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Paweł Duch · Massimiliano Gubinelli · Paolo Rinaldi

Parabolic stochastic quantisation of the fractional Φ^4_3 model in the full subcritical regime

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Abstract. We present a construction of the measure of the fractional Φ^4 Euclidean quantum field theory on $\mathbb{R}^3$ in the full subcritical regime via parabolic stochastic quantisation. Our approach is based on a truncated flow equation for the effective description of the model at sufficiently small scales and on coercive estimates for the nonlinear stochastic partial differential equation describing the interacting field. The constructed measure is translation invariant, reflection positive, and has quartic exponential tails.

Keywords: stochastic quantisation, renormalisation group, fractional Laplacian, singular SPDEs.

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Paweł Duch: Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, ul. Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland; pawel.duch@amu.edu.pl
Author IDs: ORCID 0000-0002-2445-2189

Massimiliano Gubinelli: Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, OX2 6GG Oxford, United Kingdom; gubinelli@maths.ox.ac.uk
Author IDs: ORCID 0000-0002-4014-2949

Paolo Rinaldi: Institute for Applied Mathematics and Hausdorff Center for Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany; rinaldi@iam.uni-bonn.de
Author IDs: ORCID 0000-0003-1578-2677

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1. Introduction

For $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$ let

$$\mathbb{R}^d_\varepsilon := (\varepsilon\mathbb{Z})^d, \qquad \mathbb{T}^d_{\varepsilon,M} := (\varepsilon\mathbb{Z})^d/(M\mathbb{Z})^d, \qquad \Omega_{\varepsilon,M} := \{\varphi : \mathbb{T}^d_{\varepsilon,M} \to \mathbb{R}\}.$$

Define a probability measure $\nu_{\varepsilon,M}$ on $\Omega_{\varepsilon,M}$ by

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) := \frac{\exp(-S_{\varepsilon,M}(\varphi))}{Z_{\varepsilon,M}} \prod_{x\in\mathbb{T}^d_{\varepsilon,M}} \mathrm{d}\varphi(x), \tag{1.1}$$

where

$$S_{\varepsilon,M}(\varphi) := 2\varepsilon^d \sum_{x\in\mathbb{T}^d_{\varepsilon,M}} \left[\frac{1}{2}\varphi(x)(-\Delta_\varepsilon)^s\varphi(x) + \frac{m^2}{2}\varphi(x)^2 + \frac{\lambda}{4}\varphi(x)^4 - \frac{r_{\varepsilon,M}}{2}\varphi(x)^2\right]. \tag{1.2}$$

The normalisation constant $Z_{\varepsilon,M} > 0$ is chosen so that the total mass of $\nu_{\varepsilon,M}$ equals one. The parameters $m > 0, \lambda > 0, r_{\varepsilon,M} \in \mathbb{R}$ are referred to as *mass*, *coupling constant* and *mass renormalisation*, respectively. The operator $(-\Delta_\varepsilon)^s$ is the discrete fractional Laplacian of order $s \in (0, 1]$ defined via functional calculus as the s-th power of the discrete nearest-neighbour Laplacian $-\Delta_\varepsilon$. In order to pass to the limit, we define a probability measure $\hat{\nu}_{\varepsilon,M}$ on $\mathcal{S}'(\mathbb{R}^3)$ by embedding sample paths of $\nu_{\varepsilon,M}$ in $\mathcal{S}'(\mathbb{R}^3)$ via a suitable Fourier multiplier, see (2.37) below.

For the sake of clarity, we restrict our considerations to $d = 3$. Then $s = 1$ corresponds to the standard Φ^4_3 model, while for fractional exponents $s > s_c := 3/4$ the model is subcritical, meaning that the nonlinear part can be treated as a perturbation of the Gaussian measure at small scales. Since reflection positivity is expected to hold only for $s \leqslant 1$, we restrict our

analysis to $s \in (s_c, 1]$. For further discussion of the measure constructed in Theorem 1.1, we refer to [24]. In what follows, we present a detailed analysis of the fractional regime $s \in (3/4, 1)$. The case $s = 1$, corresponding to the classical Laplacian, can be handled by the same strategy with substantial simplifications. The main result of this paper is the following theorem.

Theorem 1.1. *Let $d = 3$ and fix $s \in (3/4, 1]$, $m > 0$, $\lambda > 0$. There exists a choice of mass renormalisation $(r_{\varepsilon,M})_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+}$ such that $(\hat{\nu}_{\varepsilon,M})_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+}$ is a tight family of probability measures on $\mathcal{S}'(\mathbb{R}^3)$. Any accumulation point ν of this family is non-Gaussian, invariant under translations, reflection positive, and satisfies*

$$\int e^{\theta \|(1+|\bullet|)^{-b}(1-\Delta)^{-a}\phi\|^4_{L^2}} \nu(\mathrm{d}\phi) < \infty, \tag{1.3}$$

for sufficiently large $a, b > 0$ and sufficiently small $\theta > 0$.

Proof. Tightness is established in Section 2.4 and the bound (1.3) is proved in Section 2.5. The reflection positivity and translation invariance of any accumulation point follow as in [24], since these properties are inherited from the corresponding properties of the approximate measures $(\nu_{\varepsilon,M})_{\varepsilon,M}$. Non-Gaussianity follows from (1.3), as Gaussian measures cannot integrate functions exhibiting super-exponential growth. ∎

We expect that any accumulation point is invariant under all Euclidean transformations. Unfortunately, a direct characterisation of the limiting measures remains unavailable. As a result, we must deduce the properties of the accumulation points indirectly, relying on certain features of the approximating measures. Notably, since the measures on lattices are not rotation invariant, establishing rotation invariance of the continuum limit poses a particularly challenging problem. It is conceivable that this issue could be addressed using the technique developed in [14] in the context of the $P_2(\Phi)$ model. However, pursuing this direction lies beyond the scope of the present paper.

Let us also mention the recent work [15], where uniqueness of the limiting measure is established in the small-coupling regime for the standard Φ^4_3 model. A similar strategy might extend to the fractional variant, providing a potential route toward a full verification of the OS axioms, at least in the regime of small coupling.

Our proof strategy introduces a novel combination of renormalisation group ideas and PDE techniques which we believe may be useful more broadly in the context of the theory of subcritical singular SPDEs. The proof also applies to the vector version of the model where the field takes values in $\mathbb{R}^n$ for $n > 1$ and the functional $S_{\varepsilon,M}(\varphi)$ depends on φ in an $O(n)$-symmetric way. We briefly discuss how to adapt the proof to the vector case in Section 2.6.

Theorem 1.1 gives a construction of a model of Euclidean quantum field theory (EQFT) commonly referred to in the literature as the *fractional* Φ^4_3 model. This terminology stems from the form of the density appearing in the formula for the measure (1.1) and the fact that the underlying space is three-dimensional. In the case $s = 1$, the model reduces to the classical Φ^4_3 theory, which has long been regarded as a crucial benchmark in constructive

quantum field theory. Foundational results by Glimm and Jaffe [19], Feldman and Osterwalder [17, 18] and other pioneers of EQFT laid the groundwork for demonstrating the existence of models satisfying the Wightman axioms for local relativistic QFT through probabilistic methods based on the Euclidean framework [20]. In the fractional regime, that is, for $s \in (s_c, 1)$, the model we consider was introduced by Brydges, Mitter, and Scoppola [4] as a rigorous foundation for proving the existence of a nontrivial infrared fixed point via an ϵ-expansion for sufficiently small $\epsilon = s - s_c > 0$. See also [3] for an analogous result concerning a related four-dimensional model.

Our proof also extends to the cases $d = 1$ or $d = 2$ and $s \in (d/4, 1]$ without requiring any substantial modifications. We stress that we crucially need the condition of subcriticality (*i.e.* super-renormalisability), which translates into $s > d/4$, where s is the power of the fractional Laplacian. In particular, if $d \geqslant 4$ this would require $s > 1$. However, it is known that the fractional Laplacian $(-\Delta)^s$ with $s > 1$ does not satisfy the maximum principle, and as a result, our a priori estimate does not apply in this regime. Moreover, for $s > 1$ the measure is not expected to be reflection positive—since the Gaussian part itself lacks this property—and is therefore of limited interest from the perspective of constructive quantum field theory. Finally, we note that for $d \leqslant 4$ it was shown in [37] that at criticality $s = d/4$ as well as in the supercritical regime $s < d/4$, any possible continuum limit is trivial, in the sense that it is Gaussian.

In recent years there has been a renewed interest in EQFTs due to the development of an alternative approach to proving theorems such as Theorem 1.1. This new approach is grounded in the basic ideas of stochastic calculus and it is usually called *stochastic quantisation*. This term was introduced by Parisi and Wu [38] to describe the quantisation of gauge theories via the construction of a stochastic process that evolves in fictitious time and whose stationary distribution is the target Euclidean QFT. This stochastic evolution is a nonlinear stochastic partial differential equation of a singular kind, for which a particular procedure of renormalisation is needed to give it a precise meaning. The analysis of such equations requires a mix of probabilistic and analytic arguments that go beyond the usual framework of Itô's stochastic differential equations. For this reason it took some time before the SPDE community learned how to handle such singular equations and discovered methods such as regularity structures [27], paracontrolled calculus [5,25], and the renormalisation group [30], which finally made it possible to tackle the problem of the stochastic quantisation of the Φ^4_3 model. Gubinelli and Hofmanová [24] obtained the analogue of Theorem 1.1 with $s = 1$ and a small range of values below that using a mix of paracontrolled calculus for the small scale singularities of the equation and coercive estimates to tame the large-scale fluctuations. We refer the reader to the introduction to [24] for a deeper review of the literature and the history of constructive QFT, and also to contextualise the meaning and consequences of Theorem 1.1.

The probability measure $\nu_{\varepsilon,M}$ in (1.1) is the equilibrium distribution of the Langevin dynamics governed by the finite system of SDEs

$$\mathcal{L}_\varepsilon \phi^{(\varepsilon,M)} + \lambda(\phi^{(\varepsilon,M)})^3 - r_{\varepsilon,M}\phi^{(\varepsilon,M)} = \xi^{(\varepsilon,M)}, \tag{1.4}$$

on $\Lambda_{\varepsilon,M} := \mathbb{R} \times \mathbb{T}^d_{\varepsilon,M}$, where

$$\mathcal{L}_\varepsilon := \partial_t + (-\Delta_\varepsilon)^s + m^2$$

and $\xi^{(\varepsilon,M)}$ is a spacetime white noise satisfying

$$\mathbb{E}[\xi^{(\varepsilon,M)}(t,x)\xi^{(\varepsilon,M)}(s,y)] = \delta(t-s)\,\varepsilon^{-d}\mathbb{1}_{x=y}, \qquad (t,x),(s,y) \in \Lambda_{\varepsilon,M}.$$

The constants $m > 0, \lambda > 0, r_{\varepsilon,M} \in \mathbb{R}$ coincide with the parameters appearing in (1.2). By standard stochastic analysis arguments, there exists a unique stationary solution $\phi^{(\varepsilon,M)}$ of (1.4) and we have $\mathrm{Law}(\phi^{(\varepsilon,M)}(t)) = \nu_{\varepsilon,M}$ for all $t \in \mathbb{R}$. In what follows, we identify $\phi^{(\varepsilon,M)}$ and $\xi^{(\varepsilon,M)}$ with periodic functions on $\Lambda_\varepsilon := \mathbb{R} \times \mathbb{R}^d_\varepsilon$.

The nontrivial step is now to control the solutions to the stochastic quantisation equation (1.4) uniformly as $\varepsilon \to 0$ and $M \to \infty$. At small scales, it is expected that the nonlinear term in the dynamics is a perturbation of the linearised equation driven by spacetime white noise. Consequently, in the continuum limit, the solutions $\phi^{(\varepsilon,M)}$ are expected to converge to random distributions belonging to Besov spaces of negative regularity slightly worse than $s - d/2$, which is the regularity of the Gaussian free field. This low regularity presents a major analytical challenge, as it complicates the control of the nonlinear term.

Inspired by the works of Wilson [41,42] and Polchinski [39] on the continuous renormalisation group and by the more recent approach introduced by one of the authors in [11,13], we use a flow equation to effectively describe the solution $\phi^{(\varepsilon,M)}$ of the SPDE at some spatial scale much larger than ε (see Kupiainen [30] for a discrete counterpart). Let $\phi^{(\varepsilon,M)}_\sigma$ denote a description of the solution at a scale of order

$$[\![\sigma]\!] := (1-\sigma) \gg \varepsilon > 0$$

for some $\sigma \in (0,1)$. The flow equation approach consists in deriving a parabolic equation for $\phi^{(\varepsilon,M)}_\sigma$ of the form

$$\mathcal{L}_\varepsilon \phi^{(\varepsilon,M)}_\sigma = F^{(\varepsilon,M)}_\sigma(\phi^{(\varepsilon,M)}_\sigma). \tag{1.5}$$

Here, $\psi \mapsto F^{(\varepsilon,M)}_\sigma(\psi)$ is an analytic functional depending on the noise $\xi^{(\varepsilon,M)}$, called the *effective force*, such that

$$F^{(\varepsilon,M)}_1(\psi) = -\lambda\psi^3 + r_{\varepsilon,M}\psi + \xi^{(\varepsilon,M)}. \tag{1.6}$$

In particular, (1.4) can be recovered from (1.5) for $\sigma = 1$ and $\phi^{(\varepsilon,M)}_1 = \phi^{(\varepsilon,M)}$. The functional $F^{(\varepsilon,M)}_\sigma$ can be obtained by solving a *flow equation*

$$\partial_\sigma F^{(\varepsilon,M)}_\sigma = \mathbb{B}_\sigma(F^{(\varepsilon,M)}_\sigma, F^{(\varepsilon,M)}_\sigma), \tag{1.7}$$

backwards for $\sigma \in (\mu, 1]$ with the final condition (1.6) and where $\mathbb{B}_\sigma$ is an appropriate bilinear operator. The parameter $\sigma \in [0,1]$ does not have any specific physical meaning and the spatial scale of the decomposition is fixed conventionally to be of order $[\![\sigma]\!]$, that is $\phi^{(\varepsilon,M)}_\sigma$ is expected to fluctuate at spatial scales of order $[\![\sigma]\!]$ or larger, and in particular to be a locally bounded function on $\Lambda_0 := \mathbb{R} \times \mathbb{R}^d$, when extended in some reasonable way from

the lattice Λ_ε to the continuum. A key ingredient is the control of the stochastic process $(F_\sigma^{(\varepsilon,M)})_\sigma$ solving the flow equation (1.7). Following a simple but crucial observation of [11, 13], this control can be obtained by studying the evolution in the scale parameter of the cumulants $(\mathcal{F}_\sigma^{(\varepsilon,M)})_\sigma$ of the process $(F_\sigma^{(\varepsilon,M)})_\sigma$, which themselves satisfy a kind of higher-order deterministic flow equation

$$\partial_\sigma \mathcal{F}_\sigma^{(\varepsilon,M)} = \mathcal{A}_\sigma(\mathcal{F}_\sigma^{(\varepsilon,M)}) + \mathcal{B}_\sigma(\mathcal{F}_\sigma^{(\varepsilon,M)}, \mathcal{F}_\sigma^{(\varepsilon,M)}), \tag{1.8}$$

with prescribed final condition $\mathcal{F}_1^{(\varepsilon,M)}$. Upon choosing this initial condition appropriately by tuning the parameter $r_{\varepsilon,M}$ in (1.6), one can prove estimates uniform in ε and M for the cumulants $(\mathcal{F}_\sigma^{(\varepsilon,M)})_\sigma$ and therefore, by a Kolmogorov-type argument, suitable bounds on the effective force $(F_\sigma^{(\varepsilon,M)})_\sigma$ uniform as $\varepsilon \to 0$ and $M \to \infty$.

The flow equation (1.7) is bilinear and therefore solvable in general only in a perturbative regime, e.g. in a small interval $I = [\bar{\sigma}, 1]$ around the initial condition at $\sigma = 1$ or for small data (or small time). The size of this perturbative region depends crucially on the noise $\xi^{(\varepsilon,M)}$. Although this dependence can be controlled uniformly in ε and M, large noise fluctuations can make the region arbitrarily small and restrict the resulting existence theory to *local* results. A similar limitation is present in the work of Kupiainen [30] who, instead, uses a discrete renormalisation group (RG) iteration, and more generally in all the other approaches which use an expansion of solutions in order to resolve the singular terms and control the limit as $\varepsilon \to 0$, e.g. in regularity structures and also in paracontrolled calculus. This difficulty is the signal of the "large field problem", well known in constructive EQFT.

From the point of view of the stochastic quantisation equation, the large-field problem can be addressed by exploiting the coercivity of the nonlinear term which pulls the solution back from infinity. While this observation is standard in PDE theory, it still requires some nontrivial adaptation to be effective for singular SPDEs. Mourrat and Weber [35] were the first to solve this problem in their proof of global existence for the Φ^4_2 dynamics on the full space with the usual Laplacian diffusion term and subsequently Gubinelli and Hofmanová in the context of paracontrolled analysis of Φ^4 models [23, 24] including the three-dimensional parabolic setting. Moinat and Weber [34] proved the so-called spacetime localisation property for the dynamical Φ^4_3 model in the framework of regularity structures. This result was further extended by Chandra, Moinat, and Weber [7] to cover the full subcritical regime. In the latter work, the authors modify the covariance of the noise, rather than the diffusion term, in order to explore regularities arbitrarily close to the critical threshold. Although their estimates suffice to establish tightness of the invariant measures associated with their SPDE, these measures are not explicit, and it is not clear whether they are reflection positive. Consequently, their role in the stochastic quantisation of Euclidean QFTs is, at present, not fully understood.

In the broader context of global solutions for singular SPDEs, we also mention recent work of Chandra, Feltes, and Weber [6], which establishes results for the stochastic quantisation of the two-dimensional sine-Gordon model on a periodic domain for parameter values slightly above the first threshold. This has since been extended up to the second

threshold in [2]. Moreover, [1] proves long-time well-posedness of the two-dimensional Abelian Higgs model.

During the revision of the present paper, a new preprint [16] appeared in which a priori bounds for the fractional Φ^4_3 model on the three-dimensional torus were established in the full subcritical regime. The main distinctions between our results and those of [16] are that we employ the flow-equation framework and work on an infinite lattice, whereas the authors of [16] use the regularity structures approach and study the continuum model on a torus with mollified noise. As far as we know, mollification of the noise is not a feasible strategy for establishing stochastic quantisation of an EQFT, in particular with respect to reflection positivity.

The main contribution of our work is the identification of a framework in which the flow equation method is combined with PDE estimates for the dynamics. This hybrid approach yields a powerful variant of the renormalisation group (RG). Instead of requiring an exact solution to the flow equation (1.7), it suffices to construct a suitable approximate solution $(F_\sigma)_\sigma = (F_\sigma^{(\varepsilon,M)})_\sigma$ satisfying (1.6) for which the quantity

$$H_\sigma := \partial_\sigma F_\sigma + \mathbb{B}_\sigma(F_\sigma, F_\sigma), \tag{1.9}$$

is small enough in an appropriate sense. The price to pay for this approximation is a *remainder* term $R_\sigma^{(\varepsilon,M)}$ in the SPDE which now reads as a *system* of two equations:

$$\begin{cases} \mathcal{L}_\varepsilon \phi_\sigma = J_\sigma[F_\sigma(\phi_\sigma) + R_\sigma], \\ \partial_\sigma R_\sigma = -H_\sigma(\phi_\sigma) - \mathrm{D}F_\sigma(\phi_\sigma)((\partial_\sigma G_\sigma) R_\sigma), \qquad R_1 = 0, \end{cases} \tag{1.10}$$

for the pair of scale-dependent functions

$$(\phi_\sigma, R_\sigma)_\sigma = (\phi_\sigma^{(\varepsilon,M)}, R_\sigma^{(\varepsilon,M)})_\sigma.$$

Here $G := \mathcal{L}_\varepsilon^{-1}$ is the fractional heat kernel, with $\mathcal{L}_\varepsilon^{-1}$ denoting the inverse of $\mathcal{L}_\varepsilon$ as defined in (1.19) below, $(J_\sigma)_\sigma$ is a family of smoothing operators (see Definition 1.13) and

$$(G_\sigma)_{\sigma\in(0,1)} := (J_\sigma G)_{\sigma\in(0,1)},$$

is a scale decomposition of G. Moreover, $\mathrm{D}F(\psi)\tilde{\psi}$ denotes the functional derivative of a functional F at ψ in the direction of $\tilde{\psi}$.

One can prove that the term $F_\sigma(\phi_\sigma)$ retains the coercive structure of (1.6), that is,

$$J_\sigma F_\sigma(\phi_\sigma) = -\lambda\phi_\sigma^3 + Q_\sigma(\phi_\sigma),$$

where $Q_\sigma(\phi_\sigma)$ is "smaller" than the cubic contribution provided $[\![\sigma]\!] \ll 1$. This, together with the linearity in $R_\sigma^{(\varepsilon,M)}$ of the second equation of (1.10), makes this system amenable to standard PDE techniques: by choosing $[\![\sigma]\!] \ll 1$ one can control the noncoercive part $Q_\sigma(\phi_\sigma)$ of the effective force using the coercive part $-\lambda\phi_\sigma^3$, thereby resolving the large-field problem. At the same time, for $[\![\sigma]\!] > 0$, we have uniform estimates for $Q_\sigma, \mathrm{D}F_\sigma, H_\sigma$ as $\varepsilon \to 0$ and $M \to \infty$, provided the renormalisation constant $r_{\varepsilon,M}$ appearing in the boundary condition (1.6) for the effective force is chosen appropriately. This allows full control

of (1.4) and the proof of tightness of the laws of the processes $(\phi^{(\varepsilon,M)})_{\varepsilon,M}$, and therefore of the family of measures $(\nu_{\varepsilon,M})_{\varepsilon,M}$.

The implementation of this plan has to deal with two main technical difficulties:

(a) We decompose the fractional heat kernel G scale by scale (cf. (1.20)), using

$$(\partial_\sigma G_\sigma)_\sigma.$$

This produces kernels with limited spacetime decay (see Lemma A.11), reflecting the restricted smoothness of G away from the origin. This algebraic decay of $\partial_\sigma G_\sigma$ propagates to the kernels of the effective force, necessitating a careful choice of weighted spaces for both the solutions and the kernels. To handle this limited decay, we localise the relevant operators and kernels at several points in the proof. The main instance is the localisation procedure for the relevant cumulants, which governs the flow of the renormalisation constants; see Appendix B.2. It would be interesting to devise an alternative strategy to bypass this problem with some other scale decomposition (or an additional localisation procedure).

(b) The natural setting of the analysis provides only negative spacetime regularity for the solutions of the SPDE. Such regularity is insufficient for stochastic quantisation, since one must be able to compute the marginal of the solution at a given time. To address this, it is necessary to work with distributional norms defined via smoothing operators $(K_\mu)_\mu$ that provide only limited time regularisation. Suitable Schauder estimates for these operators can be established (see Lemma A.24). These smoothing operators also influence the definition of the norms for the kernels of the effective force (cf. Definition 4.6) and require a careful adaptation of the Kolmogorov-type argument used in Lemma 4.21.

Comparison with other approaches. The possibility of working with an approximate flow equation makes it easier to compare the RG approach advocated in this paper (and originally proposed in [30] and [11,13]) with regularity structures [27] and paracontrolled distributions [25]. There is a clear parallel among the various approaches. The flow equation constructs a random object $F_\sigma^{(\varepsilon,M)}$ — the scale-dependent *effective force* — which encapsulates the influence of the noise and takes the form of a finite polynomial built from the noise and the linear part of the equation. This object corresponds respectively to the *model* in the theory of regularity structures, the *enhanced noise* in paracontrolled calculus, or the *rough path* in rough path theory.

While $F_\sigma^{(\varepsilon,M)}$ is obtained through a probabilistic construction, the remainder term $R_\sigma^{(\varepsilon,M)}$ is defined analytically in terms of $F_\sigma^{(\varepsilon,M)}$. This deterministic component mirrors the analytic machinery of regularity structures, the paracontrolled operators in the paracontrolled calculus, and the sewing lemma in rough path analysis.

When the parameter s is near its critical value of $s_c = 3/4$ the number of terms which have to be accounted for in the approximation $F_\sigma^{(\varepsilon,M)}$ of the solution of the flow equation grows in an unbounded manner. A notable advantage of the flow equation approach, however, lies in its relative insensitivity to this increasing complexity: the analysis remains compact and largely independent of the distance to criticality. This analytical efficiency —

the ability to capture the nonlinear propagation of randomness with minimal combinatorial overhead — was first observed by Polchinski [39] in his proof of perturbative renormalisability of the Euclidean ϕ^4_4 QFT. For a modern account of this approach to perturbation theory of QFTs, the reader can consult the book of Salmhofer [40] or Kopper [29].

As we already noted, the application of RG ideas to SPDEs is made efficient by the observation of one of the authors [11, 13] that flow equations can also be used to estimate cumulants directly, thereby avoiding explicit and cumbersome inductive arguments on trees — much as Polchinski's method circumvents the inductive structure of BPHZ renormalisation. There are further conceptual similarities with recent developments by Otto, Weber, and collaborators [32, 36], who use PDE-based arguments to derive probabilistic estimates for the modes in regularity structures. The flow equation framework, however, offers an additional advantage: renormalisation conditions naturally appear as boundary conditions for the corresponding flow equation.

The combination of the flow equation approach with stochastic quantisation in the context of the construction of EQFT has recently been exploited by Meyer and one of the authors in [26] to study the sine–Gordon model in the full space up to the second threshold and, by De Vecchi, Fresta and one of the authors, in [9], to develop a new approach to Euclidean Fermionic theories. In both papers the stochastic quantisation is obtained by using a forward-backward stochastic differential equation together with an approximate analysis of Polchinski's flow equation. Finally, we mention the work [10], where the Polchinski equation was employed to construct the Gross–Neveu model, a critical fermionic model of Euclidean quantum field theory.

Conclusions. Despite the technical difficulties due to the analysis of the fractional heat equation, we would like to stress that the present paper is self-contained and presents complete arguments for all proofs. Compared with the few existing works on singular SPDEs in the full subcritical regime, we believe our work is the first to address, in a unified framework, several intricate aspects of the problem:

(a) we present the entire argument — both analytic and probabilistic — in full detail;

(b) we work with an extremely nonlocal SPDE posed on a (semi-)discrete space;

(c) we establish an a priori bound in the full subcritical regime, valid globally in spacetime;

(d) we obtain optimal tail estimates for the solutions.

We hope that this work illustrates the strength and flexibility of the stochastic quantisation approach in tackling, in a genuinely nonperturbative manner, the construction of Euclidean quantum field models in the subcritical regime.

Plan of the paper. In Section 2 we introduce the main objects of our analysis: the scale decomposition, the spacetime weighted norms which will be used to control the large fields and all the intermediate results which are needed in the proof of Theorem 1.1. The coercive estimates will be proven in Section 3, while the approximate flow equation for the effective force will be analysed in Section 4 via the flow equation for the cumulants. Appendix A contains some technical lemmas and Schauder estimates tailored to our setting

while Appendix B contains the detailed definition of the various contributions to the flow equations for the cumulants and their analytic estimates.

1.1. Preliminaries and notation

In this section we introduce the main notation that we are going to use throughout the paper.

We let $\mathbb{R}_0 := \mathbb{R}, \mathbb{R}_\varepsilon := \varepsilon\mathbb{Z}, \mathbb{T}_{\varepsilon,M} := (\varepsilon\mathbb{Z})/(M\mathbb{Z})$ and define the corresponding spacetime domains

$$\Lambda_\varepsilon := \mathbb{R} \times \mathbb{R}^d_\varepsilon, \qquad \Lambda_{\varepsilon,M} := \mathbb{R} \times \mathbb{T}^d_{\varepsilon,M}, \qquad d = 3, \quad \varepsilon \in 2^{-\mathbb{N}_0}, \quad M \in \mathbb{N}_+.$$

The assumption that $\varepsilon \in 2^{-\mathbb{N}_0}$ ensures that $M\mathbb{Z} \subset \varepsilon\mathbb{Z}$ for every $M \in \mathbb{N}_+$. These domains are continuous in the time direction and discrete in the d spatial dimensions. For a measurable function $f : \Lambda_\varepsilon \to \mathbb{R}^n$, we write

$$\|f\| := \|f\|_{L^\infty(\Lambda_\varepsilon)}.$$

Given a nonnegative weight $w \in C(\Lambda_0)$ we denote by $C(\Lambda_\varepsilon, w)$ the space of continuous functions $f : \Lambda_\varepsilon \to \mathbb{R}$ such that

$$\|f\|_{L^\infty(w)} := \sup_{z\in\Lambda_\varepsilon} |w(z)f(z)| < \infty.$$

We always identify functions $f : \Lambda_{\varepsilon,M} \to \mathbb{R}$ with their spatially periodic extensions $f : \Lambda_\varepsilon \to \mathbb{R}$. In particular, we write $C(\Lambda_{\varepsilon,M}, w)$ for the subspace of $C(\Lambda_\varepsilon, w)$ consisting of functions that are periodic in space with period M. We denote by $\mathcal{S}(\Lambda_\varepsilon)$ the space of smooth rapidly decreasing functions over Λ_ε. The Fourier transform of $f \in \mathcal{S}(\Lambda_\varepsilon)$ is defined as

$$\hat{f}(\omega, k) := \int_{\Lambda_\varepsilon} e^{-i(\omega t + k\cdot x)} f(t,x)\,\mathrm{d}t\mathrm{d}x, \qquad (\omega, k) \in \Lambda^*_\varepsilon := \mathbb{R} \times (\mathbb{R}^d_\varepsilon)^*,$$

where

$$(\mathbb{R}^d_\varepsilon)^* := (-\pi/\varepsilon, \pi/\varepsilon]^d$$

is the dual of the group $\mathbb{R}^d_\varepsilon$. The integral over $\mathbb{R}^d_\varepsilon$ is understood with respect to the counting measure (still written as $\mathrm{d}x$) with normalisation

$$\int_{\mathbb{R}^d_\varepsilon} f(x)\,\mathrm{d}x := \varepsilon^d \sum_{x\in\mathbb{R}^d_\varepsilon} f(x),$$

which ensures weak convergence to the Lebesgue measure as $\varepsilon \to 0$. As usual, the Fourier transform extends to $\mathcal{S}'(\Lambda_\varepsilon)$ by duality. The inverse Fourier transform is given by

$$f(t,x) = \int_{\Lambda^*_\varepsilon} \hat{f}(\omega,k) e^{i(\omega t + k\cdot x)} \frac{\mathrm{d}\omega\mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x) \in \Lambda_\varepsilon.$$

For $\varepsilon \geqslant 0$, we define the Laplacian Δ_ε as the Fourier multiplier on $\mathcal{S}'(\Lambda_\varepsilon)$ with symbol

$$(\mathbb{R}^d_\varepsilon)^* \ni k \mapsto -q^2_\varepsilon(k) \in \mathbb{R},$$

where

$$q_0(k) = |k|, \qquad q_\varepsilon(k) := \left[\sum_{i=1}^d \left(\frac{2}{\varepsilon}\sin\left(\frac{\varepsilon k_i}{2}\right)\right)^2\right]^{1/2}, \qquad \varepsilon > 0. \tag{1.11}$$

Note that for $\varepsilon > 0$, the operator Δ_ε coincides with the standard nearest-neighbour discrete Laplacian.

The fractional Laplacian. For $s \in (0,1)$ the fractional Laplacian $(-\Delta_\varepsilon)^s$ is defined as the Fourier multiplier with symbol $(\mathbb{R}^d_\varepsilon)^* \ni k \mapsto q^{2s}_\varepsilon(k)$. In particular it is self-adjoint and positive in $L^2(\mathbb{R}^d_\varepsilon)$ and for $s \in (0,1)$ it has the (discrete, when $\varepsilon > 0$) heat–kernel representation [31]

$$(-\Delta_\varepsilon)^s f = C_s \int_{\mathbb{R}_+} (f - e^{\theta\Delta_\varepsilon} f)\theta^{-1-s}\mathrm{d}\theta, \tag{1.12}$$

with the constant $C_s = |\Gamma(-s)|^{-1}$. In the continuum, the fractional Laplacian has, for $s \in (0,1)$, the integral representation [31]:

$$(-\Delta_0)^s f(x) = C_{d,s}\mathrm{PV}\int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+2s}}\mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{1.13}$$

where $C_{d,s}$ is a universal constant. In the discrete setting a similar formula holds [8]:

$$(-\Delta_\varepsilon)^s f(x) = \varepsilon^d \sum_{y \in \mathbb{R}^d_\varepsilon | y \neq x} H^{(\varepsilon)}_s(x-y)(f(x) - f(y)), \qquad x \in \mathbb{R}^d_\varepsilon, \tag{1.14}$$

where the kernel $H^{(\varepsilon)}_s : \mathbb{R}^d_\varepsilon \to \mathbb{R}$ is positive and such that $H^{(\varepsilon)}_s(0) = 0$, $H^{(\varepsilon)}_s(x) = H^{(\varepsilon)}_s(-x)$ and

$$|H^{(\varepsilon)}_s(x)| \leqslant C'_{d,s}|x|^{-d-2s}, \qquad x \in \mathbb{R}^d_\varepsilon,$$

uniformly in $\varepsilon > 0$ for some constant $C'_{d,s} > 0$. Note that in our notation (1.14) can be equivalently written as

$$(-\Delta_\varepsilon)^s f(x) = \int_{\mathbb{R}^d_\varepsilon} H^{(\varepsilon)}_s(x-y)(f(x) - f(y))\,\mathrm{d}y.$$

For $\varepsilon > 0$, we can encode the representation (1.14) of the fractional Laplacian, via a positive measure $\mu^{(\varepsilon)}_s$ on $\Lambda_\varepsilon \times \Lambda_\varepsilon$ for which

$$\langle f, (-\Delta_\varepsilon)^s g\rangle = \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} f(z)(g(z) - g(z'))\mu^{(\varepsilon)}_s(\mathrm{d}z\mathrm{d}z').$$

We also define the kernel $\mu^{(\varepsilon)}_s(z, \mathrm{d}z')$ arising from the disintegration of $\mu^{(\varepsilon)}_s$ with respect to the measure $\mathrm{d}z$ on Λ_ε. Specifically, we have

$$\begin{aligned} \mu^{(\varepsilon)}_s(\mathrm{d}z\mathrm{d}z') &= \delta(t-u)H^{(\varepsilon)}_s(x-y)\,\mathrm{d}z\mathrm{d}z', \\ \mu^{(\varepsilon)}_s(z, \mathrm{d}z') &= \delta(t-u)H^{(\varepsilon)}_s(x-y)\,\mathrm{d}z', \end{aligned} \tag{1.15}$$

where $z = (t, x)$ and $z' = (u, y)$. The kernel $\mu_s^{(\varepsilon)}$ is symmetric, that is,

$$\mu_s^{(\varepsilon)}(z, \mathrm{d}z')\,\mathrm{d}z = \mu_s^{(\varepsilon)}(z', \mathrm{d}z)\,\mathrm{d}z'.$$

With this notation in place, the following Leibniz-type formula with remainder holds:

$$\begin{aligned} I_s^{(\varepsilon)}(f, g) &:= f(-\Delta_\varepsilon)^s g + g(-\Delta_\varepsilon)^s f - (-\Delta_\varepsilon)^s (fg) \\ &= \int_{\Lambda_\varepsilon} (f(\bullet) - f(z))(g(\bullet) - g(z))\mu_s^{(\varepsilon)}(\bullet, \mathrm{d}z). \end{aligned} \tag{1.16}$$

Let us also introduce the fractional difference operator

$$\mathfrak{D}_s^{(\varepsilon)}(f)(z) := I_s^{(\varepsilon)}(f, f)^{1/2}(z) = \left[\int_{\Lambda_\varepsilon} [f(z') - f(z)]^2 \mu_s^{(\varepsilon)}(z, \mathrm{d}z')\right]^{1/2}, \tag{1.17}$$

which behaves like a derivative of order s. In particular, using that, for all $\delta > 0$, one has the estimates

$$\begin{aligned} \int_{B(z,\delta)} [f(z') - f(z)]^2 \mu_s^{(\varepsilon)}(z, \mathrm{d}z') &\lesssim \|\nabla_\varepsilon f\|^2 \delta^{2-2s}, \\ \int_{B(z,\delta)^c} [f(z') - f(z)]^2 \mu_s^{(\varepsilon)}(z, \mathrm{d}z') &\lesssim \|f\|^2 \delta^{-2s}, \end{aligned}$$

where

$$B(z, \delta) := \{z' \in \Lambda_\varepsilon \,|\, |z - z'|_s \leqslant \delta\}$$

denotes the ball of radius $\delta > 0$. The discrete gradient is defined as

$$\nabla_\varepsilon f = (\partial_\varepsilon^{1+} f, \partial_\varepsilon^{1-} f, \ldots, \partial_\varepsilon^{d+} f, \partial_\varepsilon^{d-} f),$$

where, for $i \in \{1, \ldots, d\}$,

$$\partial_\varepsilon^{i\pm} f(x) := \pm \frac{f(x \pm \varepsilon e_i) - f(x)}{\varepsilon}, \qquad x \in \mathbb{R}_\varepsilon^d.$$

For a spacetime multi-index

$$A \in \mathbb{N}_0^{\{0, 1\pm, \ldots, d\pm\}},$$

we write its spatial part as

$$\bar{A} := (A_{i,+}, A_{i,-})_{i=1}^d \in \mathbb{N}_0^{\{1\pm, \ldots, d\pm\}},$$

and set

$$|\bar{A}| := \sum_{i=1}^d (A_{i,+} + A_{i,-}), \qquad |A|_s := 2sA_0 + |\bar{A}|.$$

We then define

$$\partial_\varepsilon^{\bar{A}} := \prod_{i=1}^d (\partial_\varepsilon^{i+})^{A_{i,+}} (\partial_\varepsilon^{i-})^{A_{i,-}}, \qquad \partial_\varepsilon^A := \partial_t^{A_0} \partial_\varepsilon^{\bar{A}}.$$

Whenever the lattice spacing ε is fixed and no confusion can arise, we suppress the subscript ε and write $\partial^{i\pm}$, $\partial^{\bar{A}}$, and ∂^{A}. Finally, $\|\nabla_\varepsilon f\|$ denotes the supremum norm. Choosing $\delta = \|f\|/\|\nabla_\varepsilon f\|$ yields

$$\mathfrak{D}_s^{(\varepsilon)}(f)(z) \lesssim \|\nabla_\varepsilon f\|\delta^{1-s} + \|f\|\delta^{-s} \lesssim \|\nabla_\varepsilon f\|^s \|f\|^{1-s}. \tag{1.18}$$

Remark 1.2. Although the kernel representation of the fractional Laplacian in the continuum case $\varepsilon = 0$ requires the presence of the principal value, the above considerations and the results presented below extend to the continuum with only minor modifications in the proofs, or simply by taking the $\varepsilon \to 0$ limit in the relevant inequalities. We emphasise, however, that the main results of the paper do not rely on these continuum extensions. In particular, the limit $\varepsilon \to 0$ in Theorem 1.1 is obtained via tightness, using the a priori estimates established uniformly for $\varepsilon \in 2^{-\mathbb{N}_0}$.

Remark 1.3. A basic observation is that the fractional Laplacian (whether in the continuum or on the lattice) satisfies an inequality under the action of convex functions. Let $s \in (0, 1]$, $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function and Φ' one of its sub-differentials, then, for any $\varepsilon \geqslant 0$ and $u \in C(\Lambda_\varepsilon)$, we have

$$(-\Delta_\varepsilon)^s \Phi(u) \leqslant \Phi'(u)(-\Delta_\varepsilon)^s u.$$

Indeed, let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\Phi(a) - \Phi(b) \leqslant \Phi'(a)(a - b), \qquad a, b \in \mathbb{R},$$

so if $u : \Lambda_\varepsilon \to \mathbb{R}$ is a continuous and bounded function, we have

$$\Phi(u) - e^{\theta\Delta_\varepsilon}\Phi(u) \leqslant \Phi'(u)(u - e^{\theta\Delta_\varepsilon}u),$$

since $e^{\theta\Delta_\varepsilon}$ has a positive definite probability kernel. The claimed inequality now follows from (1.12) in the case $s \in (0, 1)$. The case $s = 1$ is elementary. For $s > 1$ the result is not true. It is clear that the same proof works for $\varepsilon = 0$ with some additional regularity assumption. We will incorporate this idea in the proof of the key Lemma 3.3, below.

The operator $G = G^{(\varepsilon)} = \mathcal{L}_\varepsilon^{-1}$ is defined as

$$(\mathcal{L}_\varepsilon^{-1} f)(t, \bullet) := \int_{-\infty}^{t} e^{-(m^2 + (-\Delta_\varepsilon)^s)(t-u)} f(u, \bullet)\, \mathrm{d}u, \qquad t \in \mathbb{R}, \tag{1.19}$$

and will be applied to continuous functions on Λ_ε with at most a limited polynomial growth in spacetime. Indeed, on account of Lemma 5.4 of [21] together with the argument of Section 1 of [22], the kernel $G(t, x)$ of G satisfies

$$G(t, x) \lesssim \mathbb{1}_{t\geq 0}\, \varepsilon^{-d}\, \mathbb{1}_{x=0}\, e^{-m^2 t - ct/\varepsilon^{2s}} + \frac{\mathbb{1}_{t\geq 0}\, t\, e^{-m^2 t}}{\big(t^{1/(2s)} + \varepsilon + |x|\big)^{d+2s}}, \tag{1.20}$$

uniformly in $\varepsilon \geqslant 0$. Here $\mathbb{1}_{t\geqslant 0} = 0$ if $t < 0$ and 1 if $t \geqslant 0$. If $s = 1$, then the above estimate is not optimal and the following bound

$$G(t,x) \lesssim \mathbb{1}_{t\geq 0}\, \varepsilon^{-d}\, \mathbb{1}_{x=0} e^{-m^2 t - ct/\varepsilon^2} + \frac{\mathbb{1}_{t\geq 0}\, t\, e^{-m^2 t}}{(t^{1/2}+\varepsilon)^{d+2}} \exp\left(-c\frac{|x|^2}{t+\varepsilon|x|}\right). \tag{1.21}$$

holds true uniformly in $\varepsilon \geqslant 0$.

Parameters. During the subsequent analysis, we shall introduce several parameters

$$\alpha, \beta, \gamma, \delta, \theta, \vartheta, \varrho, \nu, \kappa, \bar{\kappa}, \kappa_{\mathfrak{o}}, \bar{\ell}, \bar{k}, a, \mathfrak{b}. \tag{1.22}$$

Although their precise values are not specified at this stage, all these parameters are to be regarded as fixed once and for all. Their choice depends solely on the power $s \in (3/4, 1)$ of the fractional Laplacian. Since the specific constraints determining these values will emerge later in the analysis, we postpone their detailed specification to Sections 4.4 and 4.9 below.

Space-time weights.

Definition 1.4. We define the fractional parabolic norm by

$$|z|_s := |z_0|^{1/2s} + |\bar{z}|, \qquad z = (z_0, \bar{z}) \in \Lambda_0 = \mathbb{R}\times\mathbb{R}^3, \tag{1.23}$$

where $|\bullet|$ denotes the usual Euclidean norm on $\mathbb{R}$ and $\mathbb{R}^3$. For $\varkappa > 0$ and $z = (z_0, \bar{z}) \in \Lambda_0$ we denote by

$$\varkappa.z := (\varkappa^{2s} z_0, \varkappa\bar{z}) \in \Lambda_0$$

the fractional parabolic rescaling, satisfying

$$|[\![\mu]\!]^a.z|_s = [\![\mu]\!]^a |z|_s.$$

Definition 1.5. We introduce the following Japanese brackets:

$$\langle z\rangle_s := (1 + |z_0|^{1/s} + |\bar{z}|^2)^{1/2} \qquad \text{and} \qquad \langle \bar{z}\rangle := (1 + |\bar{z}|^2)^{1/2} \tag{1.24}$$

for $z = (z_0, \bar{z}) \in \Lambda_0$, where, as in the previous definition, $|\bullet|$ denotes the usual Euclidean norm on $\mathbb{R}$ and $\mathbb{R}^3$.

(a) Let $a > 1$ and $\nu \in (0, 1/3)$.

(b) Let $(\chi_i : \Lambda_0 \to \mathbb{R}_+)_{i\geqslant -1}$ be a dyadic partition of unity on Λ_0 with χ_i supported on an annulus of radius $\sim 2^{ai}$ for $i \geqslant 0$, χ_{-1} supported in a ball of radius ≈ 1 and $\sum_{i\geqslant -1}\chi_i = 1$.

(c) Define $(\mu_i)_{i\geqslant -1} \subseteq [1/2, 1)$ by $[\![\mu_i]\!] = 2^{-i-2}$, $i \geqslant -1$.

(d) Let $\zeta : \Lambda_0 \to \mathbb{R}$ be a weight defined by

$$\zeta(z) := \langle z\rangle_s^{-1}, \qquad z \in \Lambda_0.$$

For $\mu \in [0, 1]$, we also introduce the associated rescaled weights

$$\zeta_\mu(z) := \langle [\![\mu]\!]^a.z\rangle_s^{-1}, \qquad \rho_\mu := \zeta_\mu^\nu = \langle [\![\mu]\!]^a.z\rangle_s^{-\nu}, \qquad z \in \Lambda_0.$$

Remark 1.6. Concerning the parameters ν and a introduced above, we impose the relation

$$a\nu = \gamma, \tag{1.25}$$

where the parameter $\gamma \in (0, s)$ will be fixed in Section 4.9. In particular, throughout our analysis we shall be concerned with very small values of $\nu > 0$ and with $a > 1$.

Remark 1.7. The form of the Japanese bracket (1.24) is motivated by the requirement that the weight function ζ (and its powers) be C^1 in the time variable and C^2 in the spatial variables. This regularity is essential in the a priori estimates established below (see Section 3). Indeed, one readily verifies that

$$\partial_t \zeta(t,x) = -\frac{|t|^{(1-s)/s}\partial_t|t|}{2s(1+|t|^{1/s}+|x|^2)^{3/2}}, \qquad (t,x) \in \Lambda_0,$$

which is continuous for $s < 1$. An analogous argument applies to all spatial derivatives.

Remark 1.8. Our weights satisfy the following properties.

(a) We have

$$\zeta(z)\zeta^{-1}(z_1) \lesssim \zeta^{-1}(z - z_1),$$

uniformly over $z, z_1 \in \Lambda_0$. Consequently,

$$\zeta_\mu(z)\zeta_\mu^{-1}(z_1) \lesssim \zeta_\mu^{-1}(z - z_1) \leqslant \zeta^{-1}(z - z_1),$$
$$\rho_\mu(z)\rho_\mu^{-1}(z_1) \lesssim \rho_\mu^{-1}(z - z_1) \lesssim \rho_0^{-1}(z - z_1),$$

uniformly over $z, z_1 \in \Lambda_0$ and $\mu \in [0, 1]$.

(b) We have

$$\chi_i\zeta_\mu^{-1} \lesssim (1 + (\llbracket\mu\rrbracket^a \llbracket\mu_i\rrbracket^{-a})^2)^{1/2} \lesssim 1,$$

uniformly in $i \geqslant -1$ and $\mu \geqslant \mu_i$.

Scale decomposition. Let us introduce a scale decomposition of spacetime functions parametrised by $\sigma \in [0, 1]$, where we set

$$\llbracket\sigma\rrbracket := (1 - \sigma)$$

for convenience. The value $\sigma = 1$ corresponds to allowing fluctuations at all scales, whereas $\sigma \in (1/2, 1)$ corresponds to allowing fluctuations only at spatial scales $\gtrsim \llbracket\sigma\rrbracket$ or equivalently at Fourier scales $\lesssim \llbracket\sigma\rrbracket^{-1}$.

Definition 1.9. Consider a smooth and compactly supported function $j : \mathbb{R} \to \mathbb{R}_+$ such that

$$j(\eta) = \begin{cases} 1 & \text{if} \quad |\eta| \leqslant 1, \\ 0 & \text{if} \quad |\eta| \geqslant 2. \end{cases}$$

For $\ell = 0, 1, 2, \ldots$ denote

$$j_{\sigma,\ell}(\eta) := j(2^{-\ell}\sigma^{-1}\llbracket\sigma\rrbracket\eta), \qquad \eta \in \mathbb{R},$$

and let $j_\sigma := j_{\sigma,0}$ and $\tilde{j}_\sigma := j_{\sigma,1}$.

Remark 1.10. Note that $j_{\sigma,\ell}(\eta) j_{\sigma,\ell'}(\eta) = j_{\sigma,\ell}(\eta)$ for $0 \leqslant \ell < \ell'$.

Definition 1.11. The family $(J_\sigma)_{\sigma\in(0,1)} = (J_\sigma^{(\varepsilon)})_{\sigma\in(0,1)}$ of Fourier multipliers acting on distributions is defined as

$$J_\sigma f(t,x) := \int_{\Lambda_\varepsilon^*} j_\sigma(|\omega|^{1/2s}) j_\sigma(q_\varepsilon(k)) \hat{f}(\omega,k) e^{i(\omega t + k\cdot x)} \frac{\mathrm{d}\omega \mathrm{d}k}{(2\pi)^{d+1}}, \tag{1.26}$$

for $(t,x) \in \Lambda_\varepsilon$ and $f \in \mathcal{S}'(\Lambda_\varepsilon)$, where $q_\varepsilon^2(k)$ is the symbol of the Laplacian introduced in (1.11). In addition, we define $(\tilde{J}_{\sigma,\ell})_{\sigma\in(0,1),\ell=0,1,2,\dots}$ by (1.26) with the function j_σ replaced by $j_{\sigma,\ell}$. In particular, $J_\sigma = \tilde{J}_{\sigma,0}$. Moreover, we let $\tilde{J}_\sigma := \tilde{J}_{\sigma,1}$ and $\dot{J}_\sigma := \partial_\sigma J_\sigma$.

Note that $J_\sigma f \to f$ as $\sigma \nearrow 1$ in $\mathcal{S}'(\Lambda_\varepsilon)$. We let $q_0(k) := |k|$, so that $q_\varepsilon(k) \to q_0(k)$ pointwise for $k \in \mathbb{R}^d$ as $\varepsilon \to 0$. We observe that, on account of the above definitions, for any $\ell \in \mathbb{N}_+$ and $\sigma \in (0,1)$, we have

$$\tilde{J}_{\sigma,\ell} J_\sigma = J_\sigma, \qquad \tilde{J}_{\sigma,\ell+1} \tilde{J}_{\sigma,\ell} = \tilde{J}_{\sigma,\ell}. \tag{1.27}$$

Furthermore, since $J_\sigma, \tilde{J}_{\sigma,\ell}$ as well as $\mathcal{L}_\varepsilon$ are Fourier multipliers, they all commute. The operators $(J_\sigma)_\sigma$ are used to define the scale decomposition

$$\sigma \mapsto G_\sigma := J_\sigma G$$

of the Green function G. The operators $(\tilde{J}_\sigma)_\sigma$ play an auxiliary role and will be used frequently in the estimates. We extend the notation continuously to the endpoint by setting

$$J_1 = \tilde{J}_1 = 1, \qquad G_1 = G.$$

Remark 1.12. We shall use the fact that if $\sigma < \mu_i$, where μ_i is as in Definition 1.5, then $J_\sigma J_{\mu_{i+1}} = J_\sigma$. This is a consequence of the definitions of J_σ and μ_i, and of the fact that, for $\sigma < \mu_i$,

$$2\sigma [\![\sigma]\!]^{-1} \leqslant 2\mu_i [\![\mu_i]\!]^{-1} = 2(2^{i+2} - 1) < (2^{i+3} - 1) = \mu_{i+1} [\![\mu_{i+1}]\!]^{-1},$$

which implies $j_\sigma(\eta) j_{\mu_{i+1}}(\eta) = j_\sigma(\eta)$ for all $\eta \in \mathbb{R}$.

Smoothing operators. To establish a suitable Schauder estimate, we will employ smoothing operators $(K_\sigma)_\sigma$ that possess only limited regularising effects in time. It is convenient to choose these operators as inverses of differential operators.

Definition 1.13. For $\sigma, \eta \in (0,1)$ let

$$\begin{aligned} L_\sigma &:= (1 + [\![\sigma]\!]^{2s} \partial_t)(1 - [\![\sigma]\!]^2 \Delta_\varepsilon)^2, \\ K_\sigma &:= L_\sigma^{-1} = (1 + [\![\sigma]\!]^{2s} \partial_t)^{-1} (1 - [\![\sigma]\!]^2 \Delta_\varepsilon)^{-2}, \\ K_{\eta,\sigma} &:= L_\sigma K_\eta. \end{aligned}$$

In the tightness argument, we will also employ a Littlewood–Paley decomposition acting solely on the spatial variables.

Definition 1.14. (Spatial LP blocks) Let $(\hat{\Delta}_i : [0,\infty) \to \mathbb{R}_+)_{i\geqslant -1}$ be a dyadic partition of unity on $[0,\infty)$, where $\hat{\Delta}_{-1}$ is supported in $[0,1]$, $\hat{\Delta}_0$ is supported in $[1/2, 3/2]$ and $\hat{\Delta}_i(\bullet) := \hat{\Delta}_0(2^{-i}\bullet)$ for $i \geqslant 1$. We define spatial Littlewood-Paley blocks $(\bar{\Delta}_i)_{i\geqslant -1}$ as the Fourier multipliers on $\mathcal{S}'(\mathbb{R}^d_\varepsilon)$ associated with the symbols $k \mapsto \hat{\Delta}_i(q_\varepsilon(k))$.

Remark 1.15. By extension, we identify the fractional Laplacian $(-\Delta_\varepsilon)^s$ and the spatial Littlewood-Paley blocks $(\bar{\Delta}_i)_{i\geqslant -1}$ with the corresponding operators on $C(\Lambda_\varepsilon, \zeta)$ acting trivially on the time variable.

Convolution operators. We call T a convolution operator on $C(\Lambda_\varepsilon, w)$ if there exists a signed measure $m(\mathrm{d}z)$ such that

$$(Tf)(z) = \int_{\Lambda_\varepsilon} m(\mathrm{d}z') f(z - z').$$

By slight abuse of notation, we usually denote the measure $m(\mathrm{d}z)$ associated to T by $T(\mathrm{d}z)$. We call $T(\mathrm{d}z)$ the kernel of the operator T. We denote by $|T(\mathrm{d}z)|$ the variation of the measure $T(\mathrm{d}z)$. Given a nonnegative weight $w \in C(\Lambda_0)$ we write

$$\|T\|_{\mathrm{TV}(w)} := \int_{\Lambda_\varepsilon} w(z)|T(\mathrm{d}z)|$$

for the weighted total variation norm of the kernel of T. If $T(\mathrm{d}z)$ is absolutely continuous with respect to the measure $\mathrm{d}z$ on Λ_ε, we write $T(z)$ for its density, *i.e.* $T(\mathrm{d}z) = T(z)\,\mathrm{d}z$. For $p \in [1,\infty]$, we denote by $\|T\|_{L^p(w)}$ the weighted L^p norm of the density of the kernel of T, *i.e.*

$$\|T\|_{L^p(w)} := \|z \mapsto w(z)T(z)\|_{L^p},$$

where $\|\bullet\|_{L^p}$ is the usual L^p norm on Λ_ε with respect to the measure $\mathrm{d}z$. Given a weight w, a convolution operator T and a function f we write

$$(wTf)(z) := w(z)(Tf)(z). \tag{1.28}$$

Note that the operators

$$G, G_\sigma, K_\sigma, K_{\eta,\sigma}, J_\sigma, \tilde{J}_\sigma, (-\Delta_\varepsilon)^s, \bar{\Delta}_i$$

introduced above are all convolution operators on $C(\Lambda_\varepsilon, \zeta^\alpha)$ for every $\alpha \in [0, 2s)$. The kernels of the operators $G_\sigma, J_\sigma, \tilde{J}_\sigma$ have smooth densities. The kernels of G, K_σ have densities of limited regularity. The kernels of $K_{\eta,\sigma}, (-\Delta_\varepsilon)^s, \bar{\Delta}_i$ do not possess densities. In particular, the kernels of $(-\Delta_\varepsilon)^s, \bar{\Delta}_i$ are proportional to the Dirac delta in time.

The following lemmas collect the fundamental properties of these convolution operators, which will be used repeatedly throughout our analysis.

Definition 1.16. For $\omega \in \mathbb{R}$ and $\mu \in [0,1)$, we write

$$w^\omega_\mu(z) := (1 + [\![\mu]\!]^{-1}|z|_s)^\omega, \qquad z \in \Lambda_0.$$

Lemma 1.17. *For all $\flat \in (0, 2s)$, we have*

(a) $\|K_\sigma\|_{\mathrm{TV}(w^2_\sigma)} \vee \|K_{\sigma,\eta}\|_{\mathrm{TV}(w^2_\sigma)} \lesssim 1$,
(b) $\|J_\sigma\|_{\mathrm{TV}(w^2_\sigma)} \vee \|\tilde{J}_\sigma\|_{\mathrm{TV}(w^2_\sigma)} \vee \|L^2_\sigma J_\sigma\|_{\mathrm{TV}(w^2_\sigma)} \vee \|L^2_\sigma \tilde{J}_\sigma\|_{\mathrm{TV}(w^2_\sigma)} \lesssim 1$,
(c) $\|\partial_t \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-2s}$ *and* $\|\nabla_\varepsilon \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-1}$,
(d) $\|G\|_{\mathrm{TV}(w^\flat_0)} \lesssim 1$ *and* $\|\dot{G}_\sigma\|_{\mathrm{TV}(w^\flat_\sigma)} \vee \|L^3_\sigma \dot{G}_\sigma\|_{\mathrm{TV}(w^\flat_\sigma)} \lesssim [\![\sigma]\!]^{2s-1}$,

uniformly over $\varepsilon \in 2^{-\mathbb{N}_0}$ *and* $1/2 \leqslant \sigma \leqslant \eta < 1$.

Remark 1.18. The kernels of the operators K_σ, $K_{\sigma,\eta}$, J_σ and $\tilde{J}_\sigma$ exhibit rapid decay in spacetime. In particular, these operators have finite total variation norms $\|\bullet\|_{\mathrm{TV}(w^\omega_\sigma)}$ for all $\omega \geqslant 0$. In contrast, for fractional Laplacians of order $s \in (0,1)$, the kernels of the operators G and $\dot{G}_\sigma$ decay much more slowly. Their total variation norms $\|\bullet\|_{\mathrm{TV}(w^\omega_\sigma)}$ are finite only for $\omega \in [0, 2s)$. This slow decay of G and $\dot{G}_\sigma$ constitutes a major analytical difficulty and is one of the main reasons why PDEs involving fractional Laplacians are particularly challenging to study.

Remark 1.19. Note that all of the above bounds remain valid if one replaces the weights by $\zeta^{-\alpha}$ or $\zeta^{-\alpha}_\mu$ with any $\mu \in (0,1)$ and $\alpha \in [0,1]$, since $1 < 2s < 2$ and

$$\zeta^{-\alpha}_\mu \leqslant \zeta^{-\alpha} \lesssim w^1_\sigma \leqslant w^2_\sigma,$$

for all $\mu, \sigma \in (0,1)$.

Proof. Item a) is proved in Lemma A.4. Items b) and c) follow from Lemma A.9. Item d) follows from the estimates for $\dot{G}_\sigma$ established in Lemma A.10. ∎

Lemma 1.20. *We have*

$$\|\mathcal{L}_\varepsilon \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-2s}$$

uniformly over $\varepsilon \in 2^{-\mathbb{N}_0}$ *and* $1/2 \leqslant \sigma < 1$.

Proof. First note that

$$\|(-\Delta_\varepsilon)^s \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \leqslant \sum_{i \geqslant -1} \|(-\Delta_\varepsilon)^s \bar{\Delta}_i \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})}.$$

Next, observe that $\bar{\Delta}_i \tilde{J}_\sigma = 0$, unless $[\![\mu_i]\!] \geqslant 2^{-10}[\![\sigma]\!]$. Consequently, we have

$$\begin{aligned}\|(-\Delta_\varepsilon)^s \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} &\leqslant \sum_{i \,|\, [\![\mu_i]\!] \geqslant 2^{-10}[\![\sigma]\!]} \|(-\Delta_\varepsilon)^s \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \|\tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \\ &\lesssim \sum_{i \,|\, [\![\mu_i]\!] \geqslant 2^{-10}[\![\sigma]\!]} [\![\mu_i]\!]^{-2s} \lesssim [\![\sigma]\!]^{-2s},\end{aligned}$$

where the second estimate follows from Lemmas 1.17 and A.22. Since $\|\tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1$ and $\|\partial_t \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-2s}$ by Lemma 1.17, the proof is complete. ∎

The following lemma combines the estimates from the preceding lemmas with Young's convolution inequality. The resulting bounds will be used frequently in the subsequent analysis, often without explicit reference.

Lemma 1.21. *For all $\alpha \in [0,1]$, the following estimates hold:*

(a)
$$\|\zeta_\mu^\alpha K_\sigma f\| \lesssim \|\zeta_\mu^\alpha K_\eta f\| \lesssim \|\zeta_\mu^\alpha f\|.$$

(b)
$$\|\zeta_\mu^\alpha J_\sigma f\| \lesssim \|\zeta_\mu^\alpha \tilde{J}_\sigma f\| \lesssim \|\zeta_\mu^\alpha K_\sigma f\| \lesssim \|\zeta_\mu^\alpha f\|.$$

(c)
$$\begin{aligned}\|\zeta_\mu^\alpha \mathcal{L}_\varepsilon \tilde{J}_\sigma f\| &\lesssim [\![\sigma]\!]^{-2s}\|\zeta_\mu^\alpha f\|,\\ \|\zeta_\mu^\alpha \partial_t \tilde{J}_\sigma f\| &\lesssim [\![\sigma]\!]^{-2s}\|\zeta_\mu^\alpha f\|,\\ \|\zeta_\mu^\alpha \nabla_\varepsilon \tilde{J}_\sigma f\| &\lesssim [\![\sigma]\!]^{-1}\|\zeta_\mu^\alpha f\|.\end{aligned}$$

(d)
$$\begin{aligned}\|\zeta_\mu^\alpha G f\| &\lesssim \|\zeta_\mu^\alpha f\|,\\ \|\zeta_\mu^\alpha \dot{G}_\sigma f\| \vee \|\zeta_\mu^\alpha L_\sigma^3 \dot{G}_\sigma f\| &\lesssim [\![\sigma]\!]^{2s-1}\|\zeta_\mu^\alpha f\|.\end{aligned}$$

The estimates are uniform over

$$\varepsilon \in 2^{-\mathbb{N}_0}, \qquad 0 \leqslant \mu \leqslant 1, \qquad \frac{1}{2} \leqslant \sigma \leqslant \eta < 1, \qquad f \in \mathcal{S}'(\Lambda_\varepsilon).$$

Here $\|\bullet\|$ denotes the L^∞ norm over Λ_ε, and we use the notation introduced in (1.28).

Proof. The stated bounds are consequences of the previous lemmas, together with Young's inequality for convolution, the identities

$$K_\sigma = K_{\sigma,\eta} K_\eta, \qquad J_\sigma = J_\sigma \tilde{J}_\sigma, \qquad \tilde{J}_\sigma = L_\sigma^2 \tilde{J}_\sigma K_\sigma K_\sigma$$

and Remarks 1.8 a) and 1.19. For example, to prove the first bound, we note that

$$\begin{aligned}\|\zeta_\mu^\alpha K_\sigma f\| &\leqslant \sup_z \int_{\Lambda_\varepsilon} \zeta_\mu^\alpha(z)|(K_\eta f)(z - z_1)||K_{\sigma,\eta}(\mathrm{d}z_1)|\\ &\lesssim \sup_z \int_{\Lambda_\varepsilon} \zeta_\mu^\alpha(z - z_1)|(K_\eta f)(z - z_1)|\zeta^{-1}(z_1)|K_{\sigma,\eta}(\mathrm{d}z_1)|\\ &\leqslant \|\zeta_\mu^\alpha (K_\eta f)\| \, \|K_{\sigma,\eta}\|_{\mathrm{TV}(\zeta^{-1})}\\ &\lesssim \|\zeta_\mu^\alpha f\|.\end{aligned}$$

The proof of the remaining bounds proceeds analogously. ∎

2. Stochastic quantisation

In this section we lay out the main steps in the proof of Theorem 1.1, starting from the effective equation at (fractional, parabolic) spacetime scale $[\![\sigma]\!] := 1 - \sigma$, $\sigma \in [0,1]$, obtained through a scale decomposition and the introduction of the approximate effective force. Since our analysis primarily concerns scales close to 1, we shall restrict all scale parameters, such as σ, to the interval $[1/2, 1]$.

2.1. Scale decomposition

Let $\phi^{(\varepsilon,M)}$ be a stationary solution to the finite system of SDEs (1.4) and define the scale-dependent field

$$\phi_\sigma^{(\varepsilon,M)} := J_\sigma \phi^{(\varepsilon,M)}$$

localised at (fractional, parabolic) spacetime scales $\gtrsim [\![\sigma]\!]$. Then we have

$$\mathcal{L}_\varepsilon \phi_\sigma^{(\varepsilon,M)} = J_\sigma F^{(\varepsilon,M)}(\phi^{(\varepsilon,M)}), \qquad \sigma \in [1/2, 1],$$

where

$$F^{(\varepsilon,M)}(\phi) := -\lambda\phi^3 + r_{\varepsilon,M}\phi + \xi^{(\varepsilon,M)}. \tag{2.1}$$

We call $F^{(\varepsilon,M)}$ the force. Let $\mathcal{E} := \bigcap_{\alpha>0} C(\Lambda_{\varepsilon,M}, \zeta^\alpha)$ be the space of continuous functions on Λ_ε that are periodic in space and exhibit subpolynomial growth in time at infinity. Denote by $\hat{\mathcal{E}} \subset \mathcal{S}'(\Lambda_{\varepsilon,M})$ the sum of $\mathcal{E}$ and the image of $\mathcal{E}$ under ∂_t, interpreted as the time derivative operator on $\mathcal{S}'(\Lambda_{\varepsilon,M})$. Note that the white noise $\xi^{(\varepsilon,M)}$ on $\Lambda_{\varepsilon,M}$ belongs a.s. to $\hat{\mathcal{E}}$ and the stochastic convolution $G\xi^{(\varepsilon,M)}$ belongs a.s. to $\mathcal{E}$. Consider a family of functionals, referred to as the *effective force*,

$$\left(F_\sigma^{(\varepsilon,M)} : \mathcal{E} \to \hat{\mathcal{E}}\right)_{\sigma\in[1/2,1]},$$

which is differentiable in $\sigma \in (1/2, 1)$ and satisfies the final condition

$$F_1(\psi) = F^{(\varepsilon,M)}(\psi). \tag{2.2}$$

Using the identity $\phi_1^{(\varepsilon,M)} = \phi^{(\varepsilon,M)}$, we obtain

$$F^{(\varepsilon,M)}(\phi^{(\varepsilon,M)}) = F_1(\phi_1^{(\varepsilon,M)}) = F_\mu(\phi_\mu^{(\varepsilon,M)}) + R_\mu^{(\varepsilon,M)}, \tag{2.3}$$

for all $\mu \in [1/2, 1]$, where

$$R_\mu^{(\varepsilon,M)} := \int_\mu^1 [\partial_\sigma F_\sigma^{(\varepsilon,M)}(\phi_\sigma^{(\varepsilon,M)}) + \mathrm{D}F_\sigma^{(\varepsilon,M)}(\phi_\sigma^{(\varepsilon,M)})(\partial_\sigma \phi_\sigma^{(\varepsilon,M)})]\,\mathrm{d}\sigma.$$

Here $\mathrm{D}F(\psi)\psi'$ denotes the Fréchet derivative of a functional F in the direction of $\psi' \in \mathcal{E}$ at the point $\psi \in \mathcal{E}$. Moreover, we have

$$\partial_\sigma \phi_\sigma^{(\varepsilon,M)} = \dot{G}_\sigma(F^{(\varepsilon,M)}(\phi^{(\varepsilon,M)})) = \dot{G}_\sigma(F_\sigma^{(\varepsilon,M)}(\phi_\sigma^{(\varepsilon,M)}) + R_\sigma^{(\varepsilon,M)}),$$

where $\dot{G}_\sigma := \mathcal{L}_\varepsilon^{-1}\dot{J}_\sigma$ and $\dot{J}_\sigma := \partial_\sigma J_\sigma$. Consequently, we deduce that for any choice of the effective force

$$(F_\sigma)_{\sigma\in[1/2,1]} = (F_\sigma^{(\varepsilon,M)})_{\sigma\in[1/2,1]},$$

satisfying the conditions specified above, the pair

$$(\phi_\mu, R_\mu)_{\mu\in[1/2,1]} = (\phi_\mu^{(\varepsilon,M)}, R_\mu^{(\varepsilon,M)})_{\mu\in[1/2,1]}$$

satisfies the system of equations

$$\begin{cases} \mathcal{L}_\varepsilon \phi_\mu = J_\mu(F_\mu(\phi_\mu) + R_\mu) \\ R_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\,\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F_\sigma(\phi_\sigma)(\dot{G}_\sigma R_\sigma)]\,\mathrm{d}\sigma, \end{cases} \tag{2.4}$$

where

$$H_\sigma(\psi) := \partial_\sigma F_\sigma(\psi) + \mathrm{D}F_\sigma(\psi)(\dot{G}_\sigma F_\sigma(\psi)). \tag{2.5}$$

The first equation in (2.4) is the effective stochastic PDE at scale μ, while the second propagates the error made by replacing the original force by the scale-dependent effective force. The term H_σ measures the error of F_σ in satisfying the exact flow equation. If $H_\sigma = 0$, the second equation generates no remainder and one recovers $R_\mu = 0$.

Our main goal will be to show that this system allows for good a priori estimates for a suitably chosen effective force $(F_\sigma)_{\sigma\in[1/2,1]}$.

Remark 2.1. Except for Sections 2.4 and 2.5, we shall almost always suppress the explicit dependence on $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$. To avoid repetition, we will not restate this each time, but **all estimates should be understood as uniform in $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$**, unless explicitly stated otherwise. In particular, we shall write Λ for Λ_ε and Λ_M for $\Lambda_{\varepsilon,M}$. Likewise, we write $(-\Delta)^s$ for the discrete fractional Laplacian.

2.2. *Overview of the strategy*

The goal of this section is to provide a blueprint that guides the reader through the technical aspects of the proofs and highlights the underlying heuristics. We will not present rigorous arguments here. Our intention is instead to convey intuition and the global structure of the analysis. Readers who find the informal reasoning confusing can safely skip this section without loss of logical continuity.

Our approach to obtaining a priori global spacetime estimates for solutions to the system (2.4) relies on several new conceptual and technical ideas:

(a) the introduction of a remainder term R_σ, which circumvents the need to explicitly solve the flow equation for F_σ (unlike in the original approach of [11, 13]);

(b) the use of new weighted norms that control the solution across the entire space, inspired by the spatial decomposition introduced in [23];

(c) a stopping argument for the effective force, allowing us to close nonlinear estimates;

(d) a collection of technical innovations addressing the difficulties arising from the limited spacetime decay of the fractional heat kernel.

In what follows, we focus primarily on the new ideas required to handle the large-field problem. For a pedagogical overview of the flow equation approach, we refer the reader to the lecture notes [12].

To keep the exposition clear, let us ignore the remainder R_σ and model the original equation by

$$\mathcal{L}\phi_\sigma - \lambda\phi_\sigma^3 \approx F_\sigma(\phi_\sigma) - \lambda\phi_\sigma^3, \tag{2.6}$$

where $F_\sigma(\phi_\sigma)$ is a polynomial in the field ϕ_σ. To measure the spacetime growth of fields, we use the weight

$$\zeta_\sigma(z) \approx (1 + [\![\sigma]\!]^a |z|_s)^{-1}, \qquad z \in \Lambda,$$

where $|z|_s$ is the fractional parabolic norm on Λ and $a > 1$ is an exponent chosen to balance the scale behaviour at large distances. Its precise value will later be crucial to closing our nonlinear estimates.

Pathwise bounds for the random effective force $(F_\sigma)_\sigma$ are obtained by analysing the flow equation for its probabilistic cumulants. The outcome is that $F_\sigma(\phi_\sigma)$ behaves as a random nonlocal polynomial in ϕ_σ whose coefficients are localised in regions of size $\approx [\![\sigma]\!]$ and scale roughly as

$$[\![\sigma]\!]^{(k-3)\beta+\delta\ell},$$

where $k \in \mathbb{N}_0$ is the monomial degree, $\beta > \gamma$ encodes the field scaling, $\delta > 0$ measures the distance to criticality, and ℓ is the perturbative order. Due to a Kolmogorov-type argument needed to extract the almost sure behaviour of the force F_σ from its moments, we also lose a bit in the spacetime growth, which will be modelled by a weight $\zeta_\sigma^{-\kappa_0(\ell+1)}$, where $\kappa_0 > 0$ is an arbitrarily small exponent. Overall we have, schematically,

$$F_\sigma(\phi_\sigma) \approx \xi + \bar{r}\phi_\sigma - \lambda\phi_\sigma^3 + \sum_{\ell=1}^{\bar{\ell}} \sum_{k=0}^{\bar{k}} \zeta_\sigma^{-\kappa_0(\ell+1)} [\![\sigma]\!]^{(k-3)\beta+\delta\ell} \phi_\sigma^k, \tag{2.7}$$

where the sums over k and ℓ are finite. The cutoffs $\bar{k}$ and $\bar{\ell}$ are chosen so that the equation for the remainder R_σ can be solved, a technical aspect we omit in this heuristic discussion.

To estimate the size of the solution to (2.6), we introduce a constant $C_\Phi = \|\phi\|$ (cf. Definition 2.2) such that

$$|\phi_\sigma(z)| = |(J_\sigma\phi)(z)| \leqslant C_\Phi \zeta_\sigma^{-1/3}(z) [\![\sigma]\!]^{-\gamma}, \qquad z \in \Lambda, \tag{2.8}$$

valid for all $\sigma \geqslant \bar{\mu}$, where $\bar{\mu}$, chosen so that $[\![\bar{\mu}]\!] \ll 1$, is a random scale to be chosen later.

For moderate distances $|z| \lesssim [\![\sigma]\!]^{-a}$, the spatial weight $\zeta_\sigma^{-1/3}(z)$ is of order one and we are describing the distributional nature of the solution, growing like $[\![\sigma]\!]^{-\gamma}$ as $[\![\sigma]\!] \searrow 0$ for some $\gamma > 0$. For large distances $|z| \gg [\![\sigma]\!]^{-a}$, the spatial growth can be improved as follows. Let $[\![\hat{\mu}]\!] \ll [\![\sigma]\!]$ be such that $|z| \approx [\![\hat{\mu}]\!]^{-a}$. Observing that $\phi_\sigma = J_\sigma\phi_{\hat{\mu}}$ due to the properties of the smoothing operators, we obtain

$$\begin{aligned} |\phi_\sigma(z)| &\approx |(J_\sigma\phi_{\hat{\mu}})(z)| \approx |\phi_{\hat{\mu}}(z)| \\ &\lesssim C_\Phi (1 + [\![\hat{\mu}]\!]^a |z|)^{1/3} [\![\hat{\mu}]\!]^{-\gamma} \\ &\approx C_\Phi |z|^{\gamma/a} \approx C_\Phi (1 + [\![\sigma]\!]^a |z|)^{\gamma/a} [\![\sigma]\!]^{-\gamma}. \end{aligned} \tag{2.9}$$

This improved spatial growth replaces $\zeta_\sigma^{-1/3}$ with $\zeta_\sigma^{-\gamma/a}$, and by taking a large we can make the effective growth arbitrarily mild, an essential feature for closing nonlinear bounds.

Since higher-order monomials ($k > 3$) in (2.7) are accompanied by small coefficients $[\![\sigma]\!]^{(k-3)\beta+\delta\ell} \ll 1$, a coercive bound for (2.6) yields, schematically,

$$|\phi_\sigma|^3 \approx |F_\sigma(\phi_\sigma) + \lambda\phi_\sigma^3| \lesssim \zeta_\sigma^{-(\bar\ell+1)\kappa_0} \sum_{k,\ell} [\![\sigma]\!]^{(k-3)\beta+\delta\ell} |\phi_\sigma|^k. \tag{2.10}$$

We estimated the spacetime growth by taking the worst possible weight $\zeta_\sigma^{-(\bar\ell+1)\kappa_0}$, where $\bar\ell$ denotes the largest perturbative order that needs to be considered. Substituting (2.9) in (2.10), and ignoring the mild nonlocality of the effective force, we obtain

$$|\phi_\sigma|^3 \approx |F_\sigma(\phi_\sigma) + \lambda\phi_\sigma^3| \lesssim \zeta_\sigma^{-(\bar\ell+1)\kappa_0} \left[\sum_{k,\ell} [\![\sigma]\!]^{(k-3)\beta-k\gamma+\delta\ell} \zeta_\sigma^{-k\gamma/a} C_\Phi^k \right].$$

The terms with $\ell = 0$, namely the first two terms on the right-hand side of (2.7), are explicit and yield improved estimates, so that in total we arrive at

$$|\phi_\sigma|^3 \lesssim [\![\sigma]\!]^{-3\beta+\delta} \zeta_\sigma^{-(\bar\ell+1)\kappa_0 - \bar k\gamma/a} (1 + C_\Phi)^{\bar k}, \tag{2.11}$$

where $\bar k$ is the maximal degree of the monomials. Here we used the fact that $\beta > \gamma$, which allows the field amplitude to be compensated by the kernel size. The constant C_Φ can then be estimated as

$$C_\Phi \approx \sup_{\sigma \geqslant \bar\mu} [\![\sigma]\!]^\gamma \zeta_\sigma^{1/3} |\phi_\sigma| \lesssim \sup_{\sigma \geqslant \bar\mu} [\![\sigma]\!]^{\gamma-\beta+\delta/3} \zeta_\sigma^{(1-(\bar\ell+1)\kappa_0 - \bar k\gamma/a)/3} (1 + C_\Phi)^{\bar k/3}.$$

Choosing $\gamma \leqslant \beta$ such that $\gamma - \beta + \delta/3 \geqslant \vartheta > 0$ and then taking a large enough such that

$$1 - (\bar\ell + 1)\kappa_0 - \bar k\gamma/a \geqslant 0,$$

we obtain the bound

$$C_\Phi \lesssim [\![\bar\mu]\!]^\vartheta (1 + C_\Phi)^{\bar k/3}.$$

It follows that, for sufficiently small $[\![\bar\mu]\!] \ll 1$, the nonlinear estimate closes (after selecting a small branch with the use of a stopping-time argument) and yields $C_\Phi \approx 1$.

Let us now address the treatment of the nonlocality of the effective force kernels. The main technical challenges in this work stem from the limited decay of the slice propagator $\dot G_\sigma$ associated with the fractional parabolic operator $\mathcal{L}$. Roughly speaking, we only have algebraic decay of the form (see Lemma A.11):

$$|\dot G_\sigma(z)| \lesssim [\![\sigma]\!]^{-d-1} (1 + |z|_s / [\![\sigma]\!])^{-d-2s+\epsilon}, \qquad z \in \Lambda, \tag{2.12}$$

where $\epsilon > 0$ is arbitrarily small. This behaviour contrasts sharply with the standard heat kernel, which has stretched-exponential decay, and also with the fractional Laplacian case appearing in the usual ("static") renormalisation group analysis of the fractional Φ^4 model [4]. The difference arises from the limited smoothness of the symbol of the fractional heat operator $\mathcal{L}$.

The main consequence of (2.12) is that the effective force kernels, obtained by solving a flow equation driven by $\dot{G}_\sigma$, inherit a similar algebraic decay. The monomials which appear in $F_\sigma(\phi_\sigma)$ take the schematic form

$$F_\sigma(\phi_\sigma)(z) \approx \sum_{k,\ell} \int F_\sigma^{[\ell],(k)}(z; z_1, \dots, z_k) \prod_{j=1}^{k} \phi_\sigma(z_j) \mathrm{d}z_j,$$

where $F_\sigma^{[\ell],(k)}$ are random kernels. Ignoring their distributional nature and thinking of them as bona fide functions, their spatial nonlocality and spatial growth can be modelled as (cf. Definitions 4.3 and 4.6)

$$F_\sigma^{[\ell],(k)}(z, z_1, \dots, z_k) \approx \zeta_\sigma^{-(\ell+1)\kappa_0}(z)(1 + [\![\sigma]\!]^{-1} \operatorname{St}(z, z_1, \dots, z_k))^{-(\mathfrak{b}-\ell\kappa_0)},$$

where $\operatorname{St}(z, z_1, \dots, z_k)$ measures the diameter of the set $\{z, z_1, \dots, z_k\}$. The initial decay exponent $\mathfrak{b} \approx 2s$ follows from (2.12), while the additional loss $\ell\kappa_0$ reflects the growth at spacetime infinity of the kernel in its output variable, induced by the similar growth of the noise. It is now clear that, to proceed as in (2.11), we must be able to compensate for the spatial growth of the fields ϕ_σ by exploiting the limited decay of the kernels away from the diagonal.

The remainder of this section makes this argument rigorous and establishes further properties of the solution. Section 3 develops the coercive estimate required in (2.10), while Section 4 contains the detailed analysis of the random force coefficients and the derivation of the precise form of estimate (2.7).

2.3. *Main estimate*

In this section we introduce a family of weighted norms that will be used to measure the size of fields over spacetime uniformly across scales. These norms depend on an exponent $\gamma > 0$, to be fixed in Section 4.9, and a terminal scale $\bar{\mu} \in [1/2, 1)$, which will later be tuned according to the noise amplitude.

Definition 2.2. For $\psi \in \mathcal{S}'(\Lambda)$ and $f \in C([1/2, 1), C(\Lambda, \zeta))$, we set

$$|\!|\!|\psi|\!|\!| = |\!|\!|\psi|\!|\!|_{\bar{\mu}} := \sup_{\bar{\mu} \leqslant \sigma < 1} [\![\sigma]\!]^{\gamma} \|\zeta_\sigma^{1/3} J_\sigma \psi\|, \tag{2.13}$$

$$|\!|\!|f|\!|\!|_{\#} = |\!|\!|f|\!|\!|_{\#,\bar{\mu}} := \sup_{\bar{\mu} \leqslant \sigma < 1} [\![\sigma]\!]^{3\gamma} \|\zeta_\sigma f_\sigma\|. \tag{2.14}$$

The powers in these norms are adapted to the cubic coercivity. Roughly, a bound of size $[\![\sigma]\!]^{-\gamma}$ for ϕ_σ produces a bound of size $[\![\sigma]\!]^{-3\gamma}$ for its cubic force. Likewise, the weight $\zeta_\sigma^{1/3}$ on the field becomes the full weight ζ_σ after cubing.

The above "triple norms" exhibit a specific behaviour with respect to the weight. In particular, the following lemma shows that changing the weight yields equivalent norms.

Lemma 2.3. *For $\alpha \in [3\nu, 1]$ the following bounds*

$$\sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\phi_\sigma\| \lesssim |\!|\!|\phi|\!|\!|_{\bar\mu} \lesssim \sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\phi_\sigma\|, \tag{2.15}$$

$$\sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu} [\![\sigma]\!]^{3\gamma} \|\zeta_\mu^{\alpha}\mathcal{L}\phi_\sigma\| \lesssim |\!|\!|\sigma\mapsto\mathcal{L}\phi_\sigma|\!|\!|_{\#,\bar\mu} \lesssim \sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu} [\![\sigma]\!]^{3\gamma} \|\zeta_\mu^{\alpha}\mathcal{L}\phi_\sigma\|, \tag{2.16}$$

hold uniformly in $\bar\mu \in [1/2, 1)$ and $\phi \in \mathcal{S}'(\Lambda)$, where $\phi_\sigma := J_\sigma\phi$ for $\sigma \in [1/2, 1)$.

Proof. Let us prove the first inequality in (2.15). We start by observing that, on account of the support properties of $\{\chi_i\}_i$, we have the following decomposition

$$\|\zeta_\mu^{\alpha/3}\phi_\sigma\| \leqslant \sup_i \|\zeta_\mu^{\alpha/3}(\chi_i + \chi_{i-1} + \chi_{i+1})\phi_\sigma\| \lesssim \sup_i \|\zeta_\mu^{\alpha/3}\chi_i\phi_\sigma\|.$$

Let us consider separately the cases $\mu_i \leqslant \sigma$ and $\mu_i \geqslant \sigma$. In the first case, we use the bound

$$\|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_i}^{-1/3}\| \lesssim 1$$

to obtain

$$\begin{aligned}\sup_{i|\mu_i\leqslant\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\phi_\sigma\| &\lesssim \sup_{i|\mu_i\leqslant\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_i}^{-1/3}\| \|\zeta_{\mu_i}^{1/3}\phi_\sigma\| \\ &\lesssim [\![\sigma]\!]^\gamma \sup_{i|\mu_i\leqslant\sigma} \|\zeta_{\mu_i}^{1/3}\phi_\sigma\| \\ &\leqslant [\![\sigma]\!]^\gamma \|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim |\!|\!|\phi|\!|\!|.\end{aligned}$$

Let us turn to the case $\mu_i > \sigma$. Note that

$$\|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_{i+1}}^{-1/3}\| \lesssim (1 + [\![\mu]\!]^a [\![\mu_{i+1}]\!]^{-a})^{-\alpha/3} \lesssim [\![\mu]\!]^{-a\alpha/3} [\![\mu_{i+1}]\!]^{\alpha a/3}.$$

Moreover, on account of Remarks 1.12, 1.8 and Lemma 1.17, we have

$$\|\zeta_{\mu_{i+1}}^{1/3}\phi_\sigma\| = \|\zeta_{\mu_{i+1}}^{1/3}J_\sigma\phi_{\mu_{i+1}}\| \lesssim \|J_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\| \lesssim \|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\|.$$

Combining the above estimates, we deduce that

$$\begin{aligned}&\sup_{i|\mu_i>\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\phi_\sigma\| \\ &\lesssim \sup_{i|\mu_i>\sigma} [\![\sigma]\!]^\gamma \|\zeta_\mu^{\alpha/3}\chi_i\zeta_{\mu_{i+1}}^{-1/3}\| \|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\| \\ &\lesssim [\![\sigma]\!]^\gamma [\![\mu]\!]^{-a\alpha/3} \left(\sup_{i|\mu_i>\sigma} [\![\mu_{i+1}]\!]^{\alpha a/3} [\![\mu_{i+1}]\!]^{-\gamma}\right) \left(\sup_{i|\mu_i>\sigma} [\![\mu_{i+1}]\!]^\gamma \|\zeta_{\mu_{i+1}}^{1/3}\phi_{\mu_{i+1}}\|\right) \\ &\lesssim [\![\sigma]\!]^{\alpha a/3} [\![\mu]\!]^{-a\alpha/3} |\!|\!|\phi|\!|\!| \\ &\lesssim |\!|\!|\phi|\!|\!|,\end{aligned}$$

where we used that $\alpha a/3 \geqslant \nu a = \gamma$ and $\sigma \geqslant \mu$. This proves the first inequality in (2.15). The second inequality in (2.15) is a direct consequence of the fact that $\zeta_\sigma^{1/3} \leqslant \zeta_\sigma^{\alpha/3}$. The inequalities (2.16) are proved similarly. ∎

The two norms just defined fix the analytical setting for the global analysis of the SPDE (2.4). In Section 3, we will prove suitable coercive estimates for the fractional parabolic equation with cubic nonlinearity, which allow us to control the large values of the fields. Below we state a direct consequence of this estimate.

Theorem 2.4. *(Coercive estimate) For $\phi \in \mathcal{S}'(\Lambda)$ and $\sigma \in [1/2, 1)$ define*

$$f_\sigma := \mathcal{L}\phi_\sigma + \lambda\phi_\sigma^3, \qquad \phi_\sigma := J_\sigma \phi, \qquad \mathcal{L} := \partial_t + (-\Delta)^s + m^2.$$

Then the following bound

$$|||\phi|||_{\bar\mu} \lesssim \lambda^{-1/2}[\![\bar\mu]\!]^\gamma + \lambda^{-1/3}|||f|||_{\#,\bar\mu}^{1/3}$$

holds uniformly in $\phi \in \mathcal{S}'(\Lambda)$, $\bar\mu \in [1/2, 1)$ and $\lambda \in (0, \infty)$.

Proof. We apply the a priori estimates from Theorem 3.1 to $u = \phi_\sigma$ and $\rho = \zeta_\sigma^{1/3}$. The constants A and B appearing in the theorem can be bounded as follows:

$$A = \|(-\Delta)^s \zeta_\sigma^{2/3}\| + \|\zeta_\sigma^{1/3}(\partial_t \zeta_\sigma^{1/3})\| \lesssim [\![\sigma]\!]^{2sa},$$

and

$$\begin{aligned} B &= \|\zeta_\sigma^{1/3} u\| \left(\|\zeta_\sigma^{1/3}(\partial_t \zeta_\sigma^{1/3})\| + \|\zeta_\sigma^{1/3}(-\Delta)^s \zeta_\sigma^{1/3}\| + \left\|\zeta_\sigma^{2/3}\mathfrak{D}_s\left(\zeta_\sigma^{-1/3}\right)\mathfrak{D}_s(\zeta_\sigma^{1/3})\right\| \right) \\ &\quad + \left\|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{1/3})\mathfrak{D}_s(\zeta_\sigma^{1/3}u)\right\| \\ &\lesssim Q_\sigma^2 \|\zeta_\sigma^{1/3}\phi_\sigma\| + Q_\sigma \|\mathfrak{D}_s(\zeta_\sigma^{1/3}\phi_\sigma)\|. \end{aligned}$$

Here we define

$$Q_\sigma^2 := \|\partial_t \zeta_\sigma^{1/3}\| + \|(-\Delta)^s \zeta_\sigma^{1/3}\| + \|\mathfrak{D}_s(\zeta_\sigma^{1/3})\|^2 + \|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})\|^2 \lesssim [\![\sigma]\!]^{2sa}.$$

The bound $\|(-\Delta)^s \zeta_\sigma^\alpha\| \lesssim [\![\sigma]\!]^{2sa}$ is proved in Lemma A.12. To estimate $\|\mathfrak{D}_s(\zeta_\sigma^{1/3})\|^2$ we used (1.18), while the bound on $\|\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})\|^2$ follows from Lemma A.13. Moreover, using the bound (1.18), we obtain

$$\|\mathfrak{D}_s(\zeta_\sigma^{1/3}\phi_\sigma)\| \lesssim \|\nabla_\varepsilon(\zeta_\sigma^{1/3}\phi_\sigma)\|^s \|\zeta_\sigma^{1/3}\phi_\sigma\|^{1-s}. \tag{2.17}$$

We note that

$$\|\nabla_\varepsilon(\zeta_\sigma^{1/3}\phi_\sigma)\| \lesssim \|(\nabla_\varepsilon \zeta_\sigma^{1/3})\phi_\sigma\| + \|\zeta_\sigma^{1/3}(\nabla_\varepsilon \phi_\sigma)\|.$$

Hence, using the fact that $|\nabla_\varepsilon \zeta_\sigma^{1/3}| \lesssim \zeta_\sigma^{1/3}$ and $\phi_\sigma = \tilde J_\sigma \phi_\sigma$ as well as $\|\nabla_\varepsilon \tilde J_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim [\![\sigma]\!]^{-1}$, we arrive at

$$\begin{aligned} \|\nabla_\varepsilon(\zeta_\sigma^{1/3}\phi_\sigma)\| &\lesssim \|\zeta_\sigma^{1/3}\phi_\sigma\| + \|\zeta_\sigma^{1/3}(\nabla_\varepsilon \tilde J_{\sigma,1})\phi_\sigma\| \\ &\lesssim \|\zeta_\sigma^{1/3}\phi_\sigma\| + [\![\sigma]\!]^{-1}\|\zeta_\sigma^{1/3}\phi_\sigma\| \\ &\lesssim [\![\sigma]\!]^{-1}\|\zeta_\sigma^{1/3}\phi_\sigma\|. \end{aligned}$$

Together with (2.17), this proves that

$$\|\mathfrak{D}_s(\zeta_\sigma^{1/3}\phi_\sigma)\| \lesssim [\![\sigma]\!]^{-s}\|\zeta_\sigma^{1/3}\phi_\sigma\|,$$

uniformly in $\sigma \geqslant \bar{\mu}$. Therefore,

$$B \lesssim ([\![\sigma]\!]^{2sa} + [\![\sigma]\!]^{sa}[\![\sigma]\!]^{-s})\|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim [\![\sigma]\!]^{sa-s}\|\zeta_\sigma^{1/3}\phi_\sigma\|.$$

By Theorem 3.1 and Young's inequality, we obtain

$$\begin{aligned}\|\zeta_\sigma^{1/3}\phi_\sigma\| &\leqslant \lambda^{-1/2}C[\![\sigma]\!]^{as} + \lambda^{-1/3}(\|\zeta_\sigma f_\sigma\| + C[\![\sigma]\!]^{as-s}\|\zeta_\sigma^{1/3}\phi_\sigma\|)^{1/3}\\ &\leqslant \lambda^{-1/2}C[\![\sigma]\!]^{as} + \lambda^{-1/3}\|\zeta_\sigma f_\sigma\|^{1/3} + C\lambda^{-1/3}[\![\sigma]\!]^{(as-s)/3}\|\zeta_\sigma^{1/3}\phi_\sigma\|^{1/3}\\ &\leqslant \lambda^{-1/2}C[\![\sigma]\!]^{as} + \lambda^{-1/3}\|\zeta_\sigma f_\sigma\|^{1/3} + C'\lambda^{-1/2}[\![\sigma]\!]^{(as-s)/2} + \frac{1}{2}\|\zeta_\sigma^{1/3}\phi_\sigma\|.\end{aligned}$$

As a result, we arrive at

$$\|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim \lambda^{-1/2}[\![\sigma]\!]^{(as-s)/2} + \lambda^{-1/3}\|\zeta_\sigma f_\sigma\|^{1/3}.$$

This allows us to deduce that

$$|\!|\!|\phi|\!|\!|_{\bar{\mu}} = \sup_{\sigma\geqslant\bar{\mu}}[\![\sigma]\!]^{\gamma}\|\zeta_\sigma^{1/3}\phi_\sigma\| \lesssim \sup_{\sigma\geqslant\bar{\mu}}[\![\sigma]\!]^{\gamma+(as-s)/2}\lambda^{-1/2} + \lambda^{-1/3}\sup_{\sigma\geqslant\bar{\mu}}([\![\sigma]\!]^{3\gamma}\|\zeta_\sigma f_\sigma\|)^{1/3}.$$

Hence, provided $\gamma + (as-s)/2 \geqslant \gamma \geqslant 0$, which follows from $a > 1$, we conclude our claim. ∎

The coercivity estimate derived above provides the key ingredient for deriving a priori bounds for solutions to system (2.4), as stated in the following theorem.

Theorem 2.5. *Let $\phi = \phi^{(\varepsilon,M)}$ be a solution of the stochastic quantisation equation (1.4) and*

$$\left(F_\sigma = F_\sigma^{(\varepsilon,M)} : \mathcal{E} \to \hat{\mathcal{E}}\right)_{\sigma\in[1/2,1]}$$

be a family of functionals differentiable in $\sigma \in (1/2,1)$ and satisfying the final condition (2.2). For $\sigma \in [1/2,1]$ define

$$\phi_\sigma := J_\sigma\phi, \qquad R_\sigma := F_1(\phi) - F_\sigma(\phi_\sigma).$$

Suppose that there exist constants $S \geqslant 3, C_F \geqslant 1$ and $\vartheta, \bar{\kappa} > 0$ such that

$$\bar{\kappa} \in [\nu, 1/3), \qquad \vartheta \leqslant \gamma \wedge (2s-\gamma), \qquad (s+\gamma)\bar{\kappa}/(1-\bar{\kappa}) \leqslant \vartheta/4, \tag{2.18}$$

and $(F_\sigma)_\sigma$ satisfies the following estimates

$$\begin{aligned}&\|\zeta_\mu[J_\sigma F_\sigma(\psi_\sigma) - (-\lambda\psi_\sigma^3)]\|\\ &\quad\leqslant [\![\sigma]\!]^{-3\gamma+\vartheta}[C_F(1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^S + (1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^2|\!|\!|\mathcal{L}\psi_\bullet|\!|\!|_{\#,\bar{\mu}}],\\ &\|\zeta_\mu K_\sigma H_\sigma(\psi_\sigma)\| \leqslant C_F[\![\sigma]\!]^{\vartheta-1}(1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^S,\\ &\|\zeta_\mu^{\bar{\kappa}}K_\sigma F_\sigma(\psi_\sigma)\| \leqslant C_F[\![\sigma]\!]^{-3\gamma}(1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^S,\\ &\|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot{G}_\sigma\hat{\psi}))\| \leqslant C_F[\![\sigma]\!]^{\vartheta-1}(1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^S\|\zeta_\mu^{1-\bar{\kappa}}\tilde{J}_\sigma^2\hat{\psi}\|,\end{aligned} \tag{2.19}$$

for all $\bar\mu \in [1/2, 1), \mu \in [\bar\mu, 1), \sigma \in [\mu, 1)$ *and* $\psi, \hat\psi \in \mathcal{S}'(\Lambda)$*, where we denote* $\psi_\sigma := J_\sigma \psi$ *and* $(\mathcal{L}\psi)_\sigma := \mathcal{L}\psi_\sigma$*, the functional* H_σ *is defined by* (2.5)*.*

Then there exists a universal constant $\hat C > 0$ *such that for all* $\bar\mu \in [1/2, 1)$ *satisfying*

$$[\![\bar\mu]\!]^\vartheta \leqslant \hat C((\lambda^{-1}+1)C_F)^{-2}, \tag{2.20}$$

we have

$$|\!|\!|\phi|\!|\!|_{\bar\mu} \leqslant 1, \qquad |\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{-\vartheta/2}, \qquad |\!|\!|K_\bullet R_\bullet|\!|\!|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{\vartheta/2}.$$

Remark 2.6. By the elementary argument presented in Section 2.1, our assumptions on ϕ and $(F_\mu)_\mu$ imply that $(\phi_\mu, R_\mu)_\mu$ is a solution of (2.4). The four estimates in (2.19) have distinct roles. The first extracts the coercive cubic part of the effective force, the second controls the error H_σ in the flow equation, the third bounds the effective force in the weaker weight required in the field equation, and the fourth is the Lipschitz estimate used to propagate the remainder R_σ.

Proof. Define

$$\Phi = \Phi_{\bar\mu} := 1 + |\!|\!|\phi|\!|\!|_{\bar\mu} + \theta_{\mathcal{L}}^{-1}|\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_{\#,\bar\mu} + \theta_R^{-1}|\!|\!|K_\bullet R_\bullet|\!|\!|_{\#,\bar\mu}, \tag{2.21}$$

for constants $\theta_{\mathcal{L}}, \theta_R > 0$ to be fixed later. These constants normalise the three quantities entering the bootstrap so that the field equation and the remainder equation can be closed in a single inequality. Our goal is to bound each term on the right-hand side by an increasing function of Φ itself and then apply a continuity argument to establish the uniform bounds claimed. We omit the index $\bar\mu \in [1/2, 1)$, as the estimates hold uniformly in this parameter.

Let $C_Q = C_F(1 + |\!|\!|\phi|\!|\!|)^S$. We first prove a bound for $|\!|\!|K_\bullet R_\bullet|\!|\!|_\#$. To this end, we use Lemma 2.8 below, which shows that we can control $\|\zeta_\mu K_\sigma R_\sigma\|$ in terms of $\|\zeta_\mu^{\bar\kappa}\tilde J_\sigma^2 R_\sigma\|$. Note that

$$\tilde J_\sigma^2 R_\sigma = \tilde J_\sigma^2(\mathcal{L}\phi - F_\sigma(\phi_\sigma)),$$

which follows from (1.4) and (2.3). Since $\bar\kappa \in [\nu, 1)$ and $\vartheta \in (0, \gamma]$, by Lemmas 1.20 and 2.3, we have

$$\begin{aligned}
\|\zeta_\mu^{\bar\kappa}\tilde J_\sigma^2 \mathcal{L}\phi\| &= \|\zeta_\mu^{\bar\kappa}(\mathcal{L}\tilde J_\sigma)\tilde J_\sigma\phi\| \\
&\lesssim \|\mathcal{L}\tilde J_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\|\zeta_\mu^{\bar\kappa}\tilde J_\sigma\phi\| \\
&\lesssim [\![\sigma]\!]^{-2s}[\![\sigma]\!]^{-\gamma}|\!|\!|\phi|\!|\!| \\
&\lesssim C_Q[\![\bar\mu]\!]^\vartheta[\![\sigma]\!]^{-2(s+\gamma)}.
\end{aligned}$$

Since $\vartheta \in (0, 2s-\gamma]$, by Lemma 1.17 and the estimates (2.19), we have

$$\|\zeta_\mu^{\bar\kappa}\tilde J_\sigma^2 F_\sigma(\phi_\sigma)\| \lesssim \|\zeta_\mu^{\bar\kappa}K_\sigma F_\sigma(\phi_\sigma)\| \lesssim C_Q[\![\sigma]\!]^{-3\gamma} \lesssim C_Q[\![\bar\mu]\!]^\vartheta[\![\sigma]\!]^{-2(s+\gamma)}.$$

Altogether, this yields

$$\sup_{\sigma,\mu\,|\,\sigma\geqslant\mu\geqslant\bar\mu}[\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\mu^{\bar\kappa}\tilde J_\sigma^2 R_\sigma\| \lesssim C_Q[\![\bar\mu]\!]^\vartheta. \tag{2.22}$$

Using (2.19), the factorisation $K_\mu = K_{\mu,\eta}K_\eta$ for $\eta \geqslant \mu$, and Lemma 1.17 we also have

$$\sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu}\left\|\zeta_\mu K_\mu\int_\sigma^1 H_\eta(\phi_\eta)\,\mathrm{d}\eta\right\| \lesssim \int_{\bar\mu}^1 \|\zeta_\mu K_{\mu,\eta}K_\eta H_\eta(\phi_\eta)\|\,\mathrm{d}\eta \lesssim \int_{\bar\mu}^1 \|\zeta_\mu K_\eta H_\eta(\phi_\eta)\|\,\mathrm{d}\eta \lesssim C_Q[\![\bar\mu]\!]^\vartheta.$$

By Lemma 2.8, it then follows that

$$\sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu}\|\zeta_\mu K_\sigma R_\sigma\| \lesssim \exp(CC_Q[\![\bar\mu]\!]^{\vartheta/2}),$$

possibly after adjusting the constant C. Hence, by (2.14),

$$|\!|\!|K_\bullet R_\bullet|\!|\!|_\# \lesssim [\![\bar\mu]\!]^{3\gamma}\exp(CC_F[\![\bar\mu]\!]^{\vartheta/2}(1+|\!|\!|\phi|\!|\!|)^S) \lesssim [\![\bar\mu]\!]^{3\gamma}\exp(CC_F[\![\bar\mu]\!]^{\vartheta/2}\Phi^S). \tag{2.23}$$

The control of $|\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_\#$ is obtained from (2.19)

$$\begin{aligned}|\!|\!|\mathcal{L}\phi_\bullet + \lambda\phi_\bullet^3|\!|\!|_\# &\leqslant |\!|\!|(J_\bullet F_\bullet(\phi_\bullet) + \lambda\phi_\bullet^3)|\!|\!|_\# + |\!|\!|J_\bullet R_\bullet|\!|\!|_\# \\ &\lesssim [\![\bar\mu]\!]^\vartheta C_F(1+|\!|\!|\phi|\!|\!|)^S + [\![\bar\mu]\!]^\vartheta(1+|\!|\!|\phi|\!|\!|)^2|\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_\# + |\!|\!|K_\bullet R_\bullet|\!|\!|_\#,\end{aligned}$$

using in particular Lemma 1.17 to bound the contribution from R. From this it follows that

$$|\!|\!|\mathcal{L}\phi_\bullet + \lambda\phi_\bullet^3|\!|\!|_\# \lesssim ([\![\bar\mu]\!]^\vartheta C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta + \theta_R)\Phi^S \tag{2.24}$$

and

$$|\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_\# \leqslant |\!|\!|\lambda\phi_\bullet^3|\!|\!|_\# + |\!|\!|\mathcal{L}\phi_\bullet + \lambda\phi_\bullet^3|\!|\!|_\# \lesssim (\lambda + [\![\bar\mu]\!]^\vartheta C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta + \theta_R)\Phi^S, \tag{2.25}$$

by the triangle inequality and

$$|\!|\!|\lambda\phi_\bullet^3|\!|\!|_\# = \lambda\sup_{\sigma\geqslant\bar\mu}[\![\sigma]\!]^{3\gamma}\|\zeta_\sigma\phi_\sigma^3\| \lesssim \lambda[\sup_{\sigma\geqslant\bar\mu}[\![\sigma]\!]^\gamma\|\zeta_\sigma^{1/3}\phi_\sigma\|]^3 = \lambda|\!|\!|\phi|\!|\!|_{\bar\mu}^3.$$

Next, the a priori estimates of Theorem 2.4 and (2.24) give

$$\begin{aligned}|\!|\!|\phi|\!|\!| &\lesssim \lambda^{-1/2}[\![\bar\mu]\!]^\gamma + \lambda^{-1/3}|\!|\!|\mathcal{L}\phi_\bullet + \lambda\phi_\bullet^3|\!|\!|_\#^{1/3} \\ &\lesssim \lambda^{-1/2}[\![\bar\mu]\!]^\gamma + \lambda^{-1/3}([\![\bar\mu]\!]^\vartheta C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta + \theta_R)^{1/3}\Phi^{S/3}.\end{aligned} \tag{2.26}$$

Gathering (2.23), (2.25) and (2.26), we obtain

$$\Phi_{\bar\mu} \leqslant 1 + C[\tau(\bar\mu) + \tau(\bar\mu)^{1/3}\Phi_{\bar\mu}^{S/3} + \tau(\bar\mu)\Phi_{\bar\mu}^S + \tau(\bar\mu)\exp(C\tau(\bar\mu)\Phi_{\bar\mu}^S)], \tag{2.27}$$

where

$$\begin{aligned}\tau(\bar\mu) := {}& \lambda^{-1/2}[\![\bar\mu]\!]^{\gamma/2} + \lambda^{-1}([\![\bar\mu]\!]^\vartheta C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta + \theta_R) \\ &+ \theta_{\mathcal{L}}^{-1}(\lambda + [\![\bar\mu]\!]^\vartheta C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^\vartheta + \theta_R) + \theta_R^{-1}[\![\bar\mu]\!]^{3\gamma} + C_F[\![\bar\mu]\!]^{\vartheta/2}.\end{aligned}$$

Choose

$$\theta_R = [\![\bar\mu]\!]^{\vartheta/2}\lambda, \qquad \theta_{\mathcal{L}} = [\![\bar\mu]\!]^{-\vartheta/2}\lambda. \tag{2.28}$$

Then $\bar\mu \mapsto \tau(\bar\mu)$ is a decreasing function and

$$\begin{aligned}\tau(\bar\mu) &\leqslant [\![\bar\mu]\!]^{\vartheta/2}(\lambda^{-1/2}[\![\bar\mu]\!]^{(\gamma-\vartheta)/2} + (2\lambda^{-1}+1)C_F + 5 + \lambda^{-1}[\![\bar\mu]\!]^{3\gamma-\vartheta}) \\ &\leqslant 10[\![\bar\mu]\!]^{\vartheta/2}(\lambda^{-1}+1)C_F.\end{aligned}$$

Fix $\tau_* > 0$ small enough such that

$$C[\tau_* + \tau_*^{1/3}4^{S/3} + \tau_* 4^S + \tau_* \exp(C\tau_* 4^S)] \leqslant 1, \tag{2.29}$$

and define $\bar\mu_* = \bar\mu_*(\lambda, C_F) \in (0,1)$ as the (unique) solution to

$$10\,[\![\bar\mu_*]\!]^{\vartheta/2}(\lambda^{-1}+1)C_F = \tau_*.$$

Then for all $\bar\mu \in [\bar\mu_*, 1)$ we have $\tau(\bar\mu) \leqslant \tau_*$ and as a consequence of (2.27) and (2.29),

$$\Phi_{\bar\mu} \leqslant 4 \Rightarrow \Phi_{\bar\mu} \leqslant 2.$$

Define the set

$$A := \{\bar\mu \in [\bar\mu_*, 1) | \Phi_{\bar\mu} \leqslant 4\} \subseteq [\bar\mu_*, 1).$$

Note that $A \neq \varnothing$ since for $\bar\mu \nearrow 1$ we have $\Phi_{\bar\mu} \searrow 1$. As the map $\bar\mu \mapsto \Phi_{\bar\mu}$ is continuous, the set A is closed in $[\bar\mu_*, 1)$. Hence, to prove that $A = [\bar\mu_*, 1)$, it is enough to show that A is open in $[\bar\mu_*, 1)$. If $\mu \in A$, then $\Phi_\mu \leqslant 2$ and by continuity, there exists a neighbourhood of μ within $[\bar\mu_*, 1)$ on which the function $\bar\mu \mapsto \Phi_{\bar\mu}$ takes values not exceeding 4. We conclude that $A = [\bar\mu_*, 1)$ and therefore that $\Phi_{\bar\mu_*} \leqslant 2$. Using (2.28) this implies that for $\bar\mu \geqslant \bar\mu_*$, we have

$$|\!|\!|\phi|\!|\!|_{\bar\mu} \leqslant 1, \qquad |\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{-\vartheta/2}, \qquad |\!|\!|K_\bullet R_\bullet|\!|\!|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{\vartheta/2},$$

with

$$[\![\bar\mu_*]\!]^{-1} = (\tau_*^{-1}10(\lambda^{-1}+1)C_F)^{2/\vartheta} \lesssim ((\lambda^{-1}+1)C_F)^{2/\vartheta},$$

uniformly in C_F and $\lambda > 0$. ∎

To apply the previous theorem to the analysis of (2.4), we need to construct a suitable approximate solution to the flow equation

$$\partial_\sigma F_\sigma + \mathrm{D}F_\sigma(\dot G_\sigma F_\sigma) = 0, \tag{2.30}$$

for the effective force

$$(F_\sigma)_{\sigma\in[1/2,1]} = (F_\sigma^{(\varepsilon,M)})_{\sigma\in[1/2,1]},$$

subject to the final condition (2.2). Section 4 will be devoted to the construction of such an effective force $(F_\sigma)_\sigma$, as formulated in the following theorem.

Theorem 2.7. *There exists a choice of the deterministic parameters*

$$(r_{\varepsilon,M})_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+}$$

and a family of random, scale-dependent functionals

$$\left(F_\sigma = F_\sigma^{(\varepsilon,M)} : \mathcal{E} \to \hat{\mathcal{E}}\right)_{\sigma\in[1/2,1]}$$

such that:

(a) *The map $\sigma \mapsto F_\sigma$ is differentiable for $\sigma \in (1/2, 1)$.*

(b) *The final condition (2.2) holds.*

(c) *The estimates (2.19) are satisfied with*

$$C_F = 1 + \|F^{\mathfrak{A}}\|^2$$

and $\|F^{\mathfrak{A}}\|$ introduced in Definition 4.6.

(d) *For all $N \geqslant 1$, we have*

$$\sup_{\varepsilon \in 2^{-\mathbb{N}_0}, M \in \mathbb{N}_+} \mathbb{E}[\|F^{\mathfrak{A}}\|^N] < \infty. \tag{2.31}$$

Proof. Choose $\bar{r}$ and $(r^{[\ell]}_{\varepsilon,M})_{\ell=1,\dots,\hat{\ell}}$ as in Lemma 4.15, set

$$r_{\varepsilon,M} := \bar{r} + \sum_{\ell=1}^{\hat{\ell}} r^{[\ell]}_{\varepsilon,M},$$

and let

$$F_\sigma := \sum_{\ell=0}^{\bar{\ell}} F^{[\ell]}_\sigma$$

be the corresponding truncated solution of the perturbative flow equation (4.1) introduced in Section 4.1. By construction, $\sigma \mapsto F_\sigma$ is differentiable on $(1/2, 1)$, and the final data prescribed in (4.41) yield the final condition (2.2). Next, Lemmas 4.15 and 4.19 yield Item d). Finally, Item c) follows from the post-processing estimates established later in the paper: the first and third bounds in (2.19) are given by Lemmas 4.24, 4.25 and 4.26, the fourth one by Lemma 4.27, and the bound on H_σ by Lemma 4.28. Thus all four items hold with $C_F = 1 + \|F^{\mathfrak{A}}\|^2$. ∎

We complete this section with the proof of the following auxiliary lemma, used in the proof of Theorem 2.5.

Lemma 2.8. *Suppose that there exist constants $S \in \mathbb{N}_+, C_F \geqslant 1$ and $\vartheta, \bar{\kappa} > 0$ such that*

$$\bar{\kappa} \in [\nu, 1/2), \qquad (s + \gamma)\bar{\kappa}/(1 - \bar{\kappa}) \leqslant \vartheta/4$$

and $(F_\sigma)_\sigma$ satisfies the following estimate

$$\|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot{G}_\sigma\hat{\psi}))\|_{L^\infty} \leqslant C_F[\![\sigma]\!]^{\vartheta-1}(1 + |\!|\!|\psi|\!|\!|_{\bar{\mu}})^S \|\zeta_\mu^{1-\bar{\kappa}} \tilde{J}^2_\sigma \hat{\psi}\|_{L^\infty}, \tag{2.32}$$

for all $\bar{\mu} \in [1/2, 1)$, $\mu \in [\bar{\mu}, 1)$, $\sigma \in [\mu, 1)$, and $\psi, \hat{\psi} \in \mathcal{S}'(\Lambda)$. Here we set $\psi_\sigma := J_\sigma \psi$. Fix a function $[\bar{\mu}, 1) \ni \mu \mapsto \mathcal{H}_\mu \in \mathcal{S}'(\Lambda)$ and consider the linear equation

$$R_\mu = \mathcal{H}_\mu + \int_\mu^1 \mathrm{D}F_\sigma(\phi_\sigma)(\dot{G}_\sigma R_\sigma)\,\mathrm{d}\sigma, \qquad \mu \geqslant \bar{\mu}. \tag{2.33}$$

Then there exists a constant $C > 0$ such that

$$\|\zeta_\eta K_\mu R_\mu\| \leqslant \exp(CC_Q[\![\bar\mu]\!]^{\vartheta/2}) \left[\sup_{\sigma\geqslant\mu}\|\zeta_\eta K_\sigma \mathcal{H}_\sigma\| + \sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\eta^{\bar\kappa}\tilde J^2_\sigma R_\sigma\|\right], \tag{2.34}$$

for $\mu \geqslant \eta \geqslant \bar\mu$, where $C_Q = C_F(1 + |\!|\!|\phi|\!|\!|)^S$.

Proof. Start by observing that, by interpolation, for any $\alpha \in [0,1]$ and $\beta \in \mathbb{R}$, we have by Young's inequality

$$\begin{aligned}
&[\![\sigma]\!]^{\beta(1-\alpha)}\|\zeta_\eta^{\alpha+\bar\kappa(1-\alpha)}\tilde J^2_\sigma R_\sigma\| \\
&\quad\lesssim [\![\sigma]\!]^{\beta(1-\alpha)}\|\zeta_\eta \tilde J^2_\sigma R_\sigma\|^\alpha \|\zeta_\eta^{\bar\kappa}\tilde J^2_\sigma R_\sigma\|^{1-\alpha} \\
&\quad\lesssim \|\zeta_\eta \tilde J^2_\sigma R_\sigma\| + [\![\sigma]\!]^\beta \|\zeta_\eta^{\bar\kappa}\tilde J^2_\sigma R_\sigma\|.
\end{aligned}$$

Choose $\alpha = (1-2\bar\kappa)/(1-\bar\kappa)$. Then $\alpha + \bar\kappa(1-\alpha) = 1-\bar\kappa$ and $1-\alpha = \bar\kappa/(1-\bar\kappa)$. Consequently,

$$[\![\sigma]\!]^{\beta\bar\kappa/(1-\bar\kappa)}\|\zeta_\eta^{1-\bar\kappa}\tilde J^2_\sigma R_\sigma\| \lesssim \|\zeta_\eta \tilde J^2_\sigma R_\sigma\| + [\![\sigma]\!]^\beta\|\zeta_\eta^{\bar\kappa}\tilde J^2_\sigma R_\sigma\|. \tag{2.35}$$

Then, by (2.32), we have

$$\|\zeta_\eta K_\sigma(\mathrm{D}F_\sigma(\phi_\sigma)(\dot G_\sigma R_\sigma))\| \lesssim [\![\sigma]\!]^{\vartheta-1} C_Q \|\zeta_\eta^{1-\bar\kappa}\tilde J^2_\sigma R_\sigma\|,$$

and by (2.35) with $\beta = 2(s+\gamma)$, and Lemma 1.17, we have

$$\begin{aligned}
&\|\zeta_\eta K_\sigma(\mathrm{D}F_\sigma(\phi_\sigma)(\dot G_\sigma R_\sigma))\| \\
&\quad\lesssim C_Q[\![\sigma]\!]^{\vartheta-1}[\![\sigma]\!]^{-2(s+\gamma)\bar\kappa/(1-\bar\kappa)} \\
&\qquad\times\big(\|\zeta_\eta K_\sigma R_\sigma\| + [\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\eta^{\bar\kappa}\tilde J^2_\sigma R_\sigma\|\big).
\end{aligned} \tag{2.36}$$

At this point, to deduce the desired estimate, we apply Gronwall's inequality to

$$\begin{aligned}
\|\zeta_\eta K_\mu R_\mu\| \leqslant{}& \|\zeta_\eta K_\mu \mathcal{H}_\mu\| \\
&+ CC_Q\int_\mu^1 [\![\sigma]\!]^{\vartheta-1-2(s+\gamma)\bar\kappa/(1-\bar\kappa)} \\
&\qquad\times\big(\|\zeta_\eta K_\sigma R_\sigma\| + [\![\sigma]\!]^{2(s+\gamma)}\|\zeta_\eta^{\bar\kappa}\tilde J^2_\sigma R_\sigma\|\big)\,\mathrm{d}\sigma,
\end{aligned}$$

which follows from (2.33) and (2.36), and use the inequality $\vartheta - 2(s+\gamma)\bar\kappa/(1-\bar\kappa) \geqslant \vartheta/2$. ∎

2.4. Tightness

In this section, we apply Theorem 2.5, together with the effective force constructed in Theorem 2.7, to prove tightness of the sequence of measures $(\hat\nu_{\varepsilon,M})_{\varepsilon,M}$. This constitutes the first step in the proof of our main result, Theorem 1.1. In order to pass to the limit

$\varepsilon \to 0$ and $M \to \infty$, we embed all the random spatially periodic fields $\phi = \phi^{(\varepsilon,M)}$ in the same space by extending them from Λ_ε to Λ_0. Let

$$\phi^{[\varepsilon,M]}(t,x) := \int_{\Lambda_\varepsilon^*} \hat\theta(\varepsilon|k|)\hat\phi^{(\varepsilon,M)}(\omega,k)e^{i(\omega t+k\cdot x)}\frac{\mathrm{d}\omega\mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x)\in\Lambda_0, \quad (2.37)$$

where $\hat\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth bump function such that $\hat\theta(0) = 1$ and $\hat\theta(\eta) = 0$ for $|\eta| > 1$. The random fields $\phi^{[\varepsilon,M]}$ now all live in the continuum domain Λ_0 for any $\varepsilon \in 2^{-\mathbb{N}_0}$. To extract information about the EQFT, we must evaluate the fixed-time marginal of the SPDE solution. This is not possible when controlling only spacetime distributional norms such as $\|\!|\bullet|\!\|_{\bar\mu}$. The necessary temporal regularity can, however, be recovered by means of a Schauder estimate adapted to our weighted norms. Set $\rho = \rho_0$. Indeed, by Lemma A.24 and using $\rho \leqslant \rho_{\bar\mu}$, we obtain

$$\sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi\| \lesssim \sup_i 2^{-i\gamma}\|\rho_{\bar\mu}\bar\Delta_i\phi\| \lesssim [\![\bar\mu]\!]^{-\gamma}[\|\!|\phi|\!\|_{\bar\mu} + \|\!|K_\bullet\mathcal{L}\phi|\!\|_{\#,\bar\mu}].$$

By Theorem 2.5 there exists $\bar\mu \in [1/2, 1)$ such that $[\![\bar\mu]\!]^{-\vartheta/2}$ is of order $C_F = 1 + \|F^{\mathfrak{A}}\|^2$ and

$$\|\!|\phi|\!\|_{\bar\mu} \leqslant 1, \qquad \|\!|\mathcal{L}\phi_\bullet|\!\|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{-\vartheta/2}, \qquad \|\!|K_\bullet R_\bullet|\!\|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{\vartheta/2}.$$

Recalling that $\mathcal{L}\phi = F_\mu(\phi_\mu) + R_\mu$, we deduce from the above bounds that

$$\|\!|K_\bullet\mathcal{L}\phi|\!\|_{\#,\bar\mu} \leqslant \|\!|K_\bullet F_\bullet(\phi_\bullet)|\!\|_{\#,\bar\mu} + \|\!|K_\bullet R_\bullet|\!\|_{\#,\bar\mu} \lesssim C_F(1 + \|\!|\phi|\!\|_{\bar\mu})^S + \lambda[\![\bar\mu]\!]^{\vartheta/2} \lesssim C_F. \quad (2.38)$$

Combining this with the previous Schauder estimate yields

$$\sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi^{(\varepsilon,M)}\|_{L^\infty(\Lambda_\varepsilon)} = \sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi\| \lesssim [\![\bar\mu]\!]^{-\gamma}C_F \lesssim C_F^{2\gamma/\vartheta+1}. \quad (2.39)$$

To derive a meaningful estimate, we need to control a suitable weighted Besov norm of $\phi^{[\varepsilon,M]}$ in terms of the left-hand side of the bound above. The argument of [33, Lemma 2.24] gives, uniformly in ε and M,

$$\sup_{t\in\mathbb{R}}\|\phi^{[\varepsilon,M]}(t)\|_{C^{-\gamma}(\mathbb{R}^d,\rho(t))} \lesssim \sup_i 2^{-i\gamma}\|\rho\bar\Delta_i\phi^{(\varepsilon,M)}\|_{L^\infty(\Lambda_\varepsilon)}. \quad (2.40)$$

Although the lattice blocks in that reference are obtained by restricting the continuum dyadic partition, the same proof applies to the present blocks: on the support of the embedding multiplier one has $q_\varepsilon(k) \asymp |k|$, and Lemma A.2 provides the required uniform multiplier bounds. As a result, we have

$$\sup_{t\in\mathbb{R}}\|\phi^{[\varepsilon,M]}(t)\|_{C^{-\gamma}(\mathbb{R}^d,\rho(t))} \lesssim C_F^{2\gamma/\vartheta+1} \lesssim 1 + \|F^{\mathfrak{A}}\|^{4\gamma/\vartheta+2},$$

where $C^\alpha(\mathbb{R}^d, w)$ stands for the usual Hölder–Besov norm with the weight w and the regularity index α and here $\rho = \rho_0$, so that $\rho(t) = \rho(t,\bullet)$. Let $\bar\rho(x) := \rho(0,x)$. Combining

the above estimate with Theorem 2.7, we arrive at

$$\begin{aligned}
&\sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \int \|\varphi\|^N_{C^{-\gamma}(\mathbb{R}^d,\bar\rho)}\, \hat\nu_{\varepsilon,M}(\mathrm{d}\varphi) \\
&\quad\leqslant \sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \mathbb{E}\left[\sup_{t\in\mathbb{R}} \|\phi^{[\varepsilon,M]}(t)\|^N_{C^{-\gamma}(\mathbb{R}^d,\rho(t))}\right] \\
&\quad\lesssim \sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \mathbb{E}\left[1 + \|F^{\mathfrak{A}}\|^{4\gamma N/\vartheta+2N}\right] < \infty,
\end{aligned} \tag{2.41}$$

for all $N \in \mathbb{N}_+$. The bound (2.41) implies the tightness of the family $(\hat\nu_{\varepsilon,M})_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+}$ in $\mathcal{S}'(\mathbb{R}^d)$. This proves the first part of Theorem 1.1.

2.5. *Integrability*

In order to complete the proof of Theorem 1.1 it remains to establish the integrability property of the measures $\hat\nu_{\varepsilon,M}$ uniformly in $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$ and obtain the bound (1.3) for any limit point. We aim to estimate quantities of the form

$$Z_{\varepsilon,M,\theta} := \int \exp\Big(\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big)\, \nu_{\varepsilon,M}(\mathrm{d}\varphi),$$

where $\theta > 0$ is a small parameter. Here

$$Q_\varepsilon := (1 - \Delta_\varepsilon)^{-A/2} \tag{2.42}$$

is a regularising kernel. For $M \geq 1$, let

$$|x|_M := \min_{k\in M\mathbb{Z}^d} |x + k|, \qquad h(x) := (1 + |x|_M)^{-B}. \tag{2.43}$$

Thus h is the M-periodic extension of $x \mapsto (1 + |x|)^{-B}$ from the centred fundamental domain $[-M/2, M/2)^d$. Whenever a non-periodic weight such as ρ_σ occurs inside an integral over $\mathbb{T}^d_{\varepsilon,M}$, the integral is understood over this centred fundamental domain. The constants A and B are chosen big enough according to Lemma 2.10 below.

The main tool for this purpose is the Hairer–Steele argument [28], which provides optimal estimates with respect to the growth of the function. We introduce a new *tilted* probability measure

$$\nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi) := \frac{\exp\Big(\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big)\, \nu_{\varepsilon,M}(\mathrm{d}\varphi)}{Z_{\varepsilon,M,\theta}}. \tag{2.44}$$

To prove that this measure is well defined we take advantage of the presence of the coercive term $\|\varphi\|^4_{L^4(\mathbb{T}^d_{\varepsilon,M})}$ in the action functional (1.2) defining the original measure $\nu_{\varepsilon,M}$ and use the inequality

$$\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \leqslant \frac{\lambda}{4} \|\varphi\|^4_{L^4(\mathbb{T}^d_{\varepsilon,M})},$$

for all ε and M as soon as $\theta > 0$ is small enough and A, B are big enough. Observe that Jensen's inequality gives

$$\begin{aligned} 1 &= \int \nu_{\varepsilon,M}(\mathrm{d}\varphi) \\ &= Z_{\varepsilon,M,\theta} \int \exp\Big(-\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}\Big)\, \nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi) \\ &\geqslant Z_{\varepsilon,M,\theta} \exp\left[-\theta \int \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} &\log \int \exp[\theta \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})}] \nu_{\varepsilon,M}(\mathrm{d}\varphi) \\ &\qquad = \log Z_{\varepsilon,M,\theta} \leqslant \theta \int \|hQ_\varepsilon\varphi\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi). \end{aligned} \tag{2.45}$$

The problem of controlling the size of $Z_{\varepsilon,M,\theta}$ is, by virtue of (2.45), reduced to estimating certain polynomial moments of φ under the tilted measure $\nu_{\varepsilon,M,\theta}(\mathrm{d}\varphi)$. Through stochastic quantisation, this measure can be identified with the marginal law of a stationary solution to the SPDE

$$\mathcal{L}_\varepsilon \phi + \lambda \phi^3 - r_{\varepsilon,M}\phi = O(\phi) + \xi^{(\varepsilon,M)}, \tag{2.46}$$

where the additional perturbation $O(\phi)$ is given by

$$\begin{aligned} O(\phi)(t,\bullet) &= \frac{1}{2}\theta \frac{\delta}{\delta\phi} \|hQ_\varepsilon\phi(t,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \\ &= 2\theta \|hQ_\varepsilon\phi(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (Q_\varepsilon h^2 Q_\varepsilon \phi)(t,\bullet). \end{aligned}$$

We use the notation

$$Q_\varepsilon h^2 Q_\varepsilon \phi = Q_\varepsilon(h^2 Q_\varepsilon \phi), \qquad (h^i Q_\varepsilon \phi)(t,x) = (h(x))^i (Q_\varepsilon \phi)(t,x), \quad i \in \{1,2\}.$$

We exploited above the fact that the operator Q_ε is self-adjoint and bounded. Since Q_ε has a uniformly rapidly decaying periodised lattice kernel, $Q_\varepsilon\phi$ is a well-defined periodic lattice function with mild polynomial growth. Hence, $hQ_\varepsilon\phi(t,\bullet)$ is in $L^2(\mathbb{T}^d_{\varepsilon,M})$.

Uniform estimates for this new equation in $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$ can be obtained by modifying our previous arguments. For $\mu \in [1/2, 1]$ set

$$R^O_\mu := F_1(\phi) + O(\phi) - F_\mu(\phi_\mu).$$

As before we rewrite (2.46) as a system of equations

$$\begin{cases} \mathcal{L}_\varepsilon \phi_\mu = J_\mu(F_\mu(\phi_\mu) + R^O_\mu) \\ R^O_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\,\mathrm{d}\sigma + O(\phi) + \int_\mu^1 \mathrm{D}F_\sigma(\phi_\sigma)(\dot{G}_\sigma R^O_\sigma)\,\mathrm{d}\sigma \end{cases} \tag{2.47}$$

for

$$(\phi_\mu, R^O_\mu)_\mu = (\phi^{(\varepsilon,M)}_\mu, R^{(\varepsilon,M),O}_\mu)_\mu.$$

Applying estimate (2.34) with $\eta = \mu$ to the equation for $R^O_\bullet$, we obtain

$$\begin{aligned}
&[\![\mu]\!]^{3\gamma}\|\zeta_\mu K_\mu R^O_\mu\| \\
&\quad\leqslant \exp(CC_Q[\![\bar\mu]\!]^{\vartheta/2}) \\
&\qquad\times\Big[\sup_{\sigma\geqslant\mu}[\![\mu]\!]^{3\gamma}\|\zeta_\mu K_\sigma\mathcal{H}_\sigma\| + [\![\bar\mu]\!]^{3\gamma}\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{2(s+\gamma)}\|\zeta^{\bar\kappa}_\mu\tilde J^2_\sigma R^O_\sigma\|\Big],
\end{aligned}$$

with

$$\mathcal{H}_\mu = O(\phi) + \int_\mu^1 H_\sigma(\phi_\sigma)\mathrm{d}\sigma.$$

The term with H_σ can be estimated as previously from (2.19), while to estimate $O(\phi)$ we use the uniform boundedness of $\|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})}$ and Lemma 2.10 below. We obtain that

$$\sup_{\sigma|\sigma\geqslant\mu}\|\zeta_\mu K_\sigma\mathcal{H}_\sigma\| \lesssim C_F[\![\bar\mu]\!]^\vartheta(1+|\!|\!|\phi|\!|\!|)^S + \theta[\![\mu]\!]^{-3\gamma}[|\!|\!|\phi|\!|\!| + |\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_\#]^3,$$

uniformly in $\mu\geqslant\bar\mu$. We also observe that

$$\sup_{\sigma,\mu|\sigma\geqslant\mu\geqslant\bar\mu}[\![\sigma]\!]^{2(s+\gamma)}\|\zeta^{\bar\kappa}_\mu\tilde J^2_\sigma R^O_\sigma\| \lesssim [\![\bar\mu]\!]^\vartheta C_F(1+|\!|\!|\phi|\!|\!|)^S,$$

uniformly in $\mu\geqslant\bar\mu$, by an argument analogous to the one leading to (2.22). Therefore, combining the above estimates, we obtain

$$|\!|\!|K_\bullet R^O_\bullet|\!|\!|_\# \lesssim \exp(CC_Q[\![\bar\mu]\!]^{\vartheta/2})[C_F[\![\bar\mu]\!]^{3\gamma+\vartheta}(1+|\!|\!|\phi|\!|\!|)^S + \theta[|\!|\!|\phi|\!|\!| + |\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_\#]^3].$$

We need a good bound for $|\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_\#$ in terms of $\Phi = \Phi_{\bar\mu}$ defined by (2.21) with R replaced by R^O. Observe that, by (4.73) below, the decomposition of the force, and Lemma 1.17, we have

$$\begin{aligned}
|\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_\# &\lesssim |\!|\!|K_\bullet F(\phi_\bullet)|\!|\!|_\# + |\!|\!|K_\bullet R^O_\bullet|\!|\!|_\# \\
&\lesssim \lambda\Phi^3 + [\![\bar\mu]\!]^\vartheta C_F\Phi^S + |\!|\!|K_\bullet R^O_\bullet|\!|\!|_\# \\
&\lesssim \lambda\Phi^3 + [\![\bar\mu]\!]^\vartheta C_F\Phi^S + \theta_R\Phi.
\end{aligned} \tag{2.48}$$

As a result, for $\theta\in(0,1]$, we obtain

$$\begin{aligned}
|\!|\!|K_\bullet R^O_\bullet|\!|\!|_\# &\lesssim \exp(CC_F\Phi^S[\![\bar\mu]\!]^{\vartheta/2}) \\
&\qquad\times[C_F\Phi^S[\![\bar\mu]\!]^{3\gamma+\vartheta} + \theta(1+\lambda)^3\Phi^{3S} + \theta(C_F[\![\bar\mu]\!]^\vartheta\Phi^S)^3 + \theta\theta_R^3\Phi^3] \\
&\lesssim \exp(CC_F\Phi^S[\![\bar\mu]\!]^{\vartheta/2}) \\
&\qquad\times[[\![\bar\mu]\!]^{\vartheta/2} + \theta(1+\lambda)^3 + \theta\theta_R^3]\Phi^{3S},
\end{aligned} \tag{2.49}$$

with possibly different constants C. In the last inequality we absorbed polynomial factors of $C_F[\![\bar\mu]\!]^{\vartheta/2}\Phi^S$ into the exponential. Indeed, $x + x^3 \lesssim \exp(cx)$ for $x\geqslant 0$. Gathering (2.49), (2.25) and (2.26) we arrive at

$$\Phi_{\bar\mu} \leqslant 1 + C[\tau(\bar\mu) + \tau(\bar\mu)^{1/3}\Phi^{S/3}_{\bar\mu} + \tau(\bar\mu)\Phi^S_{\bar\mu} + \tau(\bar\mu)\Phi^{3S}_{\bar\mu}\exp(C\tau(\bar\mu)\Phi^S_{\bar\mu})], \tag{2.50}$$

where

$$\begin{aligned}\tau(\bar\mu) :=& \lambda^{-1/2}[\![\bar\mu]\!]^{\gamma/2} + \lambda^{-1}([\![\bar\mu]\!]^{\vartheta} C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^{\vartheta} + \theta_R)\\ &+ \theta_{\mathcal{L}}^{-1}(\lambda + [\![\bar\mu]\!]^{\vartheta} C_F + \theta_{\mathcal{L}}[\![\bar\mu]\!]^{\vartheta} + \theta_R)\\ &+ \theta_R^{-1}[[\![\bar\mu]\!]^{\vartheta/2} + \theta(1+\lambda)^3 + \theta\theta_R^3] + C_F[\![\bar\mu]\!]^{\vartheta/2}\end{aligned}$$

and

$$\theta_R = \lambda\theta^{1/2}, \qquad \theta_{\mathcal{L}} = [\![\bar\mu]\!]^{-\vartheta/2}\lambda. \tag{2.51}$$

Then $\bar\mu \mapsto \tau(\bar\mu)$ is non-increasing and, for $\theta \in (0,1]$, we have

$$\tau(\bar\mu) \leqslant [\![\bar\mu]\!]^{\vartheta/2}[(3+\lambda^{-1/2}) + (1+2\lambda^{-1})C_F + \lambda^{-1}\theta^{-1/2}] + \theta^{1/2}[2 + \lambda^{-1}(1+\lambda)^3 + \lambda^2].$$

Fix $\tau_* > 0$ so that

$$C[\tau_* + \tau_*^{1/3}4^{S/3} + \tau_* 4^S + \tau_* 4^{3S}\exp(C\tau_* 4^S)] \leqslant 1.$$

Let $\theta_* = \theta_*(\lambda)$ be such that

$$\theta_*^{1/2}[2 + \lambda^{-1}(1+\lambda)^3 + \lambda^2] = \tau_*/2.$$

If needed we reduce τ_* further, to ensure that $\theta_* \leqslant 1$ and that the tilted measure with $\theta = \theta_*$ is well defined. Define $\bar\mu_* = \bar\mu_*(\lambda, \theta_*, C_F)$ by

$$[\![\bar\mu_*]\!]^{\vartheta/2}[(3+\lambda^{-1/2}) + (1+2\lambda^{-1})C_F + \lambda^{-1}\theta_*^{-1/2}] = \tau_*/2.$$

Then, for $\theta = \theta_*$ and $\bar\mu \geqslant \bar\mu_*$, we have $\tau(\bar\mu) \leqslant \tau_*$, and the continuity argument from Theorem 2.5 can be applied to obtain the desired estimates. We conclude that $\Phi_{\bar\mu_*} \leqslant 2$. By (2.51) this implies that for $\bar\mu = \bar\mu_*$ and $\theta = \theta_*$, we have

$$|\!|\!|\phi|\!|\!|_{\bar\mu} \leqslant 1, \qquad |\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_{\#,\bar\mu} \leqslant \lambda[\![\bar\mu]\!]^{-\vartheta/2}, \qquad |\!|\!|K_\bullet R_\bullet^O|\!|\!|_{\#,\bar\mu} \leqslant \lambda\theta_*^{1/2},$$

with

$$\begin{aligned}[\![\bar\mu]\!]^{-1} &= [2\tau_*^{-1}((3+\lambda^{-1/2}) + (1+2\lambda^{-1})C_F + \lambda^{-1}\theta_*^{-1/2})]^{2/\vartheta}\\ &\lesssim [(1+\lambda^{-1})C_F + \lambda^{-1}\theta_*^{-1/2}]^{2/\vartheta},\end{aligned}$$

uniformly in $C_F = 1 + \|F^{\mathfrak{A}}\|^2$. By Lemma 2.10, (2.48) and (2.51), we also have

$$\begin{aligned}\sup_{t\in\mathbb{R}} \rho(t,0)^4 \|hQ_\varepsilon\phi^{(\varepsilon,M)}(t)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} &\lesssim [\![\bar\mu]\!]^{-4\gamma}[|\!|\!|\phi|\!|\!| + |\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_{\#}]^4\\ &\lesssim [\![\bar\mu]\!]^{-4\gamma}[\Phi + \lambda\Phi^3 + [\![\bar\mu]\!]^{\vartheta} C_F\Phi^S + \theta_R\Phi]^4\\ &\lesssim_\lambda [\![\bar\mu]\!]^{-4\gamma} C_F \lesssim 1 + \|F^{\mathfrak{A}}\|^{16\gamma/\vartheta+2}.\end{aligned}$$

As a consequence of Theorem 2.7, we obtain

$$\begin{aligned}&\sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \int \|hQ_\varepsilon\varphi\|^{4N}_{L^2(\mathbb{T}^d_{\varepsilon,M})} \nu_{\varepsilon,M,\theta_*}(\mathrm{d}\varphi)\\ &\quad= \sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \mathbb{E}[\|hQ_\varepsilon\phi^{(\varepsilon,M)}(t)\|^{4N}_{L^2(\mathbb{T}^d_{\varepsilon,M})}]\\ &\quad\lesssim \sup_{\varepsilon\in 2^{-\mathbb{N}_0}, M\in\mathbb{N}_+} \mathbb{E}[1 + \|F^{\mathfrak{A}}\|^{16\gamma N/\vartheta+2N}] < \infty,\end{aligned}$$

for all $N \in \mathbb{N}_+$ and $t \in \mathbb{R}$. From this and (2.45), we deduce easily that any accumulation point ν of the sequence $(\hat{\nu}_{\varepsilon,M})_{\varepsilon,M}$ satisfies (1.3) provided $\theta > 0$ is small enough. This proves the exponential integrability, required for the Osterwalder–Schrader reconstruction, completing the proof of Theorem 1.1.

Remark 2.9. The choice of the norm used to verify exponential integrability is quite arbitrary. Since the corresponding tilt must be incorporated into an SPDE, we choose a differentiable norm. In general, we could replace the L^2 norm by any finite L^{2n} norm and similarly use a different spatial weight h and smoothing operator Q_ε, as long as they remain compatible with our Schauder estimate.

Lemma 2.10. *Suppose that the parameters A and B in (2.42) and (2.43) satisfy $A > \gamma$ and $B \geqslant d + 1$. Then*

$$\|\zeta_\sigma O(\phi)\| \lesssim \theta [\![\sigma]\!]^{-3\gamma} [\|\!|\phi|\!\| + \|\!|K_\bullet \mathcal{L}\phi|\!\|_\#]^3$$

and

$$\sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|hQ_\varepsilon\phi(t)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim [\![\bar{\mu}]\!]^{-4\gamma} [\|\!|\phi|\!\| + \|\!|K_\bullet \mathcal{L}\phi|\!\|_\#]^4.$$

Proof. Let us first observe that

$$\begin{aligned}
\|\zeta_\sigma O(\phi)\| &\lesssim \theta \|(t,x) \mapsto \zeta_\sigma(t,x) \|h(Q_\varepsilon\phi)(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (Q_\varepsilon h^2 Q_\varepsilon\phi)(t,x)\| \\
&= \theta \Big\| (t,x) \mapsto \big\| h\rho_\sigma^{-1}(t,\bullet)\rho_\sigma(t,\bullet)(Q_\varepsilon\phi)\,(t,\bullet)\big\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} \\
&\qquad \times (\zeta_\sigma \rho_\sigma^{-1} \rho_\sigma Q_\varepsilon h^2 Q_\varepsilon\phi)(t,x) \Big\| \\
&\lesssim \theta \|(t,x) \mapsto \|h\rho_\sigma^{-1}(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (\zeta_\sigma\rho_\sigma^{-1})(t,x)\| \\
&\quad \times \|\rho_\sigma Q_\varepsilon\phi\|^2 \|\rho_\sigma Q_\varepsilon h^2 Q_\varepsilon\phi\|.
\end{aligned}$$

By Remark 1.8 (a) we have

$$\rho_\sigma^{-1}(t,x) \lesssim \rho_\sigma^{-1}(t,0)\rho_\sigma^{-1}(0,x).$$

Hence, restricting to the centred fundamental domain, where $h(x) = (1+|x|)^{-B}$, and using $B \geqslant d+1$, we arrive at

$$\|h\rho_\sigma^{-1}(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim \rho_\sigma^{-2}(t,0) \|h\rho_\sigma^{-1}(0,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim \rho_\sigma^{-2}(t,0).$$

Since $\rho_\sigma^{-2}(t,0) \leqslant \rho_\sigma^{-2}(t,x)$ for any $x \in \mathbb{R}^d_\varepsilon$ and $\rho_\sigma = \zeta_\sigma^\nu$ with $\nu \in (0,1/3)$ we obtain

$$\begin{aligned}
&\|(t,x) \mapsto \|h\rho_\sigma^{-1}(t,\bullet)\|^2_{L^2(\mathbb{T}^d_{\varepsilon,M})} (\zeta_\sigma\rho_\sigma^{-1})(t,x)\| \\
&\quad \lesssim \|(t,x) \mapsto \rho_\sigma^{-2}(t,0)(\zeta_\sigma\rho_\sigma^{-1})(t,x)\| \lesssim \|\zeta_\sigma\rho_\sigma^{-3}\| \lesssim 1.
\end{aligned}$$

Since $A > \gamma$, Lemma A.24, together with $\|\rho_\sigma Q_\varepsilon f\| \lesssim \|\rho_\sigma f\|$, yields

$$\begin{aligned}
\|\rho_\sigma Q_\varepsilon h^2 Q_\varepsilon\phi\| &\lesssim \|\rho_\sigma h^2 Q_\varepsilon\phi\| \lesssim \|\rho_\sigma Q_\varepsilon\phi\| \\
&\lesssim \sum_i 2^{-iA} \|\rho_\sigma \bar{\Delta}_i\phi\| \lesssim [\![\sigma]\!]^{-\gamma} [\|\!|\phi|\!\| + [\![\bar{\mu}]\!]^{2s-2\gamma} \|\!|K_\bullet \mathcal{L}\phi|\!\|_\#],
\end{aligned}$$

uniformly over $\sigma \geqslant \bar{\mu}$. Combining the above estimates we obtain

$$\|\zeta_\sigma O(\phi)\| \lesssim \theta[\![\sigma]\!]^{-3\gamma}[\|\!|\phi|\!\| + \|\!|K_\bullet \mathcal{L}\phi|\!\|_\#]^3.$$

Using Remark 1.8 a) and $B \geqslant d+1$ we get

$$\sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|h\rho_0^{-1}(t,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim \|h\rho_0^{-1}(0,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \lesssim 1.$$

Since

$$\|\rho_0 Q_\varepsilon \phi\| \lesssim \|\rho_{\bar{\mu}} Q_\varepsilon \phi\| \lesssim \sum_i 2^{-iA} \|\rho_{\bar{\mu}} \bar{\Delta}_i \phi\| \lesssim [\![\bar{\mu}]\!]^{-\gamma}[\|\!|\phi|\!\| + [\![\bar{\mu}]\!]^{2s-2\gamma}\|\!|K_\bullet \mathcal{L}\phi|\!\|_\#],$$

we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|hQ_\varepsilon\phi(t)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \\
&\quad\lesssim \sup_{t\in\mathbb{R}} \rho_0(t,0)^4 \|h\rho_0^{-1}(t,\bullet)\|^4_{L^2(\mathbb{T}^d_{\varepsilon,M})} \|\rho_0 Q_\varepsilon\phi\|^4 \\
&\quad\lesssim [\![\bar{\mu}]\!]^{-4\gamma}[\|\!|\phi|\!\| + \|\!|K_\bullet \mathcal{L}\phi|\!\|_\#]^4.
\end{aligned}$$

This finishes the proof. ∎

2.6. *The vector model*

In this final subsection, we discuss the modifications required to extend our results to the vector-valued model, where the field $\phi = \phi^{(\varepsilon,M)}$ takes values in the Euclidean space $\mathbb{R}^n$ for some $n > 1$. We denote by $(\phi^a)_{a\in\{1,\dots,n\}}$ the components of the field ϕ in the canonical basis. The dynamics reads

$$\mathcal{L}_\varepsilon \phi^a + \lambda|\phi|^2\phi^a - r_{\varepsilon,M}\phi^a = \xi^a, \qquad a = 1,\dots,n, \tag{2.52}$$

on $\Lambda_{\varepsilon,M}$, where

$$|\phi|^2 := \sum_a (\phi^a)^2.$$

Here

$$\xi = (\xi^a)_{a\in\{1,\dots,n\}} = (\xi^{(\varepsilon,M),a})_{a\in\{1,\dots,n\}} \tag{2.53}$$

is a vector-valued, spacetime white noise on $\Lambda_{\varepsilon,M}$ such that

$$\mathbb{E}[\xi^a(t,x)\xi^b(s,y)] = \delta(t-s)\delta_{a,b}\,\varepsilon^{-d}\mathbb{1}_{x=y}, \qquad (t,x),(s,y)\in\Lambda_{\varepsilon,M}, \quad a,b = 1,\dots,n.$$

As before, we identify ϕ and ξ with spatially periodic functions on Λ_ε.

Our proof of the main result in the scalar-valued case extends straightforwardly to the vector-valued case. The main subtlety lies in obtaining the analogue of Theorem 2.4 in this setting, which, however, follows directly from Theorem 3.1 below. Beyond this adjustment, the subsequent analysis carries over with minimal changes, as it does not depend on the scalar nature of the equation until the classification of the relevant cumulants in

Section 4.5. At that stage, one must exploit the $O(n)$-symmetry of the noise (2.53) to deduce that the cumulants are likewise symmetric. Consequently, the only contribution to the mass renormalisation is diagonal in the vector indices, and can therefore be absorbed into a redefinition of the renormalisation constant $r_{\varepsilon,M} \in \mathbb{R}$.

3. A priori estimates

This section is devoted to establishing weighted estimates for classical solutions of a fractional parabolic equation with a cubic coercive term. To the best of our knowledge, this result is new, although the proof closely follows the argument developed in the case of the standard Laplacian (see, for instance, [23]). Furthermore, we present the estimates directly in a vector-valued setting, as no additional difficulties arise in this more general formulation.

Theorem 3.1. *Let $n \in \mathbb{N}_+$, $\lambda > 0$, $\bar{\nu} < 2s$ and $u \in C^1_t C^2_x(\Lambda_\varepsilon, \mathbb{R}^n)$ be such that $|\zeta^{\bar{\nu}} u|$ is a bounded function, where the weight ζ was introduced in Definition 1.5. Then u belongs to the domain of the operator $\partial_t + (-\Delta)^s$. Define $f \in C(\Lambda_\varepsilon, \mathbb{R}^n)$ by*

$$f^a := \partial_t u^a + (-\Delta)^s u^a + m^2 u^a + \lambda |u|^2 u^a, \qquad a \in \{1, \dots, n\}, \tag{3.1}$$

where $|u|^2 := \sum_a (u^a)^2$.

For any positive weight $\rho \in C^1_t C^2_x(\Lambda_0)$ such that for some $\tilde{\nu} \in (\bar{\nu}, 2s)$ the functions $\zeta^{-\tilde{\nu}}\rho$, $\zeta^{\tilde{\nu}}\rho^{-1}$ are bounded we have

$$\|\rho u\| \leqslant 2\lambda^{-1/2} A^{1/2} + \lambda^{-1/3}(\|\rho^3 f\| + B)^{1/3}, \tag{3.2}$$

where

$$A := \|(-\Delta)^s \rho^2\| + \|\rho(\partial_t \rho)\|$$

and

$$B := \|\rho u\|(\|\rho(\partial_t \rho)\| + \|\rho(-\Delta)^s \rho\| + \|\rho^2 \mathfrak{D}_s(\rho^{-1})\mathfrak{D}_s(\rho)\|) + \|\rho \mathfrak{D}_s(\rho)\mathfrak{D}_s(\rho u)\|.$$

Remark 3.2. If the weight ρ were constant, both A and B would vanish and the estimate would reduce to the usual maximum-principle bound with cubic damping. Here A measures the cost of applying the time derivative and the fractional Laplacian to the weight, whereas B contains the genuinely nonlocal commutator errors produced by the interaction of $(-\Delta)^s$ with ρ.

Proof. We assume that $\|\rho^3 f\| + A + B < \infty$ as otherwise there is nothing to prove. For $N \in \mathbb{N}_+$ and $L > 0$, we define a convex function $\Phi = \Phi_{L,N} \in C(\mathbb{R})$ by

$$\Phi(\eta) := (\eta - L)_+^N, \qquad \Phi'(\eta) := N(\eta - L)_+^{N-1}, \qquad \eta \in \mathbb{R},$$

where Φ' denotes the derivative of Φ and $(\eta)_+ := \mathbb{1}_{\eta \geqslant 0}\eta$. We fix the parameter $N > 1$ such that $2N(\bar{\nu} - \bar{\nu}) > d + 2s$. The parameter L will be fixed later. Let $\hat{u} := \rho u$. After testing (3.1) with $\Phi'(|\hat{u}|^2)\hat{u}^a\rho^3$ and summing over $a \in \{1, \dots, n\}$, we obtain

$$0 = \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)[\rho^3\hat{u} \cdot \partial_t u + \rho^3\hat{u} \cdot (-\Delta)^s u + m^2\rho^2|\hat{u}|^2 + \lambda(|\hat{u}|^2)^2 - \rho^3 f \cdot \hat{u}].$$

Observe that

$$\begin{aligned}
\int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)\rho^3\hat{u} \cdot \partial_t u &= \frac{1}{2}\int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)\rho^2\partial_t|\hat{u}|^2 - \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)(\rho\partial_t\rho)|\hat{u}|^2 \\
&= \frac{1}{2}\int_{\Lambda_\varepsilon} \rho^2\partial_t\Phi(|\hat{u}|^2) - \int_{\Lambda_\varepsilon} (\rho\partial_t\rho)\Phi'(|\hat{u}|^2)|\hat{u}|^2 \\
&= -\frac{1}{2}\int_{\Lambda_\varepsilon} \partial_t(\rho^2)\Phi(|\hat{u}|^2) - \int_{\Lambda_\varepsilon} (\rho\partial_t\rho)\Phi'(|\hat{u}|^2)|\hat{u}|^2 \\
&= -\int_{\Lambda_\varepsilon} (\rho\partial_t\rho)\Phi(|\hat{u}|^2) - \int_{\Lambda_\varepsilon} (\rho\partial_t\rho)\Phi'(|\hat{u}|^2)|\hat{u}|^2 \\
&\geqslant -\|\rho(\partial_t\rho)\| \int_{\Lambda_\varepsilon} \Phi(|\hat{u}|^2) - \|\hat{u}\|^2_{L^\infty}\|\rho(\partial_t\rho)\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).
\end{aligned}$$

By Lemma 3.3 below,

$$\sum_a \langle \Phi'(|\hat{u}|^2)\hat{u}^a, \rho^3(-\Delta)^s u^a\rangle \geqslant -\frac{1}{2}\|(-\Delta)^s\rho^2\| \int_{\Lambda_\varepsilon} \Phi(|\hat{u}|^2) - \overline{B}\|\hat{u}\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).$$

We also have

$$\begin{aligned}
\langle \Phi'(|\hat{u}|^2), m^2\rho^2|\hat{u}|^2 + \lambda(|\hat{u}|^2)^2\rangle &\geqslant \langle \Phi'(|\hat{u}|^2), \lambda(|\hat{u}|^2)^2\rangle \\
&\geqslant \lambda N\langle (|\hat{u}|^2 - L)_+^{N-1}, ((|\hat{u}|^2 - L)_+ + L)L\rangle \\
&\geqslant \lambda L \int_{\Lambda_\varepsilon} \Phi(|\hat{u}|^2) + \lambda L^2 \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2)
\end{aligned}$$

and

$$-\langle \Phi'(|\hat{u}|^2), \rho^3 f \cdot \hat{u}\rangle \geqslant -\|\rho^3 f\|\|\hat{u}\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).$$

We conclude that

$$0 \geqslant [\lambda L - A] \int_{\Lambda_\varepsilon} \Phi(|\hat{u}|^2) + [\lambda L^2 - (\|\rho^3 f\| + B)\|\hat{u}\|] \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).$$

Taking

$$L > L_* := \max(\lambda^{-1}A, \lambda^{-1/2}[\|\rho^3 f\| + B]^{1/2}\|\hat{u}\|^{1/2}),$$

we deduce that

$$\int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2) = \int_{\Lambda_\varepsilon} \Phi(|\hat{u}|^2) = 0,$$

which implies that $|\hat{u}|^2 \leqslant L$ a.e. on Λ_ε. Thus,

$$\|\hat{u}\|^2 \leqslant \inf_{L > L_*} L = L_* \leqslant \lambda^{-1}A + \lambda^{-1/2}[\|\rho^3 f\| + B]^{1/2}\|\hat{u}\|^{1/2}.$$

This implies that

$$\|\hat{u}\| \leqslant \lambda^{-1/2} A^{1/2} + \lambda^{-1/4} \|\hat{u}\|^{1/4} [\|\rho^3 f\| + B]^{1/4}.$$

By Young's inequality, we have

$$\lambda^{-1/4} \|\hat{u}\|^{1/4} [\|\rho^3 f\| + B]^{1/4} \leqslant \frac{\|\hat{u}\|}{4} + \frac{3}{4} \lambda^{-1/3} [\|\rho^3 f\| + B]^{1/3},$$

and hence

$$\|\hat{u}\| \leqslant 2\lambda^{-1/2} A^{1/2} + \lambda^{-1/3} [\|\rho^3 f\| + B]^{1/3},$$

as claimed. ∎

The following lemma completes the proof.

Lemma 3.3. *Suppose that $u \in C^1_t C^2_x(\Lambda_\varepsilon, \mathbb{R}^n)$ is such that $|\zeta^{\bar{\nu}} u|$ is a bounded function for some $\bar{\nu} < 2s$ and $\rho \in C^1_t C^2_x(\Lambda_\varepsilon)$ is a positive weight such that for some $\tilde{\nu} \in (\bar{\nu}, 2s)$ the functions $\zeta^{-\tilde{\nu}} \rho$, $\zeta^{\tilde{\nu}} \rho^{-1}$ are bounded. Let $\Phi = \Phi_{L,N} \in C(\mathbb{R})$ with $N > 1$ fixed as in the proof of the previous theorem and arbitrary $L > 0$. Then we have*

$$\sum_a \langle \Phi'(|\hat{u}|^2) \hat{u}^a, \rho^3 (-\Delta)^s u^a \rangle \geqslant -\frac{1}{2} \|(-\Delta)^s \rho^2\| \int_{\Lambda_\varepsilon} \Phi(|\hat{u}|^2) - \bar{B} \|\hat{u}\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2),$$

where $\hat{u} := \rho u$ and

$$\bar{B} := \|\hat{u}\| (\|\rho (-\Delta)^s \rho\| + \|\rho^2 \mathfrak{D}_s(\rho^{-1}) \mathfrak{D}_s(\rho)\|) + \|\rho \mathfrak{D}_s(\rho) \mathfrak{D}_s(\hat{u})\|.$$

Proof. The following decomposition separates the favourable convexity contribution from the terms in which the fractional operator interacts with the weight. Leaving the sum over a implicit, we have

$$\begin{aligned}
&\langle \Phi'(|\hat{u}|^2) \hat{u}^a, \rho^3 (-\Delta)^s u^a \rangle \\
&= \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z \mathrm{d}z') \Phi'(|\hat{u}(z)|^2) \hat{u}^a(z) \rho^3(z) (u^a(z) - u^a(z')) \\
&= \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z \mathrm{d}z') \Phi'(|\hat{u}(z)|^2) \hat{u}^a(z) \rho^2(z) (\hat{u}^a(z) - \hat{u}^a(z')) \qquad (=: (\mathbb{I})) \\
&\quad + \int_{\Lambda_\varepsilon \times \Lambda_\varepsilon} \mu_s(\mathrm{d}z \mathrm{d}z') \Phi'(|\hat{u}(z)|^2) \hat{u}^a(z) \rho^2(z) (\rho(z') - \rho(z)) u^a(z') \quad (=: (\mathbb{II}))
\end{aligned}$$

Let $V_\tau := \tau \hat{u}(z') + (1 - \tau) \hat{u}(z)$. Since $\tau \in [0, 1] \mapsto \Phi(|V_\tau|^2)$ is a convex function (as a composition of a convex function with an affine one),

$$\Phi(|V_1|^2) - \Phi(|V_0|^2) \geqslant \partial_\tau \Phi(|V_\tau|^2)|_{\tau=0},$$

that is,

$$\Phi(|\hat{u}(z')|^2) - \Phi(|\hat{u}(z)|^2) \geqslant 2\Phi'(|\hat{u}(z)|^2) \sum_a \hat{u}^a(z) (\hat{u}^a(z') - \hat{u}^a(z)).$$

Using the above inequality we obtain

$$\begin{aligned}\mathbb{I} &\geqslant \frac{1}{2}\int_{\Lambda_\varepsilon\times\Lambda_\varepsilon} \mu_s(\mathrm{d}z\mathrm{d}z')\rho^2(z)[\Phi(|\hat{u}(z)|^2)-\Phi(|\hat{u}(z')|^2)]\\ &= \frac{1}{2}\int_{\Lambda_\varepsilon\times\Lambda_\varepsilon} \mu_s(\mathrm{d}z\mathrm{d}z')\Phi(|\hat{u}(z)|^2)[\rho^2(z)-\rho^2(z')]\\ &\geqslant -\frac{1}{2}\|(-\Delta)^s\rho^2\|\int_{\Lambda_\varepsilon}\Phi(|\hat{u}|^2),\end{aligned}$$

where the equality in the second line follows by an integration by parts of the fractional Laplacian, which is a symmetric operator in L^2. Let us now consider $(\mathbb{III})$. We first split the factor $\rho^{-1}(z')$ as

$$\rho^{-1}(z') = \big(\rho^{-1}(z')-\rho^{-1}(z)\big)+\rho^{-1}(z).$$

This gives

$$\begin{aligned}(\mathbb{III}) &= \int_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\mu_s(\mathrm{d}z\,\mathrm{d}z')\,\Phi'(|\hat{u}(z)|^2)\hat{u}^a(z)\rho^2(z)\big(\rho(z')-\rho(z)\big)\rho^{-1}(z')\hat{u}^a(z')\\ &= \int_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\mu_s(\mathrm{d}z\,\mathrm{d}z')\,\Phi'(|\hat{u}(z)|^2)\hat{u}^a(z)\rho^2(z)\big(\rho(z')-\rho(z)\big)\big(\rho^{-1}(z')-\rho^{-1}(z)\big)\hat{u}^a(z')\\ &\quad+\int_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\mu_s(\mathrm{d}z\,\mathrm{d}z')\,\Phi'(|\hat{u}(z)|^2)\hat{u}^a(z)\rho^2(z)\big(\rho(z')-\rho(z)\big)\rho^{-1}(z)\hat{u}^a(z').\end{aligned}$$

We call the first term $(\mathbb{III}_2)$. In the second term we split

$$\hat{u}^a(z') = \hat{u}^a(z)+\big(\hat{u}^a(z')-\hat{u}^a(z)\big).$$

Hence

$$\begin{aligned}(\mathbb{III}) &= \int_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\mu_s(\mathrm{d}z\,\mathrm{d}z')\,\Phi'(|\hat{u}(z)|^2)|\hat{u}(z)|^2\rho(z)\big(\rho(z')-\rho(z)\big)\\ &\quad+\int_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\mu_s(\mathrm{d}z\,\mathrm{d}z')\,\Phi'(|\hat{u}(z)|^2)\hat{u}^a(z)\rho^2(z)\big(\rho(z')-\rho(z)\big)\big(\rho^{-1}(z')-\rho^{-1}(z)\big)\hat{u}^a(z')\\ &\quad+\int_{\Lambda_\varepsilon\times\Lambda_\varepsilon}\mu_s(\mathrm{d}z\,\mathrm{d}z')\,\Phi'(|\hat{u}(z)|^2)\hat{u}^a(z)\rho(z)\big(\rho(z')-\rho(z)\big)\big(\hat{u}^a(z')-\hat{u}^a(z)\big)\\ &= (\mathbb{III}_1)+(\mathbb{III}_2)+(\mathbb{III}_3).\end{aligned}$$

For $(\mathbb{III}_1)$ we have

$$(\mathbb{III}_1)\geqslant -\|\hat{u}\|^2\|\rho(-\Delta)^s\rho\|\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}(z)|^2).$$

Next,

$$\begin{aligned}(\mathbb{III}_2) &\geqslant -\|\hat{u}\|^2\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}(z)|^2)\rho^2(z)\\ &\qquad\times\bigg(\int_{\Lambda_\varepsilon}|\rho(z')-\rho(z)||\rho^{-1}(z')-\rho^{-1}(z)|\,\mu_s(z,\mathrm{d}z')\bigg)\mathrm{d}z\\ &\geqslant -\|\hat{u}\|^2\|\rho^2\mathfrak{D}_s(\rho^{-1})\mathfrak{D}_s(\rho)\|\int_{\Lambda_\varepsilon}\Phi'(|\hat{u}|^2).\end{aligned}$$

Finally, using the Cauchy–Schwarz inequality, we obtain

$$(\mathbb{I}_3) \geqslant -\|\hat{u}\|\|\rho\mathfrak{D}_s(\rho)\mathfrak{D}_s(\hat{u})\| \int_{\Lambda_\varepsilon} \Phi'(|\hat{u}|^2).$$

Combining the lower bounds for $(\mathbb{I})$ and $(\mathbb{I}_j)$, $j = 1, 2, 3$, gives the desired estimate. ∎

4. Analysis of the flow equation

In this section, we establish the estimates used in the proof of Theorem 2.7. More precisely, we analyse the existence of an approximate solution to the flow equation (2.30) for the effective force $(F_\sigma)_\sigma$, with well-controlled bounds encoded by (2.19). These bounds are achievable only because we can "tune" the boundary condition (2.2) using the ε- and M-dependent renormalisation term $r_{\varepsilon,M}$.

Conceptually, we are dealing with a random bilinear equation whose solution is analysed via the evolution equation for its cumulants. The flow equation for cumulants has a similar structure to the flow equation for the effective force and propagates comparable bounds backwards from the final condition at $\sigma = 1$, except in a low-dimensional (so-called relevant) subspace, where the bounds must be propagated forwards from small to large σ. This procedure requires tuning an appropriate final condition so that the solution lies on a trajectory with controlled bounds. To simplify this tuning, we decompose the flow equation to reduce the relevant subspace to one dimension.

Once bounds for the cumulants are established, a Kolmogorov-type argument allows us to deduce pathwise bounds on the effective force. The section concludes with a technical "post-processing" step, which extracts the coercive term essential for the global a priori estimates and verifies the conditions (2.19).

4.1. Random flow equation

To study approximate solutions $(F_\sigma)_{\sigma\in[1/2,1]}$ of the flow equation (2.30) we need to set up the appropriate spaces. Recall that

$$\mathcal{E} := \bigcap_{\alpha>0} C(\Lambda_M, \zeta^\alpha), \qquad \hat{\mathcal{E}} := \mathcal{E} + \partial_t\mathcal{E}.$$

As we shall argue, the flow equation can be approximately solved in the space $\mathcal{P}(\mathcal{E})$ of polynomial functions on $\mathcal{E}$ with values in $\hat{\mathcal{E}}$. We say that a functional F belongs to $\mathcal{P}(\mathcal{E})$ if for some $\bar{k} \in \mathbb{N}_+$ there exist kernels $(F^{(k)})_{k\in\{0,\dots,\bar{k}\}}$ of operators $\mathcal{E}^{\otimes k} \to \hat{\mathcal{E}}$ such that

$$F(\phi) = \sum_{k=0}^{\bar{k}} F^{(k)}(\phi) = \sum_{k=0}^{\bar{k}} \int_{\Lambda^k} F^{(k)}(\bullet; \mathrm{d}z_1, \dots, \mathrm{d}z_k)\phi(z_1)\cdots\phi(z_k) \in \hat{\mathcal{E}},$$

for all $\phi \in \mathcal{E}$. In order to construct a suitable approximate solution of (2.30) we introduce a formal parameter $\hbar$ and make the ansatz

$$^{\hbar}F_\sigma = \sum_{\ell\geqslant 0} \hbar^\ell F_\sigma^{[\ell]}.$$

Moreover, we assume that the final condition is of the form

$$ {}^{\hbar}F(\phi) = -\lambda\phi^3 + {}^{\hbar}r_{\varepsilon,M}\phi + \xi^{(\varepsilon,M)}, \qquad {}^{\hbar}r_{\varepsilon,M} = \bar{r} + \sum_{\ell\geqslant 1}\hbar^{\ell} r^{[\ell]}_{\varepsilon,M}. $$

We are led to look for solutions of the *perturbative flow equation*

$$ \partial_\sigma {}^{\hbar}F_\sigma + \hbar \mathrm{D}^{\hbar}F_\sigma(\dot{G}_\sigma {}^{\hbar}F_\sigma) = 0, \qquad {}^{\hbar}F_1 = {}^{\hbar}F, \tag{4.1} $$

in the space $\mathcal{P}(\mathcal{E})[\![\hbar]\!]$ of formal power series in $\hbar$ with coefficients in $\mathcal{P}(\mathcal{E})$. This setup has the advantage that now the flow equation has a unique global solution which can be determined by induction on the order in $\hbar$. An approximate solution to (2.30) is obtained by fixing an integer $\bar{\ell} \geqslant 0$ and letting

$$ F_\sigma := \sum_{\ell=0}^{\bar{\ell}} F^{[\ell]}_\sigma. $$

The choice of the value of $\bar{\ell}$ will be discussed in Section 4.9 below. We observe that, thanks to (4.1), this truncation implies the existence of a maximal polynomial order $\bar{k} \in \mathbb{N}_+$ in the fields for the kernels. We decompose the force as

$$ F_\mu(\phi) = \sum_{\ell=0}^{\bar{\ell}} F^{[\ell]}_\mu(\phi) = \sum_{\ell=0}^{\bar{\ell}}\sum_{k=0}^{\bar{k}} F^{[\ell],(k)}_\mu(\phi), $$

where ℓ measures the perturbative order in $\hbar$ while k is the polynomial degree in ϕ.

Let us now introduce a condensed notation to manipulate these kernels. Let

$$ \mathfrak{A} := \{(\ell,k) | 0 \leqslant \ell \leqslant \bar{\ell}, 0 \leqslant k \leqslant \bar{k}\}, \qquad \mathfrak{A}_{\mathrm{ext}} := \mathfrak{A} \cup \{(\ell,k) | \bar{\ell}+1 \leqslant \ell \leqslant 2\bar{\ell}+1, 0 \leqslant k \leqslant 2\bar{k}\}. $$

For $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$ with $\mathfrak{a} = (\ell, k)$ we let $k(\mathfrak{a}) := k$, $\ell(\mathfrak{a}) := \ell$ and write

$$ [\mathfrak{a}] := -\alpha + \delta\ell(\mathfrak{a}) + \beta k(\mathfrak{a}), \tag{4.2} $$

for suitable positive parameters α, δ and β whose values will be fixed later. The set $\mathfrak{A}$ indexes the kernels retained in the truncated effective force, whereas the additional indices in $\mathfrak{A}_{\mathrm{ext}} \setminus \mathfrak{A}$ will be used only to label the terms in its flow-equation error. For $\mathfrak{a} \in \mathfrak{A}$, we say that a kernel

$$ F^{\mathfrak{a}}(z; \mathrm{d}z_1, \cdots, \mathrm{d}z_k) := F^{[\ell],(k)}(z; \mathrm{d}z_1, \cdots, \mathrm{d}z_k) $$

is *relevant* if $[\mathfrak{a}] < 0$, *marginal* if $[\mathfrak{a}] = 0$ and *irrelevant* if $[\mathfrak{a}] > 0$. We refer to z as the *output* variable, and to $z_1, \ldots, z_k$ as the *input* variables of the kernel $F^{\mathfrak{a}}(z; \mathrm{d}z_1, \ldots, \mathrm{d}z_k)$. Throughout, the kernels $F^{[\ell],(k)}(z; \mathrm{d}z_1, \ldots, \mathrm{d}z_k)$ are taken to be symmetric in their k input variables. To simplify the notation, we usually ignore the fact that the kernel $F^{\mathfrak{a}}(z; \mathrm{d}z_1, \ldots, \mathrm{d}z_k)$ is generally not well-defined pointwise in the output variable z.

4.2. Norms for kernels

To introduce suitable norms for the effective force kernels

$$(F^{\mathfrak{a}}_{\sigma})_{\mathfrak{a}\in\mathfrak{A},\sigma\in[1/2,1]},$$

we first need some notation and preliminary definitions for weights and smoothing operators.

Definition 4.1. We denote by $\mathrm{St}(z_1,\dots,z_n)$ the Steiner diameter of the set $\{z_1,\dots,z_n\}$ with respect to the parabolic distance introduced in Definition 1.4, *i.e.* the minimum of the lengths of trees with nodes at the points $\{z_1,\dots,z_n\}$ and possibly other points $\{z'_1,\dots,z'_m\}$.

Remark 4.2. Our fractional parabolic distance (1.23) satisfies the triangle inequality

$$|z_1+z_2|_s \leqslant |z_1|_s + |z_2|_s, \qquad z_1,z_2\in\Lambda_0.$$

Moreover, for $n\in\mathbb{N}_+, m\in\{1,\dots n-1\}$ and $z,z_1,\dots,z_n\in\Lambda_0$, the Steiner diameter satisfies

$$\mathrm{St}(z_1,\dots,z_n) \leqslant \mathrm{St}(z_1,\dots,z_m,z) + \mathrm{St}(z,z_{m+1},\dots,z_n),$$

which, noting that $\mathrm{St}(z_1,z_2)=|z_1-z_2|_s$, can be seen as a generalisation of the triangle inequality for the fractional parabolic distance.

Definition 4.3. Let $\mathfrak{b}\in(1,2s)$ be a constant close to $2s$ and $\kappa_{\mathfrak{o}}\in(0,(2-\mathfrak{b})/(2\bar{\ell}+2))$ be a small constant, to be fixed later.

(a) For $m\in\mathbb{N}_+$ and $\omega\in\mathbb{R}$, the weight $w^{(1+m),\omega}_{\mu}\in C(\Lambda_0^{(1+m)})$ is defined by

$$w^{(1+m),\omega}_{\mu}(z,z_1,\dots,z_m) := (1+[\![\mu]\!]^{-1}\,\mathrm{St}(z,z_1,\dots,z_m))^{\omega}, \qquad z,z_1,\dots,z_m\in\Lambda_0.$$

We write $w^{(1),\omega}_{\mu}=1\in C(\Lambda_0)$.

(b) For $m\in\mathbb{N}_+$, the weight $v^{(1+m)}_{\mu}\in C^{\infty}(\Lambda_0^{(1+m)})$ is defined by

$$v^{(m+1)}_{\mu}(z,z_1,\dots,z_m) := v^{(m+1)}([\![\mu]\!]^{-1}.z,[\![\mu]\!]^{-1}.z_1,\dots,[\![\mu]\!]^{-1}.z_m),$$

where $v^{(m+1)}\in C^{\infty}(\Lambda_0^{(1+m)},[0,1])$ is a fixed function such that

$$v^{(m+1)}(z,z_1,\dots,z_m) = \begin{cases} 1 \text{ if } \mathrm{St}(z,z_1,\dots,z_m)\leqslant 1,\\ 0 \text{ if } \mathrm{St}(z,z_1,\dots,z_m)\geqslant 2,\end{cases}$$

and

$$v^{(m+1)}(z,z_1,\dots,z_m) = v^{(m+1)}(0,z_1-z,\dots,z_m-z).$$

We write $v^{(1)}_{\mu}=1\in C(\Lambda_0)$.

(c) The weight $\mathfrak{o}\in C(\Lambda_0)$ is defined by

$$\mathfrak{o}(z) := \zeta(z)^{\kappa_{\mathfrak{o}}} = \langle z\rangle_s^{-\kappa_{\mathfrak{o}}}, \qquad z\in\Lambda_0.$$

(d) For $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$, we write

$$w_\mu^{\mathfrak{a}} := w_\mu^{(1+k(\mathfrak{a})),\mathfrak{b}}, \qquad \tilde{w}_\mu^{\mathfrak{a}} := w_\mu^{(1+k(\mathfrak{a})),\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}},$$
$$v_\mu^{\mathfrak{a}} := v_\mu^{(1+k(\mathfrak{a}))}, \qquad \mathfrak{o}^{\mathfrak{a}} := \mathfrak{o}^{1+\ell(\mathfrak{a})}.$$

(e) The weight $h_\mu \in C(\Lambda_0^2)$ is defined by

$$h_\mu(z, z') := (1 + [\![\mu]\!]^{-2}|z - z'|_s^2)^{-1}.$$

Recall that

$$\langle z \rangle_s := (1 + |z_0|^{1/s} + |\bar{z}|^2)^{1/2}.$$

The weights $w_\mu^{\mathfrak{a}}$ and $\tilde{w}_\mu^{\mathfrak{a}}$ will control the approximate localisation of the effective force kernels $F_\mu^{\mathfrak{a}}(z; \mathrm{d}z_1, \dots, \mathrm{d}z_k)$ near the diagonal $\{z = z_1 = \cdots = z_k\}$. The weights $\mathfrak{o}^{\mathfrak{a}}$ will instead be used to control the growth at spacetime infinity of the kernels in the output variable z, reflecting the corresponding growth inherited from the white noise. These three families of weights are constructed to be compatible with the estimates for the bilinear terms appearing in the flow equation, as detailed in the lemma below. The additional weights $v_\mu^{(m+1)}$ and h_μ will be used only in intermediate estimates and play an auxiliary role. Some of their useful properties are summarised in Appendix A.2. Finally, observe that

$$w_\mu^{(2),\mathfrak{b}}(z, z') = w_\mu^{\mathfrak{b}}(z - z') = (1 + [\![\mu]\!]^{-1}|z - z'|_s)^{\mathfrak{b}},$$

where w_μ^{ω} is the weight introduced in Definition 1.16. The exponent $\mathfrak{b} - \ell(\mathfrak{a})\kappa_{\mathfrak{o}}$ in the definition of the weight $\tilde{w}_\mu^{\mathfrak{a}}$ is designed in such a way that the bound (4.4) below holds true.

Lemma 4.4. *For all $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$ and $\mathfrak{b}, \mathfrak{c} \in \mathfrak{A}$ such that $k(\mathfrak{b}) + k(\mathfrak{c}) = k(\mathfrak{a}) + 1$ and $\ell(\mathfrak{b}) + \ell(\mathfrak{c}) = \ell(\mathfrak{a}) - 1$, the bounds*

$$w_\mu^{\mathfrak{a}}(z, z_1, \dots, z_{k(\mathfrak{a})}) \lesssim w_\mu^{\mathfrak{b}}(z, z_1, \dots, z_{k(\mathfrak{b})-1}, \hat{z}) w_\mu^{\mathfrak{b}}(\hat{z} - \tilde{z}) w_\mu^{\mathfrak{c}}(\tilde{z}, z_{k(\mathfrak{b})}, \dots, z_{k(\mathfrak{a})}), \tag{4.3}$$

$$\begin{aligned} \mathfrak{o}^{\mathfrak{a}}(z)\tilde{w}_\mu^{\mathfrak{a}}(z, z_1, \dots, z_{k(\mathfrak{a})}) \lesssim\ & \mathfrak{o}^{\mathfrak{b}}(z)\tilde{w}_\mu^{\mathfrak{b}}(z, z_1, \dots, z_{k(\mathfrak{b})-1}, \hat{z}) w_\mu^{\mathfrak{b}}(\hat{z} - \tilde{z}) \\ & \times \mathfrak{o}^{\mathfrak{c}}(\tilde{z})\tilde{w}_\mu^{\mathfrak{c}}(\tilde{z}, z_{k(\mathfrak{b})}, \dots, z_{k(\mathfrak{a})}), \end{aligned} \tag{4.4}$$

hold uniformly in $\mu \in (0, 1)$ and $z, \hat{z}, \tilde{z}, z_1, \dots, z_{k(\mathfrak{a})} \in \Lambda_0$. For all $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$, the bounds

$$\mathfrak{o}^{\mathfrak{a}}(z) \lesssim (\mathfrak{o}^{\mathfrak{a}}(z - z'))^{-1}\mathfrak{o}^{\mathfrak{a}}(z') \lesssim w_\mu^{2-\mathfrak{b}}(z - z')\mathfrak{o}^{\mathfrak{a}}(z'), \tag{4.5}$$

$$\begin{aligned} w_\mu^{\mathfrak{a}}(z, z_1, \dots, z_{k(\mathfrak{a})}) \lesssim\ & w_\mu^{\mathfrak{b}}(z - z') w_\mu^{\mathfrak{b}}(z_1 - z_1') \dots w_\mu^{\mathfrak{b}}(z_{k(\mathfrak{a})} - z_{k(\mathfrak{a})}') \\ & \times w_\mu^{\mathfrak{a}}(z', z_1', \dots, z_{k(\mathfrak{a})}'), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathfrak{o}^{\mathfrak{a}}(z)\tilde{w}_\mu^{\mathfrak{a}}(z, z_1, \dots, z_{k(\mathfrak{a})}) \lesssim\ & w_\mu^{2}(z - z') w_\mu^{\mathfrak{b}}(z_1 - z_1') \dots w_\mu^{\mathfrak{b}}(z_{k(\mathfrak{a})} - z_{k(\mathfrak{a})}') \\ & \times \mathfrak{o}^{\mathfrak{a}}(z')\tilde{w}_\mu^{\mathfrak{a}}(z', z_1', \dots, z_{k(\mathfrak{a})}'), \end{aligned} \tag{4.7}$$

hold uniformly in $\mu \in (0, 1)$ and $z, z_1, \dots, z_{k(\mathfrak{a})}, z', z_1', \dots, z_{k(\mathfrak{a})}' \in \Lambda_0$.

Proof. To prove the claim we apply iteratively the inequalities from Remark 4.2 and use the bounds

$$1 + \sum_{i\in\{1,\dots,n\}} a_i \leqslant \prod_{i\in\{1,\dots,n\}} (1+a_i), \qquad (1+a_i)^{\omega} \leqslant (1+a_i)^{\bar\omega},$$

valid for all $a_1,\dots,a_n \geqslant 0$ and $0 \leqslant \omega \leqslant \bar\omega$. To prove (4.5) and (4.6), we use furthermore that

$$(1+\ell(\mathfrak{a}))\kappa_{\mathfrak{o}} \leqslant (2\bar\ell+2)\kappa_{\mathfrak{o}} \leqslant 2-\mathfrak{b},$$

which follows from the assumption $\kappa_{\mathfrak{o}} \in (0,(2-\mathfrak{b})/(2\bar\ell+2))$. ∎

Definition 4.5. For $n,m \in \mathbb{N}_0$ and $\sigma,\eta \in (1/2,1)$ we set

$$K_\sigma^{n,m} := 1^{\otimes n} \otimes (K_\sigma)^{\otimes m}, \qquad L_\sigma^{n,m} := 1^{\otimes n} \otimes (L_\sigma)^{\otimes m}, \qquad K_{\eta,\sigma}^{n,m} := L_\sigma^{n,m} K_\eta^{n,m},$$

$$\tilde K_\sigma^{n,m} := K_\sigma^{\otimes n} \otimes (K_\sigma^2)^{\otimes m}, \qquad \tilde L_\sigma^{n,m} := L_\sigma^{\otimes n} \otimes (L_\sigma^2)^{\otimes m}, \qquad \tilde K_{\eta,\sigma}^{n,m} := \tilde L_\sigma^{n,m} \tilde K_\eta^{n,m},$$

where $L_\sigma, K_\sigma, K_{\eta,\sigma}$ are introduced in Definition 1.13. Given $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$ and $\mu \in (1/2,1)$, we write

$$\tilde K_\mu^{\mathfrak{a}} := \tilde K_\mu^{1,k(\mathfrak{a})} = K_\mu^{\otimes(1+k(\mathfrak{a}))} K_\mu^{1,k(\mathfrak{a})}.$$

Definition 4.6. The norm of a kernel F associated with an operator $\mathcal{E}^{\otimes k} \to \mathcal{E}$ is defined by

$$\|F\| := \sup_{z\in\Lambda} \int_{\Lambda^k} |F(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k)|. \tag{4.8}$$

Given weights $\mathfrak{o} \in C(\Lambda_0)$ and $w \in C(\Lambda_0^{1+k})$, we write

$$(\mathfrak{o}\cdot F\cdot w)(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k) := \mathfrak{o}(z) F(z;\mathrm{d}z_1,\cdots,\mathrm{d}z_k) w(z,z_1,\cdots,z_k).$$

Let $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$. The norm of a kernel $F^{\mathfrak{a}}$ of an operator $\mathcal{E}^{\otimes k(\mathfrak{a})} \to \hat{\mathcal{E}}$ at scale $\sigma \in (1/2,1)$ is defined by

$$\|F^{\mathfrak{a}}\|_\sigma := \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K_\sigma^{\mathfrak{a}} F^{\mathfrak{a}}] \cdot \tilde w_\sigma^{\mathfrak{a}}\|,$$

where $\tilde K_\sigma^{\mathfrak{a}} F^{\mathfrak{a}}$ denotes the convolution of $F^{\mathfrak{a}}$, viewed as an element of $\mathcal{S}'(\Lambda^{1+k(\mathfrak{a})})$, with the kernel of $\tilde K_\sigma^{\mathfrak{a}}$. We further introduce a norm for the family of effective force kernels

$$F^{\mathfrak{A}} := (F_\sigma^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A},\sigma\in(1/2,1)},$$

defined by

$$\|F^{\mathfrak{A}}\| := \left(\sup_{\mathfrak{a}\in\mathfrak{A}} \sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\mathfrak{a}]} \|F_\sigma^{\mathfrak{a}}\|_\sigma \right) \vee \left(\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{d/2+s+2\kappa} \|F_\sigma^{[0],(0)}\|_\sigma \right), \tag{4.9}$$

where $[\mathfrak{a}]$ is defined by (4.2).

Remark 4.7. Inspecting the parameter choices discussed in Sections 4.4 and 4.9, and using (4.2), shows that $[(0,0)] < -d/2 - s - 2\kappa$. Consequently, the second term on the right-hand side of (4.9) provides a stronger bound on $F_\sigma^{[0],(0)} = \xi^{(\varepsilon,M)}$ than the first term. For technical reasons, this stronger control will be required in Section 4.9 to establish the estimates stated in (2.19).

Remark 4.8. By Young's inequality, (4.7) and Lemma 1.17, we have

$$\begin{aligned}\|F^{\mathfrak{a}}\|_\mu &= \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K^{1,k(\mathfrak{a})}_\mu F^{\mathfrak{a}}] \cdot \tilde w^{\mathfrak{a}}_\mu\| \\ &= \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K^{1,k(\mathfrak{a})}_{\mu,\sigma} \tilde K^{1,k(\mathfrak{a})}_\sigma F^{\mathfrak{a}}] \cdot \tilde w^{\mathfrak{a}}_\mu\| \\ &\leqslant \|K_{\mu,\sigma}\|_{\mathrm{TV}(w^2_\mu)} \|K^2_{\mu,\sigma}\|^{k(\mathfrak{a})}_{\mathrm{TV}(w^{\mathfrak{b}}_\mu)} \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K^{1,k(\mathfrak{a})}_\sigma F^{\mathfrak{a}}] \cdot \tilde w^{\mathfrak{a}}_\mu\| \\ &\lesssim \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K^{1,k(\mathfrak{a})}_\sigma F^{\mathfrak{a}}] \cdot \tilde w^{\mathfrak{a}}_\sigma\| \\ &= \|F^{\mathfrak{a}}\|_\sigma,\end{aligned} \tag{4.10}$$

uniformly in $\mu \in [1/2, 1)$ and $\sigma \in [\mu, 1)$.

Given kernels F' and F'' of operators $\mathcal{E}^{\otimes k'} \to \mathcal{E}$ and $\mathcal{E}^{\otimes k''} \to \mathcal{E}$ respectively, we denote by

$$F = \mathfrak{C}(\dot G_\sigma)(F' \otimes F'') \tag{4.11}$$

the kernel of the operator $\mathcal{E}^{\otimes k} \to \mathcal{E}$, with $k = k' + k'' - 1$, defined by

$$\begin{aligned}F(z; \mathrm{d}z_1, \cdots, \mathrm{d}z_k) := \frac{1}{k!} \sum_{\pi \in \mathcal{P}_k} \int_{\Lambda^2} F'(z; \mathrm{d}z_{\pi(1)}, \cdots, \mathrm{d}z_{\pi(k'-1)}, \mathrm{d}z') \\ \times \dot G_\sigma(z' - z'') F''(z''; \mathrm{d}z_{\pi(k')}, \cdots, \mathrm{d}z_{\pi(k)}) \mathrm{d}z' \mathrm{d}z'',\end{aligned} \tag{4.12}$$

where $\mathcal{P}_k$ denotes the set of permutations of $\{1, \dots, k\}$ and the integral is over the variables z' and z''. Note that $\mathfrak{C}(\dot G_\sigma)$ applies $\dot G_\sigma$ to the output variable of the kernel F'' and plugs the result into the last input variable of the kernel F'. The above definition extends naturally to kernels of operators $\mathcal{E}^{\otimes k'} \to \hat{\mathcal{E}}$ and $\mathcal{E}^{\otimes k''} \to \hat{\mathcal{E}}$.

The flow equation (4.1) for the effective force can be rewritten as the following flow equation for the kernels

$$\partial_\sigma F^{\mathfrak{a}}_\sigma = -\sum_{\mathfrak{b},\mathfrak{c}} B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma), \qquad F^{\mathfrak{a}}_1 = F^{\mathfrak{a}}, \tag{4.13}$$

where the operators $B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}$ are implicitly defined by

$$\sum_{\mathfrak{b},\mathfrak{c}} B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma) := \sum_{\ell'=0}^{\ell(\mathfrak{a})-1} \sum_{k'=0}^{k(\mathfrak{a})} (k'+1) \mathfrak{C}(\dot G_\sigma)(F^{[\ell(\mathfrak{a})-1-\ell'],(k'+1)}_\sigma \otimes F^{[\ell'],(k(\mathfrak{a})-k')}_\sigma), \tag{4.14}$$

and in particular $B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}} = 0$ unless

$$\begin{cases} \ell(\mathfrak{a}) = \ell(\mathfrak{b}) + \ell(\mathfrak{c}) + 1, \\ k(\mathfrak{a}) = k(\mathfrak{b}) + k(\mathfrak{c}) - 1, \\ -[\mathfrak{a}] + [\mathfrak{b}] + [\mathfrak{c}] = -\alpha + \beta - \delta. \end{cases} \tag{4.15}$$

Lemma 4.9. *For all $\mathfrak{a} \in \mathfrak{A}_{\mathrm{ext}}$ and $\mathfrak{b}, \mathfrak{c} \in \mathfrak{A}$, the following bound*

$$[\![\sigma]\!]^{-[\mathfrak{a}]} \|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)\|_\sigma \lesssim [\![\sigma]\!]^{-[\mathfrak{b}]-[\mathfrak{c}]-1} \|F^{\mathfrak{b}}_\sigma\|_\sigma \|F^{\mathfrak{c}}_\sigma\|_\sigma$$

holds uniformly in $\sigma \in [1/2, 1)$*, provided*

$$2s - \alpha + \beta - \delta \geqslant 0. \tag{4.16}$$

Proof. By Definitions 4.6 and 4.5, together with (4.14), (4.12) and the identity $L_\sigma K_\sigma = 1$, we obtain

$$\begin{aligned}
\|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot{G}_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)\|_\sigma &= \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\sigma B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot{G}_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)] \cdot \tilde{w}^{\mathfrak{a}}_\sigma\| \\
&= \|\mathfrak{o}^{\mathfrak{a}} \cdot [B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(L^3_\sigma \dot{G}_\sigma, \tilde{K}^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma, \tilde{K}^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma)] \cdot \tilde{w}^{\mathfrak{a}}_\sigma\| \\
&\leqslant \|\mathfrak{o}^{\mathfrak{a}} \cdot [B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(|L^3_\sigma \dot{G}_\sigma|, |\tilde{K}^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma|, |\tilde{K}^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma|)] \cdot \tilde{w}^{\mathfrak{a}}_\sigma\|.
\end{aligned}$$

Hence, by (4.4), (4.14), (4.12), Definition 4.6 and Lemma 1.17, we arrive at

$$\begin{aligned}
\|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot{G}_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)\|_\sigma &\lesssim \|[B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(|L^3_\sigma \dot{G}_\sigma| \cdot w^{\flat}_\sigma, \mathfrak{o}^{\mathfrak{b}} \cdot |\tilde{K}^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma| \cdot \tilde{w}^{\mathfrak{b}}_\sigma, \mathfrak{o}^{\mathfrak{c}} \cdot |\tilde{K}^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma| \cdot \tilde{w}^{\mathfrak{c}}_\sigma)]\| \\
&\lesssim \|L^3_\sigma \dot{G}_\sigma\|_{\mathrm{TV}(w^{\flat}_\sigma)} \|\mathfrak{o}^{\mathfrak{b}} \cdot |\tilde{K}^{\mathfrak{b}}_\sigma F^{\mathfrak{b}}_\sigma| \cdot \tilde{w}^{\mathfrak{b}}_\sigma\| \|\mathfrak{o}^{\mathfrak{c}} \cdot |\tilde{K}^{\mathfrak{c}}_\sigma F^{\mathfrak{c}}_\sigma| \cdot \tilde{w}^{\mathfrak{c}}_\sigma\| \\
&\lesssim \|L^3_\sigma \dot{G}_\sigma\|_{\mathrm{TV}(w^{\flat}_\sigma)} \|F^{\mathfrak{b}}_\sigma\|_\sigma \|F^{\mathfrak{c}}_\sigma\|_\sigma \\
&\lesssim [\![\sigma]\!]^{2s-1} \|F^{\mathfrak{b}}_\sigma\|_\sigma \|F^{\mathfrak{c}}_\sigma\|_\sigma.
\end{aligned}$$

Moreover, combining (4.15) and (4.16) yields

$$-[\mathfrak{a}] + 2s - 1 = -[\mathfrak{b}] - [\mathfrak{c}] - \alpha + \beta - \delta + 2s - 1 \geqslant -[\mathfrak{b}] - [\mathfrak{c}] - 1.$$

This finishes the proof. ∎

4.3. Norms for cumulants

In order to prove probabilistic bounds for the moments of the norm $\|F^{\mathfrak{A}}\|$, we start by establishing bounds for the cumulants of the family $(F^{\mathfrak{a}}_\sigma)_{\mathfrak{a}\in\mathfrak{A}, \sigma\in[1/2,1]}$ of the effective force kernels. To this end, let us first introduce a useful notation. For

$$\boldsymbol{a} \in \boldsymbol{A} := \{(\mathfrak{a}_1, \dots, \mathfrak{a}_n) | \mathfrak{a}_k \in \mathfrak{A}, L(\boldsymbol{a}) \leqslant 2\bar{\ell}\},$$

we write

$$n(\boldsymbol{a}) := n, \qquad L(\boldsymbol{a}) := \ell(\mathfrak{a}_1) + \cdots + \ell(\mathfrak{a}_n), \qquad K(\boldsymbol{a}) := k(\mathfrak{a}_1) + \cdots + k(\mathfrak{a}_n).$$

For random variables $X_1, \dots, X_n$, we denote by $\mathfrak{K}_n(X_1, \dots, X_n)$ their n-th order joint cumulant. For $\boldsymbol{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n) \in \boldsymbol{A}$ and $\sigma \in [1/2, 1]$, we use the following shorthand notation

$$\mathcal{F}^{\boldsymbol{a}}_\sigma(z_1, \mathrm{d}Z_1, \dots, z_n, \mathrm{d}Z_n) := \mathfrak{K}_n(F^{\mathfrak{a}_1}_\sigma(z_1, \mathrm{d}Z_1), \cdots, F^{\mathfrak{a}_n}_\sigma(z_n, \mathrm{d}Z_n)),$$

where

$$Z_i = (z_{i,1}, \dots, z_{i,k(\mathfrak{a}_i)}) \in \Lambda^{k(\mathfrak{a}_i)}, \qquad i \in \{1, \dots n\}, \quad n = n(\boldsymbol{a}), \tag{4.17}$$

are the input variables of the i-th effective force kernel involved in the joint cumulant. Since $F^{\mathfrak{a}_i}_\sigma$ is a kernel of an operator $\mathcal{E}^{\otimes k(\mathfrak{a}_i)} \to \hat{\mathcal{E}}$, it is natural to view $\mathcal{F}^{\boldsymbol{a}}_\sigma$ as a kernel of an operator

$$\mathcal{E}^{\otimes K(\boldsymbol{a})} \to \hat{\mathcal{E}}^{\otimes n(\boldsymbol{a})}. \tag{4.18}$$

We define the global homogeneity of the kernel $\mathcal{F}^{\boldsymbol{a}}_\sigma$ as

$$[\boldsymbol{a}] := -\varrho + n(\boldsymbol{a})(\theta + \alpha) + [\mathfrak{a}_1] + \cdots + [\mathfrak{a}_n],$$

for suitable parameters ϱ and θ whose values will be fixed in Section 4.4. By (4.2), we have

$$[\boldsymbol{a}] = -\varrho + \theta n(\boldsymbol{a}) + \delta L(\boldsymbol{a}) + \beta K(\boldsymbol{a}). \tag{4.19}$$

We say that a cumulant $\mathcal{F}^{\boldsymbol{a}}$ is *relevant* if $[\boldsymbol{a}] < 0$, *marginal* if $[\boldsymbol{a}] = 0$ and *irrelevant* if $[\boldsymbol{a}] > 0$.

Definition 4.10. For $\boldsymbol{a} \in \boldsymbol{A}$ and $\sigma, \mu \in (0,1)$ we define

$$w^{\boldsymbol{a}}_\mu := \bigotimes_{i=1}^{n(\boldsymbol{a})} w^{\mathfrak{a}_i}_\mu, \qquad K^{\boldsymbol{a}}_\mu := \bigotimes_{i=1}^{n(\boldsymbol{a})} K^{1,k(\mathfrak{a}_i)}_\mu, \qquad L^{\boldsymbol{a}}_\sigma := \bigotimes_{i=1}^{n(\boldsymbol{a})} L^{1,k(\mathfrak{a}_i)}_\sigma, \qquad K^{\boldsymbol{a}}_{\eta,\sigma} := L^{\boldsymbol{a}}_\sigma K^{\boldsymbol{a}}_\eta,$$

where $w^{\mathfrak{a}_i}_\mu$ is introduced in Definition 4.3, and $K^{1,k}_\mu$ and $L^{1,k}_\sigma$ are introduced in Definition 4.5

The space of cumulants is endowed with the norm $|||\bullet|||$ defined by taking the L^∞ norm on the first output variable and the total variation norm in all other variables, with the output variables restricted to the first period in space. The restriction of the integration of the output variables to the first period is natural since cumulants are kernels of operators acting between spaces of periodic functions, cf. (4.18).

Definition 4.11. Let $\boldsymbol{a} \in \boldsymbol{A}$, $n = n(\boldsymbol{a})$ and $k = K(\boldsymbol{a})$. The norm of a kernel $\mathcal{F}^{\boldsymbol{a}}$ associated with an operator $\mathcal{E}^{\otimes k} \to \hat{\mathcal{E}}^{\otimes n}$ is defined by

$$|||\mathcal{F}^{\boldsymbol{a}}||| := \sup_{z_1 \in \Lambda} \int_{\Lambda^{n-1}_M} \left(\int_{\Lambda^k} |\mathcal{F}^{\boldsymbol{a}}(z_1, \mathrm{d}Z_1, \ldots, z_n, \mathrm{d}Z_n)| \right) \mathrm{d}z_2 \cdots \mathrm{d}z_n, \tag{4.20}$$

with the notation as in (4.17). Given a weight $w \in C(\Lambda^{n+k}_0)$, we write

$$(\mathcal{F} \cdot w)(z_1, \mathrm{d}Z_1, \ldots, z_n, \mathrm{d}Z_n) := \mathcal{F}(z_1, \mathrm{d}Z_1, \ldots, z_n, \mathrm{d}Z_n) w(z_1, Z_1, \cdots, z_n, Z_n).$$

The norm of $\mathcal{F}^{\boldsymbol{a}}$ at scale $\sigma \in (1/2, 1)$ is defined by

$$|||\mathcal{F}^{\boldsymbol{a}}|||_\sigma = |||[K^{\boldsymbol{a}}_\sigma \mathcal{F}^{\boldsymbol{a}}] \cdot w^{\boldsymbol{a}}_\sigma|||, \tag{4.21}$$

where $K^{\boldsymbol{a}}_\sigma \mathcal{F}^{\boldsymbol{a}}$ denotes the convolution of $\mathcal{F}^{\boldsymbol{a}}$, viewed as an element of $\mathcal{S}'(\Lambda^{n(\boldsymbol{a})+K(\boldsymbol{a})})$, with the kernel of $K^{\boldsymbol{a}}_\sigma$. We further introduce a norm for the family of cumulants of effective force kernels

$$\mathcal{F}^{\boldsymbol{A}} := (\mathcal{F}^{\boldsymbol{a}}_\sigma)_{\boldsymbol{a} \in \boldsymbol{A}, \sigma \in (1/2,1)}$$

defined by

$$|||\mathcal{F}^{\boldsymbol{A}}||| := \sup_{\boldsymbol{a}\in\boldsymbol{A}}\Big(\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]}|||\mathcal{F}^{\boldsymbol{a}}_{\sigma}|||_{\sigma}\Big)^{1/n(\boldsymbol{a})}. \tag{4.22}$$

Here $\boldsymbol{A}$ denotes the set of all finite tuples satisfying $L(\boldsymbol{a}) \le 2\bar{\ell}$. Although this set is infinite, the induction below shows that, at each level $L \le 2\bar{\ell}$, only finitely many cumulants are nonzero. All suprema below may therefore be restricted to this finite set.

Remark 4.12. Analogously to Remark 4.8, by $K^{\boldsymbol{a}}_{\mu} = K^{\boldsymbol{a}}_{\mu,\sigma}K^{\boldsymbol{a}}_{\sigma}$, Young's inequality, (4.6) and Lemma 1.17, we obtain

$$\begin{aligned}
|||\mathcal{F}^{\boldsymbol{a}}|||_{\mu} = |||[K^{\boldsymbol{a}}_{\mu}\mathcal{F}^{\boldsymbol{a}}]\cdot w^{\boldsymbol{a}}_{\mu}||| &= |||[K^{\boldsymbol{a}}_{\mu,\sigma}K^{\boldsymbol{a}}_{\sigma}\mathcal{F}^{\boldsymbol{a}}]\cdot w^{\boldsymbol{a}}_{\mu}||| \\
&\lesssim \|K_{\mu,\sigma}\|^{K(\boldsymbol{a})}_{\mathrm{TV}(w^{\flat}_{\mu})}|||[K^{\boldsymbol{a}}_{\sigma}\mathcal{F}^{\boldsymbol{a}}]\cdot w^{\boldsymbol{a}}_{\mu}||| \\
&\lesssim |||[K^{\boldsymbol{a}}_{\sigma}\mathcal{F}^{\boldsymbol{a}}]\cdot w^{\boldsymbol{a}}_{\sigma}||| \\
&= |||\mathcal{F}^{\boldsymbol{a}}|||_{\sigma},
\end{aligned}$$

uniformly in $\mu \in [1/2, 1)$ and $\sigma \in [\mu, 1)$.

In Section 4.8, we will pass from estimates on the norm $|||\mathcal{F}^{\boldsymbol{A}}|||$ for the family of cumulants to estimates on the norm $\|F^{\mathfrak{N}}\|$ for the corresponding family of kernels, using a Kolmogorov-type argument. Following [11, 13], we introduce a flow equation for cumulants, which will allow us to control the norm $|||\mathcal{F}^{\boldsymbol{A}}|||$.

Lemma 4.13. *The cumulants satisfy the following flow equation:*

$$\partial_{\sigma}\mathcal{F}^{\boldsymbol{a}}_{\sigma} = -\sum_{\boldsymbol{b}}\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_{\sigma}, \mathcal{F}^{\boldsymbol{b}}_{\sigma}) - \sum_{\boldsymbol{b},\boldsymbol{c}}\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_{\sigma}, \mathcal{F}^{\boldsymbol{b}}_{\sigma}, \mathcal{F}^{\boldsymbol{c}}_{\sigma}), \tag{4.23}$$

where the multilinear operators $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$ are defined in Appendix B.1. We have $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}} = 0$ unless

$$\begin{cases} n(\boldsymbol{a}) = n(\boldsymbol{b}) - 1, \\ L(\boldsymbol{a}) = L(\boldsymbol{b}) + 1, \\ K(\boldsymbol{a}) = K(\boldsymbol{b}) - 1, \\ [\boldsymbol{a}] = [\boldsymbol{b}] - \theta + \delta - \beta, \end{cases} \tag{4.24}$$

and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}} = 0$ unless

$$\begin{cases} n(\boldsymbol{a}) = n(\boldsymbol{b}) + n(\boldsymbol{c}) - 1, \\ L(\boldsymbol{a}) = L(\boldsymbol{b}) + L(\boldsymbol{c}) + 1, \\ K(\boldsymbol{a}) = K(\boldsymbol{b}) + K(\boldsymbol{c}) - 1, \\ [\boldsymbol{a}] = [\boldsymbol{b}] + [\boldsymbol{c}] + \varrho - \theta + \delta - \beta. \end{cases} \tag{4.25}$$

Moreover, the operators $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$ satisfy the following bounds:

$$\begin{aligned}
[\![\sigma]\!]^{-[\boldsymbol{a}]}|||\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_{\sigma}, \mathcal{F}^{\boldsymbol{b}}_{\sigma})|||_{\sigma} &\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1}|||\mathcal{F}^{\boldsymbol{b}}|||_{\sigma}, \\
[\![\sigma]\!]^{-[\boldsymbol{a}]}|||\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_{\sigma}, \mathcal{F}^{\boldsymbol{b}}_{\sigma}, \mathcal{F}^{\boldsymbol{c}}_{\sigma})|||_{\sigma} &\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}|||\mathcal{F}^{\boldsymbol{b}}_{\sigma}|||_{\sigma}|||\mathcal{F}^{\boldsymbol{c}}_{\sigma}|||_{\sigma},
\end{aligned}$$

provided the following compatibility conditions hold:

$$\theta + \beta - \delta - d \geqslant 0, \qquad -\varrho + \theta + \beta - \delta + 2s \geqslant 0. \tag{4.26}$$

Proof. The derivation of the flow equation is a direct consequence of the definition of cumulants, see [11,13]. The explicit form of the operators is not essential for the subsequent discussion and is provided in Appendix B.1, where the stated bounds are also proved (see Lemma B.2 and Lemma B.3). ∎

This general structure of the flow equation (4.23) allows us to propagate estimates for the cumulants of the form

$$\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]} |\!|\!| \mathcal{F}^{\boldsymbol{a}}_\sigma |\!|\!|_\sigma \lesssim 1.$$

However, depending on the sign of $[\boldsymbol{a}]$, we shall handle the cumulants differently: in particular, for $[\boldsymbol{a}] > 0$, namely for irrelevant cumulants, the flow equation can be solved *backward* starting from the final condition at $\sigma = 1$. On the other hand, this approach does not work for cumulants for which $[\boldsymbol{a}] < 0$ as in this case the flow equation cannot be integrated close to $\sigma = 1$. As we shall see in Section 4.6, we will solve the flow equation for this class of cumulants, called *relevant cumulants*, by integrating it *forward*.

Remark 4.14. Before proceeding with the analysis of the flow equations for cumulants, we record a few remarks about symmetries. First, we observe that the SPDE under consideration, namely (1.4), is invariant under the transformation

$$\phi \mapsto -\phi, \qquad \xi \mapsto -\xi,$$

which also preserves the law of the noise $\xi = \xi^{(\varepsilon,M)}$. By the discussion in Section 4.4, this invariance implies that $\mathcal{F}^{\boldsymbol{a}}_\sigma = 0$ for all $\sigma \in [1/2, 1]$ and $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = 0$ and

$$n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

To extend this property to higher levels $L(\boldsymbol{a}) > 0$, we exploit the fact that it is preserved by the flow equation (4.23) thanks to the compatibility conditions (4.24) and (4.25). Indeed, fix an arbitrary $\ell \in \mathbb{N}_+$ and assume inductively that

$$\mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \forall \sigma \in [1/2, 1], \boldsymbol{a} \in \boldsymbol{A} \quad \text{such that} \quad L(\boldsymbol{a}) < \ell \quad \text{and} \quad n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

Then from the flow equation (4.23) and the conditions (4.24) and (4.25), it follows that

$$\partial_\sigma \mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \quad \forall \sigma \in [1/2, 1), \boldsymbol{a} \in \boldsymbol{A} \quad \text{such that} \quad L(\boldsymbol{a}) = \ell \quad \text{and} \quad n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

Since by construction $\mathcal{F}^{\boldsymbol{a}}_{\sigma=1} = 0$ for all $\boldsymbol{a} \in \boldsymbol{A}$ such that $n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1$, we conclude that

$$\mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \forall \sigma \in [1/2, 1], \boldsymbol{a} \in \boldsymbol{A} \quad \text{such that} \quad n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N}_0 + 1.$$

A further symmetry is given by spatial reflection, that is, the transformation

$$\phi(t,x) \mapsto \phi(t,-x), \qquad \xi(t,x) \mapsto \xi(t,-x),$$

which also leaves the law of ξ invariant. By an argument analogous to the one above, we infer that for any $\boldsymbol{a} \in \boldsymbol{A}$, the cumulant $\mathcal{F}^{\boldsymbol{a}}_{\sigma}$ is symmetric under spatial reflections. Finally, the law of the noise and the deterministic kernels in the final condition are invariant under spacetime translations, and the operators in the cumulant flow equation (4.23) preserve this invariance; see also the explicit formulas (B.1) and (B.2). Consequently, all cumulants are invariant under simultaneous spacetime translations of their output and input variables. In particular, if $n(\boldsymbol{a}) = K(\boldsymbol{a}) = 1$, then

$$z \longmapsto \int_{\Lambda} \mathcal{F}^{\boldsymbol{a}}_{\sigma}(z, \mathrm{d}z_1) \quad \text{is constant.} \tag{4.27}$$

4.4. Bounds on parameters

We shall now fix the parameters

$$\beta, \theta, \varrho, \alpha, \delta, \mathfrak{b}$$

introduced in the analysis of the kernels. We have to choose these parameters so that (4.26) is satisfied. Another constraint comes from the requirement that $\|\!|\mathcal{F}^{\boldsymbol{a}}_{\sigma}|\!|_{\sigma} \lesssim [\![\sigma]\!]^{[\boldsymbol{a}]}$ for $\boldsymbol{a} \in \boldsymbol{A}$ with $L(\boldsymbol{a}) = 0$. Note that for $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = 0$, we have $\partial_{\sigma}\mathcal{F}^{\boldsymbol{a}}_{\sigma} = 0$ and $\mathcal{F}^{\boldsymbol{a}}_{\sigma} = \mathcal{F}^{\boldsymbol{a}}_{1}$. Using the equality

$$F^{[0]}_{\sigma}(\varphi) = F^{[0]}_{1}(\varphi) = -\lambda\varphi^3 + \bar{r}\varphi + \xi^{(\varepsilon,M)}$$

and the fact that the noise $\xi^{(\varepsilon,M)}$ is Gaussian, one shows that for $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = 0$ the cumulant $\mathcal{F}^{\boldsymbol{a}}_{\sigma}$ is nonzero only if:

(a) it is the covariance of the noise, that is $n(\boldsymbol{a}) = 2$, $K(\boldsymbol{a}) = 0$ and

$$\mathcal{F}^{\boldsymbol{a}}_{\sigma}(z,z') = \mathfrak{K}_2(F^{[0],(0)}(z), F^{[0],(0)}(z')) = \mathbb{E}[\xi^{(\varepsilon,M)}(z)\xi^{(\varepsilon,M)}(z')] = \delta^M(z-z'),$$

where δ^M is the periodisation in space of the Dirac delta with period M, or

(b) it is the expected value of the (deterministic) kernel $F^{[0],(3)}_{1}$, that is $n(\boldsymbol{a}) = 1$, $K(\boldsymbol{a}) = 3$ and

$$\mathcal{F}^{\boldsymbol{a}}_{\sigma}(z,z_1,z_2,z_3) = -\lambda\delta(z-z_1)\delta(z-z_2)\delta(z-z_3),$$

or

(c) it is the expected value of the (deterministic) kernel $F^{[0],(1)}_{1}$, that is $n(\boldsymbol{a}) = 1$, $K(\boldsymbol{a}) = 1$ and

$$\mathcal{F}^{\boldsymbol{a}}_{\sigma}(z,z_1) = \bar{r}\delta(z-z_1).$$

In case (a), using the fact that $w^{\boldsymbol{a}}_{\sigma} = 1$, we conclude that

$$\|\!|\mathcal{F}^{\boldsymbol{a}}_{1}|\!|_{\sigma} = \int_{\Lambda_M} \delta^M(\mathrm{d}z) = 1. \tag{4.28}$$

As a consequence, we have to require that

$$[\boldsymbol{a}] = -\varrho + 2\theta \leqslant 0. \tag{4.29}$$

In case (b), we observe that

$$(K^{\boldsymbol{a}}_\sigma \mathcal{F}^{\boldsymbol{a}}_\sigma)(z; z_1, z_2, z_3) = -\lambda \prod_{i=1}^{3} K_\sigma(z - z_i),$$

and thus $\|\mathcal{F}^{\boldsymbol{a}}_\sigma\|_\sigma \lesssim 1$ uniformly in $\sigma \in (1/2, 1)$ by Lemma 1.17. Consequently, we have to require that

$$[\boldsymbol{a}] = -\varrho + \theta + 3\beta \leqslant 0. \tag{4.30}$$

In case (c), we have

$$(K^{\boldsymbol{a}}_\sigma \mathcal{F}^{\boldsymbol{a}}_\sigma)(z; z_1) = \bar{r} K_\sigma(z - z_1),$$

and thus $\|\mathcal{F}^{\boldsymbol{a}}_\sigma\|_\sigma \lesssim 1$ uniformly in $\sigma \in (1/2, 1)$. Consequently, we have to require that

$$[\boldsymbol{a}] = -\varrho + \theta + \beta \leqslant 0.$$

Note that since $\beta > 0$ the last condition is implied by the bound (4.30).

We collect all constraints here and fix the parameters once and for all, so that the later arguments can refer directly to this table.

$[B]$	Flow kernels (4.16)	$2s - \alpha + \beta - \delta \geqslant 0$
$[\mathbb{A}]$	Flow cumulants $\mathcal{A}$ (4.26)	$\theta + \beta - \delta - d \geqslant 0$
$[\mathbb{B}]$	Flow cumulants $\mathcal{B}$ (4.26)	$2s - \varrho + \theta + \beta - \delta \geqslant 0$
$[\Xi]$	Initial condition $F^{[0],(0)}$ (4.29)	$-\varrho + 2\theta \leqslant 0$
$[\Phi^3]$	Initial condition $F^{[0],(3)}$ (4.30)	$-\varrho + \theta + 3\beta \leqslant 0$
$[K]$	Kolmogorov (4.60)	$\alpha - \frac{\varrho}{2} + \theta - \frac{d+2s}{2} - \kappa > 0$
$[\bar{F}]$	Kolmogorov $\bar{F}$ (4.63)	$\alpha - \varrho + \theta - \kappa \geqslant 0$

(4.31)

The parameter $\kappa > 0$ quantifies the loss of regularity in the Kolmogorov-type argument used to estimate the pathwise behaviour of the random kernels and will be chosen sufficiently small.

From the constraints $[\Phi^3]$ and $[\mathbb{B}]$ we have

$$\beta \leqslant s - \delta/2,$$

while from $[\Xi]$, $[\mathbb{B}]$ and this last inequality we deduce that

$$\theta \leqslant \beta - \delta + 2s \leqslant 3s - 3\delta/2.$$

Using now $[\mathbb{B}]$ again we have

$$\varrho \leqslant \theta + \beta - \delta + 2s \leqslant 6s - 3\delta.$$

And $[\mathbb{A}]$ now gives

$$0 \leqslant \theta + \beta - \delta - d \leqslant 4s - d - 3\delta = 3(\delta_\star - \delta), \tag{4.32}$$

where the strict positivity of

$$\delta_\star := \frac{4s - d}{3} > 0,$$

defines the subcritical regime of this model. We now choose the remaining parameters so as to saturate most of the inequalities above, namely:

$$\beta = s - \delta/2 - \kappa/2, \quad \theta = 3\beta, \quad \varrho = 2\theta, \quad \alpha = 3\beta + \kappa, \quad \delta = \delta_\star/2. \tag{4.33}$$

These choices are not intended to be optimal. Their purpose is to make the power counting transparent while leaving positive margins, proportional to $\delta_\star$ and κ, that can absorb the losses introduced by localisation and by the lattice-uniform Kolmogorov argument. By substituting these values into the inequalities (4.31) above, we check the following.

$$\begin{array}{lllll}
[B] & 0 \leqslant 2s - \alpha + \beta - \delta & = 2s - 2\beta - \kappa - \delta & = 0 \\
[\mathbb{A}] & 0 \leqslant \theta + \beta - \delta - d & = 4\beta - \delta - d & = \frac{3}{2}\delta_\star - 2\kappa \\
[\mathbb{B}] & 0 \leqslant -\varrho + \theta + \beta - \delta + 2s & = -2\beta - \delta + 2s & = \kappa \\
[\Xi] & 0 \geqslant -\varrho + 2\theta & = 0 & = 0 \\
[\Phi^3] & 0 \geqslant -\varrho + \theta + 3\beta & = 0 & = 0 \\
[K] & 0 < \alpha - \frac{\varrho}{2} + \theta - \frac{d+2s}{2} - \kappa & = 3\beta - \frac{d+2s}{2} & = \frac{3}{4}\delta_\star - \frac{3}{2}\kappa \\
[\bar{F}] & 0 \leqslant \alpha - \varrho + \theta - \kappa & = 0 & = 0
\end{array} \tag{4.34}$$

We used that

$$\frac{d + 2s}{2} = \frac{6s - 3\delta_\star}{2} = 3\beta - \frac{3}{2}(\delta_\star - \delta) + \frac{3}{2}\kappa. \tag{4.35}$$

All the inequalities are satisfied provided

$$\kappa \in (0, \delta_\star/2). \tag{4.36}$$

In addition, we choose κ so that

$$\frac{2\beta}{\delta} \notin \mathbb{N}_+. \tag{4.37}$$

Equivalently,

$$\delta\ell - 2\beta \neq 0 \qquad \text{for every } \ell \in \mathbb{N}_+.$$

Since $2\beta = 2s - \delta - \kappa$, the excluded values are

$$\kappa = 2s - \delta(\ell + 1), \qquad \ell \in \mathbb{N}_+.$$

They form a discrete set. Hence this condition is compatible with (4.36) and with choosing $\kappa > 0$ arbitrarily small subject to the remaining strict parameter inequalities. Moreover, in Lemma 4.18 below we use the fact that the parameter of the weight $\mathfrak{o}$ introduced in Definition 4.3 satisfies

$$\kappa_{\mathfrak{o}} < \delta \wedge (2 - \mathfrak{b})/(2\bar{\ell} + 2). \tag{4.38}$$

Finally, we explain how the value of the parameter $\flat$, introduced in Definition 4.3, is fixed. The role of this parameter is to compensate for the loss of spatial weight arising in the localisation procedure for the relevant cumulants discussed below, where the Taylor expansion allows us to gain in homogeneity and establish that only a local renormalisation is needed. In Lemma 4.15 and Lemma 4.18, we shall need $\flat > 2s - \delta$. On the other hand, as argued in Remark 1.18, we must also impose $\flat < 2s$. Thus, we fix $\flat$ such that

$$2s - \delta < \flat < 2s. \tag{4.39}$$

In Section 4.9, we shall introduce additional constraints on the parameters κ and κ_0, and we will fix all remaining parameters

$$\gamma, \vartheta, \bar{\kappa}, \bar{\ell}, \bar{k}, a, \nu, \kappa_0, \kappa,$$

which play a role in our analysis. We postpone this discussion because, apart from κ and κ_0, which must satisfy conditions (4.36) and (4.38), these parameters are only relevant for the results established in Section 4.9. We emphasise once again that all these parameters are to be regarded as fixed once and for all, and their choice depends solely on the exponent $s \in (3/4, 1)$ of the fractional Laplacian.

4.5. Classification of cumulants

Given the bounds on the parameters from the previous section, we can now examine the class of cumulants that are relevant or marginal, *i.e.*, those satisfying $[\boldsymbol{a}] \leqslant 0$. Recalling that

$$\varrho = 2\theta, \quad \text{and} \quad \theta = \varrho - 3\beta,$$

observe that the condition $[\boldsymbol{a}] \leqslant 0$ can be written as

$$[\boldsymbol{a}] = \theta(n(\boldsymbol{a}) - 2) + \beta K(\boldsymbol{a}) + \delta L(\boldsymbol{a}) = \beta(3n(\boldsymbol{a}) - 6 + K(\boldsymbol{a})) + \delta L(\boldsymbol{a}) \leqslant 0.$$

Then

(a) if $n(\boldsymbol{a}) > 2$, then all cumulants are irrelevant;

(b) if $n(\boldsymbol{a}) = 2$, then

$$[\boldsymbol{a}] = \beta K(\boldsymbol{a}) + \delta L(\boldsymbol{a}).$$

Hence the only non-positive case is $L(\boldsymbol{a}) = K(\boldsymbol{a}) = 0$, namely the covariance of the noise

$$\mathcal{F}^{\boldsymbol{a}}_\sigma = \mathfrak{K}_2(F^{[0],(0)}_\sigma, F^{[0],(0)}_\sigma).$$

This cumulant has homogeneity zero and is independent of σ.

(c) if $n(\boldsymbol{a}) = 1$, then

$$[\boldsymbol{a}] = \beta(K(\boldsymbol{a}) - 3) + \delta L(\boldsymbol{a}).$$

By Remark 4.14, the cumulants with $K(\boldsymbol{a}) = 0$ and $K(\boldsymbol{a}) = 2$ vanish. For $K(\boldsymbol{a}) = 3$, we have

$$[\boldsymbol{a}] = \delta L(\boldsymbol{a}).$$

Thus the only non-positive case is $L(\boldsymbol{a}) = 0$, namely the deterministic cubic kernel

$$\mathfrak{K}_1(F_\sigma^{[0],(3)}),$$

which is marginal and independent of σ. For $K(\boldsymbol{a}) = 1$, we have

$$[\boldsymbol{a}] = \delta L(\boldsymbol{a}) - 2\beta.$$

At $L(\boldsymbol{a}) = 0$, this is the deterministic linear kernel $\mathfrak{K}_1(F_\sigma^{[0],(1)}) = \bar{r}\,\delta$, which is relevant and independent of σ. At $L(\boldsymbol{a}) = \ell \geqslant 1$, these are the cumulants

$$\mathbb{E}F_\sigma^{[\ell],(1)}.$$

By (4.37), none of them is marginal.

Summarising, the only level-zero marginal cumulants are

$$\mathfrak{K}_2(F_\sigma^{[0],(0)}, F_\sigma^{[0],(0)}), \qquad \mathfrak{K}_1(F_\sigma^{[0],(3)}),$$

and the deterministic linear kernel

$$\mathfrak{K}_1(F_\sigma^{[0],(1)})$$

is relevant. All three have trivial flow. By (4.37), there are no marginal cumulants with $L(\boldsymbol{a}) > 0$, so the only positive-level cumulants that must be analysed in detail are

$$\mathfrak{K}_1(F_\sigma^{[\ell],(1)}) = \mathbb{E}F_\sigma^{[\ell],(1)}, \qquad \ell \in \{1, \dots, \hat{\ell}\}, \tag{4.40}$$

where

$$\hat{\ell} := \max\{\ell \in \mathbb{N}_+ : \delta\ell - 2\beta < 0\}.$$

By (4.37), we then have

$$\delta\hat{\ell} - 2\beta < 0 < \delta(\hat{\ell} + 1) - 2\beta.$$

Equivalently,

$$\hat{\ell} = \left\lfloor \frac{2\beta}{\delta} \right\rfloor.$$

Thus the positive-level relevant cumulants are exactly

$$\mathbb{E}F_\sigma^{[\ell],(1)}, \qquad 1 \leqslant \ell \leqslant \hat{\ell}.$$

Since the parameter $\hat{\ell}$ appears only in this and the following section, it is not included in the list (1.22). Note that $\bar{\ell}$ is chosen so that $\hat{\ell} \leq \bar{\ell}$.

Thus, the renormalisation problem is purely of mass type: the only relevant components are the one-point kernels that are linear in the field. All other positive-level cumulants are irrelevant and can be reconstructed by integrating the flow from their terminal condition.

4.6. Inductive procedure

The aim of this section is to derive bounds for the cumulants $\mathcal{F}^{\boldsymbol{a}}_\sigma$ using an induction on $L(\boldsymbol{a})$, based on the flow equation (4.23). More precisely, Lemma 4.13 shows that the flow equation defines a triangular system with respect to $L(\boldsymbol{a})$. For irrelevant cumulants, we have $\mathcal{F}^{\boldsymbol{a}}_{\sigma=1} = 0$, and the desired bounds follow by directly integrating the flow equation from $\sigma = 1$. For relevant cumulants $\mathcal{F}^{\boldsymbol{a}}_\sigma$, which for $L(\boldsymbol{a}) > 0$ coincide with (4.40), the bound for $\partial_\sigma \mathcal{F}^{\boldsymbol{a}}_\sigma$ is not integrable at $\sigma = 1$. To overcome this, we decompose relevant cumulants into a local relevant part and a nonlocal irrelevant part, using the procedure detailed in Appendix B.2. The irrelevant part can again be integrated from $\sigma = 1$ with zero boundary condition. For the local relevant part, we instead impose the boundary condition at $\sigma = 1/2$, which acts as the renormalisation condition fixing the mass counterterm $r_{\varepsilon,M}$. This allows the integration of the flow equation from $\sigma = 1/2$, thereby avoiding the non-integrable singularity at $\sigma = 1$.

Lemma 4.15. *For any $\bar{r} \in \mathbb{R}$ there exists a choice of constants $(r^{[\ell]}_{\varepsilon,M})_{\ell=1,\dots,\hat{\ell}}$ such that the solution*

$$F^{\mathfrak{A}} := (F^{\mathfrak{a}}_\sigma)_{\mathfrak{a}\in\mathfrak{A},\sigma\in(1/2,1)},$$

of the approximate flow equation with the final condition

$$\begin{aligned} F_1^{[0]}(\phi) &= -\lambda\phi^3 + \bar{r}\phi + \xi^{(\varepsilon,M)}, \\ F_1^{[\ell]}(\phi) &= r^{[\ell]}_{\varepsilon,M}\phi, && \ell \in \{1,\dots,\hat{\ell}\}, \\ F_1^{[\ell]}(\phi) &= 0, && \ell \in \{\hat{\ell}+1,\dots,\bar{\ell}\}, \end{aligned} \tag{4.41}$$

satisfies

$$\|\!|\mathcal{F}^{\boldsymbol{A}}|\!\| = \sup_{\boldsymbol{a}\in\boldsymbol{A}}\left[\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]}\|\!|\mathcal{F}^{\boldsymbol{a}}_\sigma|\!\|_\sigma\right]^{1/n(\boldsymbol{a})} \lesssim 1.$$

Remark 4.16. Note that

$$F(\phi) = \sum_{\ell=0}^{\bar{\ell}} F_1^{[\ell]}(\phi), \qquad r_{\varepsilon,M} = \bar{r} + \sum_{\ell=1}^{\hat{\ell}} r^{[\ell]}_{\varepsilon,M},$$

where $F = F^{(\varepsilon,M)}$ is the force (2.1) and $r_{\varepsilon,M}$ is the mass renormalisation, which appears in the expression (1.2) for the action. The parameter $\hat{\ell} \in \mathbb{N}_+$ was fixed at the end of Section 4.5, whereas the parameter $\bar{\ell} \geqslant \hat{\ell}$ will be fixed in Section 4.9.

Proof. For $\ell \in \{0,\dots,2\bar{\ell}\}$, define

$$M_\ell := 1 + \sup_{\boldsymbol{a}|L(\boldsymbol{a})\leqslant\ell}\left[\sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\boldsymbol{a}]}\|\!|\mathcal{F}^{\boldsymbol{a}}_\sigma|\!\|_\sigma\right]^{1/n(\boldsymbol{a})}.$$

We will prove by induction on ℓ that

$$M_\ell < \infty, \quad \text{and} \quad \mathcal{F}^{\boldsymbol{a}}_\sigma = 0, \qquad \text{for all } \boldsymbol{a} \text{ such that } L(\boldsymbol{a}) \leqslant \ell \text{ and } n(\boldsymbol{a}) > 2^{\ell+1}, \tag{4.42}$$

for any $\ell \in \{0, \dots, 2\bar{\ell}\}$.

Let us remark that the second part of the above statement implies that there are only finitely many nonzero cumulants $\mathcal{F}^{\boldsymbol{a}}_{\sigma}$ such that $L(\boldsymbol{a}) \leqslant \ell$. We first note that the case $\ell = 0$ was already discussed in Section 4.4, where it was proved that $M_0 < \infty$.

Let us now consider the induction step. Assume that the conditions expressed by (4.42) hold true at order $\ell - 1$, for a fixed $\ell \in \mathbb{N}_+$. We shall prove that then the same is true at order ℓ. The proof is based on the flow equation (4.23). We first note that, by Lemma 4.13, we have

$$\begin{aligned} \|\!| \mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) |\!\|_\sigma &\lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1} M_{\ell-1}^{n(\boldsymbol{b})}, \\ \|\!| \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) |\!\|_\sigma &\lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1} M_{\ell-1}^{n(\boldsymbol{b})} M_{\ell-1}^{n(\boldsymbol{c})}, \end{aligned}$$

for all $\boldsymbol{a} \in \boldsymbol{A}$ such that $L(\boldsymbol{a}) = \ell$. Recall that $n(\boldsymbol{a}) = n(\boldsymbol{b}) - 1$ and $L(\boldsymbol{b}) = L(\boldsymbol{a}) - 1 = \ell - 1$ in the first line and $n(\boldsymbol{a}) = n(\boldsymbol{b}) + n(\boldsymbol{c}) - 1$ and $L(\boldsymbol{b}) + L(\boldsymbol{c}) = L(\boldsymbol{a}) - 1 = \ell - 1$ in the second line above. Hence, using the induction hypothesis we conclude that

$$\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) = 0, \qquad \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) = 0,$$

if $n(\boldsymbol{a}) > 2^{\ell+1}$, and

$$\|\!| \mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) |\!\|_\sigma \lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1}, \qquad \|\!| \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) |\!\|_\sigma \lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1}. \tag{4.43}$$

In the rest of the proof of the induction step we treat separately two cases: $\mathcal{F}^{\boldsymbol{a}}_\sigma$ is irrelevant, *i.e.* $[\boldsymbol{a}] > 0$, or $\mathcal{F}^{\boldsymbol{a}}_\sigma$ is relevant, *i.e.*, $[\boldsymbol{a}] < 0$. Note that, by (4.37), there are no marginal cumulants at level $\ell > 0$.

If $\mathcal{F}^{\boldsymbol{a}}$ is irrelevant, *i.e.* $[\boldsymbol{a}] > 0$, then we can bound it by integrating the flow equation (4.23) *backward* from the final condition at $\sigma = 1$. We stress that for any irrelevant cumulant this final condition is vanishing, namely $\mathcal{F}^{\boldsymbol{a}}_1 = 0$. Indeed, for $1 \leqslant \ell \leqslant \hat{\ell}$ the only nonzero positive-level final kernel is

$$F_1^{[\ell],(1)}(z, z_1) = r^{[\ell]}_{\varepsilon,M} \delta(z, z_1),$$

and its one-point cumulant has homogeneity

$$\delta\ell - 2\beta < 0.$$

For $\ell > \hat{\ell}$, the final kernel vanishes by (4.41). Moreover, any cumulant of order at least two containing a deterministic positive-level final kernel vanishes. Therefore, for every irrelevant cumulant with $L(\boldsymbol{a}) > 0$, we have $\mathcal{F}^{\boldsymbol{a}}_1 = 0$. Using (4.23), Remark 4.12 and (4.43) we show easily that

$$\|\!| \mathcal{F}^{\boldsymbol{a}}_\mu |\!\|_\mu \lesssim \int_\mu^1 [\![\sigma]\!]^{[\boldsymbol{a}]-1} \mathrm{d}\sigma \lesssim [\![\mu]\!]^{[\boldsymbol{a}]}.$$

On the other hand, if $\mathcal{F}^{\boldsymbol{a}}$ is relevant, *i.e.* $[\boldsymbol{a}] < 0$, then on account of the discussion of Section 4.5 $\mathcal{F}^{\boldsymbol{a}}$ is of the form $\bar{F}^{[\ell],(1)}_\mu = \mathfrak{R}_1(F^{[\ell],(1)}_\mu)$ with $\ell \in \{1, \dots, \hat{\ell}\}$. The treatment of the relevant cumulant proceeds via a localisation procedure, which shows that only the local part of the cumulant requires renormalisation. However, in our fractional setting, where the

kernels exhibit only limited polynomial decay, this must be done with care. Specifically, we first introduce a preliminary truncation before performing a Taylor expansion to localise the fields. Consequently, to bound $\bar{F}_\mu^{[\ell],(1)}$, we shall use the following decomposition

$$\begin{aligned}\bar{F}_\mu^{[\ell],(1)} &= \bar{F}_1^{[\ell],(1)} - \int_\mu^1 [(1 - K_\sigma^{1,1})\dot{\bar{F}}_\sigma^{[\ell],(1)}]\,\mathrm{d}\sigma \\ &\quad - \int_\mu^1 [K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot(1 - h_\mu)\,\mathrm{d}\sigma \\ &\quad + \boldsymbol{L}\int_\mu^1 [K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot(1 - h_\mu)\,\mathrm{d}\sigma \\ &\quad - \boldsymbol{L}\int_\mu^1 [K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\,\mathrm{d}\sigma \\ &\quad - \boldsymbol{R}\int_\mu^1 [K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu \mathrm{d}\sigma.\end{aligned} \tag{4.44}$$

This decomposition implements the renormalisation procedure in stages. It separates the part not captured by the regularising kernel, the contribution away from the diagonal, the local Taylor component, the renormalised local coefficient, and the Taylor remainder near the diagonal. The nonlocal and remainder terms gain either a power of the ratio of scales or a power of the distance from the diagonal, while the purely local coefficient is fixed by the renormalisation condition. Here

$$\dot{\bar{F}}_\sigma^{[\ell],(1)} := \partial_\sigma \bar{F}_\sigma^{[\ell],(1)},$$

the operator $\boldsymbol{L}$ is a localisation operator and $\boldsymbol{R}$ is a remainder operator, both defined in Appendix B.2. The polynomial weight h_μ was introduced in Definition 4.3. Recall that, by Definition 4.5, in the present case $K_\sigma^{\boldsymbol{a}} = K_\sigma^{1,1} = 1 \otimes K_\sigma$. We impose the following renormalisation condition

$$\bar{F}_1^{[\ell],(1)} = \boldsymbol{L}\int_{1/2}^1 [K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\,\mathrm{d}\sigma. \tag{4.45}$$

Since

$$\bar{F}_1^{[\ell],(1)}(z, z_1) = r_{\varepsilon,M}^{[\ell]}\delta(z - z_1),$$

the above condition fixes the counterterm $r_{\varepsilon,M}^{[\ell]}$. The renormalisation condition is non-circular. Indeed, by the triangular structure of the cumulant flow,

$$\partial_\sigma \bar{F}_\sigma^{[\ell],(1)}$$

depends only on cumulants of total level $\ell - 1$. Hence, at the ℓ-th step of the induction, it depends only on the counterterms $r_{\varepsilon,M}^{[1]}, \ldots, r_{\varepsilon,M}^{[\ell-1]}$, which have already been fixed, and is independent of $r_{\varepsilon,M}^{[\ell]}$. Consequently, the right-hand side of the renormalisation condition is already known and uniquely determines the counterterm $r_{\varepsilon,M}^{[\ell]}$.

Note that the integral in (4.45) is finite for every $\varepsilon \in (0,1)$. Indeed, let $\boldsymbol{a}$ be the one-element tuple whose unique entry is $(\ell, 1)$. Applying $\boldsymbol{L}K_\sigma^{1,1}$ to the cumulant flow equation (4.23) and using Lemma B.7, we obtain

$$\boldsymbol{L}(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}) = -\boldsymbol{L}\left(K_\sigma^{1,1}\sum_{\boldsymbol{b}}\mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}})\right).$$

The kernel on the right-hand side is invariant under spatial reflections. Consequently, by (B.8), as in the proof of Lemma B.6, its local projection contains only the zeroth moment. In particular, writing $V_{\boldsymbol{b}}$ for the summands,

$$\left\|\boldsymbol{L}\sum_{\boldsymbol{b}}V_{\boldsymbol{b}}\right\| \leqslant \left\|\sum_{\boldsymbol{b}}V_{\boldsymbol{b}}\right\| \leqslant \sum_{\boldsymbol{b}}\|V_{\boldsymbol{b}}\|.$$

Since $n(\boldsymbol{a}) = 1$, the basic cumulant norm (4.20) coincides with the kernel norm (4.8). Combining the ordinary estimate (4.43), the unweighted estimate (B.5), and the induction hypothesis (4.42), we conclude that

$$\left\|\boldsymbol{L}(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)})\right\| \lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1} \wedge \varepsilon^{-\mathfrak{b}}[\![\sigma]\!]^{[\boldsymbol{a}]+\mathfrak{b}-1}. \tag{4.46}$$

Notice that (4.46) is asserted only for the local projection. Such an estimate may generally be false for the full kernel $K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}$, because its $\mathcal{B}$-contribution need not have a sub-lattice total variation gain. Consequently, the renormalisation condition (4.45) and (4.46) yield

$$\begin{aligned}
|r_{\varepsilon,M}^{[\ell]}| &\leqslant \int_{1/2}^{1}\left\|\boldsymbol{L}(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)})\right\|\mathrm{d}\sigma \\
&\lesssim \int_0^{\varepsilon}\varepsilon^{-\mathfrak{b}}[\![\sigma]\!]^{[\boldsymbol{a}]+\mathfrak{b}-1}\,\mathrm{d}[\![\sigma]\!] + \int_{\varepsilon}^{1/2}[\![\sigma]\!]^{[\boldsymbol{a}]-1}\,\mathrm{d}[\![\sigma]\!] \\
&\lesssim \varepsilon^{[\boldsymbol{a}]},
\end{aligned}$$

where we used

$$[\boldsymbol{a}] + \mathfrak{b} \geqslant \mathfrak{b} - \varrho + \theta + \delta + \beta = \mathfrak{b} - 2\beta + \delta = \mathfrak{b} - 2s + 2\delta + \kappa > 0 \tag{4.47}$$

by (4.19), (4.33) and (4.39).

Taking into account the renormalisation condition (4.45), we obtain

$$\begin{aligned}
|\!|\!|\bar{F}_\mu^{[\ell],(1)}|\!|\!|_\mu \leqslant{}& \int_\mu^1 |\!|\!|(1-K_\sigma^{1,1})\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!|\!|_\mu\mathrm{d}\sigma + \int_\mu^1 |\!|\!|[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot(1-h_\mu)|\!|\!|_\mu\mathrm{d}\sigma \\
&+ \int_\mu^1 |\!|\!|\boldsymbol{L}([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot(1-h_\mu))|\!|\!|_\mu\mathrm{d}\sigma + \int_{1/2}^{\mu} |\!|\!|\boldsymbol{L}(K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)})|\!|\!|_\mu\mathrm{d}\sigma \\
&+ \int_\mu^1 |\!|\!|\boldsymbol{R}([K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu)|\!|\!|_\mu\mathrm{d}\sigma.
\end{aligned}$$

Finally, applying Lemmas A.16, A.21, B.6 and B.8, we arrive at

$$\begin{aligned}\|\!|\bar{F}^{[\ell],(1)}_\mu|\!\|_\mu &\lesssim \llbracket\mu\rrbracket^{-2s}\int_\mu^1 \llbracket\sigma\rrbracket^{[\boldsymbol{a}]+2s-1}\mathrm{d}\sigma + \llbracket\mu\rrbracket^{-\mathfrak{b}}\int_\mu^1 \llbracket\sigma\rrbracket^{[\boldsymbol{a}]+\mathfrak{b}-1}\mathrm{d}\sigma\\ &\quad + \llbracket\mu\rrbracket^{-\mathfrak{b}}\int_\mu^1 \llbracket\sigma\rrbracket^{[\boldsymbol{a}]+\mathfrak{b}-1}\mathrm{d}\sigma + \int_{1/2}^\mu \llbracket\sigma\rrbracket^{[\boldsymbol{a}]-1}\mathrm{d}\sigma\\ &\quad + \llbracket\mu\rrbracket^{-\mathfrak{b}}\int_\mu^1 \llbracket\sigma\rrbracket^{[\boldsymbol{a}]+\mathfrak{b}-1}\mathrm{d}\sigma\\ &\lesssim \llbracket\mu\rrbracket^{[\boldsymbol{a}]},\end{aligned}$$

where we used again (4.47). This concludes the proof of the induction step. ∎

4.7. Local estimates for the flow of kernels

In this section, we prove an auxiliary result showing that the norms of the effective force kernels $F^{\mathfrak{a}}_\mu$ can be controlled in terms of the norms of the localised kernels $F^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu$ restricted to a neighbourhood of the diagonal, where $v^{\mathfrak{a}}_\mu$ is the weight introduced in Definition 4.3. We use the notation

$$\|X\|_{L^N_{\mathbb{P}}} := (\mathbb{E}|X|^N)^{1/N}.$$

Lemma 4.17. *For every $\mathfrak{a} \in \mathfrak{A}$,*

$$[\mathfrak{a}] \geqslant 0 \quad \Longrightarrow \quad F^{\mathfrak{a}}_1 = 0.$$

Proof. By (4.41), the only nonzero final kernels at level $\ell = 0$ are those with $k = 0, 1, 3$. Their homogeneities are

$$[0,0] = -\alpha < 0, \qquad [0,1] = -\alpha + \beta = -2\beta - \kappa < 0, \qquad [0,3] = -\alpha + 3\beta = -\kappa < 0.$$

At positive level, the only nonzero final kernels are

$$F^{[\ell],(1)}_1(z, z_1) = r^{[\ell]}_{\varepsilon,M}\delta(z - z_1), \qquad 1 \leqslant \ell \leqslant \hat{\ell}.$$

Their kernel homogeneity is

$$[\ell, 1] = -\alpha + \delta\ell + \beta = \delta\ell - 2\beta - \kappa < -\kappa < 0.$$

For $\ell > \hat{\ell}$, the final datum is zero. Therefore every kernel with nonnegative homogeneity has zero final datum. ∎

Lemma 4.18. *For every $\ell \in \{0, \dots, \bar{\ell}\}$ and $N \in \mathbb{N}_+$, we have*

$$\begin{aligned}&\sup_{\mathfrak{a}|\ell(\mathfrak{a})=\ell}\sup_{\mu\in(1/2,1)}\Big\|\llbracket\mu\rrbracket^{-[\mathfrak{a}]-\kappa}\|F^{\mathfrak{a}}_\mu\|_\mu\Big\|_{L^N_{\mathbb{P}}}\\ &\quad\lesssim_\ell 1 + \sup_{\substack{\mathfrak{a}|\ell(\mathfrak{a})\leqslant\ell\\ [\mathfrak{a}]<0}}\sup_{\mu\in(1/2,1)}\Big\|\llbracket\mu\rrbracket^{-[\mathfrak{a}]-\kappa}\|F^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu\Big\|^{2^\ell}_{L^{2^\ell N}_{\mathbb{P}}},\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}&\sup_{\mathfrak{a}|\ell(\mathfrak{a})=\ell}\sup_{\mu\in(1/2,1)}\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa+1}\|\partial_\mu F^{\mathfrak{a}}_\mu\|_\mu\Big\|_{L^N_{\mathbb{P}}}\\ &\lesssim_\ell 1+\sup_{\substack{\mathfrak{a}|\ell(\mathfrak{a})\leqslant\ell\\ [\mathfrak{a}]<0}}\sup_{\mu\in(1/2,1)}\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa}\|F^{\mathfrak{a}}_\mu\cdot v^{\mathfrak{a}}_\mu\|_\mu\Big\|^{2^\ell}_{L^{2^\ell N}_{\mathbb{P}}}.\end{aligned}\tag{4.49}$$

Proof. We shall prove the lemma by induction on $\ell\in\mathbb{N}_0$. First, we discuss the case $\ell=0$ for which we have $\partial_\mu F^{\mathfrak{a}}_\mu=0$ and thus $F^{\mathfrak{a}}_\mu=F^{\mathfrak{a}}_1$ for all $\mu\in(1/2,1)$. Observe that the kernels $F^{\mathfrak{a}}_1$ are local, that is, they are supported on the diagonal, as can be seen from (2.1). Hence,

$$F^{\mathfrak{a}}_\mu\cdot v^{\mathfrak{a}}_\mu=F^{\mathfrak{a}}_1\cdot v^{\mathfrak{a}}_\mu=F^{\mathfrak{a}}_1=F^{\mathfrak{a}}_\mu,\qquad \ell(\mathfrak{a})=0,$$

for all $\mu\in(1/2,1)$ since $v^{\mathfrak{a}}_\mu=1$ on the diagonal. Moreover, for kernels with $[\mathfrak{a}]\geqslant 0$, Lemma 4.17 gives $F^{\mathfrak{a}}_1=0$, hence $F^{\mathfrak{a}}_\mu=0$ for all $\mu\in(1/2,1)$. This proves the bounds (4.48) and (4.49) for $\ell=0$.

Let $\ell_0\in\mathbb{N}_0$ and suppose that the bounds (4.48) and (4.49) are true for all $\ell\leqslant\ell_0$. We shall prove the bounds for $\ell=\ell_0+1$. It follows from (4.13) and Lemma 4.9 that

$$\begin{aligned}[\![\mu]\!]^{-[\mathfrak{a}]-\kappa+1}\|\partial_\mu F^{\mathfrak{a}}_\mu\|_\mu&\lesssim\sum_{\mathfrak{b},\mathfrak{c}}[\![\mu]\!]^{-[\mathfrak{a}]-2\kappa+1}\|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\mu,F^{\mathfrak{b}}_\mu,F^{\mathfrak{c}}_\mu)\|_\mu\\ &\lesssim\sum_{\mathfrak{b},\mathfrak{c}}[\![\mu]\!]^{-[\mathfrak{b}]-\kappa}\|F^{\mathfrak{b}}_\mu\|_\mu[\![\mu]\!]^{-[\mathfrak{c}]-\kappa}\left\|F^{\mathfrak{c}}_\mu\right\|_\mu.\end{aligned}$$

Consequently, by Hölder's inequality

$$\Big\|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa+1}\|\partial_\mu F^{\mathfrak{a}}_\mu\|_\mu\Big\|_{L^N_{\mathbb{P}}}\lesssim\sum_{\mathfrak{b},\mathfrak{c}}\|[\![\mu]\!]^{-[\mathfrak{b}]-\kappa}\left\|F^{\mathfrak{b}}_\mu\|_\mu\right\|_{L^{2N}_{\mathbb{P}}}\Big\|[\![\mu]\!]^{-[\mathfrak{c}]-\kappa}\left\|F^{\mathfrak{c}}_\mu\right\|_\mu\Big\|_{L^{2N}_{\mathbb{P}}}.$$

Since $\ell(\mathfrak{b})\vee\ell(\mathfrak{c})\leqslant\ell(\mathfrak{a})-1\leqslant\ell_0$ and $\ell=\ell(\mathfrak{a})=\ell_0+1$ the above bound together with the induction hypothesis imply the bound (4.49). The dependence on ℓ in the bound (4.49) comes from the estimate on the (finite) number of terms in the sum over $\mathfrak{b},\mathfrak{c}$.

Let $F^{\mathfrak{a}}_\mu$ be non-relevant, so that $[\mathfrak{a}]\geqslant 0$. Then Lemma 4.17 yields $F^{\mathfrak{a}}_1=0$ and therefore

$$F^{\mathfrak{a}}_\mu=-\int_\mu^1\partial_\eta F^{\mathfrak{a}}_\eta\,\mathrm{d}\eta.$$

Using the monotonicity of the kernel norms (4.10), we obtain

$$\|F^{\mathfrak{a}}_\mu\|_\mu\lesssim\int_\mu^1\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta\,\mathrm{d}\eta.$$

By Minkowski's inequality, this gives

$$\begin{aligned}\Big\|\|F^{\mathfrak{a}}_\mu\|_\mu\Big\|_{L^N_{\mathbb{P}}}&\lesssim\int_\mu^1\Big\|\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta\Big\|_{L^N_{\mathbb{P}}}\,\mathrm{d}\eta\\ &\lesssim\int_\mu^1[\![\eta]\!]^{[\mathfrak{a}]+\kappa-1}\,\mathrm{d}\eta\sup_{\eta\in(\mu,1)}[\![\eta]\!]^{-[\mathfrak{a}]-\kappa+1}\Big\|\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta\Big\|_{L^N_{\mathbb{P}}}.\end{aligned}$$

Since $[\mathfrak{a}] + \kappa \geqslant \kappa > 0$, we have

$$\int_\mu^1 [\![\eta]\!]^{[\mathfrak{a}]+\kappa-1} \, \mathrm{d}\eta \lesssim [\![\mu]\!]^{[\mathfrak{a}]+\kappa}.$$

Consequently,

$$[\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \left\| \|F^{\mathfrak{a}}_\mu\|_\mu \right\|_{L^N_{\mathbb{P}}} \lesssim \sup_{\eta\in(\mu,1)} [\![\eta]\!]^{-[\mathfrak{a}]-\kappa+1} \left\| \|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \right\|_{L^N_{\mathbb{P}}}.$$

Taking the supremum over $\mu \in (1/2, 1)$, the required bound for non-relevant kernels in (4.48) follows from (4.49). This argument applies equally to marginal and irrelevant kernels in the non-relevant branch.

Let us now consider relevant kernels $F^{\mathfrak{a}}$, for which $[\mathfrak{a}] < 0$. If $k(\mathfrak{a}) = 0$, then $v^{\mathfrak{a}}_\mu F^{\mathfrak{a}}_\mu = F^{\mathfrak{a}}_\mu$ and the bound (4.48) is trivial. For $k(\mathfrak{a}) > 0$ we shall use the following decomposition

$$\begin{aligned} F^{\mathfrak{a}}_\mu \cdot (1 - v^{\mathfrak{a}}_\mu) &= F^{\mathfrak{a}}_1 \cdot (1 - v^{\mathfrak{a}}_\mu) - \int_\mu^1 \partial_\eta F^{\mathfrak{a}}_\eta \cdot (1 - v^{\mathfrak{a}}_\mu) \, \mathrm{d}\eta \\ &= -\int_\mu^1 [(1 - \tilde{K}^{\mathfrak{a}}_\eta)\partial_\eta F^{\mathfrak{a}}_\eta] \cdot (1 - v^{\mathfrak{a}}_\mu) \, \mathrm{d}\eta - \int_\mu^1 [\tilde{K}^{\mathfrak{a}}_\eta \partial_\eta F^{\mathfrak{a}}_\eta] \cdot (1 - v^{\mathfrak{a}}_\mu) \, \mathrm{d}\eta. \end{aligned}$$

Note that $F^{\mathfrak{a}}_1 \cdot (1 - v^{\mathfrak{a}}_\mu) = 0$, which follows from the fact that the kernels $F^{\mathfrak{a}}_1$ are local, that is, they are supported on the diagonal, and $v^{\mathfrak{a}}_\mu = 1$ on the diagonal. Then

$$\begin{aligned} &\|F^{\mathfrak{a}}_\mu - F^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu \\ &\quad\leqslant \int_\mu^1 \|[(1 - \tilde{K}^{\mathfrak{a}}_\eta)\partial_\eta F^{\mathfrak{a}}_\eta] \cdot (1 - v^{\mathfrak{a}}_\mu)\|_\mu \mathrm{d}\eta \\ &\qquad + \int_\mu^1 \|[\tilde{K}^{\mathfrak{a}}_\eta \partial_\eta F^{\mathfrak{a}}_\eta] \cdot (1 - v^{\mathfrak{a}}_\mu)\|_\mu \mathrm{d}\eta. \end{aligned}$$

We now observe that, on account of Lemma A.17 and Lemma A.20, we have

$$\begin{aligned} &\|F^{\mathfrak{a}}_\mu - F^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu \\ &\quad\leqslant \int_\mu^1 \left(\frac{[\![\eta]\!]^{2s}}{[\![\mu]\!]^{2s}} + \frac{[\![\eta]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}}}{[\![\mu]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}}} \right) \|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \mathrm{d}\eta \\ &\quad\lesssim \int_\mu^1 \frac{[\![\eta]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}}}{[\![\mu]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}}} \|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \mathrm{d}\eta. \end{aligned}$$

Consequently, by Minkowski's inequality

$$\left\| \|F^{\mathfrak{a}}_\mu - F^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu \right\|_{L^N_{\mathbb{P}}} \lesssim \int_\mu^1 \frac{[\![\eta]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}}}{[\![\mu]\!]^{\mathfrak{b}-\ell(\mathfrak{a})\kappa_{\mathfrak{o}}}} \left\| \|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \right\|_{L^N_{\mathbb{P}}} \mathrm{d}\eta.$$

Note that by (4.2), (4.33), (4.39) and (4.38),

$$\begin{aligned} [\mathfrak{a}] + \mathfrak{b} - \ell(\mathfrak{a})\kappa_{\mathfrak{o}} &\geqslant \mathfrak{b} - \alpha + (\delta - \kappa_{\mathfrak{o}})\ell(\mathfrak{a}) + \beta \\ &= \mathfrak{b} + (\delta - \kappa_{\mathfrak{o}})\ell(\mathfrak{a}) - 2\beta - \kappa \\ &= (\delta - \kappa_{\mathfrak{o}})\ell(\mathfrak{a}) + \delta + \mathfrak{b} - 2s > 0, \end{aligned}$$

for $k(\mathfrak{a}) > 0$ and $\ell(\mathfrak{a}) > 0$. Thus, the previous estimate and the bound (4.49) imply the bound (4.48) for the relevant kernels. This concludes the proof. ∎

4.8. From cumulants to random kernels

The bound (2.31) stated in Theorem 2.7 is proved by combining the cumulant estimates from Lemma 4.15 with the auxiliary bounds established in Lemma 4.18, via the following lemma, which shows how estimates on the cumulants translate into estimates on the effective force kernels. The remaining part of Theorem 2.7, namely, the proof that the estimates (2.19) hold with $C_F = 1 + \|F^{\mathfrak{A}}\|^2$, is presented in Section 4.9.

Our analysis relies on the decomposition of the effective force kernels into mean and fluctuation parts:

$$F^{\mathfrak{a}}_\mu = \bar{F}^{\mathfrak{a}}_\mu + \tilde{F}^{\mathfrak{a}}_\mu, \qquad \bar{F}^{\mathfrak{a}}_\mu := \mathbb{E}F^{\mathfrak{a}}_\mu, \qquad \tilde{F}^{\mathfrak{a}}_\mu := F^{\mathfrak{a}}_\mu - \mathbb{E}F^{\mathfrak{a}}_\mu.$$

Lemma 4.19. *Let $(F^{(\varepsilon,M)}_\mu)_{\mu\in(1/2,1)}$ be the solution of the approximate flow equation with final condition (4.41) with $r_{\varepsilon,M}$ as in Lemma 4.15. For every $N \in \mathbb{N}_+$, we have*

$$\mathbb{E}\|F^{\mathfrak{A}}\|^N = \Big\|\|F^{\mathfrak{A}}\|\Big\|^N_{L^N_{\mathbb{P}}} \lesssim 1, \tag{4.50}$$

where $\|F^{\mathfrak{A}}\|$ was defined in (4.9).

Proof. If $[\mathfrak{a}] \geqslant 0$, then $F^{\mathfrak{a}}_1 = 0$ by Lemma 4.17. Hence

$$F^{\mathfrak{a}}_\mu = -\int_\mu^1 \partial_\eta F^{\mathfrak{a}}_\eta \,\mathrm{d}\eta$$

and

$$\begin{aligned}
[\![\mu]\!]^{-[\mathfrak{a}]}\|F^{\mathfrak{a}}_\mu\|_\mu &\lesssim [\![\mu]\!]^{-[\mathfrak{a}]}\int_\mu^1 \|\partial_\eta F^{\mathfrak{a}}_\eta\|_\mu \mathrm{d}\eta \\
&\lesssim \int_\mu^1 [\![\eta]\!]^{-[\mathfrak{a}]}\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \mathrm{d}\eta \\
&\lesssim \int_{1/2}^1 [\![\eta]\!]^{-[\mathfrak{a}]}\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \mathrm{d}\eta.
\end{aligned}$$

On the other hand, if the kernel $F^{\mathfrak{a}}_\mu$ is relevant, that is $[\mathfrak{a}] < 0$, then $[\![\mu]\!]^{-[\mathfrak{a}]} = 0$ at $\mu = 1$. Consequently, we have $[\![\mu]\!]^{-[\mathfrak{a}]}F^{\mathfrak{a}}_\mu = 0$ at $\mu = 1$. From

$$\partial_\mu([\![\mu]\!]^{-[\mathfrak{a}]}F^{\mathfrak{a}}_\mu) = [\![\mu]\!]^{-[\mathfrak{a}]}\partial_\mu F^{\mathfrak{a}}_\mu + [\mathfrak{a}][\![\mu]\!]^{-[\mathfrak{a}]-1}F^{\mathfrak{a}}_\mu, \tag{4.51}$$

we deduce that

$$\begin{aligned}
[\![\mu]\!]^{-[\mathfrak{a}]}\|F^{\mathfrak{a}}_\mu\|_\mu &\lesssim \int_\mu^1 \big([\![\eta]\!]^{-[\mathfrak{a}]}\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\mu + [\![\eta]\!]^{-[\mathfrak{a}]-1}\|F^{\mathfrak{a}}_\eta\|_\mu\big)\,\mathrm{d}\eta \\
&\lesssim \int_{1/2}^1 \big([\![\eta]\!]^{-[\mathfrak{a}]}\|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta + [\![\eta]\!]^{-[\mathfrak{a}]-1}\|F^{\mathfrak{a}}_\eta\|_\eta\big)\,\mathrm{d}\eta.
\end{aligned}$$

Next, Minkowski's inequality gives

$$\Big\| \sup_{\mu\in(1/2,1)} [\![\mu]\!]^{-[\mathfrak{a}]} \|F^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^N_{\mathbb{P}}} \lesssim \int_{1/2}^1 \Big\| [\![\eta]\!]^{-[\mathfrak{a}]} \|\partial_\eta F^{\mathfrak{a}}_\eta\|_\eta \Big\|_{L^N_{\mathbb{P}}} \,\mathrm{d}\eta + \int_{1/2}^1 \Big\| [\![\eta]\!]^{-[\mathfrak{a}]-1} \|F^{\mathfrak{a}}_\eta\|_\eta \Big\|_{L^N_{\mathbb{P}}} \,\mathrm{d}\eta,$$

for every kernel. Since $\int_{1/2}^1 [\![\eta]\!]^{\kappa-1}\mathrm{d}\eta < \infty$, we arrive at

$$\Big\| \sup_{\mu\in(1/2,1)} [\![\mu]\!]^{-[\mathfrak{a}]} \|F^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^N_{\mathbb{P}}} \lesssim \sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{-[\mathfrak{a}]+1-\kappa} \|\partial_\mu F^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^N_{\mathbb{P}}} + \sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|F^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^N_{\mathbb{P}}}.$$

Using Lemma 4.18 and the decomposition $F^{\mathfrak{a}}_\mu = \bar{F}^{\mathfrak{a}}_\mu + \tilde{F}^{\mathfrak{a}}_\mu$ we obtain

$$\Big\| \sup_{\mu\in(1/2,1)} [\![\mu]\!]^{-[\mathfrak{a}]} \|F^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^N_{\mathbb{P}}} \lesssim 1 + \sup_{\substack{\mathfrak{b}\,|\,\ell(\mathfrak{b})\leqslant\ell(\mathfrak{a}) \\ [\mathfrak{b}]<0}} \sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{-[\mathfrak{b}]-\kappa} \|v^{\mathfrak{b}}_\mu F^{\mathfrak{b}}_\mu\|_\mu \Big\|^{2^{\ell(\mathfrak{a})}}_{L^{2^{\ell(\mathfrak{a})}N}_{\mathbb{P}}}$$

$$\lesssim 1 + \sup_{\substack{\mathfrak{b}\,|\,\ell(\mathfrak{b})\leqslant\ell(\mathfrak{a}) \\ [\mathfrak{b}]<0}} \sup_{\mu\in(1/2,1)} \Big(\Big\| [\![\mu]\!]^{-[\mathfrak{b}]-\kappa} \|v^{\mathfrak{b}}_\mu \tilde{F}^{\mathfrak{b}}_\mu\|_\mu \Big\|^{2^{\ell(\mathfrak{a})}}_{L^{2^{\ell(\mathfrak{a})}N}_{\mathbb{P}}} + \big([\![\mu]\!]^{-[\mathfrak{b}]-\kappa} \|v^{\mathfrak{b}}_\mu \bar{F}^{\mathfrak{b}}_\mu\|_\mu \big)^{2^{\ell(\mathfrak{a})}} \Big).$$

Applying Lemmas 4.21 and 4.23 and recalling that $|||\mathcal{F}^{\boldsymbol{A}}||| \lesssim 1$, on account of Lemma 4.15, we conclude that

$$\Big\| \sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{-[\mathfrak{a}]} \|F^{\mathfrak{a}}_\sigma\|_\sigma \Big\|_{L^N_{\mathbb{P}}} \lesssim 1.$$

Finally, to establish the slightly stronger bound

$$\Big\| \sup_{\sigma\in(1/2,1)} [\![\sigma]\!]^{d/2+s+2\kappa} \|F^{[0],(0)}_\sigma\|_\sigma \Big\|_{L^N_{\mathbb{P}}} \lesssim 1$$

for the noise $F^{[0],(0)}_\sigma = \xi^{(\varepsilon,M)}$, we proceed as before, and invoke the second bound stated in Lemma 4.21. ∎

The following factorisation is chosen to work uniformly above and below the lattice scale. When $[\![\mu]\!] \geqslant \varepsilon$, both the temporal and spatial variables can be smoothed as in the continuum. When $[\![\mu]\!] < \varepsilon$, the spatial structure is atomic and is therefore retained explicitly in one factor.

Definition 4.20. Let $\hat{\kappa} \in [0,1]$. For $[\![\mu]\!] \geq \varepsilon$, we set

$$\begin{aligned} \check{K}_\mu &= (1 + [\![\mu]\!]^{2s}\partial_t)^{-\hat{\kappa}} (1 - [\![\mu]\!]^2\Delta_\varepsilon)^{-\hat{\kappa}}, \\ \hat{K}_\mu &= (1 + [\![\mu]\!]^{2s}\partial_t)^{\hat{\kappa}-1} (1 - [\![\mu]\!]^2\Delta_\varepsilon)^{\hat{\kappa}-2}, \end{aligned} \tag{4.52}$$

For $[\![\mu]\!] < \varepsilon$, we set

$$\begin{aligned} \check{K}_\mu &= (1 + [\![\mu]\!]^{2s}\partial_t)^{-\hat{\kappa}}(1 - [\![\mu]\!]^2\Delta_\varepsilon)^{-2}, \\ \hat{K}_\mu &= (1 + [\![\mu]\!]^{2s}\partial_t)^{\hat{\kappa}-1} \otimes 1. \end{aligned} \tag{4.53}$$

Observe that in both regimes the product is exactly

$$\check{K}_\mu \hat{K}_\mu = (1 + [\![\mu]\!]^{2s}\partial_t)^{-1}(1 - [\![\mu]\!]^2\Delta_\varepsilon)^{-2} = K_\mu. \tag{4.54}$$

Lemma 4.21. *For all $\mathfrak{a} \in \mathfrak{A}$ and $N \in \mathbb{N}_+$, we have*

$$\begin{aligned} \sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\tilde{F}^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} &\lesssim_N |\!|\!| \mathcal{F}^{\boldsymbol{A}} |\!|\!|, \\ \sup_{\mu\in(1/2,1)} \Big\| [\![\mu]\!]^{d/2+s+\kappa} \|F^{[0],(0)}_\mu\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} &\lesssim_N 1. \end{aligned}$$

Proof. Let $k = k(\mathfrak{a})$. By Lemma A.18, there exists a weight $\tilde{V}^{\mathfrak{a}}_\mu$ with support properties analogous to those of $v^{\mathfrak{a}}_\mu$ such that

$$\|\tilde{F}^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu = \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\mu(\tilde{F}^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu)] \cdot \tilde{w}^{\mathfrak{a}}_\mu\| \lesssim \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\mu \tilde{F}^{\mathfrak{a}}_\mu] \cdot \tilde{V}^{\mathfrak{a}}_\mu \tilde{w}^{\mathfrak{a}}_\mu\|.$$

Moreover, we have

$$\sup_{z\in\Lambda} \int_{\Lambda^k} \tilde{w}^{\mathfrak{a}}_\mu(z, Z^{\mathfrak{a}}) \tilde{V}^{\mathfrak{a}}_\mu(z, Z^{\mathfrak{a}}) \mathrm{d}Z^{\mathfrak{a}} \lesssim [\![\mu]\!]^{2sk}(\varepsilon \vee [\![\mu]\!])^{dk}, \qquad Z^{\mathfrak{a}} = (z_1, \dots, z_k) \in \Lambda^k.$$

Thus, we have

$$\|\tilde{F}^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu \lesssim \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\mu \tilde{F}^{\mathfrak{a}}_\mu] \cdot \tilde{V}^{\mathfrak{a}}_\mu \tilde{w}^{\mathfrak{a}}_\mu\| \lesssim [\![\mu]\!]^{2sk}(\varepsilon \vee [\![\mu]\!])^{dk} \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\mu \tilde{F}^{\mathfrak{a}}_\mu]\|_{L^\infty},$$

where, here and below, $\|\bullet\|_{L^\infty}$ denotes the supremum norm in all the variables. Hence,

$$\Big\| \|\tilde{F}^{\mathfrak{a}}_\mu \cdot v^{\mathfrak{a}}_\mu\|_\mu \Big\|_{L^{2N}_{\mathbb{P}}} \lesssim [\![\mu]\!]^{2sk}(\varepsilon \vee [\![\mu]\!])^{dk} \Big\| \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\mu \tilde{F}^{\mathfrak{a}}_\mu]\|_{L^\infty} \Big\|_{L^{2N}_{\mathbb{P}}}. \tag{4.55}$$

To analyse the last term, we decompose the operator K_μ as

$$K_\mu = \check{K}_\mu \hat{K}_\mu,$$

where the kernels $\check{K}_\mu$ and $\hat{K}_\mu$ are introduced in Definition 4.20 and $\hat{\kappa} > 0$ is a small constant to be fixed such that

$$0 < \hat{\kappa} < \frac{1}{2 + 2\bar{k}}.$$

By Lemma A.7,

$$\|\check{K}_\mu\|_{L^{2N/(2N-1)}(1/\mathfrak{o}^{\mathfrak{a}})} \lesssim [\![\mu]\!]^{-(d+2s)/2N}, \tag{4.56}$$

provided

$$4N\hat{\kappa} > 3.$$

Note that

$$\tilde{K}^{\mathfrak{a}}_\mu = \check{K}^{\otimes(1+k)}_\mu \hat{K}^{\otimes(1+k)}_\mu K^{1,k}_\mu$$

by Definitions 1.13 and 4.5. Hence, by weighted Young's inequality, Fubini's theorem and (4.56), we have

$$\begin{aligned}
&\mathbb{E}[\|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu}]\|^{2N}_{L^\infty}] \\
&\lesssim [\![\mu]\!]^{-(d+2s)(1+k)} \int_{\Lambda^{1+k}} \mathbb{E}[(\mathfrak{o}^{\mathfrak{a}}(z)(\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu})(z;Z^{\mathfrak{a}}))^{2N}]\,\mathrm{d}z\mathrm{d}Z^{\mathfrak{a}} \\
&\lesssim [\![\mu]\!]^{-(d+2s)(1+k)} \int_{\Lambda^{1+k}} \mathbb{E}[((\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu})(z;Z^{\mathfrak{a}})w^{\mathfrak{a}}_{\mu}(z,Z^{\mathfrak{a}}))^{2N}]\left(\frac{\mathfrak{o}^{\mathfrak{a}}(z)}{w^{\mathfrak{a}}_{\mu}(z,Z^{\mathfrak{a}})}\right)^{2N}\mathrm{d}z\mathrm{d}Z^{\mathfrak{a}} \\
&\lesssim [\![\mu]\!]^{-(d+2s)(1+k)} \left\|\mathbb{E}[([\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu})^{2N}]\right\|_{L^\infty},
\end{aligned}$$

where we used that,

$$\int_{\Lambda^{1+k}} \left(\frac{\mathfrak{o}^{\mathfrak{a}}(z)}{w^{\mathfrak{a}}_{\mu}(z,Z^{\mathfrak{a}})}\right)^{2N}\mathrm{d}z\mathrm{d}Z^{\mathfrak{a}} \lesssim 1,$$

for $N \in \mathbb{N}_+$ large enough. Thus,

$$\left\|\|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu}]\|_{L^\infty}\right\|_{L^{2N}_{\mathbb{P}}} \lesssim [\![\mu]\!]^{-\frac{(d+2s)(1+k)}{2N}} \|\mathbb{E}[([\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu})^{2N}]\|^{1/2N}_{L^\infty}. \quad (4.57)$$

We need to control the $L^{2N}_{\mathbb{P}}$ norm of the centred random variable

$$(\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu})(z;Z^{\mathfrak{a}})w^{\mathfrak{a}}_{\mu}(z,Z^{\mathfrak{a}}).$$

To this end, we observe that, by induction, it is easy to show that $\tilde{F}^{\mathfrak{a}}_{\mu}$ is a polynomial of the Gaussian random field $\xi^{(\varepsilon,M)}$. Consequently, by Nelson's hypercontractivity estimate, all higher moments can be controlled by the second one. In particular, the $2N$-th moment can be bounded in terms of the second-order cumulant

$$\mathbb{E}(\tilde{F}^{\mathfrak{a}}_{\mu} \otimes \tilde{F}^{\mathfrak{a}}_{\mu}) = \mathfrak{K}_2(F^{\mathfrak{a}}_{\mu}, F^{\mathfrak{a}}_{\mu}) = \mathcal{F}^{(\mathfrak{a},\mathfrak{a})}_{\mu}.$$

Using the notation

$$w^{(\mathfrak{a},\mathfrak{a})}_{\mu} := w^{\mathfrak{a}}_{\mu} \otimes w^{\mathfrak{a}}_{\mu}, \qquad K^{(\mathfrak{a},\mathfrak{a})}_{\mu} := K^{1,k}_{\mu} \otimes K^{1,k}_{\mu},$$

we have

$$\begin{aligned}
&\|\mathbb{E}[([\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu})^{2N}]\|_{L^\infty} \\
&\quad\lesssim_N \|\mathbb{E}[([\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}\tilde{F}^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu})^{2}]\|^N_{L^\infty} \\
&\quad\lesssim \|\mathfrak{K}_2([\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}F^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu}, [\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}F^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu})\|^N_{L^\infty}.
\end{aligned} \quad (4.58)$$

Since

$$\|\!|\mathcal{F}^{(\mathfrak{a},\mathfrak{a})}_{\mu}|\!\|_{\mu} = \|\!|[K^{(\mathfrak{a},\mathfrak{a})}_{\mu}\mathcal{F}^{(\mathfrak{a},\mathfrak{a})}_{\mu}]\cdot w^{(\mathfrak{a},\mathfrak{a})}_{\mu}|\!\|,$$

by Lemma 4.22 below, we obtain

$$\begin{aligned}
&\|\mathfrak{K}_2([\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}F^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu}, [\hat{K}^{\otimes(1+k)}_{\mu} K^{1,k}_{\mu}F^{\mathfrak{a}}_{\mu}]\cdot w^{\mathfrak{a}}_{\mu})\|_{L^\infty} \\
&\quad\lesssim [\![\mu]\!]^{-2s(1+2k)}(\varepsilon \vee [\![\mu]\!])^{-d(1+2k)}\|\!|\mathcal{F}^{(\mathfrak{a},\mathfrak{a})}_{\mu}|\!\|_{\mu}.
\end{aligned}$$

Hence, by (4.58),

$$\|\mathbb{E}([\hat{K}_\mu^{\otimes(1+k)} K_\mu^{1,k} \tilde{F}_\mu^{\mathfrak{a}}] \cdot w_\mu^{\mathfrak{a}})^{2N}\|_{L^\infty}^{\frac{1}{2N}} \lesssim [\![\mu]\!]^{-s(1+2k)} (\varepsilon \vee [\![\mu]\!])^{-d(1+2k)/2} |\!|\!| \mathcal{F}_\mu^{(\mathfrak{a},\mathfrak{a})} |\!|\!|_\mu^{1/2}.$$

Overall, using (4.55), (4.57) together with the Definition (4.22) of $|\!|\!| \mathcal{F}^{\boldsymbol{A}} |\!|\!|$ and the fact that

$$[(\mathfrak{a},\mathfrak{a})] = -\varrho + 2(\theta+\alpha) + 2[\mathfrak{a}],$$

we arrive at

$$\sup_\mu \|[\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|v_\mu^{\mathfrak{a}} \tilde{F}_\mu^{\mathfrak{a}}\|_\mu\|_{L_{\mathbb{P}}^{2N}} \lesssim_N |\!|\!| \mathcal{F}^{\boldsymbol{A}} |\!|\!| \sup_\mu [\![\mu]\!]^{-\frac{\varrho}{2}+\theta+\alpha-\frac{d+2s}{2}-\frac{(d+2s)(1+k)}{2N}-\kappa}. \tag{4.59}$$

In order to conclude, we need the right-hand side above to be finite, that is, we require

$$\theta + \alpha - \frac{\varrho}{2} - \frac{d+2s}{2} - \frac{(d+2s)(1+\bar{k})}{2N} - \kappa \geqslant 0. \tag{4.60}$$

Given our choices of parameters (4.33) this bound has the form

$$\frac{3}{4}\delta_\star - \frac{(d+2s)(1+\bar{k})}{2N} - \frac{3}{2}\kappa \geqslant 0. \tag{4.61}$$

Since $\kappa \in (0, \delta_\star/2)$ by (4.36), it suffices to take $N \in \mathbb{N}_+$ large enough to satisfy this inequality and conclude our bound. The bound for smaller values of $N \in \mathbb{N}_+$ then follows immediately by Jensen's inequality.

The second of the stated bounds, namely the estimate for the noise $F^{[0],(0)} = \xi^{(\varepsilon,M)}$, is proved by the same argument as for (4.59), except that we now use the precise estimate

$$|\!|\!| \mathcal{F}_\mu^{(\mathfrak{a},\mathfrak{a})} |\!|\!|_\mu = 1, \qquad \mathfrak{a} = (0,0),$$

for the covariance of the noise

$$\mathcal{F}_\mu^{(\mathfrak{a},\mathfrak{a})} = \mathfrak{K}_2(F_\mu^{[0],(0)}, F_\mu^{[0],(0)}) = \mathfrak{K}_2(\xi^{(\varepsilon,M)}, \xi^{(\varepsilon,M)}),$$

which follows from (4.28), instead of the weaker bound

$$|\!|\!| \mathcal{F}_\mu^{(\mathfrak{a},\mathfrak{a})} |\!|\!|_\mu \leqslant [\![\mu]\!]^{[(\mathfrak{a},\mathfrak{a})]} |\!|\!| \mathcal{F}^{\boldsymbol{A}} |\!|\!|^2.$$

Specifically, in place of (4.59) we obtain

$$\begin{aligned}
&\sup_\mu \Big\| [\![\mu]\!]^{d/2+s+\kappa} \|F_\mu^{[0],(0)}\|_\mu \Big\|_{L_{\mathbb{P}}^{2N}} \\
&\quad = \sup_\mu \Big\| [\![\mu]\!]^{d/2+s+\kappa} \|\mathfrak{o} \cdot K_\mu F_\mu^{[0],(0)}\| \Big\|_{L_{\mathbb{P}}^{2N}} \\
&\quad \lesssim \sup_\mu [\![\mu]\!]^{-\frac{(d+2s)}{2N}+\kappa} \lesssim 1,
\end{aligned}$$

for $N \in \mathbb{N}_+$ large enough, depending on $\kappa > 0$. ∎

Lemma 4.22. *Let F be a random distributional field over Λ^{1+k} admitting second moment. Assume that the law of F is invariant under spacetime translations and such that*

$$z \mapsto F(z, z + z_1, \dots, z + z_k) \tag{4.62}$$

is a.s. periodic in space with period M for all $z_1, \dots, z_k \in \Lambda$. For $z, z_1, \dots, z_{2k} \in \Lambda$, define

$$\mathcal{F}(z, z_1, \dots, z_{2k}) = \mathfrak{K}_2(F(0, z_1, \dots, z_k), F(z, z_{k+1}, \dots, z_{2k})).$$

For all $\omega \geqslant 0$ and $\hat{\kappa} \in [0, 1/(2k+2))$, we have

$$\begin{aligned}
&\|\mathfrak{K}_2((\hat{K}_\mu^{\otimes(1+k)} F), (\hat{K}_\mu^{\otimes(1+k)} F)) \cdot (w_\mu^{(1+k),\omega} \otimes w_\mu^{(1+k),\omega})\|_{L^\infty} \\
&\quad \lesssim [\![\mu]\!]^{-2s(1+2k)} (\varepsilon \vee [\![\mu]\!])^{-d(1+2k)} \\
&\qquad \times \int_{\Lambda_M \times \Lambda^{2k}} w_\mu^{(1+k),\omega}(0, z_1, \dots, z_k) w_\mu^{(1+k),\omega}(z, z_{k+1}, \dots, z_{2k}) \\
&\qquad \times |\mathcal{F}(z, z_1, \dots, z_{2k})| \, \mathrm{d}z \mathrm{d}z_1 \dots \mathrm{d}z_{2k},
\end{aligned}$$

uniformly over F and $\mu \in [1/2, 1)$, where $\hat{K}_\mu$ is the kernel defined in Definition 4.20.

Proof. We claim that

$$\mathfrak{K}_2((\hat{K}_\mu^{\otimes(1+k)} F)(0, z_1, \dots, z_k), (\hat{K}_\mu^{\otimes(1+k)} F)(z, z_{k+1}, \dots, z_{2k})) = (\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k})$$

for all $z, z_1, \dots, z_{2k} \in \Lambda$, where $*$ stands for the convolution on Λ^{1+2k} and

$$\mathcal{K}_\mu(z, z_1, \dots, z_{2k}) := \int_\Lambda \hat{K}_\mu(z'') \hat{K}_\mu(z + z'') \hat{K}_\mu(z_1 + z'') \dots \hat{K}_\mu(z_{2k} + z'') \, \mathrm{d}z''.$$

Indeed, we have

$$\begin{aligned}
&(\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k}) \\
&= \int_{\Lambda^{2+2k}} \hat{K}_\mu(z'') \hat{K}_\mu(z' + z'') \hat{K}_\mu(z'_1 + z'') \dots \hat{K}_\mu(z'_{2k} + z'') \\
&\qquad \times \mathcal{F}(z - z', z_1 - z'_1, \dots, z_{2k} - z'_{2k}) \, \mathrm{d}z' \mathrm{d}z'' \mathrm{d}z'_1 \dots \mathrm{d}z'_{2k} \\
&= \int_{\Lambda^{2+2k}} \hat{K}_\mu(z'') \hat{K}_\mu(z') \hat{K}_\mu(z'_1) \dots \hat{K}_\mu(z'_{2k}) \\
&\qquad \times \mathfrak{K}_2\big(F(0 - z'', z_1 - z'_1, \dots, z_k - z'_k), \\
&\qquad\qquad F(z - z', z_{k+1} - z'_{k+1}, \dots, z_{2k} - z'_{2k})\big) \\
&\qquad \times \mathrm{d}z' \mathrm{d}z'' \mathrm{d}z'_1 \dots \mathrm{d}z'_{2k} \\
&= \mathfrak{K}_2\big((\hat{K}_\mu^{\otimes(1+k)} F)(0, z_1, \dots, z_k), (\hat{K}_\mu^{\otimes(1+k)} F)(z, z_{k+1}, \dots, z_{2k})\big),
\end{aligned}$$

where the second equality above follows from the translation invariance of the law of F.

It remains to show that

$$
\begin{aligned}
&|(\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k})| w_\mu^{(1+k),\omega}(0, z_1, \dots, z_k) w_\mu^{(1+k),\omega}(z, z_{k+1}, \dots, z_{2k}) \\
&\quad \lesssim [\![\mu]\!]^{-2s(1+2k)} (\varepsilon \vee [\![\mu]\!])^{-d(1+2k)} \\
&\qquad \times \int_{\Lambda_M \times \Lambda^{2k}} w_\mu^{(1+k),\omega}(0, z_1, \dots, z_k) w_\mu^{(1+k),\omega}(z, z_{k+1}, \dots, z_{2k}) \\
&\qquad \times |\mathcal{F}(z, z_1, \dots, z_{2k})| \mathrm{d}z \mathrm{d}z_1 \dots \mathrm{d}z_{2k}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&|(\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k})| \\
&\quad \leqslant \sum_{n \in \mathbb{Z}^d} \int_{\Lambda_M \times \Lambda^{2k}} |\mathcal{K}_\mu(z' + Mn, z'_1, \dots, z'_{2k})| \\
&\qquad \times |\mathcal{F}(z - z' - Mn, z_1 - z'_1, \dots, z_{2k} - z'_{2k})| \mathrm{d}z' \mathrm{d}z'_1 \dots \mathrm{d}z'_{2k}.
\end{aligned}
$$

Thus, by the same argument as in (4.6), we obtain

$$
\begin{aligned}
&|(\mathcal{K}_\mu * \mathcal{F})(z, z_1, \dots, z_{2k})| w_\mu^{(1+k),\omega}(0, z_1, \dots, z_k) w_\mu^{(1+k),\omega}(z, z_{k+1}, \dots, z_{2k}) \\
&\quad \lesssim \sum_{n \in \mathbb{Z}^d} \int_{\Lambda_M \times \Lambda^{2k}} w_\mu^{\omega}(z' + Mn) w_\mu^{\omega}(z'_1) \cdots w_\mu^{\omega}(z'_{2k}) |\mathcal{K}_\mu(z' + Mn, z'_1, \dots, z'_{2k})| \\
&\qquad \times w_\mu^{(1+k),\omega}(0, z_1 - z'_1, \dots, z_k - z'_k) \\
&\qquad \times w_\mu^{(1+k),\omega}(z - z' - Mn, z_{k+1} - z'_{k+1}, \dots, z_{2k} - z'_{2k}) \\
&\qquad \times |\mathcal{F}(z - z' - Mn, z_1 - z'_1, \dots, z_{2k} - z'_{2k})| \, \mathrm{d}z' \mathrm{d}z'_1 \dots \mathrm{d}z'_{2k},
\end{aligned}
$$

where for $z = (t, x) \in \Lambda$ and $n \in \mathbb{Z}^d$ we write $z + Mn = (t, x + Mn) \in \Lambda$. Using Definition 4.3 we arrive at

$$
\begin{aligned}
&\int_{\Lambda_M \times \Lambda^{2k}} w_\mu^{(1+k),\omega}(0, z_1 - z'_1, \dots, z_k - z'_k) \\
&\qquad \times w_\mu^{(1+k),\omega}(z - z' - Mn, z_{k+1} - z'_{k+1}, \dots, z_{2k} - z'_{2k}) \\
&\qquad \times \big|\mathcal{F}(z - z' - Mn, z_1 - z'_1, \dots, z_{2k} - z'_{2k})\big| \, \mathrm{d}z' \mathrm{d}z'_1 \dots \mathrm{d}z'_{2k} \\
&= \int_{\Lambda_M \times \Lambda^{2k}} w_\mu^{(1+k),\omega}(0, z'_1, \dots, z'_k) \, w_\mu^{(1+k),\omega}(0, z'_{k+1}, \dots, z'_{2k}) \\
&\qquad \times \big|\mathcal{F}(z - z' - Mn, z'_1, \dots, z'_k, z - z' - Mn + z'_{k+1}, \dots, z - z' - Mn + z'_{2k})\big| \\
&\qquad \times \mathrm{d}z' \mathrm{d}z'_1 \dots \mathrm{d}z'_{2k}.
\end{aligned}
$$

Note that by (4.62), the above expression can be rewritten as

$$\begin{aligned}
&\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1',\dots,z_k')w_\mu^{(1+k),\omega}(0,z_{k+1}',\dots,z_{2k}')\\
&\quad\times|\mathcal{F}(z',z_1',\dots,z_k',z'+z_{k+1}',\dots,z'+z_{2k}')|\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'\\
&=\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1',\dots,z_k')w_\mu^{(1+k),\omega}(z',z_{k+1}',\dots,z_{2k}')\\
&\quad\times|\mathcal{F}(z',z_1',\dots,z_k',z_{k+1}',\dots,z_{2k}')|\mathrm{d}z'\mathrm{d}z_1'\dots\mathrm{d}z_{2k}'.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|(\mathcal{K}_\mu * \mathcal{F})(z,z_1,\dots,z_{2k})|w_\mu^{(1+k),\omega}(0,z_1,\dots,z_k)w_\mu^{(1+k),\omega}(z,z_{k+1},\dots,z_{2k})\\
&\lesssim \sup_{z',z_1',\dots,z_{2k}'\in\Lambda}\sum_{n\in\mathbb{Z}^d} w_\mu^\omega(z'+Mn)w_\mu^\omega(z_1')\cdots w_\mu^\omega(z_{2k}')\\
&\quad\times|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')|\\
&\quad\times\int_{\Lambda_M\times\Lambda^{2k}} w_\mu^{(1+k),\omega}(0,z_1'',\dots,z_k'')w_\mu^{(1+k),\omega}(z'',z_{k+1}'',\dots,z_{2k}'')\\
&\quad\times|\mathcal{F}(z'',z_1'',\dots,z_{2k}'')|\,\mathrm{d}z''\mathrm{d}z_1''\dots\mathrm{d}z_{2k}''.
\end{aligned}$$

To complete the proof it remains to show that

$$\begin{aligned}
&\sup_{z',z_1',\dots,z_{2k}'\in\Lambda}\sum_{n\in\mathbb{Z}^d} w_\mu^\omega(z'+Mn)w_\mu^\omega(z_1')\cdots w_\mu^\omega(z_{2k}')\\
&\quad\times|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')|\lesssim[\![\mu]\!]^{-2s(1+2k)}(\varepsilon\vee[\![\mu]\!])^{-d(1+2k)}.
\end{aligned}$$

To this end, first note that by Lemma A.8,

$$\prod_{i=0}^{2k} w_\mu^{\omega+d+1}(z_i)\,\big|\mathcal{K}_\mu(z_0,\dots,z_{2k})\big|\lesssim[\![\mu]\!]^{-2s(1+2k)}(\varepsilon\vee[\![\mu]\!])^{-d(1+2k)}.$$

Consequently,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d} w_\mu^\omega(z'+Mn)w_\mu^\omega(z_1')\cdots w_\mu^\omega(z_{2k}')\,\big|\mathcal{K}_\mu(z'+Mn,z_1',\dots,z_{2k}')\big|\\
&\quad\lesssim[\![\mu]\!]^{-2s(1+2k)}(\varepsilon\vee[\![\mu]\!])^{-d(1+2k)}\sum_{n\in\mathbb{Z}^d} w_\mu^{-d-1}(z'+Mn)\\
&\quad\lesssim[\![\mu]\!]^{-2s(1+2k)}(\varepsilon\vee[\![\mu]\!])^{-d(1+2k)},
\end{aligned}$$

where we discarded the factors $\prod_{i=1}^{2k} w_\mu^{-d-1}(z_i')\le 1$. This proves the required bound, and finishes the proof. ∎

Lemma 4.23. *For every $\mathfrak{a}\in\mathfrak{A}$, we have*

$$[\![\mu]\!]^{-[\mathfrak{a}]-\kappa}\|\bar F_\mu^{\mathfrak{a}}\cdot v_\mu^{\mathfrak{a}}\|_\mu\lesssim|\!|\!|\mathcal{F}^{\boldsymbol{A}}|\!|\!|,$$

uniformly in $\mu\in[1/2,1)$.

Proof. By Lemma A.18, we have

$$\|\bar{F}^{\mathfrak{a}}_{\mu} \cdot v^{\mathfrak{a}}_{\mu}\|_{\mu} \lesssim \|\bar{F}^{\mathfrak{a}}_{\mu}\|_{\mu} = \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_{\mu} \bar{F}^{\mathfrak{a}}_{\mu}] \cdot \tilde{w}^{\mathfrak{a}}_{\mu}\|.$$

Moreover, by Definition 4.3,

$$\mathfrak{o}^{\mathfrak{a}} \tilde{w}^{\mathfrak{a}}_{\mu} \leqslant w^{\mathfrak{a}}_{\mu}.$$

Thus, by Lemma 1.17, (4.2) and (4.19) we obtain

$$\begin{aligned}
[\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|\bar{F}^{\mathfrak{a}}_{\mu} \cdot v^{\mathfrak{a}}_{\mu}\|_{\mu} &\lesssim [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|[\tilde{K}^{\mathfrak{a}}_{\mu} \bar{F}^{\mathfrak{a}}_{\mu}] \cdot w^{\mathfrak{a}}_{\mu}\| \\
&= [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|[K_{\mu}^{\otimes(1+k(\mathfrak{a}))} K_{\mu}^{1,k(\mathfrak{a})} \bar{F}^{\mathfrak{a}}_{\mu}] \cdot w^{\mathfrak{a}}_{\mu}\| \\
&\lesssim [\![\mu]\!]^{-[\mathfrak{a}]-\kappa} \|K_{\mu}\|^{1+k(\mathfrak{a})}_{\mathrm{TV}(w^{\flat}_{\mu})} \|[K_{\mu}^{1,k(\mathfrak{a})} \bar{F}^{\mathfrak{a}}_{\mu}] \cdot w^{\mathfrak{a}}_{\mu}\| \\
&\lesssim |\!|\!|\mathcal{F}^{\boldsymbol{A}}|\!|\!| [\![\mu]\!]^{[\boldsymbol{a}]-[\mathfrak{a}]-\kappa} \\
&\lesssim |\!|\!|\mathcal{F}^{\boldsymbol{A}}|\!|\!| [\![\mu]\!]^{\alpha-\varrho+\theta-\kappa},
\end{aligned}$$

where $\boldsymbol{a} = (\mathfrak{a})$. To conclude we use the inequality

$$\alpha - \varrho + \theta - \kappa \geqslant 0, \tag{4.63}$$

which coincides with the constraint $[\bar{F}]$ stated in (4.31). ∎

4.9. *Post-processing*

It remains to establish the bounds (2.19) with $C_F = 1 + \|F^{\mathfrak{A}}\|^2$ used above in the proof of Theorem 2.7, based on the analysis of the flow equation carried out so far. This is the content of Lemmas 4.24, 4.27 and 4.28 below.

Before continuing with the specific computations leading to (2.19) it will be useful to discuss how to fix the values of the parameters

$$\gamma, \vartheta, \bar{\kappa}, \bar{\ell}, \bar{k}, a, \nu, \kappa_{\mathfrak{o}}, \kappa.$$

The validity of the Lemmas 4.24–4.29 below, which together yield (2.19), depends on a series of conditions on these parameters, namely (4.71), (4.76), (4.79), (4.80), (4.82) and (4.83) as well as the constraints (2.18) in Theorem 2.5. The parameters $\kappa, \kappa_{\mathfrak{o}}$ are further constrained by conditions (4.36) and (4.38).

For the reader's convenience, these conditions are summarised in the following table:

$$\begin{array}{ll}
[\mathbb{A}] & \vartheta \leqslant 3(\gamma-\beta) + \delta - a\kappa_{\mathfrak{o}}(1+\bar{\ell}) - \kappa \\
[\mathbb{B}] & \vartheta \leqslant 2\gamma \wedge \left(3\gamma - \tfrac{d+2s}{2} - 2\kappa - a\kappa_{\mathfrak{o}}\right) \\
[\mathbb{C}] & \vartheta \leqslant \delta\bar{\ell} - \alpha - a\kappa_{\mathfrak{o}}(2\bar{\ell}+2) \\
[\mathbb{D}] & \bar{\kappa} \geqslant \bar{k}\nu + (1+\bar{\ell})\kappa_{\mathfrak{o}} \\
[\mathbb{E}] & \vartheta \leqslant \gamma \wedge (2s - 2\gamma) \\
[\mathbb{F}] & \vartheta \geqslant \tfrac{4(s+\gamma)\bar{\kappa}}{1-\bar{\kappa}} \\
[\mathbb{G}] & \kappa \in (0, \delta_\star/2), \qquad \tfrac{2\beta}{\delta} \notin \mathbb{N}_+ \\
[\mathbb{H}] & \kappa_{\mathfrak{o}} \leqslant \delta \wedge (2-\flat)/(2\bar{\ell}+2)
\end{array}$$

In order to satisfy all these constraints, together with the basic bounds $\vartheta > 0$ and $\delta > 0$, we proceed as follows. First, recall that

$$\beta = s - \frac{\delta_\star}{4} - \frac{\kappa}{2}, \qquad \delta = \frac{\delta_\star}{2}, \qquad \delta_\star = \frac{4s - d}{3} > 0,$$

and set

$$\gamma := \beta - \vartheta, \qquad \vartheta := \frac{1}{16}\delta_\star. \tag{4.64}$$

Next, we fix $\bar\kappa \in (0, 1/3)$ small enough so that

$$\frac{(8s + \delta_\star)\bar\kappa}{1 - \bar\kappa} \leqslant \frac{1}{16}\delta_\star.$$

Since $4(s + \gamma) < 8s + \delta_\star$, this allows us to satisfy $[\mathbb{F}]$. The constraint $[\mathbb{A}]$ is satisfied provided

$$a\kappa_{\mathfrak{o}}(1 + \bar\ell) + \kappa \leqslant \frac{1}{4}\delta_\star. \tag{4.65}$$

Since

$$\begin{aligned} 3\gamma - \frac{d + 2s}{2} - 2\kappa - a\kappa_{\mathfrak{o}} &= 3(\gamma - \beta) + 3\beta - \frac{d + 2s}{2} - 2\kappa - a\kappa_{\mathfrak{o}} \\ &= 3(\gamma - \beta) + \frac{3}{4}\delta_\star - \frac{7}{2}\kappa - a\kappa_{\mathfrak{o}}, \end{aligned}$$

the condition $[\mathbb{B}]$ is satisfied provided

$$4\vartheta \leqslant \frac{3}{4}\delta_\star - \frac{7}{2}\kappa - a\kappa_{\mathfrak{o}},$$

and

$$3\vartheta \leqslant 2\beta = 2s - \delta - \kappa = 2s - \frac{\delta_\star}{2} - \kappa,$$

that is, whenever

$$\frac{7}{2}\kappa + a\kappa_{\mathfrak{o}} \leqslant \frac{\delta_\star}{2}, \qquad \kappa \leqslant 2s - \frac{11}{16}\delta_\star. \tag{4.66}$$

Note that

$$2s - \frac{11}{16}\delta_\star \geqslant 2s - \delta_\star \geqslant 2s - \frac{4s - d}{3} = \frac{2s + d}{3} > 0,$$

so the second condition in (4.66) can indeed be satisfied. To meet condition $[\mathbb{C}]$, we shall later impose

$$2a\kappa_{\mathfrak{o}} \leqslant \frac{\delta_\star}{4}. \tag{4.67}$$

Under this constraint,

$$\left(\frac{\delta_\star}{2} - 2a\kappa_{\mathfrak{o}}\right)\bar\ell - 2a\kappa_{\mathfrak{o}} \geqslant \frac{\delta_\star}{4}\bar\ell - \frac{\delta_\star}{4}.$$

We therefore choose $\bar\ell$ large enough so that

$$\alpha + \frac{1}{16}\delta_\star \leqslant \frac{\delta_\star}{4}\bar\ell - \frac{\delta_\star}{4}.$$

Then $[\mathbb{C}]$ follows once (4.67) is imposed. Given $\bar\ell$ as above, we fix $\bar k$ large enough as discussed in Section 4.1. Then we fix a sufficiently large so that

$$a \geqslant \bar k(2s-\delta)/\bar\kappa \geqslant 2\bar k\beta/\bar\kappa \geqslant 2\bar k\gamma/\bar\kappa.$$

Consequently, relation (1.25) compels us to set $\nu = \gamma/a$. To ensure condition $[\mathbb{D}]$ we introduce a constraint

$$(1+\bar\ell)\kappa_0 \leqslant \bar\kappa/2. \tag{4.68}$$

Condition $[\mathbb{E}]$ is met if

$$\kappa \leqslant 2s - \frac{3}{4}\delta_\star = s + \frac{d}{4}. \tag{4.69}$$

Indeed, the inequality $\vartheta \leqslant 2s - 2\gamma$ is automatic, since

$$2s - 2\gamma = 2s - 2\beta + 2\vartheta = \frac{5}{8}\delta_\star + \kappa > \vartheta.$$

It remains to ensure $\vartheta \leqslant \gamma$. Since $\gamma = \beta - \vartheta$, this is equivalent to $2\vartheta \leqslant \beta$, which follows from

$$2\vartheta = \frac{\delta_\star}{8} \leqslant s - \frac{\delta_\star}{4} - \frac{\kappa}{2} = \beta$$

provided (4.69) holds. To conclude, we fix κ_0 and κ small enough as follows. We fix $\kappa_0 > 0$ and choose

$$\kappa \in (0, \delta_\star/2) \setminus \{2s - \delta(\ell+1) : \ell \in \mathbb{N}_+\}$$

sufficiently small so that the remaining constraints, namely $[\mathbb{G}]$ and $[\mathbb{H}]$, together with (4.65), (4.66), (4.67), (4.68) and (4.69), as well as (4.38), are satisfied.

Let us now detail the analysis. As a first step, we extract the coercive contribution from the effective force by defining

$$\begin{aligned} Q_\sigma(\psi_\sigma) &:= J_\sigma F_\sigma(\psi_\sigma) - (-\lambda\psi_\sigma^3) \\ &= J_\sigma F_\sigma(\psi_\sigma) - J_\sigma(-\lambda\psi_\sigma^3) + (1-J_\sigma)(\lambda\psi_\sigma^3) \\ &= J_\sigma F_\sigma^{[>0]}(\psi_\sigma) + J_\sigma F^{[0],(1)}\psi_\sigma + J_\sigma F^{[0],(0)} + (1-J_\sigma)(\lambda\psi_\sigma^3), \end{aligned} \tag{4.70}$$

where ψ is a generic field,

$$\psi_\sigma := J_\sigma\psi,$$

and

$$F^{[0],(0)} = \xi^{(\varepsilon,M)}, \qquad F^{[0],(1)}(\psi) = \bar r\psi, \qquad F_\sigma^{[>0]}(\psi) := \sum_{\mathfrak{a}\,|\,\ell(\mathfrak{a})>0} F_\sigma^{\mathfrak{a}}(\psi^{\otimes k(\mathfrak{a})}).$$

Recall that the norms $\|\!|\bullet|\!\| = \|\!|\bullet|\!\|_{\bar\mu}$ and $\|\!|\bullet|\!\|_\# = \|\!|\bullet|\!\|_{\#,\bar\mu}$ depending on the terminal scale $\bar\mu \in [1/2, 1)$ were introduced in Definition 2.2.

Lemma 4.24. *Assume* (4.76), (4.79), (4.83) *and*

$$\bar{k} \geqslant 5, \qquad \bar{\kappa} \geqslant \bar{k}\nu + (1+\bar{\ell})\kappa_{\mathfrak{o}}. \tag{4.71}$$

Then the bounds

$$\|\zeta_\mu^{\bar{\kappa}} Q_\sigma(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}[\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|)^{\bar{k}} + (1+|\!|\!|\psi|\!|\!|)^2 |\!|\!|(\mathcal{L}\psi)_\bullet|\!|\!|_\#], \tag{4.72}$$

$$\left\|\zeta_\mu^{\bar{\kappa}} K_\sigma\big(F_\sigma(\psi_\sigma) + \lambda\psi_\sigma^3\big)\right\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^{\bar{k}}, \tag{4.73}$$

$$\|\zeta_\mu^{\bar{\kappa}} K_\sigma F_\sigma(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{-3\gamma}\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|)^{\bar{k}}, \tag{4.74}$$

hold uniformly in $\bar{\mu}, \mu \in [1/2, 1)$, $\sigma \in [\mu \vee \bar{\mu}, 1)$ *and* $\psi \in \mathcal{S}'(\Lambda)$.

Proof. First, observe that, by the triangle inequality applied to the decomposition (4.70), we have

$$\begin{aligned}\|\zeta_\mu^{\bar{\kappa}} Q_\sigma(\psi_\sigma)\| &\leqslant \|\zeta_\mu^{\bar{\kappa}} J_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| + \|\zeta_\mu^{\bar{\kappa}} J_\sigma(F^{[0],(1)}\psi_\sigma + F^{[0],(0)})\| \\ &\quad + \|\zeta_\mu^{\bar{\kappa}}(1-J_\sigma)(\lambda\psi_\sigma^3)\| \\ &\lesssim \|\zeta_\mu^{\bar{\kappa}} K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| + \|\zeta_\mu^{\bar{\kappa}} K_\sigma(F^{[0],(1)}\psi_\sigma + F^{[0],(0)})\| \\ &\quad + \|\zeta_\mu^{5\nu}(1-J_\sigma)(\lambda\psi_\sigma^3)\|.\end{aligned}$$

To obtain the second estimate, we used that $\bar{\kappa} \geqslant 5\nu$, which is a consequence of (4.71), together with

$$J_\sigma = L_\sigma J_\sigma K_\sigma, \qquad \|L_\sigma J_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1.$$

The latter bound is a consequence of Lemma 1.17. The bound (4.72) now follows from Lemmas 4.25, 4.26 and 4.29.

Next, by the decomposition of the force,

$$F_\sigma(\psi_\sigma) + \lambda\psi_\sigma^3 = F_\sigma^{[>0]}(\psi_\sigma) + F^{[0],(1)}\psi_\sigma + F^{[0],(0)}.$$

Hence, by Lemmas 4.25 and 4.26,

$$\begin{aligned}&\left\|\zeta_\mu^{\bar{\kappa}} K_\sigma\big(F_\sigma(\psi_\sigma) + \lambda\psi_\sigma^3\big)\right\| \\ &\quad\leqslant \left\|\zeta_\mu^{\bar{\kappa}} K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\right\| + \left\|\zeta_\mu^{\bar{\kappa}} K_\sigma\big(F^{[0],(1)}\psi_\sigma + F^{[0],(0)}\big)\right\| \\ &\quad\lesssim [\![\sigma]\!]^{-3\gamma+\vartheta}\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|_{\bar{\mu}})^{\bar{k}}.\end{aligned}$$

This proves (4.73).

Since $\bar{\kappa} \geqslant 3\nu$ and $\|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1$ by Lemmas 1.17 and 2.3, we have

$$\|\zeta_\mu^{\bar{\kappa}} K_\sigma \psi_\sigma^3\| \lesssim \|\zeta_\mu^{3\nu} K_\sigma \psi_\sigma^3\| \lesssim \|K_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\|\zeta_\mu^{3\nu}\psi_\sigma^3\| \lesssim \|\zeta_\mu^{3\nu}\psi_\sigma^3\| \lesssim [\![\sigma]\!]^{-3\gamma}|\!|\!|\psi|\!|\!|^3.$$

The bound (4.74) now follows from the estimate

$$\|\zeta_\mu^{\bar{\kappa}} K_\sigma F_\sigma(\psi_\sigma)\| \lesssim \left\|\zeta_\mu^{\bar{\kappa}} K_\sigma\big(F_\sigma(\psi_\sigma) + \lambda\psi_\sigma^3\big)\right\| + \|\zeta_\mu^{\bar{\kappa}} K_\sigma \psi_\sigma^3\|$$

and (4.73). ∎

Lemma 4.25. *Assume (4.71). Then for every $\mathfrak{a} \in \mathfrak{A}$ such that $\ell(\mathfrak{a}) > 0$, the bound*

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma^{\otimes k(\mathfrak{a})})\| \lesssim [\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-a\kappa_{\mathfrak{o}}(1+\bar\ell)} \|F^{\mathfrak{A}}\| |\!|\!| \psi |\!|\!|^{k(\mathfrak{a})} \tag{4.75}$$

holds uniformly in $\bar\mu, \mu \in [1/2, 1)$, $\sigma \in [\mu \vee \bar\mu, 1)$ and $\psi \in \mathcal{S}'(\Lambda)$. Moreover, provided

$$\vartheta \leqslant 3(\gamma - \beta) + \delta - a\kappa_{\mathfrak{o}}(1 + \bar\ell) - \kappa, \tag{4.76}$$

the following bound

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{[>0]}(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta} \|F^{\mathfrak{A}}\| (1 + |\!|\!| \psi |\!|\!|)^{\bar k}.$$

holds uniformly in $\bar\mu, \mu \in [1/2, 1)$, $\sigma \in [\mu \vee \bar\mu, 1)$ and $\psi \in \mathcal{S}'(\Lambda)$.

Proof. We first observe that, since $\tilde J_\sigma J_\sigma = J_\sigma$ and $K_\sigma L_\sigma = 1$, we have

$$\psi_\sigma = \tilde J_\sigma \psi_\sigma = K_\sigma^2 L_\sigma^2 \tilde J_\sigma \psi_\sigma.$$

Consequently, it follows from $\tilde K_\mu^{\mathfrak{a}} = K_\mu \otimes (K_\mu^2)^{\otimes k(\mathfrak{a})}$ and Definition 4.6 of the kernel norm that

$$\|\zeta_\mu^{\bar\kappa} K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma^{\otimes k(\mathfrak{a})})\| \lesssim \|\mathfrak{w}_{\mu,\sigma}\| \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde K_\sigma^{\mathfrak{a}} F_\sigma^{\mathfrak{a}}] \cdot \tilde w_\sigma^{\mathfrak{a}}\| \|\rho_\mu L_\sigma^2 \tilde J_\sigma \psi_\sigma\|^{k(\mathfrak{a})},$$

where

$$\mathfrak{w}_{\mu,\sigma}\left(z, z_1, \dots, z_{k(\mathfrak{a})}\right) := \frac{\zeta_\mu^{\bar\kappa}(z)}{\mathfrak{o}^{\mathfrak{a}}(z) \tilde w_\sigma^{\mathfrak{a}}\left(z, z_1, \dots, z_{k(\mathfrak{a})}\right) \rho_\mu(z_1) \dots \rho_\mu(z_{k(\mathfrak{a})})},$$
$$\|\mathfrak{w}_{\mu,\sigma}\| = \|\mathfrak{w}_{\mu,\sigma}\|_{L^\infty(\Lambda^{1+k(\mathfrak{a})})}.$$

Using that the weight $\mathfrak{o}$ is at scale 1 (cf. Definition 4.3) and that the weight ζ_μ is at scale $[\![\mu]\!]^{-a}$ (cf. Definition 1.5), we deduce

$$\mathfrak{o}(z)^{-1} = \langle z \rangle_s^{\kappa_{\mathfrak{o}}} \leqslant [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}} \langle [\![\mu]\!]^a . z \rangle_s^{\kappa_{\mathfrak{o}}} = [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}} \zeta_\mu^{-\kappa_{\mathfrak{o}}}(z). \tag{4.77}$$

Noting that the Steiner diameter of a collection of points is always at least as large as the distance between any two points, we exploit the weight $(\tilde w_\sigma^a)^{-1}$ to propagate the weights ρ_μ^{-1} to the output variable. Using furthermore that $\rho_\mu = \zeta_\mu^\nu$ and $\mathfrak{o}^{\mathfrak{a}} = \mathfrak{o}^{1+\ell(\mathfrak{a})}$ as well as $\ell(\mathfrak{a}) \leqslant \bar\ell$, $k(\mathfrak{a}) \leqslant \bar k$ and (4.71), we obtain

$$\begin{aligned} \|\mathfrak{w}_{\mu,\sigma}\| &\lesssim \left\| \zeta_\mu^{\bar\kappa} \left(\mathfrak{o}^{\mathfrak{a}} \rho_\mu^{k(\mathfrak{a})}\right)^{-1} \right\|_{L^\infty(\Lambda)} \\ &\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))} \|\zeta_\mu^{\bar\kappa - \kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a})) - \nu k(\mathfrak{a})}\|_{L^\infty(\Lambda)} \\ &\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\bar\ell)} \end{aligned} \tag{4.78}$$

We also observe that, by Lemmas 1.17 and 2.3,

$$\|\rho_\mu L_\sigma^2 \tilde J_\sigma \psi_\sigma\| \lesssim \|L_\sigma^2 \tilde J_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \|\rho_\mu \psi_\sigma\|_{L^\infty} \lesssim [\![\sigma]\!]^{-\gamma} |\!|\!| \psi |\!|\!|.$$

Moreover, by Definition 4.6,

$$\|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_{\sigma} F^{\mathfrak{a}}_{\sigma}] \cdot \tilde{w}^{\mathfrak{a}}_{\sigma}\| = \|F^{\mathfrak{a}}_{\sigma}\|_{\sigma} \lesssim [\![\sigma]\!]^{[\mathfrak{a}]} \|F^{\mathfrak{A}}\|.$$

Combining the above estimates and using (4.2), we conclude that

$$\|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{\mathfrak{a}}_{\sigma}(\psi^{\otimes k(\mathfrak{a})}_{\sigma})\| \lesssim [\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-a\kappa_{\mathfrak{o}}(1+\bar{\ell})} \|F^{\mathfrak{A}}\| |\!|\!|\psi|\!|\!|^{k(\mathfrak{a})},$$

which proves (4.75).

To prove the second part, we observe that

$$\begin{aligned}
\|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{[>0]}_{\sigma}(\psi_{\sigma})\| &\leqslant \sum_{\mathfrak{a}|\ell(\mathfrak{a})>0, k(\mathfrak{a})\leqslant\bar{k}} \|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{\mathfrak{a}}_{\sigma}(\psi_{\sigma})\| \\
&\lesssim \sum_{\mathfrak{a}|\ell(\mathfrak{a})>0, k(\mathfrak{a})\leqslant\bar{k}} [\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-a\kappa_{\mathfrak{o}}(1+\bar{\ell})} \|F^{\mathfrak{A}}\| |\!|\!|\psi|\!|\!|^{k(\mathfrak{a})} \\
&\lesssim [\![\sigma]\!]^{-\alpha+\delta-a\kappa_{\mathfrak{o}}(1+\bar{\ell})} \|F^{\mathfrak{A}}\| (1 + |\!|\!|\psi|\!|\!|)^{\bar{k}},
\end{aligned}$$

where we used that $\beta \geqslant \gamma$, $1 \leqslant \ell(\mathfrak{a}) \leqslant \bar{\ell}$ and $k(\mathfrak{a}) \leqslant \bar{k}$. Since, by $\alpha = 3\beta + \kappa$ and (4.76), we have

$$3\gamma - \alpha + \delta - a\kappa_{\mathfrak{o}}(1 + \bar{\ell}) = 3(\gamma - \beta) + \delta - a\kappa_{\mathfrak{o}}(1 + \bar{\ell}) - \kappa \geqslant \vartheta,$$

the proof is complete. ∎

Lemma 4.26. *Assume* (4.71) *and*

$$\vartheta \leqslant 2\gamma \wedge \left(3\gamma - \frac{d+2s}{2} - 2\kappa - a\kappa_{\mathfrak{o}}\right). \tag{4.79}$$

Then the bound

$$\|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{[0],(0)}_{\sigma}\| \vee \|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{[0],(1)}_{\sigma}(\psi_{\sigma})\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta} \|F^{\mathfrak{A}}\| (1 + |\!|\!|\psi|\!|\!|),$$

holds uniformly in $\bar{\mu}, \mu \in [1/2, 1), \sigma \in [\mu \vee \bar{\mu}, 1)$ *and* $\psi \in \mathcal{S}'(\Lambda)$.

Proof. On account of (4.13) and (4.14), $\partial_{\sigma} F^{[0]}_{\sigma} = 0$. Thus,

$$F^{[0],(0)}_{\sigma}(z) = F^{[0],(0)}_{1}(z) = \xi^{(\varepsilon,M)}(z), \qquad F^{[0],(1)}_{\sigma}(z, z_1) = F^{[0],(1)}_{1}(z, z_1) = \bar{r}\delta(z, z_1).$$

Recall the definition (4.9) of $\|F^{\mathfrak{A}}\|$. By Lemmas 1.17 and 2.3, $\bar{\kappa} \geqslant \nu$ and $\vartheta \leqslant 2\gamma$, we obtain

$$\begin{aligned}
\|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{[0],(1)}_{\sigma}(\psi_{\sigma})\| &= |\bar{r}| \|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} \psi_{\sigma}\| \lesssim \|F^{\mathfrak{A}}\| \|K_{\sigma}\|_{\mathrm{TV}(\zeta^{-1})} \|\zeta^{\bar{\kappa}}_{\mu} \psi_{\sigma}\| \\
&\lesssim [\![\sigma]\!]^{-\gamma} \|F^{\mathfrak{A}}\| |\!|\!|\psi|\!|\!| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta} \|F^{\mathfrak{A}}\| |\!|\!|\psi|\!|\!|.
\end{aligned}$$

Moreover, by (4.79), we have

$$\begin{aligned}
\|\zeta^{\bar{\kappa}}_{\mu} K_{\sigma} F^{[0],(0)}_{\sigma}\| &\lesssim \|\zeta^{\bar{\kappa}}_{\mu} \mathfrak{o}^{-1}\| \|\mathfrak{o} K_{\sigma} F^{[0],(0)}\| \\
&\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}} [\![\sigma]\!]^{-(d+2s)/2-2\kappa} \|F^{\mathfrak{A}}\| \\
&\lesssim [\![\sigma]\!]^{-3\gamma+\vartheta} \|F^{\mathfrak{A}}\|,
\end{aligned}$$

where the bound $\|\zeta^{\bar{\kappa}}_{\mu} \mathfrak{o}^{-1}\| \lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}}$ follows from (4.77), since $\bar{\kappa} \geqslant \kappa_{\mathfrak{o}}$ by (4.71). ∎

Lemma 4.27. *Assume* (4.71) *and*

$$\vartheta \leqslant \delta - a\kappa_{\mathfrak{o}}(1+\bar{\ell}). \tag{4.80}$$

Then the bound

$$\|\zeta_\mu K_\sigma(\mathrm{D}F_\sigma(\psi_\sigma)(\dot{G}_\sigma\hat{\psi}))\| \lesssim [\![\sigma]\!]^{\vartheta-1}\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|)^{\bar{k}}\|\zeta_\mu^{1-\bar{\kappa}}\tilde{J}_\sigma^2\hat{\psi}\|,$$

holds uniformly in $\bar{\mu}\in[1/2,1)$, $\mu\in[1/2,1)$, $\sigma\in[\mu\vee\bar{\mu},1)$ *and* $\psi,\hat{\psi}\in\mathcal{S}'(\Lambda)$.

Proof. We have

$$\big\|\zeta_\mu K_\sigma\left(\mathrm{D}F_\sigma(\psi_\sigma)(\dot{G}_\sigma\hat{\psi})\right)\big\| \lesssim \sum_{\mathfrak{a}\,|\,k(\mathfrak{a})>0}\|\zeta_\mu K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma^{\otimes(k(\mathfrak{a})-1)}\otimes\dot{G}_\sigma\hat{\psi})\|.$$

Observe that since $\dot{G}_\sigma := \mathcal{L}_\varepsilon^{-1}\dot{J}_\sigma$, $\tilde{J}_\sigma^2\dot{J}_\sigma=\dot{J}_\sigma$, $\tilde{J}_\sigma J_\sigma = J_\sigma$ and $K_\sigma L_\sigma = 1$,

$$\psi_\sigma = K_\sigma^2 L_\sigma^2\tilde{J}_\sigma\psi_\sigma, \qquad \dot{G}_\sigma\hat{\psi} = \tilde{J}_\sigma^2\dot{G}_\sigma\hat{\psi} = K_\sigma^2L_\sigma^2\dot{G}_\sigma\tilde{\psi}, \qquad \tilde{\psi}=\tilde{J}_\sigma^2\hat{\psi}.$$

Using Definition 4.6 of the kernel norm, we obtain

$$\begin{aligned}\|\zeta_\mu K_\sigma F_\sigma^{\mathfrak{a}}(\psi_\sigma^{\otimes(k(\mathfrak{a})-1)}\otimes\dot{G}_\sigma\tilde{\psi})\| &\lesssim \|\mathfrak{w}_{\mu,\sigma}\|\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde{K}_\sigma^{\mathfrak{a}}F_\sigma^{\mathfrak{a}}]\cdot\tilde{w}_\sigma^{\mathfrak{a}}\| \\ &\quad\times\|\rho_\mu L_\sigma^2\tilde{J}_\sigma\psi_\sigma\|^{k(\mathfrak{a})-1}\|\zeta_\mu^{1-\bar{\kappa}}L_\sigma^2\dot{G}_\sigma\tilde{\psi}\|,\end{aligned}$$

where

$$\begin{aligned}\mathfrak{w}_{\mu,\sigma}\left(z,z_1,\dots,z_{k(\mathfrak{a})}\right) &:= \frac{\zeta_\mu(z)}{\mathfrak{o}^{\mathfrak{a}}(z)\tilde{w}_\sigma^{\mathfrak{a}}\left(z,z_1,\dots,z_{k(\mathfrak{a})}\right)\zeta_\mu^{1-\bar{\kappa}}(z_1)\rho_\mu(z_2)\dots\rho_\mu(z_{k(\mathfrak{a})})}, \\ \|\mathfrak{w}_{\mu,\sigma}\| &:= \|\mathfrak{w}_{\mu,\sigma}\|_{L^\infty(\Lambda^{1+k(\mathfrak{a})})}.\end{aligned}$$

Using furthermore that $\rho_\mu=\zeta_\mu^\nu$ and $\mathfrak{o}^{\mathfrak{a}}=\mathfrak{o}^{1+\ell(\mathfrak{a})}$ as well as $\ell(\mathfrak{a})\leqslant\bar{\ell}$, $k(\mathfrak{a})\leqslant\bar{k}$ and (4.71), we obtain

$$\begin{aligned}\|\mathfrak{w}_{\mu,\sigma}\|_{L^\infty} &\lesssim \|\zeta_\mu(\mathfrak{o}^{\mathfrak{a}}\zeta_\mu^{1-\bar{\kappa}}\rho_\mu^{(k(\mathfrak{a})-1)})^{-1}\|_{L^\infty(\Lambda)} \\ &\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))}\|\zeta_\mu^{\bar{\kappa}-\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))-\nu(k(\mathfrak{a})-1)}\|_{L^\infty(\Lambda)} \\ &\lesssim [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\bar{\ell})}.\end{aligned} \tag{4.81}$$

We also observe that, by Lemmas 1.17 and 2.3,

$$\|\rho_\mu L_\sigma^2\tilde{J}_\sigma\psi_\sigma\| \lesssim \|L_\sigma^2\tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\|\rho_\mu\psi_\sigma\|_{L^\infty} \lesssim [\![\sigma]\!]^{-\gamma}|\!|\!|\psi|\!|\!|$$

and

$$\|\zeta_\mu^{1-\bar{\kappa}}L_\sigma^2\dot{G}_\sigma\tilde{\psi}\| \lesssim \|L_\sigma^2\dot{G}_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\|\zeta_\mu^{1-\bar{\kappa}}\tilde{\psi}\| \lesssim [\![\sigma]\!]^{2s-1}\|\zeta_\mu^{1-\bar{\kappa}}\tilde{\psi}\|.$$

Moreover, by Definition 4.6,

$$\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde{K}_\sigma^{\mathfrak{a}}F_\sigma^{\mathfrak{a}}]\cdot\tilde{w}_\sigma^{\mathfrak{a}}\| = \|F_\sigma^{\mathfrak{a}}\|_\sigma \lesssim [\![\sigma]\!]^{[\mathfrak{a}]}\|F^{\mathfrak{A}}\|.$$

Combining the above estimates and using (4.2), we conclude that

$$\begin{aligned}
&\|\zeta_\mu K_\sigma F^{\mathfrak{a}}_\sigma(\psi_\sigma^{\otimes(k(\mathfrak{a})-1)} \otimes \dot G_\sigma \tilde\psi)\| \\
&\quad\lesssim [\![\sigma]\!]^{2s-\alpha+\beta+(\beta-\gamma)(k(\mathfrak{a})-1)-a\kappa_{\mathfrak{o}}(1+\bar\ell)-1}\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|)^{\bar k}\|\zeta_\mu^{1-\bar\kappa}\tilde\psi\| \\
&\quad\lesssim [\![\sigma]\!]^{\delta-a\kappa_{\mathfrak{o}}(1+\bar\ell)-1}\|F^{\mathfrak{A}}\|(1+|\!|\!|\psi|\!|\!|)^{\bar k}\|\zeta_\mu^{1-\bar\kappa}\tilde\psi\|,
\end{aligned}$$

where in the last step we used $2s-\alpha+\beta=\delta$ and that $\beta\geqslant\gamma$. This finishes the proof. ∎

Lemma 4.28. *Assume that*

$$\vartheta \leqslant \delta\bar\ell - \alpha - a\kappa_{\mathfrak{o}}(2\bar\ell+2). \tag{4.82}$$

Then

$$H_\sigma(\psi) := \partial_\sigma F_\sigma(\psi) + \mathrm{D}F_\sigma(\psi)(\dot G_\sigma F_\sigma(\psi)), \qquad \psi\in\mathcal{S}'(\Lambda),$$

satisfies the bound

$$\|\zeta_\mu K_\sigma H_\sigma(\psi_\sigma)\| \lesssim [\![\sigma]\!]^{\vartheta-1}\|F^{\mathfrak{A}}\|^2(1+|\!|\!|\psi|\!|\!|)^{2\bar k}.$$

uniformly in $\bar\mu,\mu\in[1/2,1)$, $\sigma\in[\mu\vee\bar\mu,1)$ and $\psi\in\mathcal{S}'(\Lambda)$.

Proof. We observe that, on account of the perturbative flow equation, we have

$$H_\sigma(\psi_\sigma) = \sum_{\mathfrak{a}\in\mathfrak{A}_{\mathrm{ext}}\setminus\mathfrak{A}}\sum_{\mathfrak{b},\mathfrak{c}\in\mathfrak{A}} B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)(\psi_\sigma^{\otimes k(\mathfrak{a})}).$$

Then, working as in the proof of Lemma 4.25 and using Lemma 4.9 as well as $k(\mathfrak{a}) = k(\mathfrak{b})+k(\mathfrak{c})-1\leqslant 2\bar k$, we obtain

$$\begin{aligned}
\|\zeta_\mu K_\sigma H_\sigma(\psi_\sigma)\| &\leqslant \sum_{\mathfrak{a}\in\mathfrak{A}_{\mathrm{ext}}\setminus\mathfrak{A}}\sum_{\mathfrak{b},\mathfrak{c}\in\mathfrak{A}} \|\zeta_\mu K_\sigma B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)(\psi_\sigma^{\otimes k(\mathfrak{a})})\| \\
&\lesssim \sum_{\mathfrak{a}\in\mathfrak{A}_{\mathrm{ext}}\setminus\mathfrak{A}}\sum_{\mathfrak{b},\mathfrak{c}\in\mathfrak{A}} [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))} \\
&\quad\times [\![\sigma]\!]^{-\gamma k(\mathfrak{a})}\|B^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}}(\dot G_\sigma, F^{\mathfrak{b}}_\sigma, F^{\mathfrak{c}}_\sigma)\|_\sigma |\!|\!|\psi|\!|\!|^{k(\mathfrak{a})} \\
&\lesssim \sum_{\mathfrak{a}\in\mathfrak{A}_{\mathrm{ext}}\setminus\mathfrak{A}}\sum_{\mathfrak{b},\mathfrak{c}\in\mathfrak{A}} [\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))} \\
&\quad\times [\![\sigma]\!]^{[\mathfrak{a}]-[\mathfrak{b}]-[\mathfrak{c}]-\gamma k(\mathfrak{a})-1}\|F^{\mathfrak{b}}_\sigma\|_\sigma\|F^{\mathfrak{c}}_\sigma\|_\sigma |\!|\!|\psi|\!|\!|^{k(\mathfrak{a})} \\
&\lesssim \sum_{\mathfrak{a}\in\mathfrak{A}_{\mathrm{ext}}\setminus\mathfrak{A}}\sum_{\mathfrak{b},\mathfrak{c}\in\mathfrak{A}} [\![\sigma]\!]^{[\mathfrak{a}]-\gamma k(\mathfrak{a})-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))-1} \\
&\quad\times \|F^{\mathfrak{A}}\|^2(1+|\!|\!|\psi|\!|\!|)^{2\bar k}.
\end{aligned}$$

The factor $[\![\mu]\!]^{-a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a}))}$ appearing in the second line arises due to the estimate (4.77) by a bound similar to (4.78). Now since $\beta\geqslant\gamma$ and $\ell(\mathfrak{a})\in[\bar\ell+1,2\bar\ell+1]$, we have

$$\begin{aligned}
&[\mathfrak{a}]-\gamma k(\mathfrak{a}) - a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a})) \\
&\quad= -\alpha + (\beta-\gamma)k(\mathfrak{a}) + \delta\ell(\mathfrak{a}) - a\kappa_{\mathfrak{o}}(1+\ell(\mathfrak{a})) \\
&\quad\geqslant -\alpha + \delta\bar\ell - a\kappa_{\mathfrak{o}}(2\bar\ell+2) \geqslant \vartheta,
\end{aligned}$$

where we exploited (4.2) and (4.82). This proves the claim. ∎

Lemma 4.29. *For all*

$$\vartheta \leqslant 2s - 2\gamma \tag{4.83}$$

the following bound

$$\|\zeta_\mu^{5\nu}(1 - J_\sigma)(\psi_\sigma^3)\| \lesssim [\![\sigma]\!]^{-3\gamma+\vartheta} |\!|\!|\psi|\!|\!|^2 |\!|\!|\mathcal{L}\psi_\bullet|\!|\!|_\#.$$

holds uniformly in $\bar\mu, \mu \in [1/2, 1)$, $\sigma \in [\mu \vee \bar\mu, 1)$ *and* $\psi \in \mathcal{S}'(\Lambda)$.

Proof. Let $J_{>\eta} := 1 - J_\eta$. Then

$$\begin{aligned}(1 - J_\sigma)(\psi_\sigma^3) = (1 - J_\sigma)[&(J_\eta\psi_\sigma)^3 + (J_{>\eta}\psi_\sigma)^3 \\ &+ 3(J_\eta\psi_\sigma)^2(J_{>\eta}\psi_\sigma) + 3(J_\eta\psi_\sigma)(J_{>\eta}\psi_\sigma)^2].\end{aligned}$$

Choosing $\eta = \frac{\sigma}{6-5\sigma}$, which implies $6\eta(1-\eta)^{-1} = \sigma(1-\sigma)^{-1}$, the first contribution $(1 - J_\sigma)(J_\eta\psi_\sigma)^3$ vanishes by the Fourier space support property of the product. As for the other contributions, we have, for example,

$$\begin{aligned}\|\zeta_\mu^{5\nu}(1 - J_\sigma)(J_{>\eta}\psi_\sigma)^3\| &= \|\rho_\mu^5(1 - J_\sigma)(J_{>\eta}\psi_\sigma)^3\| \\ &\lesssim \|\rho_\mu^5(J_{>\eta}\psi_\sigma)^3\| \lesssim \|\rho_\mu\psi_\sigma\|^2\|\rho_\mu^3(J_{>\eta}\psi_\sigma)\|.\end{aligned}$$

Since $\sigma \geqslant \mu$ and $[\![\eta]\!] \approx [\![\sigma]\!]$, by Lemmas 2.3 and A.25, we have

$$\|\zeta_\mu^{5\nu}(1 - J_\sigma)(J_{>\eta}\psi_\sigma)^3\| \lesssim [\![\eta]\!]^{2s}[\![\sigma]\!]^{-5\gamma} |\!|\!|\psi|\!|\!|^2 |\!|\!|\mathcal{L}\psi_\bullet|\!|\!|_\# \lesssim [\![\sigma]\!]^{2s-5\gamma} |\!|\!|\psi|\!|\!|^2 |\!|\!|\mathcal{L}\psi_\bullet|\!|\!|_\#.$$

The contributions coming from the terms $(J_\eta\psi_\sigma)^2(J_{>\eta}\psi_\sigma)$ and $(J_\eta\psi_\sigma)(J_{>\eta}\psi_\sigma)^2$ satisfy the same bound. Since $\vartheta \leqslant 2s - 2\gamma$, this finishes the proof. ∎

Lemmas 4.24 and 4.29 provide the first estimate in (2.19); Lemma 4.28 provides the second; the second bound in Lemma 4.24 provides the third; and Lemma 4.27 provides the fourth. Together with the random kernel estimates from Section 4.8, this completes the proof of Theorem 2.7.

Appendix A. Auxiliary estimates

We collect in this appendix various technical estimates of general character. Recall that $d = 3$ and $s \in (3/4, 1)$. As in the main text, all estimates are understood to hold uniformly in $\varepsilon \in 2^{-\mathbb{N}_0}$ and $M \in \mathbb{N}_+$.

A.1. Kernel estimates

Definition A.1. Recall from Section 1.1 that for a spacetime multi-index

$$A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}},$$

we denote its spatial part by

$$\bar{A} := (A_{i,+}, A_{i,-})_{i=1}^{d} \in \mathbb{N}_0^{\{1\pm,\dots,d\pm\}}$$

and write

$$|\bar{A}| := \sum_{i=1}^{d}(A_{i,+} + A_{i,-}), \qquad |A|_s := 2sA_0 + |\bar{A}|.$$

For $(k_0, k) \in \mathbb{R} \times \mathbb{R}^d$, let

$$d_\varepsilon^{i,\pm}(k) := \pm\frac{e^{\pm i\varepsilon k_i} - 1}{\varepsilon},$$

and

$$d_\varepsilon^{\bar{A}}(k) := \prod_{i=1}^{d}\big(d_\varepsilon^{i,+}(k)\big)^{A_{i,+}}\big(d_\varepsilon^{i,-}(k)\big)^{A_{i,-}}, \qquad d_\varepsilon^{A}(k_0,k) := (ik_0)^{A_0} d_\varepsilon^{\bar{A}}(k).$$

Thus

$$\widehat{\partial_\varepsilon^A f}(k_0,k) = d_\varepsilon^A(k_0,k)\hat{f}(k_0,k).$$

Lemma A.2. *Recall that*

$$q_\varepsilon(k)^2 = \frac{4}{\varepsilon^2}\sum_{j=1}^{d}\sin^2\left(\frac{\varepsilon k_j}{2}\right) = \frac{2}{\varepsilon^2}\sum_{j=1}^{d}\big(1-\cos(\varepsilon k_j)\big).$$

The following bounds hold uniformly in ε.

(a) *For every $k \in (\mathbb{R}^d_\varepsilon)^*$,*

$$q_\varepsilon(k) \asymp |k|.$$

(b) *For every $\beta \geqslant 0$ and $\tilde{a} \in \mathbb{N}_0^d$,*

$$\left|\partial_k^{\tilde{a}} q_\varepsilon^\beta(k)\right| \lesssim_{\beta,\tilde{a}} q_\varepsilon^{\beta-|\tilde{a}|}(k), \qquad q_\varepsilon(k) > 0. \tag{A.1}$$

In the special case $\beta = 2$, one has the stronger estimate

$$\left|\partial_k^{\tilde{a}} q_\varepsilon^2(k)\right| \lesssim_{\tilde{a}} \varepsilon^{(|\tilde{a}|-2)_+} q_\varepsilon^{(2-|\tilde{a}|)_+}(k) \tag{A.2}$$

for every $k \in (\mathbb{R}^d_\varepsilon)^*$, including the points at which $q_\varepsilon(k) = 0$. Equivalently,

$$\left|\partial_k^{\tilde{a}} q_\varepsilon^2(k)\right| \lesssim_{\tilde{a}} \begin{cases} q_\varepsilon^2(k), & |\tilde{a}| = 0, \\ q_\varepsilon(k), & |\tilde{a}| = 1, \\ 1, & |\tilde{a}| = 2, \\ \varepsilon^{|\tilde{a}|-2}, & |\tilde{a}| \geqslant 3. \end{cases}$$

(c) *For every* $\bar{A} \in \mathbb{N}_0^{\{1\pm,\dots,d\pm\}}$ *and* $\tilde{a} \in \mathbb{N}_0^d$,

$$\left|\partial_k^{\tilde{a}} d_\varepsilon^{\bar{A}}(k)\right| \lesssim_{\bar{A},\tilde{a}} q_\varepsilon^{|\bar{A}|-|\tilde{a}|}(k), \qquad q_\varepsilon(k) > 0. \tag{A.3}$$

Consequently, if $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$, *then, for* $a_0 \in \mathbb{N}_0$ *and* $\tilde{a} \in \mathbb{N}_0^d$,

$$\left|\partial_{k_0}^{a_0}\partial_k^{\tilde{a}} d_\varepsilon^{A}(k_0,k)\right| \lesssim_{A,\tilde{a},a_0} |k_0|^{(A_0-a_0)_+} q_\varepsilon^{|\bar{A}|-|\tilde{a}|}(k), \qquad q_\varepsilon(k) > 0, \tag{A.4}$$

with the left-hand side equal to zero if $a_0 > A_0$.

(d) *Let* $\chi \in C_c^\infty((0,\infty))$. *For every* $\beta \geqslant 0$, $\tilde{a} \in \mathbb{N}_0^d$, *and* $\eta \geqslant 1/2$,

$$\left|\partial_k^{\tilde{a}}\left(q_\varepsilon^\beta(k)\chi(\eta q_\varepsilon(k))\right)\right| \lesssim_{\beta,\tilde{a},\chi} \eta^{|\tilde{a}|-\beta} 1_{\operatorname{supp}\chi}(\eta q_\varepsilon(k)). \tag{A.5}$$

More generally, for every $\bar{A} \in \mathbb{N}_0^{\{1\pm,\dots,d\pm\}}$,

$$\left|\partial_k^{\tilde{a}}\left(q_\varepsilon^\beta(k)\chi(\eta q_\varepsilon(k)) d_\varepsilon^{\bar{A}}(k)\right)\right| \lesssim_{\beta,\tilde{a},\bar{A},\chi} \eta^{|\tilde{a}|-\beta-|\bar{A}|} 1_{\operatorname{supp}\chi}(\eta q_\varepsilon(k)). \tag{A.6}$$

(e) *For every* $R > 0$ *and* $\gamma < d$,

$$\int_{\{q_\varepsilon \leqslant R\}} q_\varepsilon^{-\gamma}(k)\,\mathrm{d}k \lesssim_\gamma R^{d-\gamma}. \tag{A.7}$$

In particular, for every $r < d + 4s$,

$$\int_{\{q_\varepsilon \leqslant 2\}} \left(1 \vee q_\varepsilon^{4s-r}(k)\right)\mathrm{d}k \lesssim_r 1. \tag{A.8}$$

Proof. The comparison $q_\varepsilon(k) \asymp |k|$ follows from

$$|\sin x| \asymp |x|, \qquad |x| \leqslant \pi/2.$$

We first prove the strengthened estimate (A.2). Since

$$q_\varepsilon^2(k) = \frac{2}{\varepsilon^2}\sum_{j=1}^d \big(1 - \cos(\varepsilon k_j)\big),$$

one has

$$\partial_k^{\tilde{a}} q_\varepsilon^2 = 0$$

whenever $\tilde{a}$ has nonzero components in at least two distinct coordinate directions. It therefore suffices to consider $\tilde{a} = ne_j$ for some $j \in \{1,\dots,d\}$ and $n \geqslant 1$. Direct differentiation gives

$$\left|\partial_{k_j}^n q_\varepsilon^2(k)\right| \leqslant 2\varepsilon^{n-2}\begin{cases} |\sin(\varepsilon k_j)|, & n \text{ odd}, \\ |\cos(\varepsilon k_j)|, & n \text{ even}. \end{cases}$$

For $n = 1$, using

$$|\sin(\varepsilon k_j)| \leqslant 2\left|\sin\left(\frac{\varepsilon k_j}{2}\right)\right| \leqslant \varepsilon q_\varepsilon(k),$$

we obtain

$$\left|\partial_{k_j} q_\varepsilon^2(k)\right| \lesssim q_\varepsilon(k).$$

For $n = 2$, the derivative is uniformly bounded, while for $n \geqslant 3$,

$$\left|\partial_{k_j}^n q_\varepsilon^2(k)\right| \lesssim_n \varepsilon^{n-2}.$$

Together with the case $n = 0$, this proves (A.2).

We next derive the weaker estimate needed for arbitrary powers. Since

$$q_\varepsilon(k) \lesssim \varepsilon^{-1},$$

estimate (A.2) implies, whenever $q_\varepsilon(k) > 0$,

$$\left|\partial_k^{\tilde{a}} q_\varepsilon^2(k)\right| \lesssim_{\tilde{a}} q_\varepsilon^{2-|\tilde{a}|}(k).$$

Indeed, this is immediate for $|\tilde{a}| \leqslant 2$, whereas for $|\tilde{a}| \geqslant 3$ one uses

$$\varepsilon^{|\tilde{a}|-2} \lesssim q_\varepsilon^{2-|\tilde{a}|}(k).$$

Now write

$$q_\varepsilon^\beta = (q_\varepsilon^2)^{\beta/2}.$$

By Faà di Bruno's formula, every term in $\partial_k^{\tilde{a}} q_\varepsilon^\beta$, with $\tilde{a} \neq 0$, is of the form

$$C(q_\varepsilon^2)^{\beta/2-m} \prod_{i=1}^m \partial_k^{\tilde{a}_i} q_\varepsilon^2, \qquad \tilde{a}_1 + \cdots + \tilde{a}_m = \tilde{a}, \qquad |\tilde{a}_i| \geqslant 1.$$

Consequently,

$$\left|(q_\varepsilon^2)^{\beta/2-m} \prod_{i=1}^m \partial_k^{\tilde{a}_i} q_\varepsilon^2\right| \lesssim (q_\varepsilon^2)^{\beta/2-m} \prod_{i=1}^m q_\varepsilon^{2-|\tilde{a}_i|} = q_\varepsilon^{\beta-|\tilde{a}|}.$$

This proves (A.1).

We next prove (A.3). For each $i \in \{1, \ldots, d\}$, each sign, and each $\tilde{r} \in \mathbb{N}_0^d$,

$$\left|\partial_k^{\tilde{r}} d_\varepsilon^{i,\pm}(k)\right| \lesssim_{\tilde{r}} q_\varepsilon^{1-|\tilde{r}|}(k), \qquad q_\varepsilon(k) > 0.$$

For $\tilde{r} = 0$, this follows from

$$|d_\varepsilon^{i,\pm}(k)| \lesssim q_\varepsilon(k).$$

For $|\tilde{r}| = 1$, it is immediate from the explicit formula for $d_\varepsilon^{i,\pm}$. For $|\tilde{r}| \geqslant 2$, the derivative vanishes unless $\tilde{r} = ne_i$, in which case it is bounded by $\varepsilon^{|\tilde{r}|-1}$. Since $q_\varepsilon(k) \lesssim \varepsilon^{-1}$, this gives

$$\varepsilon^{|\tilde{r}|-1} \lesssim q_\varepsilon^{1-|\tilde{r}|}(k).$$

The product rule now yields (A.3). Estimate (A.4) follows by differentiating the factor $(ik_0)^{A_0}$.

Estimate (A.5) follows from (A.1) and the chain rule, since on $\operatorname{supp}\chi(\eta q_\varepsilon)$ one has

$$q_\varepsilon \asymp \eta^{-1}.$$

Combining the same argument with (A.3) gives (A.6).

Finally, (A.7) follows from $q_\varepsilon \asymp |k|$. Taking $\gamma = (r-4s)_+$ gives (A.8). ∎

Lemma A.3 (Ordinary lattice heat-kernel estimates). *Let p_t^ε be the kernel of $e^{t\Delta_\varepsilon}$, with respect to ε^d-counting measure. Then $p_t^\varepsilon \geq 0$,*

$$\int_{\mathbb{R}_\varepsilon^d} p_t^\varepsilon(x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}_\varepsilon^d} |x|^2 p_t^\varepsilon(x)\,\mathrm{d}x = 2dt,$$

and

$$\|p_t^\varepsilon\|_{L^\infty} \lesssim (\sqrt{t}+\varepsilon)^{-d}. \tag{A.9}$$

For every $q \geq 2$,

$$\int_{\mathbb{R}_\varepsilon^d} |x|^q p_t^\varepsilon(x)\,\mathrm{d}x \lesssim_q t^{q/2} + t\varepsilon^{q-2}. \tag{A.10}$$

Moreover, for every $\tau \in (0,1]$ and $i \in \{1,\dots,d\}$,

$$\begin{aligned}&\|\partial^{i\pm} p_t^\varepsilon\|_{\mathrm{TV}(1+|\bullet|^2/\tau^2)}\\ &\quad\lesssim (\sqrt{t}+\varepsilon)^{-1}\left(1+\frac{t+\varepsilon^2}{\tau^2}\right)+\frac{\sqrt{t}}{\tau^2}.\end{aligned} \tag{A.11}$$

Finally, for $p \in [1,\infty)$ and $\omega \geq 0$, there exists $C = C(p,\omega,d)$ such that

$$\|(1+|\bullet|)^\omega p_t^\varepsilon\|_{L^p} \lesssim (\sqrt{t}+\varepsilon)^{-d(1-1/p)}(1+t)^C. \tag{A.12}$$

Proof. Positivity and conservation of mass follow from the Markov property. If X_t^ε denotes the continuous-time nearest-neighbour random walk with generator Δ_ε, then

$$\Delta_\varepsilon |x|^2 = 2d.$$

Dynkin's formula therefore gives $\mathbb{E}|X_t^\varepsilon|^2 = 2dt$, proving the exact second-moment identity.

The Fourier representation is

$$p_t^\varepsilon(x) = \int_{(-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-tq_\varepsilon(k)^2} e^{ik\cdot x}\frac{\mathrm{d}k}{(2\pi)^d}.$$

On the Brillouin zone, $q_\varepsilon(k)^2 \gtrsim |k|^2$. Integrating the Gaussian gives $Ct^{-d/2}$, while bounding the integrand by one gives $C\varepsilon^{-d}$. Taking the minimum proves (A.9).

Let N_t be the number of jumps up to time t. It is Poisson with parameter $2dt/\varepsilon^2$, and the quadratic variation of the martingale X^ε satisfies $[X^\varepsilon]_t = \varepsilon^2 N_t$. By the Burkholder–Davis–Gundy inequality and the standard Poisson moment estimate,

$$\mathbb{E}|X_t^\varepsilon|^q \lesssim_q \varepsilon^q \mathbb{E}N_t^{q/2} \lesssim_q t^{q/2} + t\varepsilon^{q-2}.$$

This proves (A.10).

We prove (A.11) for the forward difference. In one dimension let $P_t(n)$ be the probability that $X_t^\varepsilon = n\varepsilon$. The sequence $P_t(n)$ is symmetric and non-increasing for $n \geq 0$. Discrete summation by parts gives

$$\sum_{n\in\mathbb{Z}} \left(1 + \frac{n^2\varepsilon^2}{\tau^2}\right) |P_t(n+1) - P_t(n)| \lesssim P_t(0)\left(1 + \frac{\varepsilon^2}{\tau^2}\right) + \frac{\varepsilon^2}{\tau^2}\sum_n |n| P_t(n).$$

The on-diagonal estimate yields $P_t(0) \lesssim \varepsilon/(\sqrt{t} + \varepsilon)$, and Cauchy–Schwarz yields

$$\sum_n |n| P_t(n) \lesssim \sqrt{t}/\varepsilon.$$

Dividing by ε, as required by the discrete difference, proves the one-dimensional estimate. In dimension d, the walk is a product of one-dimensional walks. The coordinates different from i contribute their second moments, of total size $O(t/\tau^2)$, multiplied by the unweighted variation in the i-th coordinate. This gives (A.11). The backward difference is identical.

For (A.12), choose an even integer $q \geq \omega p$. Positivity gives, for $1 \leqslant p < \infty$,

$$\begin{aligned}\|(1 + |\bullet|)^\omega p_t^\varepsilon\|_{L^p}^p &\leq \|p_t^\varepsilon\|_{L^\infty}^{p-1} \int (1 + |x|)^{\omega p} p_t^\varepsilon(x)\,\mathrm{d}x \\ &\lesssim (\sqrt{t} + \varepsilon)^{-d(p-1)}(1 + t)^C.\end{aligned}$$

Taking the p-th root concludes the proof. ∎

Lemma A.4. *Let $k \in \mathbb{N}_0$ and $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ satisfy $A_0 \in \{0,1\}$ and $|\bar{A}| \leq 2$. Then*

$$\|\partial^A K_\mu\|_{\mathrm{TV}(w_\mu^2)} \lesssim [\![\mu]\!]^{-|A|_s}\left(1 + \frac{\varepsilon}{[\![\mu]\!]}\right)^{\mathbb{1}_{|\bar{A}|=1}}. \tag{A.13}$$

Moreover, uniformly in $\mu \in [0,1)$ and $\eta \in [\mu, 1)$,

$$\|K_{\mu,\eta}\|_{\mathrm{TV}(w_\mu^2)} \lesssim 1, \tag{A.14}$$

$$\|(L_\eta - 1)K_\mu\|_{\mathrm{TV}(w_\mu^2)} \lesssim [\![\mu]\!]^{-2s}[\![\eta]\!]^{2s}, \tag{A.15}$$

$$\|(\tilde{L}_\eta^{1,k} - 1)\tilde{K}_\mu^{1,k}\|_{\mathrm{TV}((w_\mu^2)^{\otimes(1+k)})} \lesssim [\![\mu]\!]^{-2s}[\![\eta]\!]^{2s}. \tag{A.16}$$

Proof. Set $\tau = [\![\mu]\!]$. Using the notation introduced in Lemma A.5 below, we write

$$K_\mu = \mathring{K}_\tau \otimes (\bar{K}_\tau * \bar{K}_\tau),$$

and observe that

$$w_\mu^2(t,x) \lesssim \mathring{w}_\tau(t)\bar{w}_\tau(x).$$

The estimates with no spatial difference and with exactly one spatial difference follow from weighted Young's inequality and Lemma A.5. A time derivative costs τ^{-2s}.

Consider now two spatial differences. Since difference operators commute with convolution, every such term can be written as a finite linear combination of kernels

$$(\partial^{i,\square}\bar{K}_\tau) * (\partial^{j,\triangle}\bar{K}_\tau), \qquad \square, \triangle \in \{+,-\}.$$

The weight satisfies

$$\bar{w}_\tau(x+y) \lesssim \bar{w}_\tau(x) + \bar{w}_\tau(y).$$

Weighted Young's inequality therefore yields

$$\begin{aligned}\tau^2\|(\partial^{i,\square}\bar{K}_\tau) * (\partial^{j,\triangle}\bar{K}_\tau)\|_{\mathrm{TV}(\bar{w}_\tau)} \\ \lesssim \big[\tau\|\partial^{i,\square}\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)}\big]\big[\tau\|\partial^{j,\triangle}\bar{K}_\tau\|_{\mathrm{TV}}\big] \\ + \big[\tau\|\partial^{i,\square}\bar{K}_\tau\|_{\mathrm{TV}}\big]\big[\tau\|\partial^{j,\triangle}\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)}\big].\end{aligned}$$

By (A.19) and (A.20), each product on the right is bounded by

$$\Big(1+\frac{\varepsilon}{\tau}\Big)\min\Big\{1,\frac{\tau}{\varepsilon}\Big\} \lesssim 1.$$

This proves (A.13) for $|\bar{A}| = 2$, including mixed spacetime derivatives.

We turn to the estimates involving two scales. Since $\eta \geq \mu$, one has $[\![\eta]\!] \leq [\![\mu]\!]$. Expand $K_{\mu,\eta} = L_\eta K_\mu$ into temporal and spatial factors. Every nonidentity temporal factor is controlled by (A.21), whereas every nonidentity spatial factor is a complete scaled Laplacian block and is controlled by (A.22). This proves (A.14).

Similarly, every term in $(L_\eta - 1)K_\mu$ contains either $[\![\eta]\!]^{2s}\partial_t$ or $[\![\eta]\!]^2\Delta_\varepsilon$. The temporal contribution is bounded by

$$[\![\eta]\!]^{2s}[\![\mu]\!]^{-2s}.$$

The spatial contribution is bounded by

$$[\![\eta]\!]^2[\![\mu]\!]^{-2} \leq [\![\eta]\!]^{2s}[\![\mu]\!]^{-2s},$$

because $s \leq 1$ and $[\![\eta]\!]/[\![\mu]\!] \leq 1$. Mixed terms are smaller. This proves (A.15).

Finally, expand $\tilde{L}_\eta^{1,k} - 1$ as a finite sum of tensor products in which at least one factor is $L_\eta - 1$. Apply (A.15) to that factor and (A.14) to all remaining factors. This proves (A.16). ∎

Lemma A.5. *For $\tau \in (0,1]$ and $\eta \in [\tau, 1]$, define*

$$\begin{aligned}\mathring{L}_\tau &:= 1 + \tau^{2s}\partial_t, & \bar{L}_\tau &:= 1 - \tau^2\Delta, \\ \mathring{K}_\tau &:= \mathring{L}_\tau^{-1}, & \bar{K}_\tau &:= \bar{L}_\tau^{-1}, \\ \mathring{K}_{\eta,\tau} &:= \mathring{L}_\tau\mathring{K}_\eta, & \bar{K}_{\eta,\tau} &:= \bar{L}_\tau\bar{K}_\eta,\end{aligned}$$

and

$$\mathring{w}_\tau(t) := 1 + \frac{|t|^2}{\tau^{4s}}, \qquad \bar{w}_\tau(x) := 1 + \frac{|x|^2}{\tau^2}.$$

Then the following estimates hold uniformly in $\varepsilon \in 2^{-\mathbb{N}_0}$.

(a) *Uniformly in $\tau \in (0,1]$,*

$$\|\mathring{K}_\tau\|_{\mathrm{TV}(\mathring{w}_\tau)} + \tau^{2s}\|\partial_t\mathring{K}_\tau\|_{\mathrm{TV}(\mathring{w}_\tau)} \lesssim 1, \tag{A.17}$$

$$\|\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} + \tau^2\|\Delta\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \lesssim 1. \tag{A.18}$$

The first spatial differences satisfy

$$\tau\|\partial^{i\pm}\bar{K}_\tau\|_{\mathrm{TV}} \lesssim \min\left\{1, \frac{\tau}{\varepsilon}\right\}, \tag{A.19}$$

$$\tau\|\partial^{i\pm}\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \lesssim 1 + \frac{\varepsilon}{\tau} \lesssim \frac{\varepsilon \vee \tau}{\tau}. \tag{A.20}$$

(b) *Uniformly in* $\tau \in (0,1]$ *and* $\eta \in [\tau, 1]$,

$$\|(1 - \mathring{L}_\tau)\mathring{K}_\eta\|_{\mathrm{TV}(\mathring{w}_\eta)} \lesssim \tau^{2s}\eta^{-2s}, \tag{A.21}$$

$$\|(1 - \bar{L}_\tau)\bar{K}_\eta\|_{\mathrm{TV}(\bar{w}_\eta)} \lesssim \tau^2\eta^{-2} \le \tau^{2s}\eta^{-2s}. \tag{A.22}$$

(c) *Uniformly in the same range,*

$$\|\mathring{K}_{\eta,\tau}\|_{\mathrm{TV}(\mathring{w}_\eta)} + \|\bar{K}_{\eta,\tau}\|_{\mathrm{TV}(\bar{w}_\eta)} \lesssim 1. \tag{A.23}$$

Proof. The temporal kernel is explicit:

$$\mathring{K}_\tau(t) = \tau^{-2s}\mathbb{1}_{\{t\ge 0\}}e^{-t/\tau^{2s}}.$$

The first term in (A.17) follows by the change of variables $t = \tau^{2s}u$. The distributional derivative is

$$\partial_t\mathring{K}_\tau = \tau^{-2s}\delta_0 - \tau^{-4s}\mathbb{1}_{\{t\ge 0\}}e^{-t/\tau^{2s}},$$

and the same change of variables proves the second term.

For the spatial kernel, the resolvent formula gives

$$\bar{K}_\tau = \int_0^\infty e^{-\lambda}e^{\lambda\tau^2\Delta_\varepsilon}\,\mathrm{d}\lambda, \qquad \bar{K}_\tau(x) = \int_0^\infty e^{-\lambda}p^\varepsilon_{\lambda\tau^2}(x)\,\mathrm{d}\lambda. \tag{A.24}$$

The kernel is nonnegative and has mass one. Using the exact second-moment identity from Lemma A.3,

$$\begin{aligned}\|\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} &= \int_0^\infty e^{-\lambda}\left(1 + \tau^{-2}\int |x|^2 p^\varepsilon_{\lambda\tau^2}(x)\,\mathrm{d}x\right)\mathrm{d}\lambda \\ &= \int_0^\infty e^{-\lambda}(1 + 2d\lambda)\,\mathrm{d}\lambda \lesssim 1.\end{aligned}$$

Since

$$(1 - \tau^2\Delta_\varepsilon)\bar{K}_\tau = \delta_0, \qquad \tau^2\Delta_\varepsilon\bar{K}_\tau = \bar{K}_\tau - \delta_0,$$

the scaled-Laplacian estimate in (A.18) follows.

We prove the weighted first-difference estimate. If $\tau \ge \varepsilon$, Minkowski's inequality, (A.24), and (A.11) give

$$\begin{aligned}&\tau\|\partial^{i\pm}\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \\ &\lesssim \int_0^\infty e^{-\lambda}\left[\frac{\tau}{\tau\sqrt{\lambda} + \varepsilon}\left(1 + \lambda + \frac{\varepsilon^2}{\tau^2}\right) + \sqrt{\lambda}\right]\mathrm{d}\lambda.\end{aligned}$$

On $0 < \lambda < (\varepsilon/\tau)^2$, use

$$\frac{\tau}{\tau\sqrt{\lambda} + \varepsilon} \le \frac{\tau}{\varepsilon};$$

the length of this interval is $(\varepsilon/\tau)^2$. On its complement, use

$$\frac{\tau}{\tau\sqrt{\lambda} + \varepsilon} \le \lambda^{-1/2}.$$

Both integrals are bounded uniformly, so

$$\tau \|\partial^{i\pm} \bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \lesssim 1 \qquad \text{when } \tau \ge \varepsilon.$$

Suppose now that $\tau < \varepsilon$. Positivity implies

$$|\partial^{i+} \bar{K}_\tau(x)| \le \varepsilon^{-1} \big(\bar{K}_\tau(x + \varepsilon e_i) + \bar{K}_\tau(x)\big).$$

Furthermore,

$$\bar{w}_\tau(y - \varepsilon e_i) \lesssim \left(1 + \frac{\varepsilon^2}{\tau^2}\right) \bar{w}_\tau(y).$$

Consequently,

$$\begin{aligned} \tau \|\partial^{i+} \bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} &\lesssim \frac{\tau}{\varepsilon} \left(1 + \frac{\varepsilon^2}{\tau^2}\right) \|\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \\ &\lesssim 1 + \frac{\varepsilon}{\tau}. \end{aligned}$$

The backward difference is identical. This proves (A.20).

For the unweighted estimate, the preceding diffusive argument gives $O(1)$ when $\tau \ge \varepsilon$. When $\tau < \varepsilon$, positivity and mass one give

$$\|\partial^{i\pm} \bar{K}_\tau\|_{\mathrm{TV}} \le 2\varepsilon^{-1}.$$

Multiplication by τ proves (A.19).

Finally,

$$(1 - \mathring{L}_\tau)\mathring{K}_\eta = -\tau^{2s} \partial_t \mathring{K}_\eta,$$

and the explicit temporal kernel at scale η gives (A.21). Similarly,

$$(1 - \bar{L}_\tau)\bar{K}_\eta = \tau^2 \Delta_\varepsilon \bar{K}_\eta = \frac{\tau^2}{\eta^2}(\bar{K}_\eta - \delta_0),$$

which proves (A.22). Since

$$\mathring{K}_{\eta,\tau} = \mathring{K}_\eta - (1 - \mathring{L}_\tau)\mathring{K}_\eta, \qquad \bar{K}_{\eta,\tau} = \bar{K}_\eta - (1 - \bar{L}_\tau)\bar{K}_\eta,$$

(A.23) follows from the preceding estimates. ∎

Remark A.6. The factor $1 + \varepsilon/\tau$ in (A.20) is necessary. Indeed, when τ/ε is small,

$$\bar{K}_\tau(0) = \varepsilon^{-d}\left(1 + O(\tau^2/\varepsilon^2)\right), \qquad \bar{K}_\tau(-\varepsilon e_i) = O\left(\varepsilon^{-d}\tau^2/\varepsilon^2\right).$$

Therefore

$$|\partial^{i+}\bar{K}_\tau(-\varepsilon e_i)| \gtrsim \varepsilon^{-d-1}.$$

Retaining this single nonnegative summand in the weighted total-variation norm gives

$$\tau\|\partial^{i+}\bar{K}_\tau\|_{\mathrm{TV}(\bar{w}_\tau)} \gtrsim \frac{\varepsilon}{\tau}.$$

Lemma A.7. *Let $N \in \mathbb{N}_+$, set*

$$p := \frac{2N}{2N-1},$$

and assume

$$\hat{\kappa} > \max\left\{\frac{1}{2N}, \frac{d}{4N}\right\}.$$

Recall Definition 4.20 of $\check{K}_\mu$. Then

$$\|\check{K}_\mu\|_{L^p(\zeta^{-1})} \lesssim [\![\mu]\!]^{-(d+2s)/(2N)} \tag{A.25}$$

uniformly in $\mu \in (0,1)$ and $\varepsilon \in 2^{-\mathbb{N}_0}$.

Proof. Put $\tau = [\![\mu]\!]$. The temporal kernel of $(1+\tau^{2s}\partial_t)^{-\hat{\kappa}}$ is

$$\check{K}_\tau(t) = \frac{\mathbb{1}_{\{t>0\}}}{\Gamma(\hat{\kappa})}\tau^{-2s\hat{\kappa}}t^{\hat{\kappa}-1}e^{-t/\tau^{2s}} = \tau^{-2s}\check{K}_1(t/\tau^{2s}).$$

For every $\omega \geq 0$,

$$\|(1+|\bullet|)^\omega\check{K}_\tau\|_{L^p(\mathbb{R})} \lesssim_\omega \tau^{-2s(1-1/p)},$$

provided $p(\hat{\kappa}-1) > -1$, equivalently $\hat{\kappa} > 1 - 1/p = 1/(2N)$.

Suppose first that $\tau \geq \varepsilon$. The spatial factor is

$$(1-\tau^2\Delta_\varepsilon)^{-\hat{\kappa}} = \frac{1}{\Gamma(\hat{\kappa})}\int_0^\infty e^{-\lambda}e^{\lambda\tau^2\Delta_\varepsilon}\lambda^{\hat{\kappa}-1}\,\mathrm{d}\lambda.$$

By Minkowski's inequality and (A.12),

$$\begin{aligned}&\|(1+|\bullet|)^\omega(1-\tau^2\Delta_\varepsilon)^{-\hat{\kappa}}\|_{L^p}\\ &\lesssim \int_0^\infty e^{-\lambda}\lambda^{\hat{\kappa}-1}(\tau\sqrt{\lambda}+\varepsilon)^{-d(1-1/p)}(1+\tau^2\lambda)^C\,\mathrm{d}\lambda.\end{aligned}$$

Since $\tau \leq 1$, the last polynomial factor is bounded by $(1+\lambda)^C$. Factoring out $\tau^{-d(1-1/p)}$ and setting $\delta = \varepsilon/\tau \in (0,1]$, it remains to control

$$\int_0^\infty e^{-\lambda}(1+\lambda)^C\lambda^{\hat{\kappa}-1}(\sqrt{\lambda}+\delta)^{-d(1-1/p)}\,\mathrm{d}\lambda.$$

Split the integral into $(0,\delta^2)$, $(\delta^2,1)$, and $(1,\infty)$. On the first interval the integral is bounded by $C\delta^{2\hat\kappa-d(1-1/p)}$; on the second it is bounded by

$$C\int_{\delta^2}^{1}\lambda^{\hat\kappa-1-d(1-1/p)/2}\,\mathrm{d}\lambda;$$

the last interval is controlled by the exponential. All three terms are uniformly bounded precisely when $2\hat\kappa > d(1-1/p)$, i.e. $\hat\kappa > d/(4N)$. Hence

$$\|(1+|\bullet|)^{\omega}(1-\tau^2\Delta_\varepsilon)^{-\hat\kappa}\|_{L^p} \lesssim_\omega \tau^{-d/(2N)}.$$

Suppose now that $\tau<\varepsilon$. The spatial factor is

$$(1-\tau^2\Delta_\varepsilon)^{-2}.$$

It is a positive probability kernel. Its L^∞-norm is bounded by $C\varepsilon^{-d}$, because its Fourier multiplier is bounded by one. Interpolation between L^1 and L^∞ gives

$$\|(1-\tau^2\Delta_\varepsilon)^{-2}\|_{L^p} \lesssim \varepsilon^{-d(1-1/p)} \le \tau^{-d(1-1/p)}.$$

To include polynomial weights, use

$$(1-\tau^2\Delta_\varepsilon)^{-2} = \int_0^\infty e^{-\lambda}\lambda e^{\lambda\tau^2\Delta_\varepsilon}\,\mathrm{d}\lambda.$$

By (A.10), for every fixed q,

$$\int (1+|x|)^q(1-\tau^2\Delta_\varepsilon)^{-2}(x)\,\mathrm{d}x \lesssim_q 1$$

uniformly when $\tau<\varepsilon\le 1$. Positivity therefore gives

$$\begin{aligned}&\|(1+|\bullet|)^{\omega}(1-\tau^2\Delta_\varepsilon)^{-2}\|_{L^p}^p\\ &\quad\le \|(1-\tau^2\Delta_\varepsilon)^{-2}\|_{L^\infty}^{p-1}\int(1+|x|)^{\omega p}(1-\tau^2\Delta_\varepsilon)^{-2}(x)\,\mathrm{d}x\\ &\quad\lesssim \varepsilon^{-d(p-1)} \le \tau^{-d(p-1)}.\end{aligned}$$

Thus the same spatial estimate holds below the lattice scale.

Choose ω large enough that

$$\zeta(t,x)^{-1} \lesssim (1+|t|)^{\omega}(1+|x|)^{\omega}.$$

The temporal and spatial kernels factorise, so Fubini gives

$$\|\check K_\mu\|_{L^p(\zeta^{-1})} \lesssim \tau^{-2s/(2N)}\tau^{-d/(2N)} = \tau^{-(d+2s)/(2N)}.$$

This finishes the proof. ∎

Lemma A.8. *Recall that $d = 3$. Let $m \in \mathbb{N}_+$ and*

$$\mathcal{K}_\mu(z_1, \dots, z_m) := \int_{\Lambda_\varepsilon} \hat{K}_\mu(z)\hat{K}_\mu(z_1 + z) \cdots \hat{K}_\mu(z_m + z)\, \mathrm{d}z, \tag{A.26}$$

where $\hat{K}_\mu$ is the kernel introduced in Definition 4.20. If

$$0 \le \hat{\kappa} < \frac{1}{m+1},$$

then $\mathcal{K}_\mu$ is continuous in the time variables and, for every $\omega \ge 0$,

$$\|(w_\mu^\omega)^{\otimes m}\mathcal{K}_\mu\|_{L^\infty} \lesssim [\![\mu]\!]^{-2sm}(\varepsilon \vee [\![\mu]\!])^{-dm}, \tag{A.27}$$

uniformly in $\mu \in (0,1)$ and $\varepsilon \in 2^{-\mathbb{N}_0}$.

Proof. Put $\tau := [\![\mu]\!]$. The kernel $\hat{K}_\mu$ factorises into its temporal and spatial parts:

$$\hat{K}_\mu(t,x) = H_\tau(t)\bar{H}_{\tau,\varepsilon}(x).$$

Accordingly, the kernel in (A.26) factorises as

$$\mathcal{K}_\mu(t_1, x_1, \dots, t_m, x_m) = \mathring{\mathcal{K}}_\mu(t_1, \dots, t_m)\bar{\mathcal{K}}_\mu(x_1, \dots, x_m).$$

The temporal kernel is the kernel of $(1 + \tau^{2s}\partial_t)^{-(1-\hat{\kappa})}$, namely

$$H_\tau(t) = \frac{\mathbb{1}_{\{t>0\}}}{\Gamma(1-\hat{\kappa})}\tau^{-2s(1-\hat{\kappa})}t^{-\hat{\kappa}}e^{-t/\tau^{2s}} = \tau^{-2s}H_1(t/\tau^{2s}).$$

Since $(m+1)\hat{\kappa} < 1$, one has $H_1 \in L^{m+1}(\mathbb{R})$. Hence Hölder's inequality gives

$$\sup_{t_1,\dots,t_m \in \mathbb{R}} \int_{\mathbb{R}} |H_1(t)H_1(t+t_1)\cdots H_1(t+t_m)|\, \mathrm{d}t < \infty.$$

The exponential decay of H_1, together with the submultiplicativity of polynomial weights, gives, for every $\omega \ge 0$,

$$\sup_{t_1,\dots,t_m \in \mathbb{R}} \prod_{i=1}^m \left(1 + \frac{|t_i|^{1/(2s)}}{\tau}\right)^\omega \left|\mathring{\mathcal{K}}_\mu(t_1, \dots, t_m)\right| \lesssim_\omega \tau^{-2sm}.$$

Continuity in $t_1, \dots, t_m$ follows from the continuity of translations in $L^{m+1}(\mathbb{R})$.

We next estimate the spatial correlation kernel. Suppose first that $\tau \ge \varepsilon$, and put

$$\delta := \frac{\varepsilon}{\tau} \in (0,1].$$

The spatial part of $\hat{K}_\mu$ is

$$(1 - \tau^2\Delta_\varepsilon)^{-(2-\hat{\kappa})}.$$

Its kernel has the Fourier representation

$$\begin{aligned}\bar{H}_{\tau,\varepsilon}(x) &= \int_{\mathbb{T}^d_\varepsilon} e^{i\tilde{k}\cdot x}(1+\tau^2 q_\varepsilon(\tilde{k})^2)^{-(2-\hat{\kappa})}\,\mathrm{d}\tilde{k}\\ &= \tau^{-d}\int_{\mathbb{T}^d_\delta} e^{i\tilde{q}\cdot(x/\tau)} a_\delta(\tilde{q})\,\mathrm{d}\tilde{q},\end{aligned}$$

where

$$a_\delta(\tilde{q}) := (1+q_\delta(\tilde{q})^2)^{-(2-\hat{\kappa})}.$$

By the strengthened $\beta = 2$ estimate in Lemma A.2, for every multi-index $A \in \mathbb{N}_0^d$,

$$\sup_{\delta\in(0,1]} \|\partial_{\tilde{q}}^A a_\delta\|_{L^1(\mathbb{T}^d_\delta)} < \infty. \tag{A.28}$$

Thus, for every $\omega \in \mathbb{N}_0$,

$$\left(1+\frac{|x|^2}{\tau^2}\right)^\omega \bar{H}_{\tau,\varepsilon}(x) = \tau^{-d}\int_{\mathbb{T}^d_\delta} e^{i\tilde{q}\cdot(x/\tau)}(1-\Delta_{\tilde{q}})^\omega a_\delta(\tilde{q})\,\mathrm{d}\tilde{q}.$$

Using (A.28), we conclude that, for every $\omega \geq 0$,

$$|\bar{H}_{\tau,\varepsilon}(x)| \lesssim_\omega \tau^{-d}\left(1+\frac{|x|}{\tau}\right)^{-\omega}, \qquad \tau \geq \varepsilon. \tag{A.29}$$

Let

$$\bar{\mathcal{K}}_\mu(x_1,\dots,x_m) := \int_{\mathbb{R}^d_\varepsilon} \bar{H}_{\tau,\varepsilon}(x)\bar{H}_{\tau,\varepsilon}(x+x_1)\cdots\bar{H}_{\tau,\varepsilon}(x+x_m)\,\mathrm{d}x.$$

For every $\bar{\omega} > d$, the discrete Riemann-sum estimate

$$\int_{\mathbb{R}^d_\varepsilon}\left(1+\frac{|x|}{\tau}\right)^{-\bar{\omega}}\mathrm{d}x \lesssim_{\bar{\omega}} \tau^d, \qquad \tau \geq \varepsilon, \tag{A.30}$$

holds uniformly in ε and τ. Moreover,

$$1+\frac{|x_i|}{\tau} \lesssim \left(1+\frac{|x|}{\tau}\right)\left(1+\frac{|x+x_i|}{\tau}\right).$$

Choose $\bar{\omega} > m\omega + d$. Inserting (A.29), transferring the weights by the preceding inequality, and then using (A.30), we obtain

$$\begin{aligned}&\prod_{i=1}^m\left(1+\frac{|x_i|}{\tau}\right)^\omega |\bar{\mathcal{K}}_\mu(x_1,\dots,x_m)|\\ &\quad\lesssim \tau^{-d(m+1)}\int_{\mathbb{R}^d_\varepsilon}\left(1+\frac{|x|}{\tau}\right)^{-\bar{\omega}+m\omega}\prod_{i=1}^m\left(1+\frac{|x+x_i|}{\tau}\right)^{-\bar{\omega}+\omega}\mathrm{d}x\\ &\quad\lesssim \tau^{-d(m+1)}\int_{\mathbb{R}^d_\varepsilon}\left(1+\frac{|x|}{\tau}\right)^{-\bar{\omega}+m\omega}\mathrm{d}x \lesssim \tau^{-dm}.\end{aligned}$$

Hence

$$\|((1+|\bullet|/\tau)^{\omega})^{\otimes m}\bar{\mathcal{K}}_\mu\|_{L^\infty} \lesssim \tau^{-dm}, \qquad \tau \geq \varepsilon. \tag{A.31}$$

Suppose now that $\tau < \varepsilon$. The spatial part of $\hat{K}_\mu$ is the identity, whose kernel is

$$\delta_0^\varepsilon(x) = \varepsilon^{-d}\mathbb{1}_{\{x=0\}}.$$

Its spatial correlation kernel is therefore

$$\bar{\mathcal{K}}_\mu(x_1,\dots,x_m) = \varepsilon^{-dm}\prod_{j=1}^{m}\mathbb{1}_{\{x_j=0\}}.$$

The spatial weights equal one on its support, and consequently

$$\|((1+|\bullet|/\tau)^{\omega})^{\otimes m}\bar{\mathcal{K}}_\mu\|_{L^\infty} = \varepsilon^{-dm}, \qquad \tau < \varepsilon. \tag{A.32}$$

Finally, by Definition 1.16,

$$w_\mu^\omega(t_i,x_i) \lesssim \left(1+\frac{|t_i|^{1/(2s)}}{\tau}\right)^{\omega}\left(1+\frac{|x_i|}{\tau}\right)^{\omega}.$$

Combining the temporal estimate with (A.31) and (A.32) gives

$$\|(w_\mu^\omega)^{\otimes m}\mathcal{K}_\mu\|_{L^\infty} \lesssim \tau^{-2sm}\begin{cases}\tau^{-dm}, & \tau \geq \varepsilon,\\ \varepsilon^{-dm}, & \tau < \varepsilon.\end{cases}$$

This is precisely (A.27). ∎

Lemma A.9. *Let $\ell \in \mathbb{N}_0$. Put*

$$\tau := [\![\sigma]\!], \qquad \rho := \tau \vee \varepsilon.$$

The following estimates hold uniformly in $\varepsilon \in 2^{-\mathbb{N}_0}$, $\sigma \in [1/2,1)$, and $(t,x) \in \Lambda_\varepsilon$.

(a) *For every $N \in \mathbb{N}_0$ and $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$,*

$$|\partial^A \tilde{J}_{\sigma,\ell}(t,x)| \lesssim_{A,N,\ell} \tau^{-2s(1+A_0)}\rho^{-d-|\bar{A}|}\left(1+\frac{|t|}{\tau^{2s}}+\frac{|x|}{\rho}\right)^{-N}, \tag{A.33}$$

$$|\partial^A\partial_\sigma \tilde{J}_{\sigma,\ell}(t,x)| \lesssim_{A,N,\ell} [\![\sigma]\!]^{-1}\tau^{-2s(1+A_0)}\rho^{-d-|\bar{A}|}\left(1+\frac{|t|}{\tau^{2s}}+\frac{|x|}{\rho}\right)^{-N}. \tag{A.34}$$

(b) *Let $a \in \mathbb{N}_0$, let $\bar{A}$ be a spatial finite-difference multi-index, and let $\omega \geq 0$. Define*

$$\mathfrak{l}_{\bar{A},\omega}(u) := \begin{cases}1, & \bar{A} = 0,\\ (1+u)^{(\omega-|\bar{A}|)_+}, & \bar{A} \neq 0.\end{cases}$$

Then

$$\|\partial_t^a \partial^{\bar{A}} \tilde{J}_{\sigma,\ell}\|_{\mathrm{TV}(w^\omega_\sigma)} \lesssim_{a,\bar{A},\omega,\ell} [\![\sigma]\!]^{-2sa-|\bar{A}|} \mathrm{I}_{\bar{A},\omega}\left(\frac{\varepsilon}{[\![\sigma]\!]}\right), \tag{A.35}$$

$$\|\partial_t^a \partial^{\bar{A}} \partial_\sigma \tilde{J}_{\sigma,\ell}\|_{\mathrm{TV}(w^\omega_\sigma)} \lesssim_{a,\bar{A},\omega,\ell} [\![\sigma]\!]^{-2sa-|\bar{A}|-1} \mathrm{I}_{\bar{A},\omega}\left(\frac{\varepsilon}{[\![\sigma]\!]}\right). \tag{A.36}$$

In particular, there is no sublattice loss when $\bar{A} = 0$, nor when $|\bar{A}| \geq 1$ and $\omega \leq |\bar{A}|$.

Consequently, for every fixed $m \in \mathbb{N}_0$,

$$\|L^m_\sigma \tilde{J}_{\sigma,\ell}\|_{\mathrm{TV}(w^2_\sigma)} \lesssim_{m,\ell} 1, \tag{A.37}$$

$$\|\partial_t \tilde{J}_{\sigma,\ell}\|_{\mathrm{TV}(w^2_\sigma)} \lesssim_\ell [\![\sigma]\!]^{-2s}, \tag{A.38}$$

$$\|\nabla_\varepsilon \tilde{J}_{\sigma,\ell}\|_{\mathrm{TV}(w^1_\sigma)} \lesssim_\ell [\![\sigma]\!]^{-1}. \tag{A.39}$$

Since $\zeta^{-1} \lesssim w^1_\sigma$, the last estimate implies

$$\|\nabla_\varepsilon \tilde{J}_{\sigma,\ell}\|_{\mathrm{TV}(\zeta^{-1})} \lesssim_\ell [\![\sigma]\!]^{-1}.$$

Proof. Fix $\ell \in \mathbb{N}_0$, and set

$$c_{\sigma,\ell} := 2^{-\ell}\sigma^{-1}.$$

Thus the symbol of $\tilde{J}_{\sigma,\ell}$ is

$$j\big(c_{\sigma,\ell}\tau|k_0|^{1/(2s)}\big) j\big(c_{\sigma,\ell}\tau q_\varepsilon(k)\big).$$

Since $\sigma \in [1/2, 1)$, one has

$$c_{\sigma,\ell} \asymp_\ell 1, \qquad c_{\sigma,\ell}\tau \asymp_\ell \tau, \qquad \left|\frac{\partial_\sigma(c_{\sigma,\ell}\tau)}{c_{\sigma,\ell}\tau}\right| = \frac{1}{\sigma\tau} \lesssim \tau^{-1}. \tag{A.40}$$

Denote by $\mathring{J}_\tau$ and $\bar{J}_{\tau,\varepsilon}$ the temporal and spatial kernels whose Fourier multipliers are, respectively,

$$k_0 \longmapsto j\big(c_{\sigma,\ell}\tau|k_0|^{1/(2s)}\big), \qquad k \longmapsto j\big(c_{\sigma,\ell}\tau q_\varepsilon(k)\big).$$

Here and below, the fixed index ℓ is suppressed from the notation. With the usual convention of denoting a convolution operator and its kernel by the same symbol, one then has

$$\tilde{J}_{\sigma,\ell}(t,x) = \mathring{J}_\tau(t)\bar{J}_{\tau,\varepsilon}(x). \tag{A.41}$$

The function

$$k_0 \longmapsto j\big(|k_0|^{1/(2s)}\big)$$

is smooth and compactly supported, since it is constant in a neighbourhood of the origin. Its inverse Fourier transform is therefore Schwartz. Using (A.40), we obtain, for every $a, N \in \mathbb{N}_0$,

$$|\partial_t^a \mathring{J}_\tau(t)| \lesssim_{a,N,\ell} \tau^{-2s(1+a)}\left(1 + \frac{|t|}{\tau^{2s}}\right)^{-N}, \tag{A.42}$$

$$|\partial_t^a \partial_\sigma \mathring{J}_\tau(t)| \lesssim_{a,N,\ell} \tau^{-2s(1+a)-1}\left(1 + \frac{|t|}{\tau^{2s}}\right)^{-N}. \tag{A.43}$$

Indeed, differentiation with respect to σ replaces $j(u)$ by a constant multiple of $uj'(u)$, together with the logarithmic derivative appearing in (A.40).

We next consider the spatial factor. Suppose first that $\tau \geq \varepsilon$, set

$$\delta := \frac{\varepsilon}{\tau} \leq 1,$$

and make the change of variables $\tilde{q} = \tau \tilde{k}$ in Fourier space. Then

$$\partial^{\bar{A}} \bar{J}_{\tau,\varepsilon}(x) = \tau^{-d-|\bar{A}|} \int_{(\mathbb{R}^d_\delta)^*} d^{\bar{A}}_\delta(\tilde{q}) j\big(c_{\sigma,\ell} q_\delta(\tilde{q})\big) e^{i\tilde{q}\cdot x/\tau} \frac{\mathrm{d}\tilde{q}}{(2\pi)^d}.$$

On the support of the integrand,

$$q_\delta(\tilde{q}) \lesssim_\ell 1,$$

and hence $|\tilde{q}| \lesssim_\ell 1$, uniformly in $\delta \leq 1$. The estimates of Lemma A.2, together with the fact that j is constant near the origin, show that the Fourier derivatives of the integrand have uniformly bounded L^1-norms. Repeated integration by parts therefore gives

$$|\partial^{\bar{A}} \bar{J}_{\tau,\varepsilon}(x)| \lesssim_{\bar{A},N,\ell} \tau^{-d-|\bar{A}|} \left(1 + \frac{|x|}{\tau}\right)^{-N}. \tag{A.44}$$

Suppose next that $\tau < \varepsilon$, set

$$\lambda := \frac{\tau}{\varepsilon} \in (0,1),$$

and rescale $\tilde{q} = \varepsilon \tilde{k}$. This gives

$$\partial^{\bar{A}} \bar{J}_{\tau,\varepsilon}(x) = \varepsilon^{-d-|\bar{A}|} \int_{(-\pi,\pi]^d} d^{\bar{A}}_1(\tilde{q}) j\big(c_{\sigma,\ell} \lambda q_1(\tilde{q})\big) e^{i\tilde{q}\cdot x/\varepsilon} \frac{\mathrm{d}\tilde{q}}{(2\pi)^d}.$$

As $\sigma \in [1/2, 1)$ and $\lambda \in [0,1]$, the multipliers in the last display form a bounded subset of $C^\infty((\mathbb{R}/2\pi\mathbb{Z})^d)$. Integration by parts on the torus yields

$$|\partial^{\bar{A}} \bar{J}_{\tau,\varepsilon}(x)| \lesssim_{\bar{A},N,\ell} \varepsilon^{-d-|\bar{A}|} \left(1 + \frac{|x|}{\varepsilon}\right)^{-N}. \tag{A.45}$$

The same arguments after differentiation with respect to σ give an additional factor τ^{-1}, by (A.40). Combining (A.41)– (A.45), and recalling that $\rho = \tau \vee \varepsilon$, proves (A.33) and (A.34).

It remains to establish the weighted total-variation estimates. The temporal bounds imply, for every $\omega \geq 0$,

$$\int_{\mathbb{R}} \left(1 + \frac{|t|^{1/(2s)}}{\tau}\right)^{\omega} |\partial_t^a \mathring{J}_\tau(t)| \, \mathrm{d}t \lesssim_{a,\omega,\ell} \tau^{-2sa}, \tag{A.46}$$

$$\int_{\mathbb{R}} \left(1 + \frac{|t|^{1/(2s)}}{\tau}\right)^{\omega} |\partial_t^a \partial_\sigma \mathring{J}_\tau(t)| \, \mathrm{d}t \lesssim_{a,\omega,\ell} \tau^{-2sa-1}. \tag{A.47}$$

We now distinguish the deep sublattice regime. Since

$$q_\varepsilon(\tilde{k}) \leq 2\sqrt{d}\, \varepsilon^{-1}, \qquad c_{\sigma,\ell} \leq 2^{1-\ell},$$

there exists $c_{d,\ell} > 0$ such that

$$j\big(c_{\sigma,\ell}\tau q_\varepsilon(\tilde{k})\big) = 1 \quad \text{for every } \tilde{k} \in (\mathbb{R}^d_\varepsilon)^*$$

whenever $\tau \le c_{d,\ell}\varepsilon$. Consequently,

$$\bar{J}_{\tau,\varepsilon} = \delta_0^\varepsilon, \qquad \partial_\sigma \bar{J}_{\tau,\varepsilon} = 0 \qquad \text{if } \tau \le c_{d,\ell}\varepsilon. \tag{A.48}$$

If $|\bar{A}| \ge 1$, the kernel $\partial^{\bar{A}}\delta_0^\varepsilon$ is supported in a ball of radius $C_{\bar{A}}\varepsilon$, and

$$\|\partial^{\bar{A}}\delta_0^\varepsilon\|_{\mathrm{TV}} \lesssim_{\bar{A}} \varepsilon^{-|\bar{A}|}.$$

Hence, in the regime $\tau \le c_{d,\ell}\varepsilon$,

$$\begin{aligned}\|\partial^{\bar{A}}\delta_0^\varepsilon\|_{\mathrm{TV}((1+|\bullet|/\tau)^\omega)} &\lesssim_{\bar{A},\omega,\ell} \varepsilon^{-|\bar{A}|}\left(1+\frac{\varepsilon}{\tau}\right)^\omega \\ &\lesssim_{\bar{A},\omega,\ell} \tau^{-|\bar{A}|}\left(1+\frac{\varepsilon}{\tau}\right)^{(\omega-|\bar{A}|)_+}.\end{aligned}$$

For $|\bar{A}| = 0$, the corresponding weighted norm of δ_0^ε is exactly one.

In the complementary regime $\tau > c_{d,\ell}\varepsilon$, one has

$$\rho = \tau \vee \varepsilon \lesssim_{d,\ell} \tau.$$

The pointwise bounds, with N sufficiently large, therefore imply

$$\begin{aligned}\|\partial^{\bar{A}}\bar{J}_{\tau,\varepsilon}\|_{\mathrm{TV}((1+|\bullet|/\tau)^\omega)} &\lesssim_{\bar{A},\omega,\ell} \tau^{-|\bar{A}|}, \\ \|\partial^{\bar{A}}\partial_\sigma\bar{J}_{\tau,\varepsilon}\|_{\mathrm{TV}((1+|\bullet|/\tau)^\omega)} &\lesssim_{\bar{A},\omega,\ell} \tau^{-|\bar{A}|-1}.\end{aligned}$$

Using the factorisation (A.41), the product rule for ∂_σ, and

$$w^\omega_\sigma(t,x) \lesssim_\omega \left(1+\frac{|t|^{1/(2s)}}{\tau}\right)^\omega \left(1+\frac{|x|}{\tau}\right)^\omega,$$

the preceding estimates prove (A.35) and (A.36).

Finally, since $\tau = [\![\sigma]\!]$,

$$L^m_\sigma = (1+\tau^{2s}\partial_t)^m(1-\tau^2\Delta_\varepsilon)^{2m}.$$

Every term in this expansion is a constant multiple of

$$\tau^{2sa+2b}\partial_t^a(-\Delta_\varepsilon)^b.$$

If $b = 0$, apply (A.35) with $|\bar{A}| = 0$ and $\omega = 2$. If $b \ge 1$, write $(-\Delta_\varepsilon)^b$ as a finite linear combination of spatial finite differences of order $2b$, and apply the same estimate with $|\bar{A}| = 2b$ and $\omega = 2$. Since $2 \le 2b$, no sublattice factor is lost. This proves (A.37).

The estimates (A.38) and (A.39) follow from (A.35) with

$$(a,|\bar{A}|,\omega) = (1,0,2) \qquad \text{and} \qquad (a,|\bar{A}|,\omega) = (0,1,1),$$

respectively. ∎

Lemma A.10. *Assume $d \geq 1$ and $s \in (3/4, 1]$. Let $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$, $\mathfrak{b} \in (0, 2s)$, and let $p \in \mathbb{N}_0$ be fixed. Then, uniformly in $\varepsilon \in 2^{-\mathbb{N}_0}$, $\sigma \in [1/2, 1)$, and every admissible spatial period $M \geq 1$,*

$$\|\partial^A \dot{G}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim_A [\![\sigma]\!]^{2s-1-|A|_s}, \tag{A.49}$$

$$\|L^p_\sigma \dot{G}_\sigma\|_{\mathrm{TV}(w^{\mathfrak{b}}_\sigma)} \lesssim_{p,\mathfrak{b}} [\![\sigma]\!]^{2s-1}, \tag{A.50}$$

$$\Big\|\Big|(L^p_\sigma \dot{G}_\sigma) w^{\mathfrak{b}}_\sigma\Big|_M\Big\|_{L^\infty} \lesssim_{p,\mathfrak{b}} [\![\sigma]\!]^{-d-1}, \tag{A.51}$$

$$\big\||L^p_\sigma \dot{G}_\sigma|_M\big\|_{L^\infty} \lesssim_p [\![\sigma]\!]^{-d-1} \wedge \varepsilon^{-d} [\![\sigma]\!]^{-1}. \tag{A.52}$$

Moreover,

$$\|\partial^A G_{1/2}\|_{\mathrm{TV}(\zeta^{-1})} \lesssim_A 1, \qquad \|G\|_{\mathrm{TV}(w^{\mathfrak{b}}_0)} \lesssim_{\mathfrak{b}} 1. \tag{A.53}$$

Proof. Put $r := [\![\sigma]\!]$. Lemma A.11 implies, for every $\omega \in [0, 2s)$,

$$\|\dot{G}_\sigma\|_{\mathrm{TV}(w^\omega_\sigma)} \lesssim_\omega r^{2s-1}, \tag{A.54}$$

$$\big\||\dot{G}_\sigma w^\omega_\sigma|_M\big\|_{L^\infty} \lesssim_\omega r^{-d-1}, \tag{A.55}$$

$$\big\||\dot{G}_\sigma|_M\big\|_{L^\infty} \lesssim r^{-d-1} \wedge r^{-1}\varepsilon^{-d}. \tag{A.56}$$

Indeed, for the first estimate choose $\bar{\alpha} \in (\omega, 2s)$ and $\mathring{\alpha}$ sufficiently large in (A.59), and integrate separately in time and space. The second estimate follows in the same way, since $\bar{\alpha} > \omega$ makes the spatial translates summable. For the last estimate, fix $\bar{\alpha} \in (0, 2s)$ and use (A.59) separately in the regimes $r \geq \varepsilon$ and $r < \varepsilon$; in the latter regime the value at the origin is bounded by $r^{-1}\varepsilon^{-d}$, and the same bound controls all the spatial translates.

The support properties of the cutoff imply

$$\tilde{J}_\sigma \dot{G}_\sigma = \dot{G}_\sigma, \qquad \tilde{J}_{1/2} G_{1/2} = G_{1/2}. \tag{A.57}$$

Consequently,

$$\partial^A \dot{G}_\sigma = (\partial^A \tilde{J}_\sigma) * \dot{G}_\sigma, \qquad L^p_\sigma \dot{G}_\sigma = (L^p_\sigma \tilde{J}_\sigma) * \dot{G}_\sigma.$$

By Lemma A.9,

$$\|\partial^A \tilde{J}_\sigma\|_{\mathrm{TV}(\zeta^{-1})} \lesssim_A r^{-|A|_s}, \qquad \|L^p_\sigma \tilde{J}_\sigma\|_{\mathrm{TV}(w^{\mathfrak{b}}_\sigma)} \lesssim_{p,\mathfrak{b}} 1.$$

Here the first estimate follows from (A.35) with $\omega = 1$: there is no sublattice loss when $|\bar{A}| = 0$, nor when $|\bar{A}| \geq 1$, since then $1 \leq |\bar{A}|$. Weighted Young inequalities, together with (A.54), prove (A.49) and (A.50). The corresponding periodised Young inequality, combined with (A.55) and (A.56), proves (A.51) and (A.52).

The first estimate in (A.53) follows in the same way from (A.58), the second identity in (A.57), and Lemma A.9. Moreover, (A.58) gives $\|G_{1/2}\|_{\mathrm{TV}(w^{\mathfrak{b}}_0)} \lesssim_{\mathfrak{b}} 1$. Finally, using

$$G = G_{1/2} + \int_{1/2}^1 \dot{G}_\sigma \, \mathrm{d}\sigma, \qquad w^{\mathfrak{b}}_0 \leq w^{\mathfrak{b}}_\sigma,$$

and integrating (A.54) in σ proves the second estimate in (A.53). ∎

Lemma A.11. *Fix $d \geq 1$ and $s \in (3/4, 1]$. For every $\mathring{\alpha} \geq 0$ and $\bar{\alpha} \in [0, 2s)$, we have*

$$\left|G_{1/2}(t,x)\right| \lesssim_{\mathring{\alpha},\bar{\alpha}} (1+|t|)^{-\mathring{\alpha}}(1+|x|)^{-d-\bar{\alpha}} \tag{A.58}$$

and

$$\left|\dot{G}_\sigma(t,x)\right| \lesssim_{\mathring{\alpha},\bar{\alpha}} [\![\sigma]\!]^{-d-1}\left(1+\frac{|t|}{[\![\sigma]\!]^{2s}}\right)^{-\mathring{\alpha}}\left(1+\frac{\varepsilon\vee|x|}{[\![\sigma]\!]}\right)^{-d}\left(1+\frac{|x|}{[\![\sigma]\!]}\right)^{-\bar{\alpha}} \tag{A.59}$$

uniformly in $(t,x) \in \Lambda$ and $\sigma \in [1/2, 1)$.

Proof. We only prove (A.59) since the proof of (A.58) follows the same lines. Observe that

$$\dot{G}_\sigma(t,x) = \int_{\Lambda_\varepsilon^*} \frac{\partial_\sigma(j_\sigma(|k_0'|^{1/2s}) j_\sigma(q_\varepsilon(\tilde{k}')))}{ik_0' + m^2 + q_\varepsilon^{2s}(\tilde{k}')} e^{i(k_0' t + \tilde{k}' x)} \frac{\mathrm{d}k_0' \mathrm{d}\tilde{k}'}{(2\pi)^{d+1}}.$$

To prove a bound for the L^∞-norm of

$$(t,x) \mapsto t^{a_0} x^{\tilde{a}} \dot{G}_\sigma(t,x),$$

it suffices to control the L^1-norm of

$$(k_0', \tilde{k}') \mapsto \partial_{k_0'}^{a_0} \partial_{\tilde{k}'}^{\tilde{a}} \left(\frac{\partial_\sigma(j_\sigma(|k_0'|^{1/2s}) j_\sigma(q_\varepsilon(\tilde{k}')))}{ik_0' + m^2 + q_\varepsilon^{2s}(\tilde{k}')} \right). \tag{A.60}$$

Since for noninteger parameters s the bound

$$|\partial_{\tilde{k}'}^{\tilde{a}} q_\varepsilon^{2s}(\tilde{k}')| \lesssim |\tilde{k}'|^{2s-|\tilde{a}|},$$

uniform in $\varepsilon \in (0,1)$, is optimal, the L^1-norm of the function (A.60) is bounded uniformly in $\varepsilon \in (0,1)$ only if $|\tilde{a}| < d + 2s$. As a result, the above simple strategy can only be used to prove the lemma for $\bar{\alpha} \in [0, \lfloor 2s \rfloor]$. To establish the claim for every $\bar{\alpha} \in [0, 2s)$ a more refined argument is needed. To this end, let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(\omega) := -\omega \partial_\omega j(\omega)$$

for $\omega \in \mathbb{R}$. Moreover, let the families of kernels $(\tilde{G}_\tau)_{\tau\in(0,\infty)}$, $(\check{G}_\tau)_{\tau\in(0,\infty)}$, $(\hat{G}_{\tau,\eta})_{\tau,\eta\in(0,\infty)}$ be defined by

$$\tilde{G}_\tau(t,x) := \tau^{-1} \int_{\Lambda_\varepsilon^*} \frac{j\left(\tau|k_0'|^{1/2s}\right) h(\tau q_\varepsilon(\tilde{k}')) + h\left(\tau|k_0'|^{1/2s}\right) j(\tau q_\varepsilon(\tilde{k}'))}{ik_0' + m^2 + q_\varepsilon^{2s}(\tilde{k}')} \times e^{i(k_0' t + \tilde{k}' x)} \frac{\mathrm{d}k_0' \mathrm{d}\tilde{k}'}{(2\pi)^{d+1}},$$

$$\hat{G}_{\tau,\eta}(t,x) := \tau^{-1}\eta^{-1} \int_{\Lambda_\varepsilon^*} \frac{q_\varepsilon^{2s}(\tilde{k}') h\left(\tau|k_0'|^{1/2s}\right) j(\tau q_\varepsilon(\tilde{k}')) h(\tau\eta q_\varepsilon(\tilde{k}'))}{(k_0' - im^2)^2 + q_\varepsilon^{4s}(\tilde{k}')} \times e^{i(k_0' t + \tilde{k}' x)} \frac{\mathrm{d}k_0' \mathrm{d}\tilde{k}'}{(2\pi)^{d+1}},$$

$$\check{G}_\tau(t,x) := \tau^{-1}\int_{\Lambda^*_\varepsilon}\left(\frac{j\left(\tau|k_0'|^{1/2s}\right)h(\tau q_\varepsilon(\tilde{k}'))}{ik_0'+m^2+q_\varepsilon^{2s}(\tilde{k}')}\right.$$
$$\left.-\frac{(ik_0'+m^2)h\left(\tau|k_0'|^{1/2s}\right)j(\tau q_\varepsilon(\tilde{k}'))}{(k_0'-im^2)^2+q_\varepsilon^{4s}(\tilde{k}')}\right)e^{i(k_0't+\tilde{k}'x)}\frac{\mathrm{d}k_0'\mathrm{d}\tilde{k}'}{(2\pi)^{d+1}}.$$

We claim that

$$\dot{G}_\sigma(t,x)=\sigma^{-2}\tilde{G}_{[\![\sigma]\!]/\sigma}(t,x),\qquad \tilde{G}_\tau=\check{G}_\tau+\int_{1/2}^\infty\hat{G}_{\tau,\eta}\mathrm{d}\eta. \tag{A.61}$$

The first identity follows from

$$\partial_\sigma j_\sigma(\omega)=[\![\sigma]\!]^{-1}\sigma^{-1}h([\![\sigma]\!]\sigma^{-1}\omega).$$

To verify the second, we use the identities

$$\frac{h\left(\tau|k_0'|^{1/2s}\right)j(\tau q_\varepsilon(\tilde{k}'))}{ik_0'+m^2+q_\varepsilon^{2s}(\tilde{k}')}$$
$$=\frac{q_\varepsilon^{2s}(\tilde{k}')h\left(\tau|k_0'|^{1/2s}\right)j(\tau q_\varepsilon(\tilde{k}'))}{(k_0'-im^2)^2+q_\varepsilon^{4s}(\tilde{k}')}-\frac{(ik_0'+m^2)h\left(\tau|k_0'|^{1/2s}\right)j(\tau q_\varepsilon(\tilde{k}'))}{(k_0'-im^2)^2+q_\varepsilon^{4s}(\tilde{k}')}$$

as well as

$$\int_{1/2}^\infty\eta^{-1}h(\tau\eta q_\varepsilon(\tilde{k}'))\,\mathrm{d}\eta=-\int_{1/2}^\infty\partial_\eta j(\tau\eta q_\varepsilon(\tilde{k}'))\,\mathrm{d}\eta=j(\tau q_\varepsilon(\tilde{k}')/2)$$

and

$$j(\tau q_\varepsilon(\tilde{k}'))j(\tau q_\varepsilon(\tilde{k}')/2)=j(\tau q_\varepsilon(\tilde{k}')).$$

Let us motivate the usefulness of the representation of $\dot{G}_\sigma$ given by (A.61). First, note that if $|\tilde{a}|<d+4s$, then the L^1-norm of the function

$$(k_0',\tilde{k}')\mapsto\partial_{k_0'}^{a_0}\partial_{\tilde{k}'}^{\tilde{a}}\left(\left(\frac{j\left(\tau|k_0'|^{1/2s}\right)h(\tau q_\varepsilon(\tilde{k}'))}{ik_0'+m^2+q_\varepsilon^{2s}(\tilde{k}')}\right.\right.$$
$$\left.\left.-\frac{(ik_0'+m^2)h\left(\tau|k_0'|^{1/2s}\right)j(\tau q_\varepsilon(\tilde{k}'))}{(k_0'-im^2)^2+q_\varepsilon^{4s}(\tilde{k}')}\right)\right),$$

is bounded uniformly in $\varepsilon\in(0,1)$. This stands in contrast to the function in (A.60), whose L^1-norm remains uniformly bounded in $\varepsilon\in(0,1)$ only if $|\tilde{a}|<d+2s$. As a result, it is possible to control the L^∞-norm of the function

$$(t,x)\mapsto t^{a_0}x^{\tilde{a}}\check{G}_\tau(t,x)$$

for $|\tilde{a}|<d+4s$. Since $\lfloor 4s\rfloor\geqslant 3>2s>\bar{\alpha}$ this indicates that $\check{G}_\tau$ indeed exhibits the desired decay at infinity. On the other hand, the L^1-norm of the function

$$(k_0',\tilde{k}')\mapsto\partial_{k_0'}^{a_0}\partial_{\tilde{k}'}^{\tilde{a}}\left(\frac{q_\varepsilon^{2s}(\tilde{k}')h\left(\tau|k_0'|^{1/2s}\right)j(\tau q_\varepsilon(\tilde{k}'))h(\tau\eta q_\varepsilon(\tilde{k}'))}{(k_0'-im^2)^2+q_\varepsilon^{4s}(\tilde{k}')}\right)$$

is bounded uniformly in $\varepsilon \in (0,1)$ for all $a_0, \tilde{a}$ and consequently $\hat{G}_{\tau,\eta}$ has good decay properties. However, the bound for the L^∞-norm of the function

$$(t,x) \mapsto t^{a_0} x^{\tilde{a}} \hat{G}_{\tau,\eta}(t,x)$$

depends on η. To complete the argument, we must therefore control the above norms uniformly in $\varepsilon \in (0,1)$, $\tau \in (0,1)$ and $\eta \in (1/2, \infty)$.

Fix an integer $N \geqslant \mathring{\alpha}$. We claim that the following bounds imply (A.59),

$$\begin{aligned}
|\tilde{G}_\tau(t,0)| &\lesssim \tau^{-1}\varepsilon^{-d}(1+|t|/\tau^{2s})^{-N},\\
|\check{G}_\tau(t,x)| &\lesssim \tau^{-d-1}(1+|x|/\tau)^{-d-3}(1+|t|/\tau^{2s})^{-N},\\
|\hat{G}_{\tau,\eta}(t,x)| &\lesssim \eta^{-2s-1}\tau^{-d-1}(1+|x|/\tau)^{-d}(1+|t|/\tau^{2s})^{-N},\\
|\hat{G}_{\tau,\eta}(t,x)| &\lesssim \eta^{1-2s}\tau^{-d-1}(1+|x|/\tau)^{-d-2}(1+|t|/\tau^{2s})^{-N}.
\end{aligned} \tag{A.62}$$

To prove this claim, first use the last two bounds to conclude that for all $\bar{\alpha} \in [0,2]$ we have

$$|\hat{G}_{\tau,\eta}(t,x)| \lesssim \eta^{\bar{\alpha}-1-2s}\tau^{-d-1}(1+|x|/\tau)^{-d-\bar{\alpha}}(1+|t|/\tau^{2s})^{-N}.$$

Consequently, the second identity in (A.61) and the above bound yield

$$|\tilde{G}_\tau(t,x)| \lesssim_{\bar{\alpha}} \tau^{-d-1}(1+|x|/\tau)^{-d-\bar{\alpha}}(1+|t|/\tau^{2s})^{-N}$$

for $\bar{\alpha} \in [0,2s)$. If $x \neq 0$, then $|x| \geqslant \varepsilon$, while for $x = 0$ we combine the above estimate with the first estimate in (A.62). We obtain

$$|\tilde{G}_\tau(t,x)| \lesssim_{\bar{\alpha}} \tau^{-d-1}(1+(\varepsilon \vee |x|)/\tau)^{-d}(1+|x|/\tau)^{-\bar{\alpha}}(1+|t|/\tau^{2s})^{-N}.$$

Since $N \geqslant \mathring{\alpha}$ and $[\![\sigma]\!]/\sigma \asymp [\![\sigma]\!]$ for $\sigma \in [1/2,1)$, the bound (A.59) follows from the first identity in (A.61).

It remains to prove (A.62). Starting from the bound for $\hat{G}_{\tau,\eta}(t,x)$, we have

$$\begin{aligned}
(\hat{G}_{\tau,\eta})(t,x) &= \tau^{-1}\eta^{-1}\int_{\Lambda^*_\varepsilon} \frac{q_\varepsilon^{2s}(\tilde{k}')}{(k_0'-im^2)^2+q_\varepsilon^{4s}(\tilde{k}')} h\left(\tau|k_0'|^{1/2s}\right) j(\tau q_\varepsilon(\tilde{k}'))\\
&\qquad \times h(\tau\eta q_\varepsilon(\tilde{k}'))e^{i(k_0't+\tilde{k}'x)}\frac{\mathrm{d}k_0'\mathrm{d}\tilde{k}'}{(2\pi)^{d+1}}\\
&= \tau^{-d-1}\eta^{-1}\int_{\Lambda^*_{\varepsilon/\tau}} \frac{q_{\varepsilon/\tau}^{2s}(\tilde{k})}{(k_0-i\tau^{2s}m^2)^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\\
&\qquad \times h\big(|k_0|^{1/2s}\big) j(q_{\varepsilon/\tau}(\tilde{k}))\\
&\qquad \times h(\eta q_{\varepsilon/\tau}(\tilde{k}))e^{i(k_0t/\tau^{2s}+\tilde{k}x/\tau)}\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}},
\end{aligned}$$

where we set $k_0 = \tau^{2s}k_0'$ and $\tilde{k} = \tau\tilde{k}'$. It follows that

$$\begin{aligned}
&(\tau^{-2s}|t|)^{a_0}(\tau^{-1}|x|)^{\tilde{a}}|(\hat{G}_{\tau,\eta})(t,x)|\\
&\lesssim \tau^{-d-1}\eta^{-1}\int_{\Lambda^*_{\varepsilon/\tau}} \left|\partial_{k_0}^{a_0}\partial_{\tilde{k}}^{\tilde{a}}\left[\frac{q_{\varepsilon/\tau}^{2s}(\tilde{k})h\left(|k_0|^{1/2s}\right) j(q_{\varepsilon/\tau}(\tilde{k}))h(\eta q_{\varepsilon/\tau}(\tilde{k}))}{(k_0-i\tau^{2s}m^2)^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\right]\right|\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}.
\end{aligned} \tag{A.63}$$

Observe the following:

- In (A.63), the factor $h\left(|k_0|^{1/2s}\right)$ restricts the integration domain to $|k_0|^{1/2s} \in [1,2]$. In this region, the denominator is never vanishing. More precisely
$$|(k_0 - i\tau^{2s}m^2)^2 + q^{4s}_{\varepsilon/\tau}(\tilde{k})|^2 \geqslant c_m > 0.$$
- Similarly, the factor $h(\eta q_{\varepsilon/\tau}(\tilde{k}))$ restricts the integration domain to those $\tilde{k}$ such that $q_{\varepsilon/\tau}(\tilde{k}) \in [\eta^{-1}, 2\eta^{-1}]$. Thus, by (A.1), on the integration domain
$$|\partial^{\tilde{a}}_{\tilde{k}} q^{2s}_{\varepsilon/\tau}(\tilde{k})| \lesssim q^{2s-|\tilde{a}|}_{\varepsilon/\tau}(\tilde{k}) \approx \eta^{|\tilde{a}|-2s}.$$

On account of these comments and (A.5), we have the following bound for the integrand
$$\begin{aligned}\left|\partial^{a_0}_{k_0}\partial^{\tilde{a}}_{\tilde{k}}\left(\frac{q^{2s}_{\varepsilon/\tau}(\tilde{k})h\left(|k_0|^{1/2s}\right)j(q_{\varepsilon/\tau}(\tilde{k}))h(\eta q_{\varepsilon/\tau}(\tilde{k}))}{(k_0 - i\tau^{2s}m^2)^2 + q^{4s}_{\varepsilon/\tau}(\tilde{k})}\right)\right| \\ \lesssim \eta^{|\tilde{a}|-2s}1_{[0,2]}(|k_0|^{1/2s})1_{[1,2]}(\eta q_{\varepsilon/\tau}(\tilde{k})),\end{aligned}$$
for all $(a_0, \tilde{a}) \in \mathbb{N}_0^{1+d}$. Taking $a_0 = p$ and summing over all $\tilde{a}$ with $|\tilde{a}| = q$, this implies that
$$\begin{aligned}&(\tau^{-2s}|t|)^p(\tau^{-1}|x|)^q|(\hat{G}_{\tau,\eta})(t,x)| \\ &\lesssim \tau^{-d-1}\eta^{q-2s-1}\int_{\Lambda^*_{\varepsilon/\tau}} 1_{[0,2]}(|k_0|^{1/2s})1_{[1,2]}(\eta q_{\varepsilon/\tau}(\tilde{k}))\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}} \\ &\lesssim \tau^{-d-1}\eta^{q-2s-d-1}\int_{\Lambda^*_{\varepsilon/(\tau\eta)}} 1_{[0,2]}(|k_0|^{1/2s})1_{[1,2]}(q_{\varepsilon/(\tau\eta)}(\tilde{k}))\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}} \\ &\lesssim \tau^{-d-1}\eta^{q-2s-d-1},\end{aligned}$$
where the last estimate follows from (A.7). For any $p, q \in \mathbb{N}_0$, taking $p \in \{0, N\}$ and $q \in \{0, d, d+2\}$ proves the last two bounds in (A.62). The proof for $\check{G}_\tau(t,x)$ is similar. Working as in the previous case, we have
$$\begin{aligned}&(\tau^{-2s}|t|)^p(\tau^{-1}|x|)^q|\check{G}_\tau(t,x)| \\ &\lesssim \tau^{-d-1}\int_{\Lambda^*_{\varepsilon/\tau}}\left|\partial^{a_0}_{k_0}\partial^{\tilde{a}}_{\tilde{k}}\left[\left(\frac{j\left(|k_0|^{1/2s}\right)h(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0 + \tau^{2s}m^2 + q^{2s}_{\varepsilon/\tau}(\tilde{k})}\right.\right.\right. \\ &\quad\left.\left.\left. - \frac{(ik_0 + \tau^{2s}m^2)h\left(|k_0|^{1/2s}\right)j(q_{\varepsilon/\tau}(\tilde{k}))}{(k_0 - i\tau^{2s}m^2)^2 + q^{4s}_{\varepsilon/\tau}(\tilde{k})}\right)\right]\right|\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}.\end{aligned}$$

As above, also here the denominators are non-vanishing on the integration domain, more precisely,
$$|ik_0 + \tau^{2s}m^2 + q^{2s}_{\varepsilon/\tau}(\tilde{k})| \geqslant 1 \qquad \text{and} \qquad |(k_0 - i\tau^{2s}m^2)^2 + q^{4s}_{\varepsilon/\tau}(\tilde{k})| \geqslant 1.$$

In the first term, this is due to the factor $h(q_{\varepsilon/\tau}(\tilde{k}))$, which restricts the integration domain to $q_{\varepsilon/\tau}(\tilde{k}) \in [1,2]$, while in the second one this is due to the factor $h\left(|k_0|^{1/2s}\right)$, which

restricts the integration domain to $|k_0|^{1/2s} \in [1,2]$. By (A.1), since $|\partial_{\tilde{k}}^{\tilde{a}} q_{\varepsilon/\tau}^{4s}(\tilde{k})| \lesssim q_{\varepsilon/\tau}^{4s-|\tilde{a}|}(\tilde{k})$, we get

$$\begin{aligned}\Big|\partial_{k_0}^{a_0}\partial_{\tilde{k}}^{\tilde{a}}\Big[\Big(&\frac{j\left(|k_0|^{1/2s}\right)h(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^2+q_{\varepsilon/\tau}^{2s}(\tilde{k})}\\&-\frac{(ik_0+\tau^{2s}m^2)h\left(|k_0|^{1/2s}\right)j(q_{\varepsilon/\tau}(\tilde{k}))}{(k_0-i\tau^{2s}m^2)^2+q_{\varepsilon/\tau}^{4s}(\tilde{k})}\Big)\Big]\Big|\\&\lesssim 1_{[0,2]}(|k_0|^{1/2s})1_{[0,2]}(q_{\varepsilon/\tau}(\tilde{k}))\,(1\vee q_{\varepsilon/\tau}^{4s-|\tilde{a}|}(\tilde{k})).\end{aligned}$$

Taking $a_0 = p$ and summing over all $\tilde{a}$ with $|\tilde{a}| = q$, this implies that, for $q < d+4s$,

$$\begin{aligned}&(\tau^{-2s}|t|)^p(\tau^{-1}|x|)^q|\check{G}_\tau(t,x)|\\&\lesssim \tau^{-d-1}\int_{\Lambda_{\varepsilon/\tau}^*}1_{[0,2]}(|k_0|^{1/2s})1_{[0,2]}(q_{\varepsilon/\tau}(\tilde{k}))(1\vee q_{\varepsilon/\tau}^{4s-q}(\tilde{k}))\mathrm{d}k_0\mathrm{d}\tilde{k}\\&\lesssim \tau^{-d-1}.\end{aligned}$$

Here the last estimate is (A.8). Since $d+4s > d+3$, taking $p \in \{0,N\}$ and $q \in \{0,d+3\}$ proves the bound for $\check{G}_\tau$. Finally, we discuss the first bound in (A.62). By changing the variables as above, we arrive at

$$\begin{aligned}&(\tau^{-2s}|t|)^p|\tilde{G}_\tau(t,0)| \leqslant \tau^{-1-d}\\&\times\int_{\Lambda_{\varepsilon/\tau}^*}\left|\partial_{k_0}^p\left[\frac{j\left(|k_0|^{1/2s}\right)h(q_{\varepsilon/\tau}(\tilde{k}))+h\left(|k_0|^{1/2s}\right)j(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^2+q_{\varepsilon/\tau}^{2s}(\tilde{k})}\right]\right|\mathrm{d}k_0\mathrm{d}\tilde{k}.\end{aligned}$$

We use the same argument as for $\check{G}_\tau$: indeed both terms in the integrand have non-vanishing denominators due to the factors $h(q_{\varepsilon/\tau}(\tilde{k}))$ and $h\left(|k_0|^{1/2s}\right)$, respectively. Analogously, we obtain

$$\begin{aligned}&\left|\partial_{k_0}^p\left[\frac{j\left(|k_0|^{1/2s}\right)h(q_{\varepsilon/\tau}(\tilde{k}))+h\left(|k_0|^{1/2s}\right)j(q_{\varepsilon/\tau}(\tilde{k}))}{ik_0+\tau^{2s}m^2+q_{\varepsilon/\tau}^{2s}(\tilde{k})}\right]\right|\\&\lesssim 1_{[0,2]}(|k_0|^{1/2s}),\end{aligned}$$

and thus

$$\begin{aligned}&(\tau^{-2s}|t|)^p|\tilde{G}_\tau(t,0)|\\&\lesssim \tau^{-1-d}\int_{\Lambda_{\varepsilon/\tau}^*}1_{[0,2]}(|k_0|^{1/2s})\frac{\mathrm{d}k_0\mathrm{d}\tilde{k}}{(2\pi)^{d+1}}\\&\lesssim \tau^{-1-d}(\varepsilon\tau^{-1})^{-d}=\tau^{-1}\varepsilon^{-d}.\end{aligned}$$

Taking $p \in \{0,N\}$ proves the first bound in (A.62). This concludes the proof. ∎

A.2. Properties of the weights

We collect here some important properties of the various weights introduced throughout the paper.

Lemma A.12. *For $\alpha \geqslant 0$, we have*

$$\|(-\Delta_\varepsilon)^s \zeta^\alpha_\sigma\| \lesssim [\![\sigma]\!]^{2sa}$$

uniformly in $\sigma \in (0,1)$.

Proof. Let $f \in C^2(\mathbb{R}^d)$. Recall that by (1.14), we have

$$(-\Delta_\varepsilon)^s f(x) = \int_{\mathbb{R}^d_\varepsilon} H^{(\varepsilon)}_s(y)(f(x) - f(x-y))\,\mathrm{d}y, \qquad x \in \mathbb{R}^d_\varepsilon,$$

where the kernel $H^{(\varepsilon)}_s : \mathbb{R}^d_\varepsilon \to \mathbb{R}$ is positive and such that $H^{(\varepsilon)}_s(0) = 0$, $H^{(\varepsilon)}_s(x) = H^{(\varepsilon)}_s(-x)$ and

$$|H^{(\varepsilon)}_s(x)| \lesssim |x|^{-d-2s}, \qquad x \in \mathbb{R}^d_\varepsilon.$$

Let $B(\delta)$ be the ball in $\mathbb{R}^d_\varepsilon$ of radius $\delta > 0$ centred at the origin and $B^c(\delta) := \mathbb{R}^d_\varepsilon \setminus B(\delta)$. For any $\delta > 0$, we have

$$\begin{aligned}|(-\Delta_\varepsilon)^s f(x)| \leq \frac{1}{2}\int_{B(\delta)} H^{(\varepsilon)}_s(y)\,|2f(x) - f(x+y) - f(x-y)|\,\mathrm{d}y \\ + \int_{B^c(\delta)} H^{(\varepsilon)}_s(y)\,|f(x) - f(x-y)|\,\mathrm{d}y,\end{aligned}$$

To bound the first term observe that

$$|2f(x) - f(x+y) - f(x-y)| \lesssim \|\nabla\nabla f\||y|^2,$$

by Taylor's theorem, where $\|\nabla\nabla f\|$ denotes the supremum norm of the Hessian of f in the continuum. As a result, we obtain

$$\begin{aligned}&\int_{B(\delta)} H^{(\varepsilon)}_s(y)\,|2f(x) - f(x+y) - f(x-y)|\,\mathrm{d}y \\ &\quad\lesssim \|\nabla\nabla f\| \int_{B(\delta)} |y|^{2-d-2s}\mathrm{d}y \lesssim \|\nabla\nabla f\|\delta^{2-2s}.\end{aligned}$$

Moreover, we have

$$\int_{B^c(\delta)} H^{(\varepsilon)}_s(y)\,|f(x) - f(x-y)|\,\mathrm{d}y \lesssim \|f\| \int_{B^c(\delta)} |y|^{-d-2s}\mathrm{d}y \lesssim \|f\|\delta^{-2s}.$$

To conclude, we apply the above estimates with $f(x) = \zeta^\alpha_\sigma(t,x)$ and choose $\delta = [\![\sigma]\!]^{-a}$. ∎

Lemma A.13. *We have*

$$\|\zeta^{1/3}_\sigma \mathfrak{D}_s(\zeta^{-1/3}_\sigma)\|^2 \lesssim [\![\sigma]\!]^{2sa},$$

uniformly in $\sigma \in (0,1)$.

Proof. Recall that μ_s denotes the kernel defined by (1.15), which is local in time. Let $B(z,\delta)$ be the ball of radius $\delta > 0$ centred at $z \in \Lambda$. By (1.17), we have

$$\begin{aligned}[\zeta_\sigma^{1/3}\mathfrak{D}_s(\zeta_\sigma^{-1/3})(z)]^2 &= \zeta_\sigma^{2/3}(z)\int [\zeta_\sigma^{-1/3}(z') - \zeta_\sigma^{-1/3}(z)]^2 \mu_s(z,\mathrm{d}z') \\ &= \zeta_\sigma^{2/3}(z)\int_{B^c(z,\delta)} [\zeta_\sigma^{-1/3}(z') - \zeta_\sigma^{-1/3}(z)]^2 \mu_s(z,\mathrm{d}z') \\ &+ \zeta_\sigma^{2/3}(z)\int_{B(z,\delta)} [\zeta_\sigma^{-1/3}(z') - \zeta_\sigma^{-1/3}(z)]^2 \mu_s(z,\mathrm{d}z') \\ &= \mathbb{I} + \mathbb{II}.\end{aligned}$$

On $B^c(z,\delta) := \Lambda_\varepsilon \setminus B(z,\delta)$ we have

$$\begin{aligned}\mathbb{I} &= \int_{B^c(z,\delta)} [\zeta_\sigma^{1/3}(z)\zeta_\sigma^{-1/3}(z') - 1]^2 \mu_s(z,\mathrm{d}z') \\ &\lesssim \int_{B^c(z,\delta)} [\zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z') + 1]\mu_s(z,\mathrm{d}z') \\ &\lesssim \int_{B^c(z,\delta)} \zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z')\mu_s(z,\mathrm{d}z') + \delta^{-2s}.\end{aligned}$$

Recall that

$$\zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z') \lesssim \zeta_\sigma^{-2/3}(z - z') = \langle [\![\sigma]\!]^a.(z - z')\rangle_s^{2/3} = (1 + [\![\sigma]\!]^{2a}|\bar z - \bar z'|^2)^{1/3},$$

uniformly in $z = (z_0, \bar z) \in \Lambda$ and $z' = (z_0, \bar z') \in \Lambda$. Since $2s > 2/3$,

$$\begin{aligned}\int_{B^c(z,\delta)} \zeta_\sigma^{2/3}(z)\zeta_\sigma^{-2/3}(z')\mu_s(z,\mathrm{d}z') &\lesssim \int_{B^c(z,\delta)} (1 + [\![\sigma]\!]^{2a}|\bar z - \bar z'|^2)^{1/3}\mu_s(z,\mathrm{d}z') \\ &\lesssim \delta^{-2s} + [\![\sigma]\!]^{2a/3}\delta^{-2s+2/3}.\end{aligned}$$

Let us bound the second term. Thanks to the smoothness of $\zeta_\sigma^{-1/3}$ in space, by Taylor's theorem

$$\mathbb{II} \lesssim \zeta_\sigma^{2/3}(z)\int_{B(z,\delta)} \|\nabla\zeta_\sigma^{-1/3}\|^2 |\bar z' - \bar z|^2 \mu_s(z,\mathrm{d}z').$$

Since $\|\nabla\zeta_\sigma^{-1/3}\|^2 \lesssim [\![\sigma]\!]^{2a}$ and $\|\zeta_\sigma^{2/3}\| \lesssim 1$, we conclude

$$\mathbb{II} \lesssim [\![\sigma]\!]^{2a}\int_{B(z,\delta)} |\bar z' - \bar z|^2 \mu_s(z,\mathrm{d}z') \lesssim \delta^{2-2s}[\![\sigma]\!]^{2a} = [\![\sigma]\!]^{2sa}.$$

Choosing now $\delta = [\![\sigma]\!]^{-a}$, we get

$$\mathbb{I} \lesssim [\![\sigma]\!]^{2as} + [\![\sigma]\!]^{2a/3}[\![\sigma]\!]^{2sa-2a/3} \lesssim [\![\sigma]\!]^{2as}, \qquad \mathbb{II} \lesssim \delta^{2-2s}[\![\sigma]\!]^{2a} = [\![\sigma]\!]^{2sa}.$$

This finishes the proof. ∎

Lemma A.14. *Recall that* $1 < \mathfrak{b} < 2s < 2$. *The bound*

$$\frac{(1 - h_\mu) w_\mu^{(2),\mathfrak{b}}}{w_\sigma^{(2),\mathfrak{b}}} \lesssim [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}}$$

holds uniformly in $\mu, \sigma \in (0, 1)$.

Proof. We have

$$\begin{aligned}
\frac{(1 - h_\mu(z, z_1)) w_\mu^{(2),\mathfrak{b}}(z, z_1)}{w_\sigma^{(2),\mathfrak{b}}(z, z_1)} &= \frac{w_\mu^{(2),\mathfrak{b}}(z, z_1)}{w_\sigma^{(2),\mathfrak{b}}(z, z_1)} \frac{[\![\mu]\!]^{-2} |z - z_1|_s^2}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\
&= \frac{w_\mu^{(2),\mathfrak{b}}(z, z_1)}{w_\sigma^{(2),\mathfrak{b}}(z, z_1)} \frac{([\![\mu]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}} ([\![\mu]\!]^{-1} |z - z_1|_s)^{2-\mathfrak{b}}}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\
&= [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}} \frac{(1 + [\![\mu]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}}}{(1 + [\![\sigma]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}}} \frac{([\![\sigma]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}} ([\![\mu]\!]^{-1} |z - z_1|_s)^{2-\mathfrak{b}}}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\
&\leqslant [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}} \frac{(1 + [\![\mu]\!]^{-1} |z - z_1|_s)^{\mathfrak{b}} ([\![\mu]\!]^{-1} |z - z_1|_s)^{2-\mathfrak{b}}}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \\
&\leqslant [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}} \frac{(1 + [\![\mu]\!]^{-1} |z - z_1|_s)^2}{1 + [\![\mu]\!]^{-2} |z - z_1|_s^2} \lesssim [\![\sigma]\!]^{\mathfrak{b}} [\![\mu]\!]^{-\mathfrak{b}},
\end{aligned}$$

which proves the claim. ∎

Lemma A.15. *For all* $\omega > 0$ *and* $m \in \mathbb{N}_0$, *the following bound*

$$\frac{(1 - v_\mu^{(1+m)}) w_\mu^{(1+m),\omega}}{w_\sigma^{(1+m),\omega}} \lesssim [\![\sigma]\!]^{\omega} [\![\mu]\!]^{-\omega}$$

holds uniformly in $\mu, \sigma \in (0, 1)$.

Proof. By Definition 4.3, $1 - v_\mu^{(m+1)} = 0$ if $[\![\mu]\!]^{-1} \operatorname{St}(z, y_1, \dots, y_m) \leqslant 1$. As a consequence,

$$\begin{aligned}
\frac{(1 - v_\mu^{(1+m)}) w_\mu^{(1+m),\omega}}{w_\sigma^{(1+m),\omega}} &= \frac{\big(1 + [\![\mu]\!]^{-1} \operatorname{St}(z, y_1, \dots, y_m)\big)^{\omega}}{\big(1 + [\![\sigma]\!]^{-1} \operatorname{St}(z, y_1, \dots, y_m)\big)^{\omega}} (1 - v_\mu^{(1+m)}) \\
&= [\![\mu]\!]^{-\omega} [\![\sigma]\!]^{\omega} \frac{\big([\![\mu]\!] + \operatorname{St}(z, y_1, \dots, y_m)\big)^{\omega}}{\big([\![\sigma]\!] + \operatorname{St}(z, y_1, \dots, y_m)\big)^{\omega}} (1 - v_\mu^{(1+m)}),
\end{aligned}$$

is non-vanishing only if $\operatorname{St}(z, y_1, \dots, y_m) > [\![\mu]\!]$. Hence

$$\frac{(1 - v_\mu^{(1+m)}) w_\mu^{(1+m),\omega}}{w_\sigma^{(1+m),\omega}} \leqslant [\![\mu]\!]^{-\omega} [\![\sigma]\!]^{\omega} \frac{(2 \operatorname{St}(z, y_1, \dots, y_m))^{\omega}}{(\operatorname{St}(z, y_1, \dots, y_m))^{\omega}} (1 - v_\mu^{(m+1)}) \lesssim [\![\mu]\!]^{-\omega} [\![\sigma]\!]^{\omega},$$

which proves the claim. ∎

A.3. Norm estimates

Lemma A.16. *The cumulant norms introduced in Definition 4.11 satisfy the bound*

$$\|\!|[K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1-h_\mu)|\!\|_\mu \lesssim [\![\mu]\!]^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}} \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma,$$

uniformly over $\sigma, \mu \in [1/2, 1)$ and $\dot{\bar{F}}^{[\ell],(1)}_\sigma$.

Proof. By (4.6), Young's inequality and Lemma A.14 we obtain

$$\begin{aligned}
\|\!|[K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1-h_\mu)|\!\|_\mu &= \|\!|[K^{1,1}_\mu([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1-h_\mu))] \cdot w^{(2),\mathfrak{b}}_\mu|\!\| \\
&\lesssim \|K^{1,1}_\mu\|_{\mathrm{TV}((w^{\mathfrak{b}}_\mu)^{\otimes 2})} \|\!|[K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1-h_\mu) w^{(2),\mathfrak{b}}_\mu|\!\| \\
&\lesssim \|w^{(2),\mathfrak{b}}_\mu (1-h_\mu)(w^{(2),\mathfrak{b}}_\sigma)^{-1}\|_{L^\infty} \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma \\
&\lesssim [\![\mu]\!]^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}} \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma.
\end{aligned}$$

This proves the statement. ∎

Lemma A.17. *For every $\mathfrak{a} \in \mathfrak{A}$, the kernel norms introduced in Definition 4.6 satisfy the bound*

$$\|[\tilde{K}^{\mathfrak{a}}_\sigma \dot{F}^{\mathfrak{a}}_\sigma] \cdot (1-v^{\mathfrak{a}}_\mu)\|_\mu \lesssim [\![\mu]\!]^{\ell(\mathfrak{a})\kappa_0 - \mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b} - \ell(\mathfrak{a})\kappa_0} \|\dot{F}^{\mathfrak{a}}_\sigma\|_\sigma,$$

uniformly over $\sigma, \mu \in [1/2, 1)$ and $\dot{F}^{\mathfrak{a}}_\sigma$.

Proof. By (4.7), Young's inequality, Lemma A.15 and $\tilde{w}^{\mathfrak{a}}_\mu := w^{(1+k(\mathfrak{a})),\mathfrak{b}-\ell(\mathfrak{a})\kappa_0}_\mu$ we obtain

$$\begin{aligned}
\|[\tilde{K}^{\mathfrak{a}}_\sigma \dot{F}^{\mathfrak{a}}_\sigma] \cdot (1-v^{\mathfrak{a}}_\mu)\|_\mu &= \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\mu([\tilde{K}^{\mathfrak{a}}_\sigma \dot{F}^{\mathfrak{a}}_\sigma] \cdot (1-v^{\mathfrak{a}}_\mu))] \cdot \tilde{w}^{\mathfrak{a}}_\mu\| \\
&\lesssim \|\tilde{K}^{\mathfrak{a}}_\mu\|_{\mathrm{TV}(\mathfrak{w}_\mu)} \|\mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_\sigma \dot{F}^{\mathfrak{a}}_\sigma] \cdot (1-v^{\mathfrak{a}}_\mu) \tilde{w}^{\mathfrak{a}}_\mu\| \\
&\lesssim \|(1-v^{\mathfrak{a}}_\mu)\tilde{w}^{\mathfrak{a}}_\mu (\tilde{w}^{\mathfrak{a}}_\sigma)^{-1}\|_{L^\infty} \|\dot{F}^{\mathfrak{a}}_\sigma\|_\sigma \\
&\lesssim [\![\mu]\!]^{\ell(\mathfrak{a})\kappa_0 - \mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b} - \ell(\mathfrak{a})\kappa_0} \|\dot{F}^{\mathfrak{a}}_\sigma\|_\sigma,
\end{aligned}$$

where

$$\mathfrak{w}_\mu(z, z_1, \dots, z_{k(\mathfrak{a})}) = w^2_\mu(z) w^{\mathfrak{b}}_\mu(z_1) \cdots w^{\mathfrak{b}}_\mu(z_{k(\mathfrak{a})}).$$

This proves the statement. ∎

Lemma A.18 (Auxiliary localisation). *Let $\mathfrak{a} \in \mathfrak{A}$ and $\mu \in [1/2, 1)$, and set*

$$k := k(\mathfrak{a}), \qquad \tau := [\![\mu]\!], \qquad \rho := \tau \vee \varepsilon, \qquad \delta := \frac{\tau}{\rho} \in (0, 1].$$

There exist nonnegative smooth functions

$$\tilde{v}^{\mathfrak{a}}_{\mu,0}, \qquad \tilde{v}^{\mathfrak{a}}_{\mu,1} \colon \Lambda^{1+k}_0 \longrightarrow [0, \infty)$$

such that, uniformly in μ, ε, and M,

$$\|\tilde{v}^{\mathfrak{a}}_{\mu,0}\|_{L^\infty} + \|\tilde{v}^{\mathfrak{a}}_{\mu,1}\|_{L^\infty} \lesssim 1. \tag{A.64}$$

Set $\rho_0 := \tau$ and $\rho_1 := \rho$. Then, for $j \in \{0,1\}$,

$$\operatorname{supp} \tilde{v}^{\mathfrak{a}}_{\mu,j} \subset \left\{ (z, z_1, \dots, z_k) : \begin{array}{l} |z_0 - z_{i,0}| \leqslant c\tau^{2s}, \\ |\bar{z} - \bar{z}_i| \leqslant c\rho_j, \quad i = 1, \dots, k, \end{array} \right\}, \tag{A.65}$$

where c is independent of μ, ε, and M.

Define

$$\tilde{V}^{\mathfrak{a}}_{\mu} := \tilde{v}^{\mathfrak{a}}_{\mu,0} + \delta^2 \tilde{v}^{\mathfrak{a}}_{\mu,1}. \tag{A.66}$$

Then, for every kernel $F^{\mathfrak{a}}$,

$$\|F^{\mathfrak{a}} v^{\mathfrak{a}}_{\mu}\|_{\mu} \lesssim \left\| \mathfrak{o}^{\mathfrak{a}} \cdot [\tilde{K}^{\mathfrak{a}}_{\mu} F^{\mathfrak{a}}] \cdot \tilde{V}^{\mathfrak{a}}_{\mu} \tilde{w}^{\mathfrak{a}}_{\mu} \right\| \lesssim \|F^{\mathfrak{a}}\|_{\mu}. \tag{A.67}$$

Moreover, for every $\omega \in [0,2]$,

$$\sup_{z \in \Lambda} \int_{\Lambda^k} w^{(1+k),\omega}_{\mu}(z, Z^{\mathfrak{a}}) \tilde{V}^{\mathfrak{a}}_{\mu}(z, Z^{\mathfrak{a}}) \, \mathrm{d}Z^{\mathfrak{a}} \lesssim \tau^{2sk} \rho^{dk}. \tag{A.68}$$

In particular, taking $\omega = \mathfrak{b} - \ell(\mathfrak{a})\kappa_{\mathfrak{o}}$ in (A.68) gives the estimate with $w^{(1+k),\omega}_{\mu} = \tilde{w}^{\mathfrak{a}}_{\mu}$.

Remark A.19 (Meaning of the auxiliary localisation). Lemma A.18 allows one to transfer the original localisation $v^{\mathfrak{a}}_{\mu}$ from the kernel $F^{\mathfrak{a}}$ to the regularised kernel $\tilde{K}^{\mathfrak{a}}_{\mu} F^{\mathfrak{a}}$. More precisely, (A.67) shows that, up to a uniform constant, multiplication by $v^{\mathfrak{a}}_{\mu}$ before applying the norm may be replaced by multiplication of $\tilde{K}^{\mathfrak{a}}_{\mu} F^{\mathfrak{a}}$ by the auxiliary localisation

$$\tilde{V}^{\mathfrak{a}}_{\mu} = \tilde{v}^{\mathfrak{a}}_{\mu,0} + \delta^2 \tilde{v}^{\mathfrak{a}}_{\mu,1}.$$

The first term is localised at the natural scale τ in space and τ^{2s} in time. The second term accounts for lattice translations generated by the spatial operator when τ lies below the lattice spacing. Its spatial support may therefore extend to the effective lattice scale $\rho = \tau \vee \varepsilon$, but it comes with the factor

$$\delta^2 = \left(\frac{\tau}{\rho} \right)^2.$$

In the regime $\tau \gtrsim \varepsilon$, one has $\rho \asymp \tau$ and $\delta \asymp 1$, so both terms are localised at the same scale. In the sublattice regime $\tau \ll \varepsilon$, the support of $\tilde{v}^{\mathfrak{a}}_{\mu,0}$ is contained in the exact spatial diagonal, whereas $\tilde{v}^{\mathfrak{a}}_{\mu,1}$ is supported on finitely many neighbouring lattice diagonals at distance $O(\varepsilon)$.

The estimate (A.68) shows that this enlargement of the support does not change the natural weighted volume of the localisation. Indeed, a spatial displacement of size $O(\varepsilon)$ may cost at most

$$\left(\frac{\rho}{\tau} \right)^{\omega} = \delta^{-\omega}$$

in the Steiner weight, while the translated part carries the compensating factor δ^2. The restriction $\omega \leqslant 2$ therefore yields

$$\delta^2 \delta^{-\omega} = \delta^{2-\omega} \leqslant 1.$$

Thus $\tilde{V}^{\mathfrak{a}}_{\mu}$ has, uniformly at and below the lattice scale, the same effective weighted volume $\tau^{2sk} \rho^{dk}$ as a cutoff localised in k input variables at temporal scale τ^{2s} and spatial scale ρ.

Proof. The case $k = 0$ is immediate: take

$$\tilde{v}^{\mathfrak{a}}_{\mu,0} = 1, \qquad \tilde{v}^{\mathfrak{a}}_{\mu,1} = 0.$$

We henceforth assume $k \geqslant 1$.

We first record a weighted Young estimate used below. Set

$$\mathfrak{w}_\mu := w_\mu^2 \otimes (w_\mu^\flat)^{\otimes k}.$$

By (4.7), whenever Q is a tensor-product convolution kernel satisfying

$$\|Q\|_{\mathrm{TV}(\mathfrak{w}_\mu)} \lesssim 1,$$

one has

$$\left\| \mathfrak{o}^{\mathfrak{a}} \cdot (Q * G) \cdot \tilde{w}^{\mathfrak{a}}_\mu \right\| \lesssim \left\| \mathfrak{o}^{\mathfrak{a}} \cdot G \cdot \tilde{w}^{\mathfrak{a}}_\mu \right\|. \tag{A.69}$$

Put

$$h := \tilde{K}^{\mathfrak{a}}_\mu F^{\mathfrak{a}}.$$

For the tensor variables (the output variable when $j = 0$, and the j-th input variable when $j = 1, \dots, k$), set

$$q_0 = 1, \qquad q_j = 2 \quad (j \geqslant 1), \qquad p_0 = 2, \qquad p_j = 4 \quad (j \geqslant 1),$$

and define

$$\mathring{\mathcal{L}}_\tau := \bigotimes_{j=0}^{k} (1 + \tau^{2s} \partial_{t_j})^{q_j}, \qquad \mathring{\mathcal{K}}_\tau := \mathring{\mathcal{L}}_\tau^{-1},$$

$$\bar{\mathcal{L}}_\tau := \bigotimes_{j=0}^{k} (1 - \tau^2 \Delta_{\varepsilon,j})^{p_j}, \qquad \bar{\mathcal{K}}_\tau := \bar{\mathcal{L}}_\tau^{-1}.$$

By Definitions 1.13 and 4.5,

$$\tilde{L}^{\mathfrak{a}}_\mu = \mathring{\mathcal{L}}_\tau \bar{\mathcal{L}}_\tau, \qquad \tilde{K}^{\mathfrak{a}}_\mu = \mathring{\mathcal{K}}_\tau \bar{\mathcal{K}}_\tau, \qquad F^{\mathfrak{a}} = \mathring{\mathcal{L}}_\tau \bar{\mathcal{L}}_\tau h. \tag{A.70}$$

Temporal reduction. We first commute the temporal part of the regularising operator through the cutoff. Let $b = \tau^{2s}$, $q \in \{1, 2\}$, and $R_{\tau,q} = (1 + b\partial_t)^{-q}$. For smooth compactly supported g, u,

$$R_{\tau,q}[g(1 + b\partial_t)^q u] = \sum_{m=0}^{q} \binom{q}{m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} [(b\partial_t)^{m-r} R_{\tau,q}] * [u(b\partial_t)^r g]. \tag{A.71}$$

Indeed, expand $(1 + b\partial_t)^q$ and integrate by parts m times in the term containing $b^m \partial_t^m u$. Moreover,

$$(b\partial_t)^n R_{\tau,q} = \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} (1 + b\partial_t)^{-(q-\ell)}, \qquad 0 \leqslant n \leqslant q. \tag{A.72}$$

The term with exponent zero is the identity kernel. The explicit exponential formula for $(1+b\partial_t)^{-1}$ shows that every kernel on the right-hand side of (A.72) has uniformly bounded weighted total variation at scale τ.

Iterating (A.71) over the tensor variables yields a finite set Γ, scaled time-derivative operators

$$D^{\gamma}_{\tau,t} := \prod_{j=0}^{k} (\tau^{2s}\partial_{t_j})^{\gamma_j}, \qquad 0 \leqslant \gamma_j \leqslant q_j,$$

and tensor-product temporal kernels $Q_{\tau,\gamma}$ such that

$$\mathring{\mathcal{K}}_\tau[g\mathring{\mathcal{L}}_\tau u] = \sum_{\gamma\in\Gamma} Q_{\tau,\gamma} * [uD^{\gamma}_{\tau,t} g] \tag{A.73}$$

and

$$\|Q_{\tau,\gamma}\|_{\mathrm{TV}(\mathfrak{w}_\mu)} \lesssim 1. \tag{A.74}$$

Since the temporal and spatial operators commute, (A.70) and (A.73), applied with $g = v^{\mathfrak{a}}_\mu$ and $u = \bar{\mathcal{L}}_\tau h$, give the common identity

$$\tilde{K}^{\mathfrak{a}}_\mu(F^{\mathfrak{a}} v^{\mathfrak{a}}_\mu) = \sum_{\gamma\in\Gamma} \bar{\mathcal{K}}_\tau Q_{\tau,\gamma} * [(\bar{\mathcal{L}}_\tau h) D^{\gamma}_{\tau,t} v^{\mathfrak{a}}_\mu]. \tag{A.75}$$

Thus the temporal part has been treated completely, and it remains only to handle the spatial operator in (A.75).

The finitely many functions $D^{\gamma}_{\tau,t} v^{\mathfrak{a}}_\mu$ are uniformly bounded and are supported where

$$|z_0 - z_{j,0}| \leqslant C\tau^{2s}, \qquad |\bar{z} - \bar{z}_j| \leqslant C\tau, \qquad j = 1,\dots,k. \tag{A.76}$$

We may therefore choose a nonnegative smooth function χ_τ, uniformly bounded and supported in the same set up to increasing C, such that

$$|D^{\gamma}_{\tau,t} v^{\mathfrak{a}}_\mu| \lesssim \chi_\tau, \qquad \gamma \in \Gamma. \tag{A.77}$$

By Lemma A.5 and weighted Young's inequality,

$$\|\bar{\mathcal{K}}_\tau\|_{\mathrm{TV}(\mathfrak{w}_\mu)} \lesssim 1. \tag{A.78}$$

Fix a sufficiently small constant $c_0 > 0$, to be chosen below. We now treat the spatial part of (A.75) in two regimes.

1. The regime $\tau \geqslant c_0\varepsilon$. In this regime $\rho \asymp \tau$. We use the exact discrete product rule

$$\Delta_\varepsilon(ug) = g\Delta_\varepsilon u + u\Delta_\varepsilon g + \sum_{i=1}^{d} \left[(\partial^{i+}u)(\partial^{i+}g) + (\partial^{i-}u)(\partial^{i-}g)\right]. \tag{A.79}$$

When a mixed first-difference term occurs, discrete summation by parts gives, for example,

$$\begin{aligned} &\tau^2 \bar{K}_\tau * [(\partial^{i+}u)(\partial^{i+}g)] \\ &\quad = [\tau\partial^{i+}\bar{K}_\tau] * [u\, T_i^{-}(\tau\partial^{i+}g)] - \bar{K}_\tau * [u\,\tau^2\partial^{i-}\partial^{i+}g], \end{aligned} \tag{A.80}$$

where $\bar{K}_\tau$ is the kernel introduced in Lemma A.5 and $T_i^{\pm}$ denotes translation of the spatial variable by $\pm\varepsilon e_i$. The backward-difference term is treated analogously. Thus every isolated first spatial difference transferred to a regularising kernel is paired with a first spatial difference of the cutoff.

Iterating these identities in all $1+k$ spatial tensor variables in (A.75) yields a finite set $\mathcal{I}$, constants $c_{A,B}$, and scaled spatial difference-translation operators $D^A_{\tau,x}, D^B_{\tau,x}$ such that

$$\tilde{K}^{\mathfrak{a}}_\mu(F^{\mathfrak{a}} v^{\mathfrak{a}}_\mu) = \sum_{\gamma\in\Gamma} \sum_{(A,B)\in\mathcal{I}} c_{A,B}[Q_{\tau,\gamma} * (D^A_{\tau,x}\bar{\mathcal{K}}_\tau)] * [h\, D^B_{\tau,x} D^\gamma_{\tau,t} v^{\mathfrak{a}}_\mu]. \tag{A.81}$$

Every spatial difference in $D^A_{\tau,x}, D^B_{\tau,x}$ is multiplied by the corresponding power of τ.

Since $\tau \geqslant c_0\varepsilon$, the kernel estimates in Lemmas A.4 and A.5, together with (A.74), imply, for every term in the finite expansion,

$$\|Q_{\tau,\gamma} * (D^A_{\tau,x}\bar{\mathcal{K}}_\tau)\|_{\mathrm{TV}(\mathfrak{w}_\mu)} \lesssim_{c_0} 1. \tag{A.82}$$

Indeed, complete second-order spatial blocks are uniformly bounded, while an isolated first difference costs at most $1+\varepsilon/\tau \lesssim_{c_0} 1$. Since $v^{\mathfrak{a}}_\mu$ is obtained from a fixed smooth function by parabolic rescaling at scale τ, all the scaled cutoff derivatives in (A.81) satisfy

$$\|D^B_{\tau,x} D^\gamma_{\tau,t} v^{\mathfrak{a}}_\mu\|_{L^\infty} \lesssim_{c_0} 1. \tag{A.83}$$

Their supports are contained in a fixed enlargement of the support of $v^{\mathfrak{a}}_\mu$. We may therefore choose a nonnegative smooth function $\tilde{v}^{\mathfrak{a}}_{\mu,0}$ satisfying

$$\|\tilde{v}^{\mathfrak{a}}_{\mu,0}\|_{L^\infty} \lesssim 1, \qquad |D^B_{\tau,x} D^\gamma_{\tau,t} v^{\mathfrak{a}}_\mu| \lesssim \tilde{v}^{\mathfrak{a}}_{\mu,0},$$

for all terms occurring in (A.81), and supported in (A.65). Set

$$\tilde{v}^{\mathfrak{a}}_{\mu,1} = 0.$$

Combining (A.69), (A.81), (A.82), and (A.83) proves the first inequality in (A.67). The second follows from (A.64).

On the support of $\tilde{v}^{\mathfrak{a}}_{\mu,0}$ one has

$$w^{(1+k),\omega}_\mu \lesssim 1 \qquad (\omega \in [0,2]),$$

because $\rho \asymp \tau$. At fixed output point, the temporal support has volume $O(\tau^{2sk})$, while its spatial support has normalised lattice volume $O(\rho^{dk})$. This proves (A.68) in the present regime.

2. The regime $\tau < c_0\varepsilon$. We now rewrite the discrete spatial derivatives as finite linear combination of lattice translations and do not commute them through the cutoff. Since $\rho = \varepsilon$ in this regime,

$$\frac{\tau^2}{\varepsilon^2} = \delta^2 < 1.$$

Put

$$\Delta^{(1)}_j := \varepsilon^2 \Delta_{\varepsilon,j}.$$

For a tensor point

$$Z = (z, z_1, \dots, z_k), \qquad z = (z_0, \bar{z}), \qquad z_j = (z_{j,0}, \bar{z}_j), \quad j = 1, \dots, k,$$

and $\nu = (\nu_0, \dots, \nu_k) \in (\mathbb{Z}^d)^{1+k}$, define the spatial translation of Z by

$$\Theta_\nu Z := \big((z_0, \bar{z} + \varepsilon\nu_0), (z_{1,0}, \bar{z}_1 + \varepsilon\nu_1), \dots, (z_{k,0}, \bar{z}_k + \varepsilon\nu_k)\big).$$

Thus Θ_ν leaves all time coordinates unchanged. We denote by T_ν the induced translation operator on functions,

$$(T_\nu f)(Z) := f(\Theta_\nu Z). \tag{A.84}$$

We use the analogous notation Θ_{j,ν_j} and T_{j,ν_j} when only the spatial component of the j-th tensor variable is translated.

For each j, the finite-range operator $(1 - \delta^2\Delta_j^{(1)})^{p_j}$ admits an expansion

$$(1 - \delta^2\Delta_j^{(1)})^{p_j} = \sum_{\nu_j \in \mathcal{N}_j} c_{j,\nu_j}(\delta^2) T_{j,\nu_j},$$

where $\mathcal{N}_j \subset \mathbb{Z}^d$ is finite and

$$|c_{j,0}(\delta^2)| \lesssim 1, \qquad |c_{j,\nu_j}(\delta^2)| \lesssim \delta^2 \quad (\nu_j \neq 0).$$

Tensorising gives

$$\bar{\mathcal{L}}_\tau = \sum_{\nu \in \mathcal{N}} c_\nu(\delta^2) T_\nu, \tag{A.85}$$

where $\mathcal{N} \subset (\mathbb{Z}^d)^{1+k}$ is finite. With

$$r(\nu) := \#\{j \in \{0, \dots, k\} : \nu_j \neq 0\},$$

one has

$$|c_\nu(\delta^2)| \lesssim \delta^{2r(\nu)}. \tag{A.86}$$

Inserting (A.85) into the common temporal reduction (A.75) gives

$$\tilde{K}^{\mathfrak{a}}_\mu(F^{\mathfrak{a}} v^{\mathfrak{a}}_\mu) = \sum_{\nu \in \mathcal{N}} \sum_{\gamma \in \Gamma} c_\nu(\delta^2) \bar{\mathcal{K}}_\tau Q_{\tau,\gamma} * \big[(T_\nu h) D^\gamma_{\tau,t} v^{\mathfrak{a}}_\mu\big]. \tag{A.87}$$

Enlarge, if necessary, the constant C in (A.76) so that χ_τ is supported where

$$|z_0 - z_{j,0}| \leqslant C\tau^{2s}, \qquad |\bar{z} - \bar{z}_j| \leqslant C\tau, \qquad j = 1, \dots, k.$$

Choose $c_0 > 0$ so small that $2Cc_0 < 1$. If $\tau < c_0\varepsilon$, then, on the support of χ_τ,

$$|\bar{z}_i - \bar{z}_j| \leqslant |\bar{z}_i - \bar{z}| + |\bar{z} - \bar{z}_j| \leqslant 2C\tau < \varepsilon,$$

where we use the convention $\bar{z}_0 := \bar{z}$. Since all spatial variables belong to $\varepsilon\mathbb{Z}^d$, this implies

$$\bar{z} = \bar{z}_1 = \cdots = \bar{z}_k. \tag{A.88}$$

For $\nu \in \mathcal{N}$, define the translated cutoff

$$\chi_{\tau,\nu} := T_{-\nu}\chi_\tau. \tag{A.89}$$

Equivalently, for a tensor point $Y = (y, y_1, \dots, y_k)$,

$$\chi_{\tau,\nu}(Y) = \chi_\tau(\Theta_{-\nu}Y). \tag{A.90}$$

We claim that

$$\left\| \mathfrak{o}^{\mathfrak{a}} \cdot (T_\nu h) D^{\gamma}_{\tau,t} v^{\mathfrak{a}}_\mu \tilde{w}^{\mathfrak{a}}_\mu \right\| \lesssim \left\| \mathfrak{o}^{\mathfrak{a}} \cdot h \cdot \chi_{\tau,\nu} \tilde{w}^{\mathfrak{a}}_\mu \right\|. \tag{A.91}$$

Indeed, by (A.77), the left-hand side is bounded by the same norm with $D^{\gamma}_{\tau,t} v^{\mathfrak{a}}_\mu$ replaced by χ_τ. Make the change of variables

$$Y = \Theta_\nu Z.$$

By (A.84),

$$(T_\nu h)(Z) = h(Y), \qquad \chi_\tau(Z) = \chi_\tau(\Theta_{-\nu}Y) = \chi_{\tau,\nu}(Y).$$

The normalised lattice measure and the supremum in the output variable are translation invariant. Writing $Y = (y, y_1, \dots, y_k)$, the output variables z and y differ by one of finitely many spatial shifts of size $O(\varepsilon)$. Since $\varepsilon \leqslant 1$,

$$\mathfrak{o}^{\mathfrak{a}}(z) \lesssim \mathfrak{o}^{\mathfrak{a}}(y).$$

It remains to compare the Steiner weights. On the support of $\chi_\tau(Z)$, all spatial components of Z coincide by (A.88). Let $\mathrm{St}_t(z_0, z_{1,0}, \dots, z_{k,0})$ denote the Steiner diameter of the time coordinates for the metric $|t - t'|^{1/(2s)}$. Since the time projection is 1-Lipschitz for the parabolic metric and Θ_ν does not change the time coordinates,

$$\mathrm{St}(Z) = \mathrm{St}_t(z_0, z_{1,0}, \dots, z_{k,0}) \leqslant \mathrm{St}(Y).$$

The exponent in $\tilde{w}^{\mathfrak{a}}_\mu$ is nonnegative, and hence

$$\tilde{w}^{\mathfrak{a}}_\mu(Z) \leqslant \tilde{w}^{\mathfrak{a}}_\mu(Y).$$

Together with the preceding observations, this proves (A.91).

Combining (A.69), (A.74), (A.78), (A.87), and (A.91), we obtain

$$\|F^{\mathfrak{a}} v^{\mathfrak{a}}_\mu\|_\mu \lesssim \sum_{\nu \in \mathcal{N}} |c_\nu(\delta^2)| \left\| \mathfrak{o}^{\mathfrak{a}} \cdot h \cdot \chi_{\tau,\nu} \tilde{w}^{\mathfrak{a}}_\mu \right\|. \tag{A.92}$$

Define

$$\tilde{v}^{\mathfrak{a}}_{\mu,0} := \sum_{r(\nu)=0} |c_\nu(\delta^2)| \chi_{\tau,\nu},$$
$$\tilde{v}^{\mathfrak{a}}_{\mu,1} := \sum_{r(\nu)\geqslant 1} \delta^{-2} |c_\nu(\delta^2)| \chi_{\tau,\nu}.$$

The coefficient estimate (A.86) and the finiteness of $\mathcal{N}$ imply (A.64). Their supports satisfy (A.65), because all shifts have size $O(\varepsilon) = O(\rho)$ and (A.76) has spatial width $O(\tau)$.

Since $\mathcal{N}$ is finite, after increasing the two majorants by a fixed multiplicative constant, each norm in (A.92) is bounded by the norm with the corresponding sum of cutoffs. Consequently, (A.92) proves the first inequality in (A.67). The second follows from (A.64) and $\delta \leqslant 1$.

It remains to prove (A.68). Fix the output point z. For every $\boldsymbol{\nu} \in \mathcal{N}$, the support of $\chi_{\tau,\boldsymbol{\nu}}(z,\cdot)$ has temporal volume $O(\tau^{2sk})$. By (A.88), each input spatial coordinate is fixed relative to the output coordinate at one of finitely many lattice offsets. Its normalised spatial volume is therefore $O(\varepsilon^{dk})$. Hence

$$\sup_z \int_{\Lambda^k} \chi_{\tau,\boldsymbol{\nu}}(z,Z)\,\mathrm{d}Z \lesssim \tau^{2sk}\varepsilon^{dk}. \tag{A.93}$$

If $r(\boldsymbol{\nu}) = 0$, then $\boldsymbol{\nu} = 0$ and $w_\mu^{(1+k),\omega} \lesssim 1$ on the support. If $r(\boldsymbol{\nu}) \geqslant 1$, the finite range of the shifts and (A.76) imply

$$\mathrm{St}(z, z_1, \dots, z_k) \lesssim \varepsilon.$$

Therefore, for every $\omega \in [0,2]$,

$$w_\mu^{(1+k),\omega}(z, z_1, \dots, z_k) \lesssim \left(\frac{\varepsilon}{\tau}\right)^{\omega} = \delta^{-\omega}. \tag{A.94}$$

Since $\rho = \varepsilon$ in this regime, (A.93) and (A.94) yield

$$\sup_z \int w_\mu^{(1+k),\omega}\tilde{v}^{\mathfrak{a}}_{\mu,0}\,\mathrm{d}Z \lesssim \tau^{2sk}\rho^{dk},$$
$$\delta^2 \sup_z \int w_\mu^{(1+k),\omega}\tilde{v}^{\mathfrak{a}}_{\mu,1}\,\mathrm{d}Z \lesssim \tau^{2sk}\rho^{dk}\,\delta^{2-\omega}.$$

Since $\omega \in [0,2]$ and $\delta \in (0,1]$, both right-hand sides are bounded by $C\tau^{2sk}\rho^{dk}$. This proves (A.68) and completes the proof. ∎

Lemma A.20. *For every $\mathfrak{a} \in \mathfrak{A}$, the kernel norms introduced in Definition 4.6 satisfy the bound*

$$\|[(1-\tilde{K}^{\mathfrak{a}}_\sigma)\dot{F}^{\mathfrak{a}}_\sigma]\cdot(1-v^{\mathfrak{a}}_\mu)\|_\mu \lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}\|\dot{F}^{\mathfrak{a}}_\sigma\|_\sigma,$$

uniformly over $\mu \in [1/2,1)$, $\sigma \in [\mu,1)$ and $\dot{F}^{\mathfrak{a}}_\sigma$.

Proof. By Lemma A.18 we have

$$\|[(1-\tilde{K}^{\mathfrak{a}}_\sigma)\dot{F}^{\mathfrak{a}}_\sigma]\cdot(1-v^{\mathfrak{a}}_\mu)\|_\mu \lesssim \|(1-\tilde{K}^{\mathfrak{a}}_\sigma)\dot{F}^{\mathfrak{a}}_\sigma\|_\mu = \|\mathfrak{o}^{\mathfrak{a}}\cdot[(\tilde{L}^{\mathfrak{a}}_\sigma - 1)\tilde{K}^{\mathfrak{a}}_\mu\tilde{K}^{\mathfrak{a}}_\sigma\dot{F}^{\mathfrak{a}}_\sigma]\cdot\tilde{w}^{\mathfrak{a}}_\mu\|.$$

By (4.7), Young's inequality, Lemma A.4 and $\tilde{w}^{\mathfrak{a}}_\mu \leqslant \tilde{w}^{\mathfrak{a}}_\sigma$ we obtain

$$\begin{aligned}\|\mathfrak{o}^{\mathfrak{a}}\cdot[(\tilde{L}^{\mathfrak{a}}_\sigma - 1)\tilde{K}^{\mathfrak{a}}_\mu\tilde{K}^{\mathfrak{a}}_\sigma\dot{F}^{\mathfrak{a}}_\sigma]\cdot\tilde{w}^{\mathfrak{a}}_\mu\| &\lesssim \|(\tilde{L}^{\mathfrak{a}}_\sigma - 1)\tilde{K}^{\mathfrak{a}}_\mu\|_{\mathrm{TV}(\mathfrak{w}_\mu)}\|\mathfrak{o}^{\mathfrak{a}}\cdot[\tilde{K}^{\mathfrak{a}}_\sigma\dot{F}^{\mathfrak{a}}_\sigma]\cdot\tilde{w}^{\mathfrak{a}}_\mu\| \\ &\lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}\|\dot{F}^{\mathfrak{a}}_\sigma\|_\sigma,\end{aligned}$$

where $\mathfrak{w}_\mu(z,z_1,\dots,z_{k(\mathfrak{a})}) = w^2_\mu(z)w^{\mathfrak{b}}_\mu(z_1)\cdots w^{\mathfrak{b}}_\mu(z_{k(\mathfrak{a})})$. ∎

Lemma A.21. *The cumulant norms introduced in Definition 4.11 satisfy the bound*

$$\|\!|(1-K_\sigma^{1,1})\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\|_\mu \lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}\|\!|\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma,$$

uniformly over $\mu \in [1/2, 1)$ *and* $\sigma \in [\mu, 1)$.

Proof. By Young's inequality, Lemma A.4 and $w_\mu^{\mathfrak{a}} \leqslant w_\sigma^{\mathfrak{a}}$ we obtain

$$\begin{aligned}\|\!|(1-K_\sigma^{1,1})\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\|_\mu &= \|\!|[(1\otimes(L_\sigma-1)K_\mu)K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot w_\mu^{(2),\mathfrak{b}}|\!\| \\ &\lesssim \|(L_\sigma-1)K_\mu\|_{\mathrm{TV}(w_\mu^{\mathfrak{b}})}\|\!|[K_\sigma^{1,1}\dot{\bar{F}}_\sigma^{[\ell],(1)}]\cdot w_\mu^{(2),\mathfrak{b}}|\!\| \\ &\lesssim [\![\mu]\!]^{-2s}[\![\sigma]\!]^{2s}\|\!|\dot{\bar{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma,\end{aligned}$$

as claimed. ∎

A.4. Schauder estimates

In this section, we establish estimates that characterise the spacetime regularity of distributions ϕ in terms of the size of $\mathcal{L}\phi$.

Lemma A.22. *For all* $\bar{A} \in \mathbb{N}_0^{\{1\pm,\dots,d\pm\}}$ *the following bounds*

$$\begin{aligned}\|\partial^{\bar{A}}\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim [\![\mu_i]\!]^{-|\bar{A}|}, \\ \|(1-\Delta_\varepsilon)^{-1}\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim [\![\mu_i]\!]^{2}, \\ \|(-\Delta_\varepsilon)^{s}\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim [\![\mu_i]\!]^{-2s}\end{aligned}$$

hold uniformly in $i \geqslant -1$.

Proof. The estimates follow from the identity $[\![\mu_i]\!] = 2^{-i-2}$ and Definition 1.14 of the spatial Littlewood-Paley blocks. ∎

Lemma A.23. *For all* $A \in \mathbb{N}_0^{\{0,1\pm,\dots,d\pm\}}$ *the following bounds*

$$\begin{aligned}\|\bar{\Delta}_i G\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim [\![\mu_i]\!]^{2s}, \\ \|\partial^A\dot{G}_\eta\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim [\![\eta]\!]^{2s-1-|A|_s}, \\ \|(G-G_\eta)\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim [\![\eta]\!]^{2s},\end{aligned}$$

hold uniformly in $i \geqslant -1$ *and* $\eta \in (1/2, 1)$.

Proof. Recall that $\bar{\Delta}_i$ is the spatial Littlewood-Paley block, $\zeta(z) := (1+|z|_s^2)^{-1/2}$ and note that

$$G = \int_{\mu_i}^1 \dot{G}_\mu \mathrm{d}\mu + \int_{1/2}^{\mu_i} \dot{G}_\mu \mathrm{d}\mu + G_{1/2}.$$

Taking into account Remark 1.8 a), we obtain

$$\begin{aligned}\|\bar\Delta_i G\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim \|\bar\Delta_i\|_{\mathrm{TV}(\zeta^{-1})}\int_{\mu_i}^1 \|\dot G_\mu\|_{\mathrm{TV}(\zeta^{-1})}\mathrm{d}\mu\\ &+ \|(1-\Delta_\varepsilon)^{-1}\bar\Delta_i\|_{\mathrm{TV}(\zeta^{-1})}\int_{1/2}^{\mu_i}\|(1-\Delta_\varepsilon)\dot G_\mu\|_{\mathrm{TV}(\zeta^{-1})}\mathrm{d}\mu\\ &+ \|(1-\Delta_\varepsilon)^{-1}\bar\Delta_i\|_{\mathrm{TV}(\zeta^{-1})}\|(1-\Delta_\varepsilon)G_{1/2}\|_{\mathrm{TV}(\zeta^{-1})}.\end{aligned}$$

Consequently, using Lemma A.10 we arrive at

$$\begin{aligned}\|\bar\Delta_i G\|_{\mathrm{TV}(\zeta^{-1})} &\lesssim \int_{\mu_i}^1 [\![\mu]\!]^{2s-1}\mathrm{d}\mu + [\![\mu_i]\!]^2\int_{1/2}^{\mu_i}[\![\mu]\!]^{2s-3}\mathrm{d}\mu + [\![\mu_i]\!]^2\times 1\\ &\lesssim [\![\mu_i]\!]^{2s} + [\![\mu_i]\!]^2[\![\mu_i]\!]^{2s-2} + [\![\mu_i]\!]^2 \lesssim [\![\mu_i]\!]^{2s}.\end{aligned}$$

This proves the first bound. The second bound is precisely (A.49). To show the third bound notice that

$$\|(G-G_\eta)\|_{\mathrm{TV}(\zeta^{-1})} \leqslant \int_\eta^1 \|\dot G_\sigma\|_{\mathrm{TV}(\zeta^{-1})}\mathrm{d}\sigma \lesssim \int_\eta^1 [\![\sigma]\!]^{2s-1}\mathrm{d}\sigma \lesssim [\![\eta]\!]^{2s}.$$

This completes the proof. ∎

Lemma A.24. *Recall that $\gamma \in (0,s)$ and $\gamma = a\nu$. Then*

$$\|\rho_\eta\bar\Delta_i\phi\| \lesssim [\![\eta]\!]^{-\gamma}[\![\bar\mu]\!]^{\gamma}[\![\mu_i\vee\bar\mu]\!]^{-\gamma}[|\!|\!|\phi|\!|\!| + [\![\bar\mu]\!]^{2s-2\gamma}|\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_\#], \tag{A.95}$$

uniformly in $\phi\in\mathcal{S}'(\Lambda)$, $i\geqslant -1$ and $\eta\in[\bar\mu,1)$, where the norms $|\!|\!|\bullet|\!|\!| = |\!|\!|\bullet|\!|\!|_{\bar\mu}$ and $|\!|\!|\bullet|\!|\!|_\# = |\!|\!|\bullet|\!|\!|_{\#,\bar\mu}$ are defined by (2.13) and (2.14). In particular, for $\hat\gamma>\gamma$, we have

$$\sum_i 2^{-\hat\gamma i}\|\rho_\eta\bar\Delta_i\phi\| \lesssim [\![\eta]\!]^{-\gamma}[|\!|\!|\phi|\!|\!| + [\![\bar\mu]\!]^{2s-2\gamma}|\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_\#],$$

uniformly in $\bar\mu\in[1/2,1)$, $\eta\in[\bar\mu,1)$ and $\phi\in\mathcal{S}'(\Lambda)$.

Proof. First observe that

$$\begin{aligned}\|\rho_\eta\bar\Delta_i\phi\| &\lesssim \sup_j\|\rho_\eta\chi_j\bar\Delta_i\phi\|\\ &\lesssim \sup_j\|\rho_\eta\chi_j\zeta^{-1}_{\mu_j\vee\bar\mu}\|\|\zeta_{\mu_j\vee\bar\mu}\bar\Delta_i\phi\|\\ &\lesssim [\![\eta]\!]^{-\gamma}\sup_j[\![\mu_j\vee\bar\mu]\!]^{\gamma}\|\zeta_{\mu_j\vee\bar\mu}\bar\Delta_i\phi\|,\end{aligned} \tag{A.96}$$

uniformly in $i\geqslant -1$ and $\eta\geqslant\bar\mu$. To obtain the last estimate we used the fact that

$$\|\rho_\eta\chi_j\zeta^{-1}_{\mu_j\vee\bar\mu}\| \lesssim \|\rho_\eta\chi_j\zeta^{-1}_{\mu_j}\| \lesssim [\![\eta]\!]^{-\gamma}[\![\mu_j\vee\bar\mu]\!]^{\gamma}. \tag{A.97}$$

Indeed note that if $\bar\mu\leqslant\eta\leqslant\mu_j$, we have

$$\|\rho_\eta\chi_j\zeta^{-1}_{\mu_j}\| \leqslant (1+[\![\eta]\!]^a[\![\mu_j]\!]^{-a})^{-\nu} \lesssim [\![\eta]\!]^{-\gamma}[\![\mu_j]\!]^{\gamma} = [\![\eta]\!]^{-\gamma}[\![\mu_j\vee\bar\mu]\!]^{\gamma}.$$

On the other hand, if $\eta \geqslant \mu_j \vee \bar{\mu}$, then $\|\rho_\eta \chi_j \zeta_{\mu_j}^{-1}\| \leqslant 1$ and $1 \leqslant [\![\eta]\!]^{-\gamma}[\![\mu_j \vee \bar{\mu}]\!]^{\gamma}$. Hence, the bound (A.97) follows. By the estimate (A.96) it suffices to establish the bound (A.95) with the weight ρ_η replaced by ζ_η. To prove the latter bound note that

$$\begin{aligned}\sup_{\eta \geqslant \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi\| &= \sup_{\bar{\mu} \leqslant \eta \leqslant \mu_i} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi\| \vee \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi\| \\ &\leqslant [\![\bar{\mu}]\!]^{\gamma} \|\zeta_{\mu_i \vee \bar{\mu}} \bar{\Delta}_i \phi\| \vee [\![\bar{\mu}]\!]^{\gamma} [\![\mu_i \vee \bar{\mu}]\!]^{-\gamma} \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi\| \\ &\leqslant [\![\bar{\mu}]\!]^{\gamma} [\![\mu_i \vee \bar{\mu}]\!]^{-\gamma} \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi\|.\end{aligned}$$

In order to bound the supremum in the last line above we shall use the following decomposition

$$\phi = \phi_\eta + (G - G_\eta)\mathcal{L}\phi, \qquad \eta \geqslant \mu_i \vee \bar{\mu}.$$

Note that

$$\sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi\| \lesssim \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi_\eta\| + \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i (G - G_\eta)\mathcal{L}\phi\|,$$

and

$$\sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \bar{\Delta}_i \phi_\eta\| \lesssim \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta \phi_\eta\| \lesssim \sup_{\eta \geqslant \mu_i \vee \bar{\mu}} [\![\eta]\!]^{\gamma} \|\zeta_\eta^{1/3} \phi_\eta\| \leqslant |\!|\!|\phi|\!|\!|.$$

To complete the proof it is now enough to show that

$$\|\zeta_\eta \bar{\Delta}_i (G - G_\eta)\mathcal{L}\phi\| \lesssim [\![\eta]\!]^{2s-3\gamma} |\!|\!| K_\bullet \mathcal{L}\phi |\!|\!|_{\#}, \tag{A.98}$$

uniformly in $\eta \geqslant \mu_i \vee \bar{\mu}$. To this end, recall that $L_\mu = (1 + [\![\mu]\!]^{2s}\partial_t)\bar{L}_\mu$, where $\bar{L}_\mu := (1 - [\![\mu]\!]^2\Delta)^2$. Now

$$\begin{aligned}\bar{\Delta}_i (G - G_\eta)\mathcal{L}\phi &= \bar{\Delta}_i (G - G_\eta) L_\eta K_\eta \mathcal{L}\phi \\ &= \bar{L}_\eta \bar{\Delta}_i (G - G_\eta)(1 + [\![\eta]\!]^{2s}\partial_t)(K_\eta \mathcal{L}\phi) \\ &= \bar{L}_\eta \bar{\Delta}_i (G - G_\eta) K_\eta \mathcal{L}\phi + [\![\eta]\!]^{2s} \bar{L}_\eta \bar{\Delta}_i \partial_t G (1 - J_\eta) K_\eta \mathcal{L}\phi \\ &= \mathbb{I} + \mathbb{II}.\end{aligned}$$

We have the following bound for the first term

$$\begin{aligned}\|\zeta_\eta \mathbb{I}\| = \|\zeta_\eta \bar{L}_\eta \bar{\Delta}_i (G - G_\eta) K_\eta \mathcal{L}\phi\| &\lesssim \|\bar{L}_\eta \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \|(G - G_\eta)\|_{\mathrm{TV}(\zeta^{-1})} \|\zeta_\eta K_\eta \mathcal{L}\phi\| \\ &\lesssim [\![\eta]\!]^{2s} \|\zeta_\eta K_\eta \mathcal{L}\phi\|.\end{aligned}$$

The above estimates follow from Remark 1.8 a), Lemma A.23 and the bound

$$\|\bar{L}_\eta \bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})} \lesssim 1,$$

which is a consequence of Lemma A.22. In order to bound $\|\zeta_\eta \mathbb{II}\|_{L^\infty}$ we first note that

$$\partial_t G = 1 - (m^2 + (-\Delta_\varepsilon)^s)G$$

and write

$$\begin{aligned}\mathbb{I} &= [\![\eta]\!]^{2s}\bar{L}_\eta\bar{\Delta}_i\partial_t G(1-J_\eta)K_\eta\mathcal{L}\phi\\ &= [\![\eta]\!]^{2s}\bar{L}_\eta\bar{\Delta}_i(1-J_\eta)K_\eta\mathcal{L}\phi - [\![\eta]\!]^{2s}(m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta\bar{\Delta}_i G(1-J_\eta)K_\eta\mathcal{L}\phi\\ &= \mathbb{I}_1+\mathbb{I}_2.\end{aligned}$$

We have the following bound for $\|\zeta_\eta\mathbb{I}_1\|_{L^\infty}$:

$$\begin{aligned}\|\zeta_\eta\mathbb{I}_1\| &= [\![\eta]\!]^{2s}\|\zeta_\eta\bar{L}_\eta\bar{\Delta}_i(1-J_\eta)K_\eta\mathcal{L}\phi\|\\ &\lesssim [\![\eta]\!]^{2s}\|\bar{L}_\eta\bar{\Delta}_i\|_{\mathrm{TV}(\zeta^{-1})}\|(1-J_\eta)\|_{\mathrm{TV}(\zeta^{-1})}\\ &\quad\times\|\zeta_\eta K_\eta\mathcal{L}\phi\|\\ &\lesssim [\![\eta]\!]^{2s}\|\zeta_\eta K_\eta\mathcal{L}\phi\|.\end{aligned}$$

For $\|\zeta_\eta\mathbb{I}_2\|$ we have, exploiting Lemma A.23,

$$\begin{aligned}\|\zeta_\eta\mathbb{I}_2\| &\leqslant [\![\eta]\!]^{2s}\|\zeta_\eta(m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta(\bar{\Delta}_{i-1}+\bar{\Delta}_i+\bar{\Delta}_{i+1})\bar{\Delta}_i G(1-J_\eta)K_\eta\mathcal{L}\phi\|\\ &\lesssim [\![\eta]\!]^{2s}\|(m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta(\bar{\Delta}_{i-1}+\bar{\Delta}_i+\bar{\Delta}_{i+1})\|_{\mathrm{TV}(\zeta^{-1})}\\ &\quad\times\|\bar{\Delta}_i G\|_{\mathrm{TV}(\zeta^{-1})}\|(1-J_\eta)\|_{\mathrm{TV}(\zeta^{-1})}\|\zeta_\eta K_\eta\mathcal{L}\phi\|\\ &\lesssim [\![\eta]\!]^{2s}[\![\mu_i]\!]^{-2s}[\![\mu_i]\!]^{2s}\|\zeta_\eta K_\eta\mathcal{L}\phi\|\\ &= [\![\eta]\!]^{2s}\|\zeta_\eta K_\eta\mathcal{L}\phi\|,\end{aligned}$$

where we used that

$$\|(m^2+(-\Delta_\varepsilon)^s)\bar{L}_\eta(\bar{\Delta}_{i-1}+\bar{\Delta}_i+\bar{\Delta}_{i+1})\|_{\mathrm{TV}(\zeta^{-1})}\lesssim [\![\mu_i]\!]^{-2s},$$

which follows from Lemma A.22. Summing up, we have

$$\begin{aligned}\big\|\zeta_\eta\bar{\Delta}_i(G-G_\eta)\mathcal{L}\phi\big\| &\lesssim [\![\eta]\!]^{2s}\big\|\zeta_\eta K_\eta\mathcal{L}\phi\big\|\\ &\lesssim [\![\eta]\!]^{2s-3\gamma}[\![\eta]\!]^{3\gamma}\big\|\zeta_\eta K_\eta\mathcal{L}\phi\big\|\\ &\lesssim [\![\eta]\!]^{2s-3\gamma}|\!|\!|K_\bullet\mathcal{L}\phi|\!|\!|_{\#}.\end{aligned}$$

This proves the bound (A.98) and completes the proof of the lemma. ∎

Lemma A.25. *Let $J_{>\eta} := 1-J_\eta$. We have*

$$\|\rho_\mu^3 J_{>\eta}\phi_\sigma\|\lesssim [\![\eta]\!]^{2s}[\![\sigma]\!]^{-3\gamma}|\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_{\#},$$

uniformly in $\bar\mu,\mu,\eta\in[1/2,1)$, $\sigma\in[\mu\vee\bar\mu,1)$ and $\phi\in\mathcal{S}'(\Lambda)$, where the norm $|\!|\!|\bullet|\!|\!|_{\#}=|\!|\!|\bullet|\!|\!|_{\#,\bar\mu}$ is defined by (2.14).

Proof. Using Remark 1.8 a) and Lemma A.23, we obtain

$$\|\rho_\mu^3(G-G_\eta)\mathcal{L}\phi_\sigma\|\lesssim\|G-G_\eta\|_{\mathrm{TV}(\zeta^{-1})}\|\rho_\mu^3\mathcal{L}\phi_\sigma\|\lesssim[\![\eta]\!]^{2s}\|\rho_\mu^3\mathcal{L}\phi_\sigma\|.$$

Next, we note that

$$\begin{aligned}\|\rho_\mu^3 J_{>\eta}\phi_\sigma\| &= \|\rho_\mu^3(G-G_\eta)\mathcal{L}\phi_\sigma\|\lesssim[\![\eta]\!]^{2s}\|\rho_\mu^3\mathcal{L}\phi_\sigma\|\\ &\lesssim[\![\eta]\!]^{2s}\|\rho_\sigma^3\mathcal{L}\phi_\sigma\|\lesssim[\![\eta]\!]^{2s}[\![\sigma]\!]^{-3\gamma}|\!|\!|\mathcal{L}\phi_\bullet|\!|\!|_{\#},\end{aligned}$$

on account of Lemma 2.3 and the identity $\rho_\sigma=\zeta_\sigma^\nu$. This finishes the proof. ∎

Appendix B. Flow equation estimates

B.1. Estimates for $\mathcal{A}$ and $\mathcal{B}$

The goal of this section is to prove the bounds for the operators $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$ appearing in the flow equation (4.23) for cumulants, thereby proving Lemma 4.13. We begin with the definition of the first operator:

$$\sum_{\boldsymbol{b}} \mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) := \sum_{i=1}^{n(\boldsymbol{a})} \sum_{\boldsymbol{b}} \mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma),$$

where

$$\begin{aligned}\sum_{\boldsymbol{b}} \mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma) := \sum_{\ell'=0}^{\ell(\mathfrak{a}_i)-1} \sum_{k'=0}^{k(\mathfrak{a}_i)} (k'+1)\mathfrak{C}_i(\dot{G}_\sigma) \\ \times \mathfrak{K}_{n(\boldsymbol{a})+1}(F^{\mathfrak{a}_1}_\sigma, \dots, F^{\mathfrak{a}_{i-1}}_\sigma, F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma, \\ F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma, F^{\mathfrak{a}_{i+1}}_\sigma, \dots, F^{\mathfrak{a}_{n(\boldsymbol{a})}}_\sigma).\end{aligned} \tag{B.1}$$

The summation index $\boldsymbol{b}$ is constrained by the allowed values of (ℓ', k') appearing on the right-hand side. For fixed $\boldsymbol{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n) \in \boldsymbol{A}$ and $i \in \{1, \dots, n\}$, we say that $\boldsymbol{b}$ is $(\boldsymbol{a}, i)$-*admissible for* $\mathcal{A}$ if there exist

$$0 \leqslant \ell' \leqslant \ell(\mathfrak{a}_i) - 1, \qquad 0 \leqslant k' \leqslant k(\mathfrak{a}_i),$$

such that

$$\boldsymbol{b} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{i-1}, (\ell(\mathfrak{a}_i) - 1 - \ell', k' + 1), (\ell', k(\mathfrak{a}_i) - k'), \mathfrak{a}_{i+1}, \dots, \mathfrak{a}_n).$$

If $\boldsymbol{b}$ is not of this form, we set $\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}} = 0$. The operator $\mathfrak{C}_i(\dot{G}_\sigma)$ acts by applying $\dot{G}_\sigma$ to the output variable of the $(i+1)$-th kernel, and by inserting the resulting function into the last input variable of the i-th kernel. This operator generalises $\mathfrak{C}(\dot{G}_\sigma)$ defined in (4.11), which appears in the flow equation for the effective force kernels (4.13).

We now proceed with the definition of the second operator:

$$\sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) := \sum_{i=1}^{n(\boldsymbol{a})} \sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma),$$

where

$$\begin{aligned}\sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) := \sum_{I_1, I_2} \sum_{\ell'=0}^{\ell(\mathfrak{a}_i)-1} \sum_{k'=0}^{k(\mathfrak{a}_i)} (k'+1)\mathfrak{C}_{|I_1|+1}(\dot{G}_\sigma) \\ \Big(\mathfrak{K}_{|I_1|+1}\Big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1}, F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma\Big) \\ \mathfrak{K}_{|I_2|+1}\Big(F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma, (F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\Big)\Big).\end{aligned} \tag{B.2}$$

Here, for a fixed $i \in \{1, \dots, n(\boldsymbol{a})\}$, the sum $\sum_{I_1, I_2}$ runs over all the partitions of the set

$$I_1 \sqcup I_2 = \{1, \dots, i-1, i+1, \dots, n(\boldsymbol{a})\}.$$

The summation indices $\boldsymbol{b}$ and $\boldsymbol{c}$ are now constrained by the allowed values of (I_1, I_2, ℓ', k') on the right-hand side of (B.2). We say that $(\boldsymbol{b}, \boldsymbol{c})$ is $(\boldsymbol{a}, i)$-*admissible for* $\mathcal{B}$ if there exist a partition

$$I_1 \sqcup I_2 = \{1, \dots, i-1, i+1, \dots, n\}$$

and integers ℓ', k' in the above ranges such that, with the elements of I_1 and I_2 ordered increasingly,

$$\boldsymbol{b} = \big((\mathfrak{a}_j)_{j \in I_1}, (\ell(\mathfrak{a}_i) - 1 - \ell', k' + 1)\big),$$

and

$$\boldsymbol{c} = \big((\ell', k(\mathfrak{a}_i) - k'), (\mathfrak{a}_j)_{j \in I_2}\big).$$

If $(\boldsymbol{b}, \boldsymbol{c})$ is not of this form, we set $\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}} = 0$. The operator $\mathfrak{C}_{|I_1|+1}(\dot{G}_\sigma)$ acts by applying $\dot{G}_\sigma$ to the output variable of the kernel $F_\sigma^{[\ell'], (k(\mathfrak{a}_i) - k')}$, and by inserting the resulting function into the last input variable of the kernel $F_\sigma^{[\ell(\mathfrak{a}_i) - 1 - \ell'], (k'+1)}$. For precise definitions of the operators $\mathcal{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$, see [11].

Remark B.1 (Periodicity and translation invariance). The estimates for the operators $\mathcal{A}$ and $\mathcal{B}$ are applied to cumulant kernels which are spatially periodic and invariant under simultaneous spacetime translations. These properties are preserved by the regularising operators, since their kernels are translation invariant, and the weights $w^{\boldsymbol{a}}_\sigma$ are likewise invariant under simultaneous translations. Although the norm in Definition 4.11 distinguishes the first output variable, this choice is immaterial for the cumulants appearing below. Indeed, let F be simultaneously translation invariant in all its output and input variables and spatially periodic with period M. Then, for every $r \in \{1, \dots, n\}$,

$$\sup_{z_r \in \Lambda} \int_{\Lambda_M^{n-1}} \left(\int_{\Lambda^k} \big| F(z_1, \mathrm{d}Z_1, \dots, z_n, \mathrm{d}Z_n) \big| \right) \prod_{q \neq r} \mathrm{d}z_q = |\!|\!| F |\!|\!|.$$

To see this, translate all variables simultaneously so that the fixed variable z_r is moved to the position of the distinguished output. Lebesgue measure is translation invariant, while a translated spatial fundamental domain may be replaced by Λ_M by periodicity.

In the estimate for $\mathcal{A}$, spatial periodicity is used to decompose the integration over the additional output variable into translates of the fundamental domain Λ_M. After translating the corresponding output–input block by an element of the spatial period lattice, the cumulant kernel and its weight are unchanged, while the sum over lattice translates is absorbed into the periodisation

$$\big| (L_\sigma \dot{G}_\sigma) w^{\flat}_\sigma \big|_M.$$

This yields the factor

$$\Big\| \big| (L_\sigma \dot{G}_\sigma) w^{\flat}_\sigma \big|_M \Big\|_{L^\infty}$$

in Lemma B.2. Translation invariance also shows that the operators $\mathcal{A}$ and $\mathcal{B}$ preserve the class of cumulants considered here. Moreover, in the estimate for $\mathcal{B}$, it allows one to choose any output variable as the distinguished variable in the cumulant norm, while periodicity permits the corresponding translated fundamental domain to be replaced by Λ_M.

Lemma B.2. *Let $\boldsymbol{a} \in \boldsymbol{A}$ and $i \leqslant n(\boldsymbol{a})$. For every $(\boldsymbol{a}, i)$-admissible $\boldsymbol{b}$ one has*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} |\!|\!| \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}) |\!|\!|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} |\!|\!| \mathcal{F}_\sigma^{\boldsymbol{b}} |\!|\!|_\sigma,$$

uniformly in $\sigma \in [1/2, 1)$ and in $\mathcal{F}_\sigma^{\boldsymbol{b}}$.

Proof. Recalling (B.1), our task is to analyse the expression

$$\mathfrak{C}_i(\dot{G}_\sigma)\mathfrak{K}_{n(\boldsymbol{a})+1}(F_\sigma^{\mathfrak{a}_1}, \dots, F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}, F_\sigma^{[\ell'],(k(\mathfrak{a}_i)-k')}, \dots, F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}).$$

Using $L_\sigma K_\sigma = 1$ and Definition 4.5, we obtain

$$\begin{aligned} &K_\sigma^{n(\boldsymbol{a}),K(\boldsymbol{a})}\big(\mathfrak{C}_i(\dot{G}_\sigma)\mathfrak{K}_{n(\boldsymbol{a})+1}(F_\sigma^{\mathfrak{a}_1}, \dots, F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}, F_\sigma^{[\ell'],(k(\mathfrak{a}_i)-k')}, \dots, F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}})\big) \\ &= \mathfrak{C}_i(L_\sigma \dot{G}_\sigma) K_\sigma^{n(\boldsymbol{a})+1,K(\boldsymbol{a})+1} \mathfrak{K}_{n(\boldsymbol{a})+1}(F_\sigma^{\mathfrak{a}_1}, \dots \\ &\qquad \dots, F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}, F_\sigma^{[\ell'],(k(\mathfrak{a}_i)-k')}, \dots, F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}). \end{aligned}$$

As a consequence,

$$K_\sigma^{\boldsymbol{a}} \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}) = \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(L_\sigma \dot{G}_\sigma, K_\sigma^{\boldsymbol{b}} \mathcal{F}_\sigma^{\boldsymbol{b}}).$$

Hence, by Definition 4.11 of the cumulant norm,

$$\begin{aligned} |\!|\!| \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}) |\!|\!|_\sigma &= |\!|\!| [K_\sigma^{\boldsymbol{a}} \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}})] \cdot w_\sigma^{\boldsymbol{a}} |\!|\!| \\ &= |\!|\!| \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(L_\sigma \dot{G}_\sigma, K_\sigma^{\boldsymbol{b}} \mathcal{F}_\sigma^{\boldsymbol{b}}) \cdot w_\sigma^{\boldsymbol{a}} |\!|\!| \\ &\leqslant |\!|\!| \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(|L_\sigma \dot{G}_\sigma|, |K_\sigma^{\boldsymbol{b}} \mathcal{F}_\sigma^{\boldsymbol{b}}|) \cdot w_\sigma^{\boldsymbol{a}} |\!|\!| \\ &\lesssim |\!|\!| \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(|L_\sigma \dot{G}_\sigma| \cdot w_\sigma^\flat, |K_\sigma^{\boldsymbol{b}} \mathcal{F}_\sigma^{\boldsymbol{b}}| \cdot w_\sigma^{\boldsymbol{b}}) |\!|\!|, \end{aligned}$$

where the last estimate follows from (4.3) and $|L_\sigma \dot{G}_\sigma| \cdot w_\sigma^\flat$ denotes the function

$$z \mapsto |(L_\sigma \dot{G}_\sigma)(z)| w_\sigma^\flat(z). \tag{B.3}$$

As a result, using Definition 4.11 one proves along the lines of Lemma 5.29 in [11] that

$$\begin{aligned} &|\!|\!| \mathcal{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathcal{F}_\sigma^{\boldsymbol{b}}) |\!|\!|_\sigma \\ &\quad \lesssim \Big\| \big| (L_\sigma \dot{G}_\sigma) \cdot w_\sigma^\flat \big|_M \Big\|_{L^\infty} |\!|\!| [K_\sigma^{\boldsymbol{b}} \mathcal{F}_\sigma^{\boldsymbol{b}}] \cdot w_\sigma^{\boldsymbol{b}} |\!|\!| \\ &\quad = \Big\| \big| (L_\sigma \dot{G}_\sigma) \cdot w_\sigma^\flat \big|_M \Big\|_{L^\infty} |\!|\!| \mathcal{F}_\sigma^{\boldsymbol{b}} |\!|\!|_\sigma, \end{aligned}$$

where $|(L_\sigma \dot{G}_\sigma) \cdot w_\sigma^\flat|_M$ denotes the periodisation in space of the function (B.3) with period M. Thus, since by Lemma A.10

$$\Big\| \big| (L_\sigma \dot{G}_\sigma) \cdot w_\sigma^\flat \big|_M \Big\|_{L^\infty} \lesssim [\![\sigma]\!]^{-1-d},$$

we obtain

$$\|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)\|_\sigma \lesssim [\![\sigma]\!]^{-1-d}\|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma.$$

It follows that

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\|\mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)\|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{a}]-1-d}\|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1}\|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma.$$

In the last step, we used (4.19) and (4.26), which imply

$$-[\boldsymbol{a}] - 1 - d = -[\boldsymbol{b}] - 1 + \theta + \beta - \delta - d \geqslant -[\boldsymbol{b}] - 1.$$

This finishes the proof. ∎

Lemma B.3. *Let $\boldsymbol{a} \in \boldsymbol{A}$ and $i \leqslant n(\boldsymbol{a})$. For every $(\boldsymbol{a}, i)$-admissible pair $(\boldsymbol{b}, \boldsymbol{c})$ one has*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma)\|_\sigma \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma\|\mathcal{F}^{\boldsymbol{c}}_\sigma\|_\sigma,$$

uniformly in $\sigma \in [1/2, 1)$ and in $\mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma$.

Proof. The proof proceeds along the same lines as the previous one. Recalling (B.2), we now need to analyse the expression

$$\mathfrak{C}_{|I_1|+1}(\dot{G}_\sigma)(\mathfrak{K}_{|I_1|+1}((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1}, F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma)\mathfrak{K}_{|I_2|+1}(F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma, (F^{\mathfrak{a}_j}_\sigma)_{j\in I_2})).$$

Using $L_\sigma K_\sigma = 1$ and Definition 4.5, we obtain

$$\begin{aligned}
&K^{\boldsymbol{a}}_\sigma\Big(\mathfrak{C}_{|I_1|+1}(\dot{G}_\sigma)\mathfrak{K}_{|I_1|+1}\big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1}, F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma\big)\mathfrak{K}_{|I_2|+1}\big(F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma, (F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\big)\Big)\\
&\quad= \mathfrak{C}_{|I_1|+1}(L_\sigma\dot{G}_\sigma)\Big(K^{|I_1|+1,k_1}_\sigma\mathfrak{K}_{|I_1|+1}\big((F^{\mathfrak{a}_j}_\sigma)_{j\in I_1}, F^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}_\sigma\big)\\
&\qquad\qquad K^{|I_2|+1,k_2}_\sigma\mathfrak{K}_{|I_2|+1}\big(F^{[\ell'],(k(\mathfrak{a}_i)-k')}_\sigma, (F^{\mathfrak{a}_j}_\sigma)_{j\in I_2}\big)\Big),
\end{aligned}$$

where

$$k_1 := \sum_{j\in I_1} k(\mathfrak{a}_j) + k' + 1, \qquad k_2 := \sum_{j\in I_2} k(\mathfrak{a}_j) + k(\mathfrak{a}_i) - k'.$$

As a consequence,

$$K^{\boldsymbol{a}}_\sigma\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) = \mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(L_\sigma\dot{G}_\sigma, K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma, K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma).$$

Hence, by Definition 4.11 of the cumulant norm,

$$\begin{aligned}
\|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma)\|_\sigma &= \|[K^{\boldsymbol{a}}_\sigma\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma)] \cdot w^{\boldsymbol{a}}_\sigma\|\\
&= \|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(L_\sigma\dot{G}_\sigma, K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma, K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma) \cdot w^{\boldsymbol{a}}_\sigma\|\\
&\leqslant \|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(|L_\sigma\dot{G}_\sigma|, |K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma|, |K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma|) \cdot w^{\boldsymbol{a}}_\sigma\|\\
&\lesssim \|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(|L_\sigma\dot{G}_\sigma| \cdot w^{\flat}_\sigma, |K^{\boldsymbol{b}}_\sigma\mathcal{F}^{\boldsymbol{b}}_\sigma| \cdot w^{\boldsymbol{b}}_\sigma, |K^{\boldsymbol{c}}_\sigma\mathcal{F}^{\boldsymbol{c}}_\sigma| \cdot w^{\boldsymbol{c}}_\sigma)\|,
\end{aligned}$$

where the last estimate follows from (4.3). As a result, by Definition 4.11 and Lemma A.10, we have

$$\begin{aligned}\|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma)\|_\sigma &\lesssim \|L_\sigma \dot{G}_\sigma\|_{\mathrm{TV}(w^{\mathfrak{b}}_\sigma)} \|K^{\boldsymbol{b}}_\sigma \mathcal{F}^{\boldsymbol{b}}_\sigma \cdot w^{\boldsymbol{b}}_\sigma\| \|K^{\boldsymbol{c}}_\sigma \mathcal{F}^{\boldsymbol{c}}_\sigma \cdot w^{\boldsymbol{c}}_\sigma\| \\ &= \|L_\sigma \dot{G}_\sigma\|_{\mathrm{TV}(w^{\mathfrak{b}}_\sigma)} \|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma \|\mathcal{F}^{\boldsymbol{c}}_\sigma\|_\sigma \\ &\lesssim [\![\sigma]\!]^{2s-1} \|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma \|\mathcal{F}^{\boldsymbol{c}}_\sigma\|_\sigma .\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathcal{B}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma)\|_\sigma &\lesssim [\![\sigma]\!]^{-[\boldsymbol{a}]} [\![\sigma]\!]^{2s-1} \|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma \|\mathcal{F}^{\boldsymbol{c}}_\sigma\|_\sigma , \\ &\leqslant [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1} \|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma \|\mathcal{F}^{\boldsymbol{c}}_\sigma\|_\sigma .\end{aligned}$$

In the last step we used (4.19) and (4.26), which imply

$$-[\boldsymbol{a}] + 2s - 1 = -[\boldsymbol{b}] - [\boldsymbol{c}] - \varrho + \theta + \beta - \delta + 2s - 1 \geqslant -[\boldsymbol{b}] - [\boldsymbol{c}] - 1.$$

This finishes the proof. ∎

Remark B.4. The unweighted L^∞ estimate used in the proof of Lemma B.2 admits an additional sublattice gain, whereas the corresponding total variation estimate used in the proof of Lemma B.3 does not. More precisely, uniformly in $\sigma \in [1/2, 1)$ and in every admissible spatial period $M \geqslant 1$,

$$\big\| \big| L_\sigma \dot{G}_\sigma \big|_M \big\|_{L^\infty} \lesssim [\![\sigma]\!]^{-d-1} \wedge \varepsilon^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}-d-1}. \tag{B.4}$$

Indeed, (A.52) with $p = 1$ gives

$$\big\| \big| L_\sigma \dot{G}_\sigma \big|_M \big\|_{L^\infty} \lesssim [\![\sigma]\!]^{-d-1} \wedge \varepsilon^{-d} [\![\sigma]\!]^{-1}.$$

Since $d = 3$ and $\mathfrak{b} < 2s \leqslant 2$ by (4.39), the second term is bounded by $\varepsilon^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}-d-1}$ when $[\![\sigma]\!] \leqslant \varepsilon$. This proves (B.4).

The first part of the following estimate follows from Lemma B.2, since the weights are bounded from below by one. Repeating its proof without an output weight and using (B.4) gives the second part. Thus, for every $(\boldsymbol{a}, i)$-admissible $\boldsymbol{b}$,

$$\begin{aligned}&[\![\sigma]\!]^{-[\boldsymbol{a}]} \|K^{\boldsymbol{a}}_\sigma \mathcal{A}^{\boldsymbol{a},(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma)\| \\ &\quad\lesssim \Big([\![\sigma]\!]^{-[\boldsymbol{b}]-1} \wedge \varepsilon^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}-[\boldsymbol{b}]-1} \Big) \|\mathcal{F}^{\boldsymbol{b}}_\sigma\|_\sigma .\end{aligned} \tag{B.5}$$

Here we also used (4.24) and the first compatibility condition in (4.26), as in the last step of the proof of Lemma B.2.

There is, in general, no analogue of (B.5) for the full $\mathcal{B}$-operator. Indeed, the atomic sublattice part of $L_\sigma \dot{G}_\sigma$ has total variation of order $[\![\sigma]\!]^{2s-1}$ and does not gain the factor $\varepsilon^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}}$. For the relevant one-point cumulants, the required improvement will instead follow from the exact cancellation of the local $\mathcal{B}$-contribution established in Lemma B.7 below.

B.2. Localisation

In this section, we introduce the Taylor expansion on the (semi)-discrete lattice $\Lambda = \Lambda_\varepsilon$. At first order, we have

$$\psi(z_1) = \psi(z) + \sum_{i\in\{0,1\pm,2\pm,\dots,d\pm\}} \int_0^1 [\mathrm{d}\rho_{z_1-z}(t)]^i (\partial^i\psi)(z+\rho_{z_1-z}(t)).$$

Here ∂^0 denotes the time derivative, while

$$\partial^{k\pm}\psi(z) := \pm\varepsilon^{-1}[\psi(z\pm\varepsilon e_k) - \psi(z)],$$

denotes the discrete forward ($k+$) and backward ($k-$) derivatives in the k-th spatial direction. For $h \in \Lambda$, the function $\rho_h : [0,1] \to \Lambda$ is a bounded variation path (left-continuous at its spatial jump) such that $\rho_h(0) = 0$ and $\rho_h(1) = h$. We use the notation

$$[\mathrm{d}\rho_{z_1-z}(t)]^0 := \mathrm{d}\rho^0_{z_1-z}(t).$$

We introduce the decomposition

$$\mathrm{d}\rho^k_{z_1-z} = (\mathrm{d}\rho^k_{z_1-z})_+ + (\mathrm{d}\rho^k_{z_1-z})_-,$$

where $(\mathrm{d}\rho^k_{z_1-z})_+$ is nonnegative and $(\mathrm{d}\rho^k_{z_1-z})_-$ is non-positive. We write

$$[\mathrm{d}\rho_{z_1-z}(t)]^{k\pm} := (\mathrm{d}\rho^k_{z_1-z}(t))_\pm.$$

Note that the path ρ_h is piecewise constant in space, so that the signed measure $\mathrm{d}\rho^i_h$ is well defined and given by a sum of delta functions multiplied by the corresponding increments. We choose it so that its total variation is bounded by a constant times $|h|$ and

$$\int_0^1 [\mathrm{d}\rho_h(t)]^{k\pm} = [h]^{k\pm} := h^k \mathbb{1}_{\pm h^k \geqslant 0}.$$

Remark B.5. Note that in the continuum we could choose

$$\rho_{z_1-z}(t) = (z_1 - z)t.$$

Moreover, since in the continuum both the right and left derivatives $\partial^{k\pm}$ coincide with ∂^k, we have

$$\psi(z_1) = \psi(z) + \sum_{k\in\{0,\dots,d\}} \int_0^1 (\partial^k\psi)(z+\rho_{z_1-z}(t))(z_1-z)^k \mathrm{d}t.$$

Since we intend to use the second-order Taylor expansion, it is convenient to choose the path such that $\rho^0_{z_1-z}(u) = 0$ for $u \in [0, 1/2]$ and

$$\rho^i_{z_1-z}(u) = \rho^i_{z_1-z}(1/2) + \mathbb{1}_{i=0}(z_1-z)^0(2u-1)$$

for $u \in [1/2, 1]$. This allows us to avoid second-order terms with one derivative in time and one in space. Note also that

$$\int_0^{1/2} |\mathrm{d}\rho_{z_1-z}(u)| \lesssim |\bar{z}_1 - \bar{z}|, \qquad \int_{1/2}^1 |\mathrm{d}\rho_{z_1-z}(u)| \leqslant |z_{1,0} - z_0|.$$

We have

$$\begin{aligned}\psi(z_1) &= \psi(z) + \sum_i \int_0^1 [\mathrm{d}\rho_{z_1-z}(t)]^i \partial^i \psi(z + \rho_{z_1-z}(t)) \\ &= \psi(z) + \sum_{i\neq 0} \int_0^{1/2} [\mathrm{d}\rho_{z_1-z}(t)]^i \partial^i \psi(z + \rho_{z_1-z}(t)) \\ &\quad + \int_{1/2}^1 \mathrm{d}\rho^0_{z_1-z}(t) \partial^0 \psi(z + \rho_{z_1-z}(t)).\end{aligned}$$

Expanding once more the spatial derivatives, we have

$$\begin{aligned}\psi(z_1) = \psi(z) &+ \sum_{i\neq 0} \partial^i \psi(z) [z_1 - z]^i \\ &+ \sum_{i,j\neq 0} \int_0^{1/2} [\mathrm{d}\rho_{z_1-z}(t)]^i \int_0^t [\mathrm{d}\rho_{z_1-z}(u)]^j \, \partial^i \partial^j \psi(z + \rho_{z_1-z}(u)) \\ &+ \int_{1/2}^1 \mathrm{d}\rho^0_{z_1-z}(t) \partial^0 \psi(z + \rho_{z_1-z}(t)).\end{aligned}$$

As a consequence, for a generic kernel $V(z, z_1)$, and symmetrising the factor $\partial^i \psi(z)$ using the relation $\partial^{i+}\psi(z) - \partial^{i-}\psi(z) = \varepsilon(\partial^{i+}\partial^{i-}\psi)(z)$ we obtain

$$\begin{aligned}V(\psi)(z) &= \int_\Lambda \mathrm{d}z_1 V(z, z_1) \psi(z_1) \\ &= \psi(z) \int_\Lambda \mathrm{d}z_1 V(z, z_1) + \sum_{k\neq 0} \frac{\partial^{k+}\psi(z) + \partial^{k-}\psi(z)}{2} \int_\Lambda \mathrm{d}z_1 (z_1 - z)^k V(z, z_1) \\ &\quad + \int_\Lambda \mathrm{d}z_1 \int_{1/2}^1 \mathrm{d}\rho^0_{z_1-z}(t) V(z, z_1) \partial^0 \psi(z + \rho_{z_1-z}(t)) \\ &\quad + \sum_k \frac{\varepsilon(\partial^{k+}\partial^{k-}\psi)(z)}{2} \int_\Lambda |(z_1 - z)^k| V(z, z_1) \, \mathrm{d}z_1 \\ &\quad + \sum_{i,j\neq 0} \int_\Lambda \mathrm{d}z_1 \int_0^{1/2} \int_0^t [\mathrm{d}\rho_{z_1-z}(u)]^j [\mathrm{d}\rho_{z_1-z}(t)]^i \\ &\qquad \times V(z, z_1) \partial^i \partial^j \psi(z + \rho_{z_1-z}(u)).\end{aligned} \tag{B.6}$$

By duality, this Taylor expansion defines operators $\boldsymbol{L}$ and $\boldsymbol{R}$ acting on the kernel V, where

$$\boldsymbol{R} := \boldsymbol{R}^{(0)} + \sum_{i,j\neq 0} \boldsymbol{R}^{(i,j)} + \sum_k \boldsymbol{R}^{(k)}_\varepsilon, \tag{B.7}$$

and these operators have the form

$$\begin{aligned}(\boldsymbol{L}V)(z,z_1) &:= \int_\Lambda \mathrm{d}\tilde{z} V(z,\tilde{z})\delta(z_1-z)\\ &\quad + \sum_{i\neq 0}\frac{1}{2}\int_\Lambda (\tilde{z}-z)^i V(z,\tilde{z})(\partial^{i+}+\partial^{i-})\delta(z_1-z)\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}(\boldsymbol{R}^{(0)}V)(z,z_1) &:= \int_\Lambda\int_{1/2}^1 \mathrm{d}\rho^0_{\tilde{z}-z}(t)V(z,\tilde{z})\\ &\quad\times\partial^0\delta(z_1-z-\rho_{\tilde{z}-z}(t))\,\mathrm{d}\tilde{z},\\ (\boldsymbol{R}^{(i,j)}V)(z,z_1) &:= \int_\Lambda\int_0^{1/2}\int_0^t [\mathrm{d}\rho_{\tilde{z}-z}(u)]^j[\mathrm{d}\rho_{\tilde{z}-z}(t)]^i\\ &\quad\times V(z,\tilde{z})\partial^i\partial^j\delta(z_1-z-\rho_{\tilde{z}-z}(u))\,\mathrm{d}\tilde{z},\\ (\boldsymbol{R}^{(k)}_\varepsilon V)(z,z_1) &:= \frac{\varepsilon}{2}\int_\Lambda |(\tilde{z}-z)^k|V(z,\tilde{z})\mathrm{d}\tilde{z}(\partial^{k+}\partial^{k-})\delta(z_1-z).\end{aligned} \tag{B.9}$$

When tested with a smooth function this gives the identity

$$V(\psi) = (\boldsymbol{L}V)(\psi) + (\boldsymbol{R}V)(\psi),$$

as seen in (B.6). We use this expansion for the analysis of relevant cumulants, namely

$$\mathcal{F}^{\boldsymbol{a}}_\mu = \bar{F}^{[\ell],(1)}_\mu = \mathbb{E}F^{[\ell],(1)}_\mu, \qquad \ell\in\{1,\dots,\hat{\ell}\}, \qquad \boldsymbol{a}=((\ell,1)).$$

Recall that we have the following decomposition (4.44),

$$\begin{aligned}\bar{F}^{[\ell],(1)}_\mu &= \bar{F}^{[\ell],(1)}_1 - \int_\mu^1 [(1-K^{1,1}_\sigma)\dot{\bar{F}}^{[\ell],(1)}_\sigma]\,\mathrm{d}\sigma - \int_\mu^1 [K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot(1-h_\mu)\,\mathrm{d}\sigma\\ &\quad + \boldsymbol{L}\int_\mu^1 [K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot(1-h_\mu)\,\mathrm{d}\sigma - \boldsymbol{L}\int_\mu^1 [K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma]\,\mathrm{d}\sigma\\ &\quad - \boldsymbol{R}\int_\mu^1 [K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu\,\mathrm{d}\sigma,\end{aligned}$$

where $\dot{\bar{F}}^{[\ell],(1)}_\sigma := \partial_\sigma\bar{F}^{[\ell],(1)}_\sigma$.

Lemma B.6. *The following bounds hold uniformly in $\sigma,\mu\in[1/2,1)$.*

$$\begin{aligned}\|\!|\boldsymbol{L}(K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma)\|\!|_\mu &\lesssim \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma\|\!|_\sigma,\\ \|\!|\boldsymbol{L}([K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot(1-h_\mu))\|\!|_\mu &\lesssim [\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma\|\!|_\sigma.\end{aligned}$$

Proof. As we argued in Remark 4.14, the cumulants are invariant under translations and spatial reflections. As a result, we conclude that

$$(\partial^{i+}+\partial^{i-})\delta(z_1-z)\int_\Lambda \mathrm{d}\tilde{z}(\tilde{z}-z)^i(K^{1,1}_\sigma\dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z}) = 0.$$

Hence, the following identity

$$(\boldsymbol{L}(K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma))(z,z_1) = \delta(z_1 - z) \int_\Lambda (K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})\,\mathrm{d}\tilde{z}$$

holds. Similarly,

$$(\boldsymbol{L}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1 - h_\mu)))(z,z_1) = \delta(z_1 - z) \int_\Lambda (K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})(1 - h_\mu(z,\tilde{z}))\,\mathrm{d}\tilde{z}.$$

Using the first identity, we obtain

$$\begin{aligned} &w^{(2),\mathfrak{b}}_\mu(z,z_1)(K^{1,1}_\mu \boldsymbol{L}[K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma])(z,z_1) \\ &\quad = w^{(2),\mathfrak{b}}_\mu(z,z_1) K_\mu(z_1 - z) \int_\Lambda (K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})\,\mathrm{d}\tilde{z}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|\!|\boldsymbol{L}(K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)|\!\|_\mu &\lesssim \sup_z \int_\Lambda |K_\mu(z_1 - z)| w^{\mathfrak{b}}_\mu(z - z_1)\mathrm{d}z_1 \int_\Lambda |(K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})|\,\mathrm{d}\tilde{z} \\ &\lesssim \|K_\mu\|_{\mathrm{TV}(w^{\mathfrak{b}}_\mu)} \sup_z \int_\Lambda |(K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})|\,\mathrm{d}\tilde{z} \\ &\lesssim \sup_z \int_\Lambda |(K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})|\,\mathrm{d}\tilde{z}, \end{aligned}$$

by Lemma 1.17. Since

$$\begin{aligned} \sup_z \int_\Lambda |K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma(z,\tilde{z})|\,\mathrm{d}\tilde{z} &\lesssim \sup_z \int_\Lambda |K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma(z,\tilde{z})|\, w^{(2),\mathfrak{b}}_\sigma(z,\tilde{z})\,\mathrm{d}\tilde{z} \\ &= \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma, \end{aligned}$$

this proves the first of the bounds stated in the lemma. By a similar argument, we get

$$\|\!|\boldsymbol{L}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1 - h_\mu))|\!\|_\mu \lesssim \sup_z \int_\Lambda |(K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde{z})|(1 - h_\mu(z,\tilde{z}))\,\mathrm{d}\tilde{z}.$$

Consequently, by Lemma A.14, we obtain

$$\begin{aligned} &\|\!|\boldsymbol{L}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma] \cdot (1 - h_\mu))|\!\|_\mu \\ &\quad \lesssim \|w^{(2),\mathfrak{b}}_\mu (1 - h_\mu)(w^{(2),\mathfrak{b}}_\sigma)^{-1}\|_{L^\infty} \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma \\ &\quad \lesssim [\![\mu]\!]^{-\mathfrak{b}} [\![\sigma]\!]^{\mathfrak{b}} \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma. \end{aligned}$$

This proves the second of the bounds stated in the lemma. ∎

Lemma B.7 (Cancellation of the local $\mathcal{B}$-contribution). *Let $\ell \in \{1,\dots,\hat{\ell}\}$ and let $\boldsymbol{a}$ be the one-element tuple whose unique entry is $(\ell,1)$. Then, uniformly in $\sigma \in [1/2,1)$,*

$$\boldsymbol{L}\left(K^{1,1}_\sigma \sum_{\boldsymbol{b},\boldsymbol{c}} \mathcal{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma)\right) = 0. \tag{B.10}$$

Proof. Since $n(\boldsymbol{a}) = 1$, in (B.2) we have $i = 1$ and $I_1 = I_2 = \varnothing$. Consider first the terms with $k' = 1$. The two cumulants appearing in such a term are indexed by the one-element tuples

$$\boldsymbol{b} = ((\ell - 1 - \ell', 2)), \qquad \boldsymbol{c} = ((\ell', 0)).$$

They vanish by Remark 4.14, since respectively $n(\boldsymbol{b}) + K(\boldsymbol{b}) = 3$ and $n(\boldsymbol{c}) + K(\boldsymbol{c}) = 1$ are odd.

It remains to consider $k' = 0$. In this case

$$\boldsymbol{b} = ((\ell - 1 - \ell', 1)), \qquad \boldsymbol{c} = ((\ell', 1)).$$

As in the proof of Lemma B.3, we use $L_\sigma K_\sigma = 1$ to write

$$\begin{aligned} &K_\sigma^{1,1} \mathcal{B}^{\boldsymbol{a},(1)}_{\boldsymbol{b},\boldsymbol{c}}(\dot G_\sigma, \mathcal{F}^{\boldsymbol{b}}_\sigma, \mathcal{F}^{\boldsymbol{c}}_\sigma) \\ &\qquad = \mathcal{B}^{\boldsymbol{a},(1)}_{\boldsymbol{b},\boldsymbol{c}}(L_\sigma \dot G_\sigma, K^{\boldsymbol{b}}_\sigma \mathcal{F}^{\boldsymbol{b}}_\sigma, K^{\boldsymbol{c}}_\sigma \mathcal{F}^{\boldsymbol{c}}_\sigma). \end{aligned}$$

Set

$$W_\sigma := K^{\boldsymbol{b}}_\sigma \mathcal{F}^{\boldsymbol{b}}_\sigma, \qquad U_\sigma := K^{\boldsymbol{c}}_\sigma \mathcal{F}^{\boldsymbol{c}}_\sigma.$$

By (4.27), and since convolution with K_σ preserves translation invariance, the quantity

$$m_{U_\sigma} := \int_\Lambda U_\sigma(z''; \mathrm{d}z_1)$$

is independent of z''. Hence, using the definition (4.12) of the contraction operator,

$$\begin{aligned} &\int_\Lambda \mathcal{B}^{\boldsymbol{a},(1)}_{\boldsymbol{b},\boldsymbol{c}}(L_\sigma \dot G_\sigma, W_\sigma, U_\sigma)(z; \mathrm{d}z_1) \\ &\qquad = \int_{\Lambda^2} W_\sigma(z; \mathrm{d}z')(L_\sigma \dot G_\sigma)(z' - z'') \left(\int_\Lambda U_\sigma(z''; \mathrm{d}z_1) \right) \mathrm{d}z'' \\ &\qquad = m_{U_\sigma} \int_\Lambda W_\sigma(z; \mathrm{d}z') \int_\Lambda (L_\sigma \dot G_\sigma)(z' - z'') \, \mathrm{d}z'' = 0. \end{aligned}$$

For the last identity, observe that

$$\int_\Lambda L_\sigma \dot G_\sigma(z) \, \mathrm{d}z = 0. \tag{B.11}$$

Indeed, by Definition 1.9 and (1.26), the Fourier multiplier of $\partial_\sigma J_\sigma$ equals zero at the zero spacetime frequency. Therefore the zero Fourier mode of $L_\sigma \dot G_\sigma = L_\sigma \partial_\sigma (J_\sigma G)$ vanishes.

We have shown that the zeroth moment in the definition (B.8) of $\boldsymbol{L}$ vanishes. Moreover, the kernel on the left-hand side of (B.10) is invariant under simultaneous translations and spatial reflections, by Remark 4.14, the corresponding symmetries of K_σ and G_σ, and the explicit formula (B.2). Translation invariance reduces it to a kernel of the relative variable, and reflection invariance makes that relative kernel even in every spatial coordinate. Its first spatial moments are therefore zero, exactly as in the expanded argument in the proof of Lemma B.6. All coefficients in its local projection are zero, and (B.10) follows. ∎

Lemma B.8. *The following bound*

$$\|\!|\boldsymbol{R}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)|\!\|_\mu \lesssim [\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}\|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma|\!\|_\sigma$$

holds uniformly in $\mu \in [1/2, 1), \sigma \in [\mu, 1)$.

Proof. By applying the triangle inequality to (B.7), we arrive at

$$\begin{aligned}\|\!|\boldsymbol{R}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)|\!\|_\mu \lesssim{}& \|\!|\boldsymbol{R}^{(0)}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)|\!\|_\mu \\ &+ \sum_{i,j\neq 0}\|\!|\boldsymbol{R}^{(i,j)}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)|\!\|_\mu \\ &+ \sum_k \|\!|\boldsymbol{R}^{(k)}_\varepsilon([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)|\!\|_\mu.\end{aligned}$$

We have

$$\begin{aligned}&(\boldsymbol{R}^{(i,j)}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu))(z,z_1)\\ &\quad= \int_\Lambda \int_0^{\frac12}\int_0^t [\mathrm{d}\rho_{\tilde z - z}(u)]^j [\mathrm{d}\rho_{\tilde z - z}(t)]^i h_\mu(z,\tilde z)\\ &\qquad\times (K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde z)\partial^i\partial^j\delta(z_1 - z - \rho_{\tilde z - z}(u))\,\mathrm{d}\tilde z.\end{aligned}$$

Thus,

$$\begin{aligned}&|w^{(2),\mathfrak{b}}_\mu(z,z_1)(\boldsymbol{R}^{(i,j)}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu))(z,z_1)|\\ &\quad\leqslant \int_\Lambda\int_0^{\frac12}\int_0^t |\mathrm{d}\rho_{\tilde z - z}(u)|\,|\mathrm{d}\rho_{\tilde z - z}(t)|\,w^{(2),\mathfrak{b}}_\mu(z,z_1)h_\mu(z,\tilde z)\\ &\qquad\times |K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma(z,\tilde z)|\,|\partial^i\partial^j K_\mu(z_1 - z - \rho_{\tilde z - z}(u))|\,\mathrm{d}\tilde z.\end{aligned}$$

Recall that

$$w^{(2),\mathfrak{b}}_\mu(z,z_1) = w^{\mathfrak{b}}_\mu(z - z_1) = (1 + [\![\mu]\!]^{-1}|z - z_1|_s)^{\mathfrak{b}}.$$

Introduce the point

$$Z_u = Z_u(z,\tilde z) := z + \rho_{\tilde z - z}(u).$$

Then

$$\begin{aligned}w^{\mathfrak{b}}_\mu(z - z_1) &\lesssim w^{\mathfrak{b}}_\mu(z - Z_u)w^{\mathfrak{b}}_\mu(Z_u - z_1)\\ &\leqslant w^{\mathfrak{b}}_\mu(z - \tilde z)w^{\mathfrak{b}}_\mu(Z_u - z_1).\end{aligned}$$

The second inequality above follows from

$$w^{\mathfrak{b}}_\mu(z - Z_u) \lesssim w^{\mathfrak{b}}_\mu(z - \tilde z),$$

which holds because, by construction, Z_u lies on the path connecting z and $\tilde{z}$. Overall, this yields

$$\begin{aligned}
&|w_\mu^\mathfrak{b}(z-z_1)(\boldsymbol{R}^{(i,j)}([K_\sigma^{1,1}\dot{\hat{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu))(z,z_1)| \\
&\quad\lesssim \int_\Lambda \int_0^{\frac{1}{2}} \int_0^t |\mathrm{d}\rho_{\tilde{z}-z}(u)|\,|\mathrm{d}\rho_{\tilde{z}-z}(t)|\, w_\mu^\mathfrak{b}(z-\tilde{z}) h_\mu(z,\tilde{z})(w_\sigma^\mathfrak{b}(z-\tilde{z}))^{-1} \\
&\qquad\times w_\sigma^\mathfrak{b}(z-\tilde{z})\,|(K_\sigma^{1,1}\dot{\hat{F}}_\sigma^{[\ell],(1)})(z,\tilde{z})| \\
&\qquad\times w_\mu^\mathfrak{b}(Z_u-z_1)\,|\partial^i\partial^j K_\mu(z_1-Z_u)|\,\mathrm{d}\tilde{z}.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\|\!|\boldsymbol{R}^{(i,j)}([K_\sigma^{1,1}\dot{\hat{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu)|\!\|_\mu \\
&\quad\lesssim \sup_{z,\tilde{z}\in\Lambda} \int_0^{\frac{1}{2}} \int_0^t |\mathrm{d}\rho_{\tilde{z}-z}(u)|\,|\mathrm{d}\rho_{\tilde{z}-z}(t)|\, w_\mu^\mathfrak{b}(z-\tilde{z}) h_\mu(z,\tilde{z})(w_\sigma^\mathfrak{b}(z-\tilde{z}))^{-1} \\
&\qquad\times \|\!|\dot{\hat{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma \sup_{z,\tilde{z}\in\Lambda} \int_\Lambda w_\mu^\mathfrak{b}(Z_u-z_1)\,|\partial^i\partial^j K_\mu(z_1-Z_u)|\,\mathrm{d}z_1.
\end{aligned}$$

Using

$$\int_0^{\frac{1}{2}} \int_0^t |\mathrm{d}\rho_{\tilde{z}-z}(u)||\mathrm{d}\rho_{\tilde{z}-z}(t)| \lesssim |\bar{\tilde{z}}-\bar{z}|^2, \tag{B.12}$$

and

$$\int_\Lambda w_\mu^\mathfrak{b}(z)|\partial^i\partial^j K_\mu(z)|\,\mathrm{d}z \lesssim [\![\mu]\!]^{-2}, \tag{B.13}$$

as well as

$$\begin{aligned}
\frac{|\bar{\tilde{z}}-\bar{z}|^2 w_\mu^\mathfrak{b}(z-\tilde{z}) h_\mu(z,\tilde{z})}{w_\sigma^\mathfrak{b}(z-\tilde{z})} &\leqslant [\![\mu]\!]^{2-\mathfrak{b}}[\![\sigma]\!]^\mathfrak{b} \frac{(1+[\![\mu]\!]^{-1}|z-\tilde{z}|_s)^\mathfrak{b}}{(1+[\![\sigma]\!]^{-1}|z-\tilde{z}|_s)^\mathfrak{b}} \frac{[\![\mu]\!]^{-(2-\mathfrak{b})}[\![\sigma]\!]^{-\mathfrak{b}}|z-\tilde{z}|_s^2}{1+([\![\mu]\!]^{-1}|z-\tilde{z}|_s)^2} \\
&\leqslant [\![\mu]\!]^{2-\mathfrak{b}}[\![\sigma]\!]^\mathfrak{b} \frac{\big(1+[\![\mu]\!]^{-1}|z-\tilde{z}|_s\big)^\mathfrak{b}\,([\![\mu]\!]^{-1}|z-\tilde{z}|_s)^{2-\mathfrak{b}}}{1+([\![\mu]\!]^{-1}|z-\tilde{z}|_s)^2} \\
&\lesssim [\![\mu]\!]^{2-\mathfrak{b}}[\![\sigma]\!]^\mathfrak{b},
\end{aligned}$$

we conclude that

$$\|\!|\boldsymbol{R}^{(i,j)}([K_\sigma^{1,1}\dot{\hat{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu)|\!\|_\mu \lesssim [\![\sigma]\!]^\mathfrak{b}[\![\mu]\!]^{-\mathfrak{b}}\|\!|\dot{\hat{F}}_\sigma^{[\ell],(1)}|\!\|_\sigma.$$

Next, note that

$$\begin{aligned}
&(\boldsymbol{R}^{(0)}([K_\sigma^{1,1}\dot{\hat{F}}_\sigma^{[\ell],(1)}]\cdot h_\mu))(z,z_1) \\
&\quad= \int_\Lambda \int_{1/2}^1 \mathrm{d}\rho^0_{\tilde{z}-z}(t)(h_\mu K_\sigma^{1,1}\dot{\hat{F}}_\sigma^{[\ell],(1)})(z,\tilde{z})\,\partial^0\delta(z_1-z-\rho_{\tilde{z}-z}(t))\,\mathrm{d}\tilde{z}.
\end{aligned}$$

Proceeding as above, we get

$$\begin{aligned}
&\|\!|\boldsymbol{R}^{(0)}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)\|\!|_\mu \\
&\quad\lesssim \sup_{z,\tilde z\in\Lambda}\int_{1/2}^1 |\mathrm{d}\rho^0_{\tilde z - z}(t)|\, w^{\mathfrak{b}}_\mu(z-\tilde z) h_\mu(z,\tilde z)\,(w^{\mathfrak{b}}_\sigma(z-\tilde z))^{-1} \\
&\qquad\times \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma\|\!|_\sigma \sup_{z,\tilde z\in\Lambda}\int_\Lambda w^{\mathfrak{b}}_\mu(Z_t - z_1)\,|\partial^0 K_\mu(z_1 - Z_t)|\,\mathrm{d}z_1.
\end{aligned}$$

Using

$$\int_{1/2}^1 |\mathrm{d}\rho^0_{\tilde z - z}(t)| \leqslant |\tilde z_0 - z_0|,$$

and

$$\int_\Lambda w^{\mathfrak{b}}_\mu(z-z')\,|\partial^0 K_\mu(z-z')|\,\mathrm{d}z \lesssim [\![\mu]\!]^{-2s},$$

as well as

$$\begin{aligned}
|\tilde z_0 - z_0|\frac{w^{\mathfrak{b}}_\mu(z-\tilde z)}{w^{\mathfrak{b}}_\sigma(z-\tilde z)} h_\mu(z,\tilde z) &\leqslant [\![\mu]\!]^{2s-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}} \frac{\big(1+[\![\mu]\!]^{-1}|z-\tilde z|_s\big)^{\mathfrak{b}}}{(1+[\![\sigma]\!]^{-1}|z-\tilde z|_s)^{\mathfrak{b}}} \frac{[\![\mu]\!]^{-(2s-\mathfrak{b})}[\![\sigma]\!]^{-\mathfrak{b}}|z-\tilde z|_s^{2s}}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2} \\
&\leqslant [\![\mu]\!]^{2s-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}} \frac{\big(1+[\![\mu]\!]^{-1}|z-\tilde z|_s\big)^{\mathfrak{b}}\,([\![\mu]\!]^{-1}|z-\tilde z|_s)^{2s-\mathfrak{b}}}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^2} \\
&\lesssim [\![\mu]\!]^{2s-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}} \frac{1}{1+([\![\mu]\!]^{-1}|z-\tilde z|_s)^{2-2s}} \\
&\lesssim [\![\mu]\!]^{2s-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}},
\end{aligned}$$

we arrive at

$$\|\!|\boldsymbol{R}^{(0)}([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)\|\!|_\mu \lesssim [\![\sigma]\!]^{\mathfrak{b}}[\![\mu]\!]^{-\mathfrak{b}}\|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma\|\!|_\sigma.$$

Finally, we discuss the term

$$\begin{aligned}
&(\boldsymbol{R}^{(k)}_\varepsilon([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu))(z,z_1) \\
&\quad= \frac{\varepsilon}{2}\int_\Lambda |(\tilde z - z)^k|(h_\mu K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma)(z,\tilde z)\,\mathrm{d}\tilde z\,(\partial^{k+}\partial^{k-})\delta(z_1 - z).
\end{aligned}$$

In this case, the proof also proceeds along the same lines. We obtain

$$\begin{aligned}
\|\!|\boldsymbol{R}^{(k)}_\varepsilon([K^{1,1}_\sigma \dot{\bar{F}}^{[\ell],(1)}_\sigma]\cdot h_\mu)\|\!|_\mu \lesssim\varepsilon\ &\sup_{z,\tilde z\in\Lambda} |\bar{\tilde z} - \bar z|\, w^{\mathfrak{b}}_\mu(z-\tilde z) h_\mu(z,\tilde z)(w^{\mathfrak{b}}_\sigma(z-\tilde z))^{-1} \\
&\times \|\!|\dot{\bar{F}}^{[\ell],(1)}_\sigma\|\!|_\sigma \\
&\times \int_\Lambda w^{\mathfrak{b}}_\mu(z-z_1)\,|\partial^{k+}\partial^{k-}K_\mu(z_1 - z)|\,\mathrm{d}z_1.
\end{aligned}$$

Note that

$$\begin{aligned}\frac{|\bar{\tilde{z}}-\bar{z}|w^{\mathfrak{b}}_{\mu}(z-\tilde{z})h_{\mu}(z,\tilde{z})}{w^{\mathfrak{b}}_{\sigma}(z-\tilde{z})} &\leqslant [\![\sigma]\!]\frac{(1+[\![\mu]\!]^{-1}|z-\tilde{z}|_s)^{\mathfrak{b}}}{(1+[\![\sigma]\!]^{-1}|z-\tilde{z}|_s)^{\mathfrak{b}}}\frac{[\![\sigma]\!]^{-1}|\tilde{z}-z|_s}{(1+([\![\mu]\!]^{-1}|z-\tilde{z}|_s)^2)}\\ &\lesssim [\![\sigma]\!]\frac{1}{(1+[\![\sigma]\!]^{-1}|\tilde{z}-z|_s)^{\mathfrak{b}-1}}\frac{1}{(1+[\![\mu]\!]^{-1}|z-\tilde{z}|_s)^{2-\mathfrak{b}}},\end{aligned}$$

Since $|\bar{\tilde{z}}-\bar{z}|\neq 0$ implies $|\tilde{z}-z|_s\geqslant\varepsilon$ we have

$$\begin{aligned}\frac{|\bar{\tilde{z}}-\bar{z}|w^{\mathfrak{b}}_{\mu}(z-\tilde{z})h_{\mu}(z,\tilde{z})}{w^{\mathfrak{b}}_{\sigma}(z-\tilde{z})} &\lesssim [\![\sigma]\!]\frac{1}{(1+[\![\sigma]\!]^{-1}\varepsilon)^{\mathfrak{b}-1}}\frac{1}{(1+[\![\mu]\!]^{-1}\varepsilon)^{2-\mathfrak{b}}}\\ &\leqslant [\![\sigma]\!][\![\sigma]\!]^{\mathfrak{b}-1}\varepsilon^{1-\mathfrak{b}}[\![\mu]\!]^{2-\mathfrak{b}}\varepsilon^{\mathfrak{b}-2}\\ &= [\![\sigma]\!]^{\mathfrak{b}}[\![\mu]\!]^{2-\mathfrak{b}}\varepsilon^{-1}.\end{aligned}$$

Hence, using again (B.13) with $i=k+$ and $j=k-$ we get

$$\begin{aligned}&|\!|\!|\boldsymbol{R}^{(k)}_{\varepsilon}([K^{1,1}_{\sigma}\dot{\tilde{F}}^{[\ell],(1)}_{\sigma}]\cdot h_{\mu})|\!|\!|_{\mu}\\ &\quad\lesssim \varepsilon[\![\mu]\!]^{-2}[\![\sigma]\!]^{\mathfrak{b}}[\![\mu]\!]^{2-\mathfrak{b}}\varepsilon^{-1}|\!|\!|\dot{\tilde{F}}^{[\ell],(1)}_{\sigma}|\!|\!|_{\sigma}\\ &\quad= [\![\mu]\!]^{-\mathfrak{b}}[\![\sigma]\!]^{\mathfrak{b}}|\!|\!|\dot{\tilde{F}}^{[\ell],(1)}_{\sigma}|\!|\!|_{\sigma}.\end{aligned}$$

This finishes the proof. ∎

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