^k_{z_1-z}(t)$. Note that the path ρ_h is piecewise constant in space, so that the signed measure $\mathrm{d}\rho^i_h$ is well defined and given by a sum of delta functions multiplied by the corresponding increments. We choose it so that its total mass is bounded by $|h|$ and

$$\int_0^1 [\mathrm{d}\rho_h(t)]^{k\pm} = [h]^{k\pm} := h^k \mathbb{1}_{\pm h^k \geqslant 0}.$$

Remark B.4. Note that in the continuum we could choose

$$\rho_{z_1-z}(t) = (z_1 - z)\, t.$$

Moreover, since in the continuum both the right and left derivatives $\partial^{k\pm}$ coincide with ∂^k, we have

$$\psi(z_1) = \psi(z) + \sum_{k \in \{0,\dots,d\}} \int_0^1 (\partial^k \psi)(z + \rho_{z_1-z}(t))\, (z_1 - z)^k \,\mathrm{d}t.$$

Since we intend to use the second-order Taylor expansion, it is convenient to choose the path such that $\rho^0_{z_1-z}(u) = 0$ for $u \in [0, 1/2]$ and

$$\rho^i_{z_1-z}(u) = \rho^i_{z_1-z}(1/2) + 1_{i=0}\,(z_1 - z)^0\,(2u - 1)$$

for $u \in [1/2, 1]$. This allows to avoid second order terms with one derivative in time and one in space. Note also that

$$\int_0^{1/2} |\mathrm{d}\rho_{z_1-z}(u)| \leqslant \int_0^{1/2} |[\mathrm{d}\rho_{z_1-z}(u)]^j| \leqslant |\bar z_1 - \bar z|, \qquad \int_{1/2}^1 |\mathrm{d}\rho_{z_1-z}(u)| \leqslant \int_{1/2}^1 |[\mathrm{d}\rho_{z_1-z}(u)]^0| \leqslant |(z_1)_0 - z_0|.$$

We have

$$\begin{aligned}\psi(z_1) &= \psi(z) + \sum_i \int_0^1 [\mathrm{d}\rho_{z_1-z}(t)]^i\, \partial^i \psi(z + \rho_{z_1-z}(t))\\ &= \psi(z) + \sum_{i \neq 0} \int_0^{1/2} [\mathrm{d}\rho_{z_1-z}(t)]^i\, \partial^i \psi(z + \rho_{z_1-z}(t)) + \int_{1/2}^1 \mathrm{d}\rho^0_{z_1-z}(t)\, \partial^0 \psi(z + \rho_{z_1-z}(t)).\end{aligned}$$

Expanding once more the spatial derivatives, we have

$$\begin{aligned}\psi(z_1) &= \psi(z)+\sum_{i\neq 0}\partial^i\psi(z)\,[z_1-z]^i+\sum_{i,j\neq 0,}\int_0^{1/2}[\mathrm{d}\rho_{z_1-z}(t)]^i\int_0^t[\mathrm{d}\rho_{z_1-z}(u)]^j\,\partial^i\partial^j\psi(z+\rho_{z_1-z}(u))+\\ &+\int_{1/2}^1\mathrm{d}\rho^0_{z_1-z}(t)\,\partial^0\psi(z+\rho_{z_1-z}(t)).\end{aligned}$$

As a consequence, for a generic kernel $V(z,z_1)$, and symmetrising the factor $\partial^i\psi(z)$ using the relation $\partial^{i-}\psi(z)-\partial^{i+}\psi(z)=\varepsilon(\partial^{i+}\partial^{i-}\psi)(z)$ we obtain

$$\begin{aligned}V(\psi)(z) &= \int_\Lambda \mathrm{d}z_1 V(z,z_1)\psi(z_1)\\ &= \psi(z)\int_\Lambda \mathrm{d}z_1 V(z,z_1)+\sum_{k\neq 0}\frac{\partial^{k+}\psi(z)+\partial^{k-}\psi(z)}{2}\int_\Lambda \mathrm{d}z_1(z_1-z)^k V(z,z_1)\\ &\quad+\int_\Lambda \mathrm{d}z_1\int_{1/2}^1\mathrm{d}\rho^0_{z_1-z}(t)V(z,z_1)\partial^0\psi(z+\rho_{z_1-z}(t))\\ &\quad-\sum_k\frac{\varepsilon(\partial^{k+}\partial^{k-}\psi)(z)}{2}\int_\Lambda|z_1-z|^k\,V(z,z_1)\mathrm{d}z_1\\ &\quad+\sum_{i,j\neq 0}\int_\Lambda \mathrm{d}z_1\int_0^{1/2}\int_0^t[\mathrm{d}\rho_{z_1-z}(u)]^j[\mathrm{d}\rho_{z_1-z}(t)]^i V(z,z_1)\partial^i\partial^j\psi(z+\rho_{z_1-z}(u)).\end{aligned}\tag{B.4}$$

By duality, we can write this Taylor expansion as an operation over the kernel V via the operators $\boldsymbol{L}$ and

$$\boldsymbol{R}:=\boldsymbol{R}^{(0)}+\sum_{i,j\neq 0}\boldsymbol{R}^{(i,j)}+\sum_k\boldsymbol{R}^{(k)}_\varepsilon,\tag{B.5}$$

of the form

$$\begin{aligned}(\boldsymbol{L}V)(z,z_1) &:= \int_\Lambda \mathrm{d}\tilde z\,V(z,\tilde z)\,\mathrm{d}\tilde z\,\delta(z_1-z)+\sum_{i\neq 0}\frac{1}{2}\int_\Lambda(\tilde z-z)^i V(z,\tilde z)\,(\partial^{i+}+\partial^{i-})\delta(z_1-z),\\ (\boldsymbol{R}^{(0)}V)(z,z_1) &:= \int_\Lambda\int_{1/2}^1\mathrm{d}\rho^0_{\tilde z-z}(t)V(z,\tilde z)\partial^0\delta(z_1-z-\rho_{\tilde z-z}(t))\,\mathrm{d}\tilde z,\\ (\mathbf{R}^{(i,j)}V)(z,z_1) &:= \int_\Lambda\int_0^{1/2}\int_0^t[\mathrm{d}\rho_{\tilde z-z}(u)]^j[\mathrm{d}\rho_{\tilde z-z}(t)]^i V(z,\tilde z)\partial^i\partial^j\delta(z_1-z-\rho_{\tilde z-z}(u))\,\mathrm{d}\tilde z,\\ (\mathbf{R}^{(k)}_\varepsilon V)(z,z_1) &:= -\frac{\varepsilon}{2}\int_\Lambda|\tilde z-z|^k\,V(z,\tilde z)\,\mathrm{d}\tilde z\,(\partial^{k+}\partial^{k-})\delta(z_1-z).\end{aligned}\tag{B.6}$$

When tested with a smooth function this gives the identity

$$V(\psi)=(\boldsymbol{L}V)(\psi)+(\boldsymbol{R}V)(\psi),$$

as seen in (B.4). We use this expansion for the analysis of relevant cumulants, namely

$$\mathcal{F}^{\boldsymbol{a}}_\mu=\bar F^{[\ell],(1)}_\mu=\mathbb{E}F^{[\ell],(1)}_\mu,\qquad \ell\in\{1,\dots,\hat\ell\}.$$

Recall that we have the following decomposition (4.42),

$$\begin{aligned}\bar F^{[\ell],(1)}_\mu &= \bar F^{[\ell],(1)}_1-\int_\mu^1\big[(1-K^{1,1}_\sigma)\dot{\bar F}^{[\ell],(1)}_\sigma\big]\,\mathrm{d}\sigma-\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\cdot(1-h_\mu)\mathrm{d}\sigma\\ &\quad-\boldsymbol{L}\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\cdot(1-h_\mu)\,\mathrm{d}\sigma-\boldsymbol{L}\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\mathrm{d}\sigma\\ &\quad-\mathbf{R}\int_\mu^1\big[K^{1,1}_\sigma\dot{\bar F}^{[\ell],(1)}_\sigma\big]\cdot h_\mu\,\mathrm{d}\sigma,\end{aligned}$$

where $\dot{\bar F}^{[\ell],(1)}_\sigma:=\partial_\sigma\bar F^{[\ell],(1)}_1$.

Lemma B.5. *The following bounds hold uniformly in $\sigma,\mu\in[1/2,1)$.*

$$\begin{aligned}\|\!|\boldsymbol{L}\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)|\!\|_\mu &\lesssim \|\!|\dot{\tilde{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma,\\ \|\!|\boldsymbol{L}\big(\big[K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big)|\!\|_\mu &\lesssim [\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\,\|\!|\dot{\tilde{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma.\end{aligned}$$

Proof. As we arged in Remark 4.14 the cumulants are invariant under spatial reflections. As a result, we conclude that

$$(\partial^{i+}+\partial^{i-})\delta(z_1-z)\int_\Lambda \mathrm{d}\tilde{z}\,(\tilde{z}-z)^i\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})=0.$$

Hence, the following identity

$$\big(\boldsymbol{L}\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)\big)(z,z_1)=\delta(z_1-z)\int_\Lambda\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\,\mathrm{d}\tilde{z}$$

holds true. Similarly,

$$\big(\boldsymbol{L}\big(\big[K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big)\big)(z,z_1)=\delta(z_1-z)\int_\Lambda\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\,(1-h_\mu(z,\tilde{z}))\,\mathrm{d}\tilde{z}.$$

In order to prove the first of the bounds stated in the lemma, we observe that

$$\big(\boldsymbol{L}\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)\big)(z,z_1)=\delta(z_1-z)\int_\Lambda\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\,\mathrm{d}\tilde{z}.$$

Thus, we obtain

$$w_\mu^{(2),\mathfrak{b}}(z,z_1)\big(K_\mu^{1,1}\boldsymbol{L}\big[K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big]\big)(z,z_1)=w_\mu^{(2),\mathfrak{b}}(z,z_1)\,K_\mu(z_1-z)\int_\Lambda\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\,\mathrm{d}\tilde{z}.$$

and

$$\begin{aligned}\|\!|\boldsymbol{L}\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)|\!\|_\mu &\lesssim \sup_z\int_\Lambda|K_\mu(z_1-z)|\,w_\mu^{\mathfrak{b}}(z-z_1)\,\mathrm{d}z_1\int_\Lambda\big|\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\big|\mathrm{d}\tilde{z}\\ &\lesssim \|K_\mu\|_{\mathrm{TV}(w_\mu^{\mathfrak{b}})}\sup_z\int_\Lambda\big|\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\big|\,\mathrm{d}\tilde{z}\\ &\lesssim \sup_z\int_\Lambda\big|\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\big|\,\mathrm{d}\tilde{z},\end{aligned}$$

by Lemma 1.17. Since

$$\sup_z\int_\Lambda\big|K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}(z,\tilde{z})\big|\,\mathrm{d}\tilde{z}\lesssim\sup_z\int_\Lambda\big|K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}(z,\tilde{z})\big|\,w_\mu^{(2),\mathfrak{b}}(z,\tilde{z})\,\mathrm{d}\tilde{z}=\|\!|\dot{\tilde{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma,$$

this proves the first of the bounds stated in the lemma. By a similar argument, we get

$$\|\!|\boldsymbol{L}\big(\big[K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big)|\!\|_\mu\lesssim\int_\Lambda\big|\big(K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde{z})\big|\,(1-h_\mu(z,\tilde{z}))\,\mathrm{d}\tilde{z}.$$

Consequently, by Lemma A.11, we obtain

$$\|\!|\boldsymbol{L}\big(\big[K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big]\cdot(1-h_\mu)\big)|\!\|_\mu\lesssim\|\,w_\mu^{(2),\mathfrak{b}}(1-h_\mu)\big(w_\mu^{(2),\mathfrak{b}}\big)^{-1}\|_{L^\infty}\|\!|\dot{\tilde{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma\lesssim[\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\|\!|\dot{\tilde{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma.$$

This proves the second of the bounds stated in the lemma. □

Lemma B.6. *The following bound*

$$\|\!|\mathbf{R}\big(\big[K_\sigma^{1,1}\dot{\tilde{F}}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)|\!\|_\mu\lesssim[\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\,\|\!|\dot{\tilde{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma$$

holds uniformly in $\mu\in[1/2,1)$, $\sigma\in[\mu,1)$.

Proof. By applying the triangular inequality to (B.5), we arrive at

$$\begin{aligned}\|\!|\mathbf{R}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)|\!\|_\mu \lesssim\ & \|\!|\boldsymbol{R}^{(0)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)|\!\|_\mu \\ &+\sum_{i,j\neq 0}\|\!|\mathbf{R}^{(i,j)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)|\!\|_\mu \\ &+\sum_k\|\!|\mathbf{R}_\varepsilon^{(k)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)|\!\|_\mu.\end{aligned}$$

We have

$$\begin{aligned}&\big(\mathbf{R}^{(i,j)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)\big)(z,z_1)= \\ &\quad=\int_\Lambda\int_0^{\frac12}\int_0^t[\mathrm{d}\rho_{\tilde z-z}(u)]^j[\mathrm{d}\rho_{\tilde z-z}(t)]^i h_\mu(z,\tilde z)\big(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde z)\,\partial^i\partial^j\delta(z_1-z-\rho_{\tilde z-z}(u))\,\mathrm{d}\tilde z.\end{aligned}$$

Thus,

$$\begin{aligned}&\big|w_\mu^{(2),\flat}(z,z_1)\big(\mathbf{R}^{(i,j)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)\big)(z,z_1)\big| \\ &\lesssim\int_\Lambda\int_0^{\frac12}\int_0^t|\mathrm{d}\rho_{\tilde z-z}(u)|\,|\mathrm{d}\rho_{\tilde z-z}(t)|\,w_\mu^{(2),\flat}(z,z_1)h_\mu(z,\tilde z)\big|K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}(z,z_1)\big|\,|\,\partial^i\partial^jK_\mu(z_1-z-\rho_{\tilde z-z}(u))|\,\mathrm{d}\tilde z.\end{aligned}$$

Recall that

$$w_\mu^{(2),\flat}(z,z_1)=w_\mu^\flat(z-z_1)=(1+[\![\mu]\!]^{-1}|z-z_1|_s)^\flat.$$

Introduce the point $\tilde z(z,u):=z+\rho_{\tilde z-z}(u)$ and note that

$$w_\mu^\flat(z-z_1)\lesssim w_\mu^\flat(z-\tilde z(z,u))w_\mu^\flat(\tilde z(z,u)-z_1)\lesssim w_\mu^\flat(z-\tilde z)w_\mu^\flat(\tilde z(z,u)-z_1).$$

The second inequality above follows from

$$w_\mu^\flat(z-\tilde z(z,u))\lesssim w_\mu^\flat(z-\tilde z),$$

which holds because, by construction, $\tilde z(z,u)$ lies on the path connecting z and $\tilde z$. Overall, this yields

$$\begin{aligned}&\big|w_\mu^\flat(z-z_1)\big(\mathbf{R}^{(i,j)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)\big)(z,z_1)\big| \\ &\quad\lesssim\int_\Lambda\int_0^{\frac12}\int_0^t|\mathrm{d}\rho_{\tilde z-z}(u)||\mathrm{d}\rho_{\tilde z-z}(t)|\,w_\mu^\flat(z-\tilde z)h_\mu(z,\tilde z)\,(w_\sigma^\flat(z-\tilde z))^{-1} \\ &\qquad\times w_\sigma^\flat(z-\tilde z)\big|\big(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}\big)(z,\tilde z)\big|\,w_\mu^\flat(\tilde z(z,u)-z_1)\,|\,\partial^i\partial^jK_\mu(z_1-\tilde z(z,u))|\,\mathrm{d}\tilde z.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}\|\!|\mathbf{R}^{(i,j)}\big([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu\big)|\!\|_\mu \lesssim\ & \sup_{z,\tilde z\in\Lambda}\int_0^{\frac12}\int_0^t|\mathrm{d}\rho_{\tilde z-z}(u)|\,|\mathrm{d}\rho_{\tilde z-z}(t)|\,w_\mu^\flat(z-\tilde z)h_\mu(z,\tilde z)(w_\sigma^\flat(z-\tilde z))^{-1} \\ &\times\|\!|\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma\sup_{z,\tilde z\in\Lambda}\int_\Lambda w_\mu^\flat(\tilde z(z,u)-z_1)\,|\,\partial^i\partial^jK_\mu(z_1-\tilde z(z,u))|\,\mathrm{d}z_1.\end{aligned}$$

Using

$$\int_0^{\frac12}\int_0^t|\mathrm{d}\rho_{\tilde z-z}(u)|\,|\mathrm{d}\rho_{\tilde z-z}(t)|\lesssim|\tilde{\bar z}-\bar z|^2,\tag{B.7}$$

and

$$\int_\Lambda w_\mu^\flat(z)|\,\partial^i\partial^jK_\mu(z)|\,\mathrm{d}z\lesssim[\![\mu]\!]^{-2},\tag{B.8}$$

as well as

$$\begin{aligned}\frac{|\bar{\tilde z}-\bar z|^2 w_\mu^\flat(z-\tilde z)h_\mu(z,\tilde z)}{w_\sigma^\flat(z-\tilde z)} &= [\![\mu]\!]^{2-\flat}[\![\sigma]\!]^\flat \frac{(1+[\![\mu]\!]^{-1}|z-\tilde z|_s)^\flat}{(1+[\![\sigma]\!]^{-1}|z-\tilde z|_s)^\flat}\frac{[\![\mu]\!]^{-(2-\flat)}[\![\sigma]\!]^{-\flat}|z-\tilde z|_s^2}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2}\\ &\le [\![\mu]\!]^{2-\flat}[\![\sigma]\!]^\flat \frac{(1+[\![\mu]\!]^{-1}|z-\tilde z|_s)^\flat([\![\mu]\!]^{-1}|z-\tilde z|_s)^{2-\flat}}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2}\\ &\lesssim [\![\mu]\!]^{2-\flat}[\![\sigma]\!]^\flat,\end{aligned} \tag{B.9}$$

we conclude that

$$\big\|\!\big|\mathbf{R}^{(i,j)}\big(\big[K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)\big\|\!\big|_\mu \lesssim [\![\sigma]\!]^\flat [\![\mu]\!]^{-\flat}\,\big\|\!\big|\dot{\tilde F}_\sigma^{[\ell],(1)}\big\|\!\big|_\sigma.$$

Next, note that

$$\big(\mathbf{R}^{(0)}\big(\big[K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)\big)(z,z_1)=\int_\Lambda\int_{1/2}^1 \mathrm{d}\rho^0_{\tilde z-z}(t)\big(h_\mu K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big)(z,\tilde z)\,\partial^0\delta(z_1-z-\rho_{\tilde z-z}(t))\,\mathrm{d}\tilde z.$$

Proceeding as above we get

$$\begin{aligned}\big\|\!\big|\mathbf{R}^{(0)}\big(\big[K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)\big\|\!\big|_\mu \lesssim{}& \sup_{z,\tilde z\in\Lambda}\int_{1/2}^1 |\mathrm{d}\rho_{\tilde z-z}(t)|\; w_\mu^\flat(z-\tilde z)h_\mu(z,\tilde z)(w_\sigma^\flat(z-\tilde z))^{-1}\times\\ &\times\big\|\!\big|\dot{\tilde F}_\sigma^{[\ell],(1)}\big\|\!\big|_\sigma \sup_{z,\tilde z\in\Lambda}\int_\Lambda w_\mu^\flat(\tilde z(z,u)-z_1)\,|\,\partial^0 K_\mu(z_1-\tilde z(z,u))|\,\mathrm{d}z_1.\end{aligned}$$

Using

$$\int_{1/2}^1 \mathrm{d}\rho^0_{\tilde z-z}(t)\le|\tilde z_0-z_0|,$$

and

$$\int_\Lambda w_\mu^\flat(z-z')\,|\partial^0 K_\mu(z-z')|\,\mathrm{d}z\lesssim[\![\mu]\!]^{-2s},$$

as well as

$$\begin{aligned}|\tilde z_0-z_0|\frac{w_\mu^\flat(z-\tilde z)}{w_\mu^\flat(z-\tilde z)}h_\mu(z,\tilde z) &\le [\![\mu]\!]^{2s-\flat}[\![\sigma]\!]^\flat\frac{(1+[\![\mu]\!]^{-1}|z-\tilde z|_s)^\flat}{(1+[\![\sigma]\!]^{-1}|z-\tilde z|_s)^\flat}\frac{[\![\mu]\!]^{-(2s-\flat)}[\![\sigma]\!]^{-\flat}|z-\tilde z|_s^{2s}}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2}\\ &\le [\![\mu]\!]^{2s-\flat}[\![\sigma]\!]^\flat\frac{(1+[\![\mu]\!]^{-1}|z-\tilde z|_s)^\flat([\![\mu]\!]^{-1}|z-\tilde z|)_s^{2s-\flat}}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2}\\ &\lesssim [\![\mu]\!]^{2s-\flat}[\![\sigma]\!]^\flat\frac{1}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^{2-2s}}\\ &\lesssim [\![\mu]\!]^{2s-\flat}[\![\sigma]\!]^\flat,\end{aligned}$$

we arrive at

$$\big\|\!\big|\mathbf{R}^{(0)}\big(\big[K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)\big\|\!\big|_\mu \lesssim [\![\sigma]\!]^\flat [\![\mu]\!]^{-\flat}\,\big\|\!\big|\dot{\tilde F}_\sigma^{[\ell],(1)}\big\|\!\big|_\sigma.$$

Finally, we discuss the term

$$\big(\mathbf{R}_\varepsilon^{(k)}\big(\big[K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)\big)(z,z_1)=\frac{\varepsilon}{2}\int_\Lambda|\tilde z-z|^k\big(h_\mu K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big)(z,\tilde z)\,\mathrm{d}\tilde z\,(\partial^{k+}\partial^{k-})\delta(z_1-z).$$

Also in this case the proof follows the same lines. We obtain

$$\begin{aligned}\big\|\!\big|\mathbf{R}_\varepsilon^{(k)}\big(\big[K_\sigma^{1,1}\dot{\tilde F}_\sigma^{[\ell],(1)}\big]\cdot h_\mu\big)\big\|\!\big|_\mu \lesssim{}& \varepsilon\sup_{z,\tilde z\in\Lambda}|\bar{\tilde z}-\bar z|\,w_\mu^2(z-\tilde z)\,h_\mu(z,\tilde z)\,(w_\sigma^\flat(z-\tilde z))^{-1}\\ &\times\big\|\!\big|\dot{\tilde F}_\sigma^{[\ell],(1)}\big\|\!\big|_\sigma\int_\Lambda w_\mu^\flat(z-z_1)\,|\,\partial^{k+}\partial^{k-}K_\mu(z_1-z)|\,\mathrm{d}z_1.\end{aligned}$$

Note that

$$\begin{aligned}\frac{|\bar{\tilde z}-\bar z|w_\mu^\flat(z-\tilde z)\,h_\mu(z,\tilde z)}{w_\sigma^\flat(z-\tilde z)} &= [\![\sigma]\!]\frac{(1+[\![\mu]\!]^{-1}|z-\tilde z|_s)^\flat}{(1+[\![\sigma]\!]^{-1}|z-\tilde z|_s)^\flat}\frac{[\![\sigma]\!]^{-1}|\tilde z-z|_s}{(1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2)}\\ &\lesssim [\![\sigma]\!]\frac{1}{(1+[\![\sigma]\!]^{-1}|\tilde z-z|_s)^{\flat-1}}\frac{1}{(1+[\![\mu]\!]^{-1}|z-\tilde z|_s)^{2-\flat}},\end{aligned}$$

Since $|\bar{\tilde{z}} - \bar{z}| \neq 0$ implies $|\tilde{z} - z|_s \geqslant \varepsilon$ we have

$$\begin{aligned}\frac{|\bar{\tilde{z}} - \bar{z}| w_\mu^\mathfrak{b}(z - \tilde{z})\, h_\mu(z, \tilde{z})}{w_\sigma^\mathfrak{b}(z - \tilde{z})} &\lesssim [\![\sigma]\!] \frac{1}{(1 + [\![\sigma]\!]^{-1}/\varepsilon)^{\mathfrak{b}-1}} \frac{1}{(1 + [\![\mu]\!]^{-1}/\varepsilon)^{2-\mathfrak{b}}} \\ &\leqslant [\![\sigma]\!]\, [\![\sigma]\!]^{\mathfrak{b}-1} \varepsilon^{1-\mathfrak{b}} [\![\mu]\!]^{2-\mathfrak{b}} \varepsilon^{\mathfrak{b}-2} \\ &= [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{2-\mathfrak{b}} \varepsilon^{-1}.\end{aligned}$$

Hence, using again (B.8) with $i = k+$ and $j = k-$ we get

$$\big\| \mathbf{R}_\varepsilon^{(k)} \big(\big[K_\sigma^{1,1} \dot{\tilde{F}}_\sigma^{[\ell],(1)} \big] \cdot h_\mu \big) \big\|_\mu \lesssim \varepsilon [\![\mu]\!]^{-2} [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{2-\mathfrak{b}} \varepsilon^{-1} \big\| \dot{\tilde{F}}_\sigma^{[\ell],(1)} \big\|_\sigma = [\![\mu]\!]^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}} \big\| \dot{\tilde{F}}_\sigma^{[\ell],(1)} \big\|_\sigma.$$

This finishes the proof. □

Bibliography

[BC24] Bjoern Bringmann and Sky Cao. Global well-posedness of the stochastic Abelian-Higgs equations in two dimensions. *arXiv preprint arXiv:2403.16878*, 2024.

[BC25] Bjoern Bringmann and Sky Cao. Global well-posedness of the dynamical sine-Gordon model up to 6π. *The Annals of Probability*, 54(3):1564–1607, 2026. doi:10.1214/25-AOP1797.

[BDH98] David Brydges, Jonathan Dimock, and Thomas R. Hurd. A non-Gaussian fixed point for ϕ^4 in $4 - \varepsilon$ dimensions. *Communications in Mathematical Physics*, 198(1):111–156, 1998. doi:10.1007/s002200050474.

[BMS03] David Brydges, Pronob K. Mitter, and Benedetto Scoppola. Critical $(\Phi^4)_{3,\varepsilon}$. *Communications in Mathematical Physics*, 240(1):281–327, 2003.

[CC18] Rémi Catellier and Khalil Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *The Annals of Probability*, 46(5):2621–2679, 2018.

[CFW24] Ajay Chandra, Guilherme de Lima Feltes, and Hendrik Weber. A priori bounds for the generalised parabolic Anderson model. *Communications on Pure and Applied Mathematics*, 79:1315–1394, 2026. doi:10.1002/cpa.70025.

[CMW23] Ajay Chandra, Augustin Moinat, and Hendrik Weber. A priori bounds for the Φ^4 equation in the full subcritical regime. *Archive for Rational Mechanics and Analysis*, 247(3):48, 2023.

[CRS+15] Óscar Ciaurri, Luz Roncal, Pablo R. Stinga, José L. Torrea, and Juan L. Varona. Fractional discrete Laplacian versus discretized fractional Laplacian. *arXiv preprint arXiv:1507.04986*, 2015.

[DDJ24] Paweł Duch, Wojciech Dybalski, and Azam Jahandideh. Stochastic quantization of two-dimensional $P(\Phi)$ quantum field theory. *Annales Henri Poincaré*, 26:1055–1086, 2025. doi:10.1007/s00023-024-01447-w.

[DFG22] Francesco C. De Vecchi, Luca Fresta, and Massimiliano Gubinelli. A stochastic analysis of subcritical Euclidean fermionic field theories. *The Annals of Probability*, 53(3):906–966, 2025. doi:10.1214/24-AOP1714.

[DHYZ25] Paweł Duch, Martin Hairer, Jaeyun Yi, and Wenhao Zhao. Ergodicity of infinite volume Φ^4_3 at high temperature. *arXiv preprint arXiv:2508.07776*, 2025.

[Duc22] Paweł Duch. Renormalization of singular elliptic stochastic PDEs using flow equation. *Probability and Mathematical Physics*, 6(1):111–138, 2025. doi:10.2140/pmp.2025.6.111.

[Duc24] Paweł Duch. Construction of Gross–Neveu model using Polchinski flow equation. *arXiv preprint arXiv:2403.18562*, 2024.

[Duc25a] Paweł Duch. Flow equation approach to singular stochastic PDEs. *Probability and Mathematical Physics*, 6(2):327–437, 2025. doi:10.2140/pmp.2025.6.327.

[Duc25b] Paweł Duch. Lecture notes on flow equation approach to singular stochastic PDEs. *arXiv preprint arXiv:2511.07120*, 2025.

[EW24] Salvador Esquivel and Hendrik Weber. A priori bounds for the dynamic fractional Φ^4 model on $\mathbb{T}^3$ in the full subcritical regime. *arXiv preprint arXiv:2411.16536*, 2024.

[Fel74] Joel Feldman. The ϕ^4_3 field theory in a finite volume. *Communications in Mathematical Physics*, 37:93–120, 1974.

[FO76] Joel S. Feldman and Konrad Osterwalder. The Wightman axioms and the mass gap for weakly coupled ϕ^4_3 quantum field theories. *Annals of Physics*, 97(1):80–135, 1976.

[GH19] Massimiliano Gubinelli and Martina Hofmanová. Global solutions to elliptic and parabolic φ^4 models in Euclidean space. *Communications in Mathematical Physics*, 368(3):1201–1266, 2019.

[GH21] Massimiliano Gubinelli and Martina Hofmanová. A PDE construction of the Euclidean Φ^4_3 quantum field theory. *Communications in Mathematical Physics*, 384(1):1–75, 2021.

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3(e6), 2015.

[GJ73] James Glimm and Arthur Jaffe. Positivity of the ϕ_3^4 Hamiltonian. *Fortschritte der Physik. Progress of Physics*, 21:327–376, 1973.

[GM24] Massimiliano Gubinelli and Sarah-Jean Meyer. The FBSDE approach to sine-Gordon up to 6π. *arXiv preprint arXiv:2401.13648*, 2024.

[Gri03] Alexander Grigor'yan. Heat kernels and function theory on metric measure spaces. In *Heat Kernels and Analysis on Manifolds*, volume 338 of *Contemporary Mathematics*, pages 143–172. American Mathematical Society, Providence, RI, 2003.

[GT01] Alexander Grigor'yan and Andras Telcs. Sub-Gaussian estimates of heat kernels on infinite graphs. *Duke Mathematical Journal*, 109(3):451–510, 2001.

[Hai14] Martin Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[HS22] Martin Hairer and Rhys Steele. The φ_3^4 measure has sub-Gaussian tails. *Journal of Statistical Physics*, 186(3):38, 2022.

[Kop07] Christoph Kopper. Renormalization theory based on flow equations. In *Rigorous Quantum Field Theory*, volume 251 of *Progr. Math.*, pages 161–174. Birkhäuser, Basel, 2007.

[Kup16] Antti Kupiainen. Renormalization group and stochastic PDEs. *Ann. Henri Poincaré*, 17(3):497–535, 2016.

[Kwa17] Mateusz Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fractional Calculus and Applied Analysis*, 20(1):7–51, 2017.

[LOTT21] Pablo Linares, Felix Otto, Markus Tempelmayr, and Pavlos Tsatsoulis. A diagram-free approach to the stochastic estimates in regularity structures. *Inventiones Mathematicae*, 237(3):1469–1565, 2024. doi:10.1007/s00222-024-01275-z.

[MW17] Jean-Christophe Mourrat and Hendrik Weber. Global well-posedness of the dynamic ϕ^4 model in the plane. *Ann. Probab.*, 45(4):2398–2476, 2017.

[MW20] Augustin Moinat and Hendrik Weber. Space-time localisation for the dynamic Φ_3^4 model. *Communications on Pure and Applied Mathematics*, 73(12):2519–2555, 2020.

[OSSW21] Felix Otto, Jonas Sauer, Scott Smith, and Hendrik Weber. A priori bounds for quasi-linear SPDEs in the full sub-critical regime. *Journal of the European Mathematical Society*, 27:71–118, 2024. doi:10.4171/jems/1574.

[Pan25] Romain Panis. Triviality of the scaling limits of critical Ising and Φ^4 models with effective dimension at least four. *The Annals of Probability*, 54(2):892–972, 2026. doi:10.1214/25-AOP1782.

[Pol84] Joseph Polchinski. Renormalization and effective Lagrangians. *Nuclear Physics B*, 231(2):269–295, 1984.

[PW81] Giorgio Parisi and Yong Shi Wu. Perturbation theory without gauge fixing. *Scientia Sinica. Zhongguo Kexue*, 24(4):483–496, 1981. MR626795.

[Sal07] Manfred Salmhofer. *Renormalization: An Introduction*. Springer, Berlin, corrected 2nd printing, 2007.

[Wil71] Kenneth G. Wilson. Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture. *Physical Review B*, 4(9):3174, 1971.

[WK74] Kenneth G. Wilson and John Kogut. The renormalization group and the ε expansion. *Physics Reports*, 12(2):75–199, 1974.