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Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions

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Abstract

We introduce a stochastic analysis of Grassmann random variables suitable for the stochastic quantization of Euclidean fermionic quantum field theories. Analysis on Grassmann algebras is developed from the point of view of quantum probability: a Grassmann random variable is a homomorphism of an abstract Grassmann algebra into a quantum probability space, i.e. the von Neumann algebra of bounded operators on a Hilbert space endowed with a suitable state. On the way we define the notion of Gaussian processes, Brownian motion and stochastic differential equations taking values in Grassmann algebras. We use them to study the long time behavior of Langevin Grassmann stochastic differential equations driven by white noise and describe their invariant measures. As an application we give a proof of the stochastic quantization and of the removal of the infrared cutoff for the Euclidean Yukawa model.

Keywords: Grassmann algebra, Euclidean Fermions, stochastic quantization, non-commutative probability, Yukawa model.

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1 Introduction

The aim of this work is to develop a stochastic quantization of Euclidean fermionic quantum field theories (QFTs). This involves defining and studying the long time behavior of stochastic equations in Grassmann algebras.

Euclidean fermion fields form a Grassmann algebra [65, 66]. Their correlation functions (Schwinger functions) are typically represented by Berezin integrals [19] of the type

$$\langle O(\psi,\bar\psi)\rangle = \frac{\int \mathrm{d}\psi\mathrm{d}\bar\psi O(\psi,\bar\psi)e^{-S_E(\psi,\bar\psi)}}{\int \mathrm{d}\psi\mathrm{d}\bar\psi e^{-S_E(\psi,\bar\psi)}} \tag{1}$$

where the fields $\psi,\bar\psi$ form a Grassmann algebra, O is a functional of ψ and $\bar\psi$ generating the Schwinger functions, and $S_E(\psi,\bar\psi)$ is the Euclidean action, usually the sum of a quadratic part and a polynomial interaction (possibility involving Boson fields, which will have to be averaged according to an appropriate probability measure) [34, 30, 57]. Without entering too much into the precise details, let us point out here that the formula (1) is a purely algebraic expression, devoid of any analytic meaning and is used as a convenient bookkeeping for various kind of expansions used to define and study the properties of the Schwinger functions. This is due to the fact that Grassmann algebras do not have a natural notion of positivity, nor a norm, associated with them.

Stochastic quantization (SQ) is an approach to the construction of the correlation functions of an Euclidean QFT that has been introduced by Parisi and Wu [67]. The basic idea is to introduce an additional variable (usually a fictitious time) and interpret the Euclidean fields φ, to be constructed, as solutions to stochastic differential equations involving the mentioned additional variable and a source of noise ξ, most commonly Gaussian. One of the advantages of this approach is that the fields of the theory are expressed via the stochastic evolutions as functionals of the noise, i.e. $\varphi = \varphi(\xi)$, which then allows to study them as functionals on ξ

exploiting detailed information about the law of ξ. In this way even interacting fields can be studied non-perturbatively exploiting the Gaussian structure of ξ. Moreover, another conceptual advantage of stochastic quantization is that it is insensitive to questions of absolute continuity with respect to a reference measure. This, on the other hand, is usually the main difficulty of the Gibbsian point of view as expressed in equation (1).

This strategy has been employed to construct bosonic theories, starting with the work [48] on the stochastic quantization of the Φ^4_2 model for a scalar particle in two dimension with quartic interaction. More recently the work of Hairer on regularity structures [42] opened the way to the study of the three dimensional Φ^4 model, see also [24, 55]. These new perspectives on this and related problems led to a series of new results on the global space-time control of the stochastic dynamics [60, 40, 5, 59] and to a novel proof of the constructions of non-Gaussian bosonic Euclidean quantum field theories in three dimensions [39].

One of main differences between the bosonic and the fermionic case is the necessity, in the fermionic setting, of considering stochastic processes in a non-commutative framework.

The interest in studying time evolutions and stochastic processes with values in non commutative algebras arose in connection with the development of quantum mechanics in the 20s–30s of last century and later in quantum field theory, particularly around the 50s. In these theories one associates physical observables with operators evolving in time and acting on a, in general infinite dimensional, complex Hilbert space. Mathematically, this lead to the representation theory of certain operator algebras (CCR-algebras resp. CAR-algebras in the case of fermions) that are particular cases of more general operator algebras.

In this line quantum fields require the study of certain infinite dimensional (non-commutative) algebras, the study of which has been particularly stimulated by developments in the period 1940-1960, where the principles of systems with an infinite number of degrees of freedom, in particular quantum fields and statistical mechanics has been worked out mathematically. Let us mention the first works on von Neumann algebras by Murray and von Neumann [61, 62, 63, 82] and the work on C^*-algebras developed originally and quite independently by I. M. Gelfand and his school (see, e.g., [36]), since the beginnings of the 40s, and by I. E. Segal and his school (see [71, 10]). Both types of algebras are associative and equipped with an involution (the adjoint operation). Whereas C^*-algebras are looked upon as abstract normed algebras, without necessarily being realized as algebras of operators acting in Hilbert spaces , the study of von Neumann algebras (also called W^*-algebras) rather involves looking at them as self-adjoint algebras of bounded operators in Hilbert spaces,closed in the weak operator topology. Both algebras can be looked upon as particular cases of normed algebras, that in the case of closedness are called Banach algebras. More precisely, C^*-algebras are normed algebras with a unit $\mathbb{I}$ and an involution $\cdot^*$, any element A of which satisfies $\|AA^*\| = \|A\|^2$. Although in principle every von Neumann algebra can be looked upon as a C^*-algebra, structural and conceptual developments in the study of von Neumann algebras involve particularly their realizations in Hilbert space. The mathematical literature on normed, Banach, C^* and W^* algebras is very rich and there are a number of excellent monographs on them, see, e.g., Naimark [64] and Zelazko [83] for normed spaces and algebras, and Kadison and Ringrose [50, 49] (and references therein) for the other type of algebras.

For applications in mathematical physics, in particular quantum mechanics, quantum field theory and statistical mechanics, there are monographs by Bratteli and Robinson [21, 22],

Emch [32], Baumgärtel [15], as well as Baumgärtel and Wollenberg [16], Segal [72]; we refer particularly to the books by Baez, Segal and Zhou [10], and Haag [41], and also the articles by Summers [78] and Doplicher [11], that surveys particularly some recent developments related to such algebras in quantum field theory.

Before describing the approach we use in this article, which is based on stochastic processes forming a Grassmann algebra, let us mention some works about stochastic quantization of fermionic fields not using Grassmann algebras but more connected with ideas of the theory of noncommutative processes and Markov semigroups. They involve noncommutative Brownian motion and stochastic analysis based on on it, in particular also SDEs but with no space variable dependence, hence no SPDEs. Pioneering work in this direction has been done by Accardi in the 70s, see e.g. [1]. Work extending SDEs with values in algebras other than Grassmann algebras, like C^* and von Neumann algebras are by authors like Streater and coworkers [14, 13, 12], Applebaum and Hudson [9, 8], Belavkin [17], Carlen and Kree [23], Gordina [37], Sinha and Goswami [77], Kümmer [54] see also the monographs by Meyer [58], Holevo [43], and the recent survey article by Cipriani [25] and the references therein.

In this paper we focus on stochastic quantization of interacting fermions, realizing the fermionic fields as non-commutative operator solution of a suitable stochastic differential equation taking values in a Grassmann sub-algebra of the Banach algebra of all bounded operators acting on a Hilbert space (see the precise definition in Section 2). Here let us mention that Grassmann algebras constitute a class of associative non commutative algebras that were discovered by Grassmann in the second half of the 19th century. They are connected with Clifford algebras, that in turn are the underlying structure to the study of CCR and CAR algebras. All such algebras are nowadays part of the general theory of algebra as pioneered by E. Noether in the 20s-40s, see e.g. Van der Waerden [80, 81].

In our paper we use stochastic processes forming a Grassmann algebra, rather than Clifford, C^* or W^* algebras, since they better relate to work done in the context of interesting models of constructive quantum field theory involving fermions, in which methods based on integrals on non commutative structures like Berezin integrals are involved. Such methods are quite essential in most oft the physics literature on fermionic quantum fields, including renormalization theory, see, e.g., Feldman, Knörrer and Trubowitz [33], Salmhofer [70] or Zinn-Justin [84]. As for fermions (i.e fields associated with particles of half integer spins, like e.g. electrons) stochastic quantization ideas have been discussed heuristically, after Parisi and Wu seminal paper, in the physics literature, starting with Kakudo et al. [51], Fukai et al. [35], Damgaard and Tsokos [26] and She-Sheng and Ting-Chang [76] up to the recent paper of Efremov [31]. These authors stress particularly the relations with Grassmann valued random variables, solving partial differential equations with Grassmann valued Gaussian white noise. The invariant measures of physical interest are here described by averages with respect to Berezin integrals, see [18] (further developed into a analysis on Grassmann algebras by the same author, in the form of super-analysis [20]; for super-algebras see also, e.g., Rogers [69] and Leĭtes [56], on a more heuristic level also DeWitt [28]). Berezin integrals have been useful in calculating fermionic systems, see e.g. Zinn-Justin [84], Izkyson and Zuber [46].

Rigorous attempts to define Grassmann stochastic quantization were undertaken in a series of papers by Scherbakov et al. [75, 74, 73] using the locally convex topology generated by considering all the possible correlation functions of the fields and the noise to generate a family of semi-norms. Convergence in this setting corresponds to convergence of all the correlation functions. This approach has the disadvantage of not allowing the introduction of norms strong enough to study

differential equations via standard approaches. However it should be said that these papers define and successfully solve equations with finitely many degrees of freedom (i.e. with small scale and large scale cut-offs) and then address the convergence when the large scale cut-off is removed (infinite volume limit) using cluster expansion techniques. Other results in this direction were obtained later on by Ignatyuk et al. [45], with applications in statistical mechanics.

Our approach here is different and, in our opinion, both simpler and more effective. We consider this problem in the context of non-commutative probability (sometimes referred to as *quantum* probability), namely we will define the relevant Grassmannian objects as non-commutative random variables. A non-commutative probability space $(\mathcal{A}, \omega)$ is given by a C^* algebra $\mathcal{A}$ and a state ω, which is a linear positive functional, on $\mathcal{A}$. Inspired by the general approach of Accardi et al. [2] (see also [1]) we will define random variables with values in a Grassmann algebra Λ as homomorphisms of Λ into $\mathcal{A}$ (note however that we will not require the homomorphisms to respect the involution, since there is no canonical candidate for that in Grassmann algebras).

It is useful to keep in mind how classical commutative random variables fit in this algebraic approach. Consider a random variable $X : \Omega \to \mathcal{M}$ taking values in a manifold $\mathcal{M}$ and let $\mathcal{B}$ be the algebra of all measurable, complex bounded function on $\mathcal{M}$. Then all the relevant probabilistic information about X is encoded in the homomorphisms of algebras $f \in \mathcal{B} \mapsto f(X) \in \mathcal{A}$ where $\mathcal{A}$ is the algebra of bounded measurable complex random variables on Ω endowed with the linear functional given by the expectation $\mathbb{E}$. In particular this is the point of view with which SDEs on manifolds are usually defined and studied [44].

The embedding of Λ into $\mathcal{A}$ allows to use the norm of $\mathcal{A}$ to do analysis on Grassmann algebras. Of course the associated analysis will not be *canonical* from the point of view of the Grassmann algebra but seems enough to allow us to obtain a satisfactory theory of stochastic quantization. Again, an analogy can be helpful to understand this setting: the standard approach to study Gaussian processes in Hilbert spaces [38] (see also, e.g., [47]) requires to have a non-canonical embedding of the Hilbert space into a larger Banach space which supports the Gaussian measure. In our case, the role of "bigger" Banach space is played by a suitable CAR sub-algebra of $\mathcal{A}$ which provides us with enough Grassmann generators to realize our homomorphisms and has a canonical notion of norm. It will be for us just "a place where to hang our (analytic) hat on".

Another approach to infinite dimensional Grassmann algebras (and the related probability theory) we are aware of is that of Rogers [68] which essentially consists in looking at all the finite dimensional sub-algebras, use the Berezin integral (see, e.g, [19]) to compute averages and require certain natural consistency conditions. It is interesting to note however that our approach follows Osterwalder and Schrader [65, 66] which were the first to do rigorous analysis of Euclidean Fermions: they were using the language of Fock space operators and not of Berezin calculus, and this allowed them to consider rigorously expressions involving infinite dimensional Grassmann algebras. The Fock space approach has also been used in discussing supersymmetry in the context of Parisi–Sourlas dimensional reduction [53, 52] and Euclidean stochastic quantization [3, 4, 27].

Plan of the paper. In Section 2 we define Grassmann random variables and various related notions. We then introduce Gaussian variables, including white noise and Brownian motion. Section 3 studies finite dimensional Grassmann differential equations with additive noise and prove certain properties of their correlation functions, showing in particular that we can construct Grassmann Langevin diffusions which have prescribed stationary correlations. Section 4 is devoted to extend the previous results to SDE in infinite dimensions of the kind relevant for stochastic quantization. This allows us to use these processes to represent the correlation

functions of Euclidean fermionic theories like (1). As an application we provide in Section 5 a construction of the fermionic sector of the massive Yukawa Euclidean quantum field theory in two dimensions, in the weak coupling region and with an ultraviolet cutoff. In Appendix A we collect various technical results about functional spaces of Banach-valued functions, while in Appendix B we include some side results about the convergence of series expansions for the solution to finite dimensional SDEs.

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Notations. We denote by $\mathcal{L}(A, B)$ the space of linear maps between vector spaces A and B, if both A, B have topologies we consider all the maps to be continuous. We let $\mathcal{L}(A) = \mathcal{L}(A, A)$. With $\mathrm{Hom}(A, B)$ we denote homomorphisms between algebras A and B.

2 Grassmann random variables

Algebraic probability. We consider a complex Hilbert space $\mathcal{H}$ (to be fixed later) and denote by $\mathcal{A} = \mathcal{L}(\mathcal{H})$ the von Neumann algebra of bounded operators with the operator norm. Moreover we assume to have a state ω on $\mathcal{A}$, that is a positive definite functional normalized by the condition that $\omega(\mathbb{I}) = 1$, where $\mathbb{I}$ is the identity in $\mathcal{L}(\mathcal{H})$. The pair $(\mathcal{A}, \omega)$ is a (non-commutative or algebraic) probability space. We do not require the state ω to be either faithful, or tracial.

Grassmann algebras. Let V be a (finite dimensional or infinite dimensional and separable) real Hilbert space. Denote by ΛV the Grassmann algebra generated by V, i.e. the exterior algebra obtained by quotienting the tensor algebra $T(V) = \oplus_{n \geqslant 0} V^{\otimes n}$ by the two-sided ideal generated by elements of the form $x \otimes x$ for $x \in V$. We denote the product of two elements $f, g \in \Lambda V$ by $f \wedge g$ (or if there is no confusion with other products simply by fg).

Let $(v_\alpha)_\alpha$ be a fixed basis of V. As a consequence ΛV is spanned by elements of the form $v_A := v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_n}$ and $A = (\alpha_1, \dots, \alpha_n)$ is a multi-index with the convention that $v_\emptyset = 1$ is the unit element in ΛV. When it is clear from the context we will denote the product in ΛV simply by $v_A = v_{\alpha_1} \dots v_{\alpha_n}$. We recall that ΛV is $\mathbb{Z}_2$ graded in the sense that it splits into odd and even parts $\Lambda V = \Lambda_{\mathrm{odd}} V \oplus \Lambda_{\mathrm{even}} V$. On ΛV there is an Hopf algebra structure with coproduct $\Delta : \Lambda V \to \Lambda V \otimes \Lambda V$ where $\Lambda V \otimes \Lambda V$ is the $\mathbb{Z}_2$-graded tensor product algebra which satisfies

$$(f \otimes g)(h \otimes k) = (-1)^{|g||h|} fh \otimes gk, \qquad f, g, h, k \in \Lambda V,$$

with $|\cdot| : \Lambda V \to \{0, 1\}$ the even/odd grading. The coproduct Δ is the algebra homomorphism such that $\Delta v = 1 \otimes v + v \otimes 1$ for all $v \in V \subseteq \Lambda V$ and the counit $\varepsilon : \Lambda V \to \mathbb{C}$ given by $\varepsilon(v_A) = \mathbb{1}_{A=\emptyset}$.

Random variables.

Definition 1 *A V-Grassmann random variable Ψ is an algebra homomorphism from the Grassmann algebra ΛV into $\mathcal{A}$. We denote by $\mathcal{G}(V) = \mathrm{Hom}(\Lambda V, \mathcal{A})$ the set of all such homomorphisms. We will always assume that $\Psi \in \mathcal{G}(V)$ comes with a unitary element U_Ψ of $\mathcal{A}$ such $U_\Psi \Psi(v) U_\Psi^* = \Psi(-v)$ for all $v \in V$. We call the law of $\Psi \in \mathcal{G}(V)$ the family of its moments $\omega^\Psi(F) := \omega(\Psi(F))$ for all $F \in \Lambda V$, also represented by the linear functional $\omega^\Psi : \Lambda V \to \mathbb{R}$.*

Note that $\Psi \in \mathcal{G}(V)$ cannot be assumed a to be $*$-algebra-homomorphism since ΛV has no (natural) $*$-operation. If $F \in \Lambda V$ has representation $F = \sum_A F_A v_A$ we shall employ the following dual notation

$$F(\Psi) := \Psi(F) = \sum_A F_A \Psi^A,$$

where $\Psi^A := \Psi(v_A)$ and $F_A \in \mathbb{R}$. Since Ψ is assumed to be an algebra homomorphism, we have e.g. $\Psi^\alpha \Psi^\beta = -\Psi^\beta \Psi^\alpha$, where $\Psi^\alpha = \Psi(v_\alpha)$ and $(v_\alpha)_\alpha$ is a fixed basis of V. Moreover, $\Psi(\Lambda V)$ is a Grassmann sub-algebra of $\mathcal{A}$ and $\Psi(F)$ has the same degree (even or odd) as F. It is important to note that, as shown by arguments in [80, 81], even if some arguments are formulated in a basis dependent way the definition of ΛV and its characterization by anti-commutation relations is basis independent.

Compatibility. When the context is clear we will abbreviate $X \in \mathcal{G}(V)$ as $X \in \mathcal{G}$. Let $X \in \mathcal{G}(V)$ and $Y \in \mathcal{G}(W)$, we say that they are *compatible* if the linear map $Z : V \oplus W \to \mathcal{A}$ given by $Z(v) = X(v)$ if $v \in V$ and $Z(w) = Y(w)$ if $w \in W$, extends to a homomorphism $Z : \Lambda(V \oplus W) \to \mathcal{A}$. In this case we write $Z \in \mathcal{G}(V \oplus W)$ or briefly $(X, Y) \in \mathcal{G}$. Note that we have a super- algebra isomorphism $\Lambda(V \oplus W) \approx \Lambda(V) \otimes \Lambda(W)$ where tensor product is in the sense of super-algebras. From this isomorphism we get in particular $Z(F) = m_{\mathcal{A}}[(X \otimes Y)(F)]$ for all $F \in \Lambda(V \oplus W)$ where $m_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication of $\mathcal{A}$. Compatibility can of course be defined for $X_1, \ldots, X_n \in \mathcal{G}$ in a corresponding way. We shall express that $X_1, \ldots, X_n \in \mathcal{G}$ are compatible by writing $(X_1, \ldots, X_n) \in \mathcal{G}$.

Remark 2 The notion of compatibility which we introduce here in not standard in algebraic probability. A related concept is that of kinematic independence (see e.g. the review of Accardi [1]).

Given $X, Y \in \mathcal{G}(V)$ which are compatible we define $X + Y \in \mathcal{G}(V)$ as

$$(X + Y)(F) = F(X + Y) := m_{\mathcal{A}}[(X \otimes Y)\Delta F], \qquad F \in \Lambda V,$$

Note indeed that by compatibility $Z = m_{\mathcal{A}} \circ (X \otimes Y) : \Lambda(V \oplus V) \approx \Lambda V \otimes \Lambda V \to \mathcal{A}$ is an algebra homomorphism and therefore

$$(X + Y)(FG) = Z(\Delta(FG)) = Z(\Delta F \Delta G) = Z(\Delta F) Z(\Delta G) = (X + Y)(F)(X + Y)(G).$$

The notation is justified from the fact that $(X + Y)(v) = X(v) + Y(v)$ for all $v \in V$.

Independence. If $(X_1, \ldots, X_n) \in \mathcal{G}(V_1 \oplus \cdots \oplus V_n)$ are compatible Grassmann variables with values in the probability space $(\mathcal{A}, \omega)$, then we say that $X_1, \ldots, X_n$ are (tensor) independent (with respect to the state ω) if, for all $F_j \in \Lambda V_j$, we have that [29]

$$\omega\left(\prod_{j=1}^{k} X_j(F_j)\right) = \prod_{j=1}^{k} \omega(X_j(F_j)).$$

For example if $(X, Y) \in \mathcal{G}(V \oplus W)$ are (compatible and) independent Grassmann random variables with values in $(\mathcal{A}, \omega)$, then for all $v \in V \approx V \oplus 0, w \in W \approx 0 \oplus W$ we have $vw \approx ((v \oplus 0) \wedge (0 \oplus w)) \in \Lambda(V \oplus W) \approx \Lambda V \otimes \Lambda W$ and

$$\omega((X \otimes Y)(vw)) = \omega(X(v)Y(w)) = \omega(X(v))\omega(Y(w)).$$

Let $X_1 \in \mathcal{G}(V_1)$, $X_2 \in \mathcal{G}(V_2)$ be two Grassmann random variables. We want to show how to construct two compatible and independent random variables $\tilde{X}_1, \tilde{X}_2$ with the same law as X_1, X_2. The random variables X_1 and X_2 have values in the non-commutative probability spaces $(\mathcal{L}(\mathcal{H}_1), \omega_1)$ and $(\mathcal{L}(\mathcal{H}_2), \omega_2)$ respectively. By a GNS argument we can assume that ω_1 is the state associated to a vector Ω_1 in $\mathcal{H}_1$. Moreover we also assume that $\mathcal{H}_1 = \overline{X_1(\Lambda V)\Omega_1}$ (the bar denotes the closure with respect to the Hilbert space topology) since this enough to compute all the quantities $\omega_1(X_1(F))$ for all $F \in \Lambda V$. Then on $\mathcal{H}_1$ we can introduce an involution R_1 which act as $R_1 X_1(F)\Omega_1 = X_1((-1)^{|F|}F)\Omega_1$ where $|\cdot|$ is the grading on ΛV. A similar construction can be done for the other space $\mathcal{H}_2$. Now we take $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\tilde{X}_1(v) = X_1(v) \otimes R_2$ for $v \in V_1$ and $\tilde{X}_2(w) = \mathbb{1}_{\mathcal{H}_1} \otimes X_2(w)$ for $w \in V_2$. Then $\tilde{X}_1(v)\tilde{X}_2(w) = -\tilde{X}_2(w)\tilde{X}_1(v)$ and since $\Lambda(V_1 \oplus V_2) \approx \Lambda V_1 \otimes \Lambda V_2$ we have also that the map $X(v \oplus w) = X_1(v) + X_2(w)$ with $v \in V_1, w \in V_2$ can be extended to a homomorphism from $\Lambda(V_1 \oplus V_2)$ to $\mathcal{A}$ in such a way that $\omega(\tilde{X}_i(F)) = \omega_i(X_i(F))$ for all $F \in \Lambda V_i$, $i = 1, 2$.

2.1 Topology and calculus on Grassmann variables

Before beginning we want to specify the topology we consider on $\mathcal{G}(V)$. We consider here only the case where V is finite dimensional, since there is a more or less unique natural topology on $\mathcal{G}(V)$. Some topologies of $\mathcal{G}(V)$ when V is infinite dimensional are discussed in Section 4.

When V is finite dimensional all norms on V are equivalent. For this reason we choose the norm induced by the pre-Hilbert space inner product $\langle \cdot, \cdot \rangle$ (that in the case of V finite dimensional is a Hilbert space inner product on V) related to the construction of V-valued Gaussian random variables X (see Section 2.2). In this case $\mathcal{G}(V)$ has a natural metric topology given by the distance

$$d_{\mathcal{G}(V)}(X, Y) := \|X - Y\|_{\mathcal{G}(V)} = \sup_{v \in V, |v|_V = 1} \|X(v) - Y(v)\|_{\mathcal{A}}, \tag{2}$$

where $\|\cdot\|_{\mathcal{A}}$ is the natural norm in the $*$-algebra $\mathcal{A}$.

Remark 3 In principle the definition of the distance (2) and the related norm $\|\cdot\|_{\mathcal{G}(V)}$ does not use in any way the fact that V is finite dimensional. For this reason in the following, when V is a (in general infinite dimensional) per-Hilbert space we will use the notation $\|X\|_{\mathcal{G}(V)}$ for the quantity $\sup_{v \in V, |v|_V = 1} \|X(v) - Y(v)\|_{\mathcal{A}}$.

When V is finite dimensional $\mathcal{G}(V)$ is a complete metric space with respect to $d_{\mathcal{G}(V)}$, in fact we have:

Lemma 4 *The metric $d_{\mathcal{G}(V)}$ makes $\mathcal{G}(V)$ a complete metric space.*

Proof We let $\|\cdot\|_{\mathcal{G}(V)}$ to be the restriction to $\mathcal{G}(V)$ of the natural norm of the Banach space $\mathcal{L}(V, \mathcal{A})$ of the continuous linear maps between V and $\mathcal{A}$, then the quantity

$$d_{\mathcal{G}(V)}(X, Y) = \|X - Y\|_{\mathcal{G}(V)}$$

where $(X - Y)(v) = X(v) - Y(v)$, satisfies the usual properties of a metric and if X_n is a Cauchy sequence with respect to $d_{\mathcal{G}(V)}$, there is an element $X \in \mathcal{L}(V, \mathcal{A})$ such that X_n converges to X in $\mathcal{L}(V, \mathcal{A})$. The only thing that we have to prove is that the linear map $X \in \mathcal{L}(V, \mathcal{A})$ can be extended to a homomorphism from ΛV to $\mathcal{A}$. This is equivalent to prove that for any $v, w \in V$

we have $X(v)X(w) = -X(w)X(v)$ where the product is the natural one in $\mathcal{A}$. On the other hand by the continuity of the product of $\mathcal{A}$ with respect to $\|\cdot\|_{\mathcal{A}}$ we have

$$X(v)X(w) = \lim_{n\to+\infty} X_n(v)X_n(w) = -\lim_{n\to+\infty} X_n(w)X_n(v) = -X(w)X(v).$$

□

Remark 5 The distance $d_{\mathcal{G}(V)}$ is not the unique reasonable choice in $\mathcal{G}(V)$ and it depends on the topology chosen on $\mathcal{A}$ (in this case we choose the topology of the uniform converge of operators). An essential choice to preserve the non-linear structure of $\mathcal{G}(V)$, exploited in the proof of Lemma 4, seems to be that the product on $\mathcal{A}$ is continuous with respect to this topology.

In the following we will denote $\|Y\|_{\mathcal{G}(V)}$ for $Y \in \mathcal{G}(V)$ the positive number

$$\|Y\|_{\mathcal{G}(V)} := \|Y\|_{\mathcal{L}(V,\mathcal{A})} = \sup_{v\in V, \|v\|=1} \|X(v)\|_{\mathcal{A}},$$

i.e. the norm of the restriction of Y to $\mathcal{L}(V,\mathcal{A})$. This is useful to formulate a Taylor formula on $\mathcal{G}(V)$. We will sue the simpler notation m for the multiplication $m_{\mathcal{A}}$ in $\mathcal{A}$.

Define the *right derivative* $\partial_R : \Lambda V \to \Lambda V \otimes V$ by

$$\partial_R F := (\mathbb{1} \otimes \Pi_V)(\Delta F) : \Lambda V \to \Lambda V \otimes V \tag{3}$$

where $\Pi_V : \Lambda V \to V$ is the projection from the tensor algebra ΛV onto V. Then

$$F(X+Y) - F(X) - m[(X\otimes Y)(\partial_R F)] = m[(X\otimes Y)(\mathbb{1} - \mathbb{1}\otimes\varepsilon - \mathbb{1}\otimes\Pi_V)\Delta F]$$

for $F \in \Lambda V$, where we used that $F(X) = m[(X\otimes Y)(\mathbb{1}\otimes\varepsilon)\Delta F]$. Given that

$$(\mathbb{1} - \mathbb{1}\otimes\varepsilon - \mathbb{1}\otimes\Pi_V)\Delta F \in \Lambda V \otimes \Lambda_{\geqslant 2} V$$

where $\Lambda_{\geqslant 2}V$ denotes the subspace of ΛV of elements of degree $\geqslant 2$. We can define recursively the k+1-th derivative as

$$\partial_R^{k+1} = (\partial_R \otimes 1 \otimes \cdots \otimes 1)\partial_R^k : \Lambda V \to \Lambda V \otimes V \otimes \cdots \otimes V$$

where ∂_R^k is the k-th order derivative. Note that the right derivative $\partial_R : \Lambda V \to \Lambda V \otimes V$ satisfies

$$\partial_R(f_1 \cdots f_n) = \sum_{k=1}^{n} (-1)^{n-k} (f_1 \cdots \not{f}_k \cdots f_n) \otimes f_k, \qquad f_1, \ldots, f_n \in V.$$

Of course one can define also a left derivative $\partial_L = (\Pi_V \otimes \mathbb{1}) \circ \Delta : \Lambda V \to V \otimes \Lambda V$ with similar properties and

$$\partial_L(f_1 \cdots f_n) = \sum_{k=1}^{n} (-1)^{k-1} (f_1 \cdots \not{f}_k \cdots f_n) \otimes f_k, \qquad f_1, \ldots, f_n \in V.$$

We will consider $\Lambda V \otimes V$ as a ΛV-bimodule and in particular we define the bilinear form $\langle\cdot,\cdot\rangle : (\Lambda V \otimes V) \otimes V \to \Lambda V$ by

$$\langle f \otimes v, w\rangle = f\langle v, w\rangle, \qquad f \in \Lambda V, v, w \in V. \tag{4}$$

Note that there is no ambiguity on whether $\langle\cdot,\cdot\rangle$ denotes this bilinear form or whether it denotes the scalar product of V.

Remark 6 The definition of derivative ∂_R^k and the bilinear for (4) do not depend on the fact that V is finite dimensional. For this reason we can define ∂_R^k and the bilinear for (4) also when V is a infinite dimensional pre-Hilbert space.

Remark 7 Note that if $F_{\rm odd} \in \Lambda_{\rm odd}V$ then $\partial_R F_{\rm odd} = \partial_L F_{\rm odd}$. On the other hand, if $F_{\rm even} \in \Lambda_{\rm even}V$ then $\partial_R F_{\rm even} = -\partial_L F_{\rm even}$. In general, if we denote by $\mathbb{P}_{\rm odd}, \mathbb{P}_{\rm even}$ respectively the projection onto the odd and even parts of ΛV, then $\partial_R = (\mathbb{P}_{\rm even} - \mathbb{P}_{\rm odd})\partial_L$.

We can now state the version of Taylor formula which we will use below.

Lemma 8 *Consider $F \in \Lambda V$ and let $X, Y \in \mathcal{G}(V)$ be two compatible Grassmann random variables such that $\|Y\|_{\mathcal{G}(V)} \leqslant 1$, then*

$$F(X+Y) = F(X) + \sum_{k=1}^{n} \frac{1}{k!} m[(X \otimes Y \otimes \cdots \otimes Y)(\partial_R^k F)] + O(\|Y\|_{\mathcal{G}(V)}^{n+1}) \tag{5}$$

Proof We have

$$F(X+Y) = m[(X \otimes Y)(\Delta F)] = \sum_{k \geqslant 0} m[(X \otimes Y)(1 \otimes \Pi_{\Lambda^k V})(\Delta F)],$$

where the sum is finite since $F \in \Lambda V$ is a finite polynomial. Equation (5) easily follows as soon as we prove that

$$m[(X \otimes Y \otimes \cdots \otimes Y)(\partial_R^k F)] = (k!) m[(X \otimes Y)(1 \otimes \Pi_{\Lambda^k V})(\Delta F)].$$

We have

$$\begin{aligned}
\partial_R^k F &= (\partial_R \otimes 1 \otimes \cdots \otimes 1) \cdots (\partial_R \otimes 1)\partial_R F \\
&= \big((1 \otimes \Pi_V)\Delta \otimes 1 \otimes \cdots \otimes 1\big) \cdots \big((1 \otimes \Pi_V)\Delta \otimes 1\big) \big(1 \otimes \Pi_V\big) \Delta F \\
&= \big((1 \otimes \Pi_V) \otimes 1 \otimes \cdots \otimes 1\big) (\Delta \otimes 1 \otimes \cdots \otimes 1) \cdots \big((1 \otimes \Pi_V) \otimes 1\big) (\Delta \otimes 1) \big(1 \otimes \Pi_V\big) \Delta F \\
&= \big((1 \otimes \Pi_V) \otimes 1 \otimes \cdots \otimes 1\big) \cdots \big((1 \otimes \Pi_V) \otimes 1\big) \big(1 \otimes \Pi_V\big) (\Delta \otimes 1 \otimes \cdots \otimes 1) \cdots (\Delta \otimes 1)\Delta F \\
&= \big(1 \otimes \Pi_V \otimes \Pi_V \otimes \cdots \otimes \Pi_V\big) (\Delta \otimes 1 \otimes \cdots \otimes 1) \cdots (\Delta \otimes 1)\Delta F.
\end{aligned}$$

Let $N := \{1, \ldots, n\}$, $\#N = n$. For $I = \{j_1, \ldots, j_k\} \subset N$, we put $v_I := v_{j_1} \cdots v_{j_k}$ for $v_j \in V, j \in N$. Let $\mathrm{sgn}(N \backslash I)$ be a sign such that $v_N = \mathrm{sgn}(N \backslash I) v_{N \backslash I} v_I$ By definition of the coproduct Δ in the exterior algebra ΛV, we have

$$\begin{gathered}
(1 \otimes \Pi_V \otimes \cdots \otimes \Pi_V)(\Delta \otimes 1 \otimes \cdots \otimes 1) \cdots (\Delta \otimes 1)(\Delta)(v_1 \cdots v_n) = \\
= \textstyle\sum_{j_1, \ldots j_n \in N} \mathrm{sgn}(N \backslash I) v_{N \backslash \{j_1, \ldots, j_n\}} \otimes v_{j_1} \otimes \cdots \otimes v_{j_n}.
\end{gathered}$$

On the other hand

$$\begin{aligned}
(1 \otimes \Pi_{\Lambda^k V})\Delta v_1 \cdots v_n &= (1 \otimes \Pi_{\Lambda^k V}) \textstyle\sum_{I \subset N} \mathrm{sgn}(N \backslash I) v_{N \backslash I} \otimes v_I \\
&= \tfrac{1}{k!} \textstyle\sum_{j_1, \ldots, j_k} \mathrm{sgn}(N \backslash I) v_{N \backslash \{j_1, \ldots, j_k\}} \otimes v_{\{j_1, \ldots, j_k\}}.
\end{aligned}$$

By linearity, we therefore have

$$(1 \otimes m^{(k)})(\partial_R^k F) = k!(1 \otimes \Pi_{\Lambda^k V})(\Delta F).$$

where $m^{(k)} : V^{\otimes k} \to \Lambda V$ is the product. Since X, Y are algebra homomorphisms they commute with the product hence we have

$$\begin{aligned} m[(X \otimes Y \otimes \cdots \otimes Y)(\partial_R^k F)] &= m[(X \otimes Y)(1 \otimes m^{(k)})(\partial_R^k F)] \\ &= k! m[(X \otimes Y)(1 \otimes \Pi_{\Lambda^k V})(\Delta F)]. \end{aligned}$$

This concludes the proof. □

2.2 Grassmann Gaussian variables

Let now V be a real pre-Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and with an antisymmetric bounded operator $C : V \to V$. We recall that, by Remark 3 and Remark 6, we can extend the definitions of $\| \cdot \|_{\mathcal{G}(V)}$, ∂_R^k and the bilinear form (4) from the finite dimensional case to the generic (infinite) pre-Hilbert space V.

Definition 9 *A (V-)Grassmann (centered) Gaussian variable with correlation C is a random variable $X \in \mathcal{G}(V)$ such that*

$$\omega(X(F)X(f)) = \omega(X(\langle \partial_R F, Cf \rangle)), \qquad F \in \Lambda V, f \in V. \tag{6}$$

We also require that $\|X\| < \infty$, i.e that the map $X : V \to \mathcal{A}$ must be continuous with respect the topology induced on V by the pre-Hilbert product structure and the uniform operator topology on $\mathcal{A}$.

If we define $\partial_R X(F) = X(\partial_R F)$, with the understanding that $X(g \otimes w) = X(g) \otimes w$, $g \in V$ and $w \in \Lambda V$, then we have the more familiar expression (similar to the bosonic counterpart)

$$\omega(F(X)X(h)) = \omega(\langle \partial_R F(X), Ch \rangle).$$

Note that the integration by parts formula determine all the moments of the Gaussians. In particular (6) implies that $\omega(X(f_1) \cdots X(f_n)) = 0$ if n is odd and if $n = 2k$ is even

$$\omega(X(f_1) \cdots X(f_{2k})) = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{k} \langle f_{\sigma(2i-1)}, C f_{\sigma(2i)} \rangle_V \tag{7}$$

where the sum runs over all the pairings σ of $\{1, \ldots, 2k\}$ and $(-1)^{\sigma}$ is an appropriate sign. The right hand side of (7) can be written as a Pfaffian:

$$\omega(X(f_1) \cdots X(f_n)) = \operatorname*{Pf}_{1 \leqslant i,j \leqslant n} \langle f_i, C f_j \rangle_V,$$

where the Pfaffian is defined, for an anti-symmetric $n \times n$ matrix M, by the relation

$$(\operatorname*{Pf}_{1 \leqslant i,j \leqslant n} M_{ij})^2 = \det_{1 \leqslant i,j \leqslant n} M_{ij}.$$

The existence of a Gaussian variable, in the above sense, implies the inequality

$$\det_{1 \leqslant i,j \leqslant n} \langle f_i, C f_j \rangle_V = [\omega(X(f_1) \cdots X(f_n))]^2 \leqslant \|X(f_1)\|^2 \cdots \|X(f_n)\|^2 \leqslant \|X\|^{2n} \|f_1\|^2 \cdots \|f_n\|^2$$

well known in the mathematical physics literature relative to fermionic expansions e.g. see [57].

Remark 10 The averages of Gaussian variables depend only on the quadratic form $(f,g) \mapsto \langle f, Cg\rangle$, however analysis on the Grassmann algebra relies on the scalar product $\langle\cdot,\cdot\rangle$ also via the requirement that $\|X(f)\|_{\mathcal{A}} \leqslant \|f\|_V$. In particular the realization of the Grassmann algebra as a family of bounded operators is not canonical. This is the reason we need to require the above continuity of the map $X : V \to \mathcal{A}$.

In order to construct Grassmann Gaussian variables, we need "a place to hang the hat on", this place will be a (canonical commuting relations) CAR algebra endowed with its vacuum state. To allow for arbitrary correlations C we can use a standard trick [66] which consists in doubling the generators of the CAR algebra with respect to the generators of the Grassmann algebra.

Lemma 11 *For every antisymmetric and bounded $C : V \to V$ there exists a (V-)Grassmann Gaussian variable X with correlation C (on a suitable probability space $(\mathcal{A},\omega)$).*

Proof Let us consider ΛV itself as a real pre-Hilbert space with respect to the scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on ΛV given by

$$\langle\!\langle v_1\wedge\cdots\wedge v_n, w_1\wedge\cdots\wedge w_m\rangle\!\rangle = \delta_{nm}\det_{1\leqslant j,k\leqslant n}\langle v_j, w_k\rangle,$$

for $v_j, w_k \in V$, $j = 1,\ldots,n$, $k = 1,\ldots,m$, where $\langle\cdot,\cdot\rangle$ denotes as usual the scalar product on V. Note that $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\Lambda^n V$ is simply the restriction of the Hilbert scalar product on the tensor product $V^{\otimes n}$. Let $\mathcal{H}$ be the completion of ΛV with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and denote by Ω the element in $\mathcal{H}$ which corresponds to $1 \in \Lambda V$. Denote by $\lambda : \Lambda V \to \mathrm{End}(\Lambda V)$ the left action of ΛV on itself given by $\lambda(F)G = F\wedge G$, where $F, G \in \Lambda V$. We show that this action extends to a representation of ΛV on $\mathcal{H}$. Indeed $\lambda(v)$, $v \in V$ corresponds to a creation operator. Let us denote by $\lambda(v)^{\mathrm{T}}$ the adjoint of the operator $\lambda(v)$ with respect to the scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. A standard computation shows that for $v \in V$ we have

$$\lambda(v)^{\mathrm{T}} w_1\wedge\cdots\wedge w_n = \sum_{\ell=1}^{n}(-1)^{\ell-1}\langle v, w_\ell\rangle w_1\wedge\cdots\wedge \not{w}_\ell\wedge\cdots\wedge w_n.$$

Namely $\lambda^T(v)x$ can be expressed in terms of the left derivative as $\lambda^T(v)x = \langle v, \partial_L x\rangle$ (recall the definition given in equation (4)) for all $x \in \Lambda V$. In particular we have

$$\{\lambda(v),\lambda(w)\} = \{\lambda(v)^{\mathrm{T}},\lambda(w)^{\mathrm{T}}\} = 0, \qquad \{\lambda(v)^{\mathrm{T}},\lambda(w)\} = \langle v, w\rangle\mathbb{I}, \qquad v, w \in V, \tag{8}$$

where as usual $\{\cdot,\cdot\}$ denotes the anticommutator. As a consequence we obtain, for any $x \in \Lambda V$

$$\langle\!\langle\lambda(v)x,\lambda(v)x\rangle\!\rangle + \langle\!\langle\lambda(v)^{\mathrm{T}}x,\lambda(v)^{\mathrm{T}}x\rangle\!\rangle = \langle\!\langle x,\{\lambda(v)^{\mathrm{T}},\lambda(v)\}x\rangle\!\rangle = \langle v, v\rangle\langle\!\langle x, x\rangle\!\rangle.$$

In particular $\lambda(v), \lambda(v)^{\mathrm{T}}, v \in V$, extend to bounded operators on $\mathcal{H}$ (still denoted by the same symbol). We now define $X : \Lambda V \to \mathcal{L}(\mathcal{H})$ to be the algebra homomorphism defined first on V by

$$X(v) = \lambda(Cv) + \lambda(v)^{\mathrm{T}}, \qquad v \in V.$$

Then X is extended in a natural way to ΛV (still denoted by the same symbol). Note that $X(x)$, $x \in \Lambda V$, is indeed a bounded operator since it is the sum of products of bounded operators. Finally we define $\omega : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ to be the state on $\mathcal{A} = \mathcal{L}(\mathcal{H})$ defined by Ω, that is

$$\omega(T) := \langle\!\langle\Omega, T\Omega\rangle\!\rangle, \qquad T \in \mathcal{L}(\mathcal{H}).$$

We claim that X is a Gaussian Grassmann random variable on $(\mathcal{L}(\mathcal{H}), \omega)$. We first show that it is Grassmann:

$$\begin{aligned}\{X(v), X(w)\} &= \{\lambda(Cv), \lambda(w)^{\mathrm{T}}\} + \{\lambda(v)^{\mathrm{T}}, \lambda(Cw)\} \\ &= \langle Cv, w\rangle + \langle v, Cw\rangle = \langle v, C^{\mathrm{T}}w\rangle + \langle v, Cw\rangle = 0,\end{aligned}$$

where $v, w \in \Lambda$, because by assumption $C^{\mathrm{T}} = -C$. Now we have also that

$$\begin{aligned}\omega(X(v)X(w)) &= \langle\!\langle \Omega, \lambda(v)^{\mathrm{T}}\lambda(Cw)\Omega\rangle\!\rangle \\ &= \langle\!\langle \lambda(v)\Omega, \lambda(Cw)\Omega\rangle\!\rangle = \langle\!\langle v, Cw\rangle\!\rangle,\end{aligned}$$

where the first equality follows from the fact that $\lambda(v)^{\mathrm{T}}\Omega = 0$. More generally using that $X(v)\Omega = \lambda(Cv)\Omega$ and the commutation relation (8) we have that, for all $f_1, \dots, f_{2k} \in V$

$$\omega(X(f_1)\cdots X(f_{2k})) = \omega(X(f_1)\cdots X(f_{2k-1})\lambda(Cf_{2k}))$$

$$= -\omega(X(f_1)\cdots\lambda(Cf_{2k})X(f_{2k-1})) + \omega(X(f_1)\cdots X(f_{2k-2}))\langle f_{2k-1}, Cf_{2k}\rangle$$

$$= \cdots = \sum_{\ell=1}^{2k-1}(-1)^{2k-1-\ell}\omega\left(X(f_1)\cdots \not{X}(f_\ell)\cdots X(f_{2k-1})\right)\langle f_\ell, Cf_{2k}\rangle$$

Therefore

$$\omega(X(F)X(f)) = \omega(X(\langle\partial_R F, Cf\rangle)), \qquad F \in \Lambda V, f \in V.$$

Note that, similarly

$$\omega(X(f)X(F)) = \omega(X(\langle f, C\partial_L F\rangle)). \tag{9}$$

□

Complex Gaussians. Later on we will need also complex Grassmann Gaussian variables. As in the commutative setting, their definition poses no particular problems. Let $V_{\mathbb{C}}$ be a complex pre-Hilbert space with a notion of complex conjugation denoted $v \mapsto \bar{v}$ and an Hermitian scalar product $(\cdot,\cdot)$ which will be assumed to be anti-linear in the left variable. Consider a complex bilinear antisymmetric form $B_{\mathbb{C}}$ on $V_{\mathbb{C}}$

$$B_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}, \quad B_{\mathbb{C}} : v \times w \mapsto B_{\mathbb{C}}(v, w), \quad v, w \in V_{\mathbb{C}}.$$

Antisymmetric here means simply that $B_{\mathbb{C}}(v, w) = -B_{\mathbb{C}}(w, v)$, $v, w \in V_{\mathbb{C}}$. Define the standard bilinear form of $V_{\mathbb{C}}$ to be

$$\langle\cdot,\cdot\rangle_{\mathbb{C}} : v \times w \mapsto \langle v, w\rangle_{\mathbb{C}} := (\bar{v}, w), \quad v, w \in V_{\mathbb{C}}.$$

The suffix $\mathbb{C}$ in $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is a reminder that this is a bilinear form with complex values. The antisymmetric complex bilinear form $B_{\mathbb{C}}$ can be written in terms of a linear operator $C_{\mathbb{C}}$ in the following way

$$B_{\mathbb{C}}(v, w) = \langle v, C_{\mathbb{C}}w\rangle_{\mathbb{C}} = (\bar{v}, C_{\mathbb{C}}w),$$

where $C_{\mathbb{C}}$ is a complex operator which is antisymmetric with respect to $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, that is, we have

$$\langle v, C_{\mathbb{C}}w\rangle = \langle C_{\mathbb{C}}^{\mathrm{T}}v, w\rangle = -\langle C_{\mathbb{C}}v, w\rangle, \quad v, w \in V_{\mathbb{C}}.$$

Definition 12 *A $V_{\mathbb{C}}$-complex Grassmann (centered) Gaussian variable with complex correlation $C_{\mathbb{C}}$ is a random variable $X \in \mathcal{G}(V_{\mathbb{C}})$ such that*

$$\omega(X(F)X(h)) = \omega(X(\langle \partial_R F, C_{\mathbb{C}} h\rangle_{\mathbb{C}})), \qquad F \in \Lambda V_{\mathbb{C}}, h \in V_{\mathbb{C}}.$$

We require that $\|X\|_{\mathcal{G}(V_{\mathbb{C}})} < \infty$, i.e that the map $X : V_{\mathbb{C}} \to \mathcal{A}$ is continuous.

White noise. A relevant example of a random variable taking values in an infinite dimensional Grassmann algebra is d-dimensional (Gaussian) white noise.

Definition 13 *A V-valued d-dimensional (Gaussian) white noise with correlation $C : V \to V$ is the Grassmann Gaussian variable $\Xi \in \mathcal{G}(L^2(\mathbb{R}^d) \otimes V)$ with correlation $\tilde{C}$ given by*

$$(\tilde{C}f)(x) = Cf(x)$$

for all $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d) \otimes V \approx L^2(\mathbb{R}^d; V)$.

Take now a one dimensional white noise Ξ with values in V and let $B_t(v) = \Xi(\mathbb{1}_{[0,t]} \otimes v)$ for $v \in V$. For fixed $t \geqslant 0$, B_t extends as homomorphism to all ΛV and therefore $B_t \in \mathcal{G}(V)$. Note also that

$$\|B_t(v) - B_s(v)\|_{\mathcal{A}} \leqslant |t-s|^{1/2}\|v\|_V, \qquad t, s \geqslant 0, v \in V, \tag{10}$$

so $B(v) \in C(\mathbb{R}_+, \mathcal{A})$ and now note that $B \in \mathrm{Hom}(\Lambda V, C(\mathbb{R}_+, \mathcal{A}))$. This in particular implies that $B_{t_1}, \dots, B_{t_n}$ is a compatible family and that $B = (B_t)_{t\in\mathbb{R}_+}$ is a Gaussian process with continuous trajectories. We have that $B_0(v) = 0$,

$$\omega(B_t(v)) = 0, \qquad \omega(B_t(v)B_s(w)) = \langle v, Cw\rangle(t \wedge s), \qquad t, s \geqslant 0, \quad v, w \in V,$$

where C is the correlation of the Grassmann Gaussian random variable Ξ. Increments of B are independent and higher order moments can be computed via Wick's rule (7). Note in particular that

$$\sup_{0\leqslant t\leqslant T} \|B_t\|_{\mathcal{G}(V)} < \infty \tag{11}$$

for all $T > 0$. Let us remark here that properties (10) and (11) are very different from the pathwise properties of the commutative (bosonic) Brownian motion.

3 Finite dimensional SDEs

We want to study simple SDEs taking values in ΛV as far as it is needed for the purpose of stochastic quantization, that is with additive white noise. We refrain to undertake here a general study of Grassmann SDEs, in particular no stochastic calculus will be needed below. It seems possible to devise such a calculus but we leave it for a future work. More precisely in this section we want to study SDEs driven by an additive Brownian motion B_t when the linear space V is finite dimensional.

Definition 14 *Let $F \in \mathcal{L}(V, \Lambda V)$ and assume that $F(v)$ is odd for all $v \in V$, and let $\Psi_0 \in \mathcal{G}(V)$ be a random variable compatible and independent of the Brownian motion $B \in \mathcal{G}(L^2(\mathbb{R}_+, V))$, then $\Psi \in C^0([0,T]; \mathcal{G}(V))$ is a solution in $[0,T]$ to the (additive) SDE driven by B with drift F and initial condition Ψ_0, if,*

$$\Psi_t(v) = \Psi_0(v) + \int_0^t \Psi_s(F(v))\mathrm{d}s + B_t(v), \qquad t \in [0,T], v \in V, \tag{12}$$

where the integral with respect to the variable s is understood in Bochner's sense (relative to the norm on $\mathcal{G}(V)$ introduced in equation (2)).

Remark 15 Note that we do not require that Ψ_t, Ψ_s are compatible for $t \neq s$, in particular

$$C^0([0,T];\mathcal{G}(V)) \neq \mathrm{Hom}(\Lambda V; C([0,T],\mathcal{A})),$$

nor we require any compatibility of Ψ and B. It will turn out that such compatibility holds in fact for the unique solution of (12), but is not necessary to put such restriction to formulate the notion of solution.

3.1 The Ornstein–Uhlenbeck motion

In this section we introduce the Grassmann analog of the Ornstein–Uhlenbeck process, that is when $F(v) = Av$ with $A : V \to V$ is a linear operator on V. In this case we can write down an explicit formula for the solution to equation (12).

Proposition 16 *Suppose that $A : V \to V$ and $F(v) = Av$. The unique solution to equation* (12) *is given by*

$$\Psi_t(v) = \Psi_0(e^{At}v) + \Xi(\mathbb{1}_{[0,t]}(\cdot)e^{A(t-\cdot)}v), \qquad t \in \mathbb{R}_+, \tag{13}$$

where Ξ is the Grassmann Gaussian noise related with the Brownian motion B_t (see Definition 13 and the discussion that follows).

Proof Let $h(t,v) \in L^2(\mathbb{R};V)$ be given by

$$h(t,v)(s) := \mathbb{1}_{[0,t]}(s)e^{A(t-s)}v, \qquad s \in \mathbb{R}.$$

Then (13) reads $\Psi_t(v) = \Psi_0(e^{At}v) + \Xi(h(t,v))$. An explicit computation using that

$$h(t,v)(s) - \mathbb{1}_{[0,t]}(s)v = \int_0^t Ah(r,v)(s)\mathrm{d}r = \int_0^t h(r,Av)(s)\mathrm{d}r, \qquad s \in \mathbb{R},$$

in $L^2(\mathbb{R};V)$ gives

$$\Xi(h(t,v)) = B_t(v) + \int_0^t \Xi(h(r,Av))\mathrm{d}r.$$

Moreover, by the definition of exponential of a matrix, $\partial_t\Psi_0(e^{At}v) = \Psi_0(e^{At}Av)$ so

$$\begin{aligned}\Psi_t(v) &= \Psi_0(v) + \int_0^t \Psi_0(e^{Ar}Av)\mathrm{d}r + B_t(v) + \int_0^t \Xi(h(r,Av))\mathrm{d}r \\ &= \Psi_0(v) + B_t(v) + \int_0^t \Psi_r(Av)\mathrm{d}r\end{aligned}$$

as required by equation (13) for our assumption on F. □

Remark 17 In particular this shows that if (Ψ_0, Ξ) is a Grassmann Gaussian process then also the solution Ψ to the SDE (12), is a Grassmann Gaussian process compatible with (Ψ_0, Ξ). In order to prove this fact it is sufficient to prove that any product of the form

$$\omega(\Psi_{t_1}(v_1)\cdots\Psi_{t_{2k}}(v_2)) \tag{14}$$

can be computed using the products (weighted with a suitable sign due to the anti-commutation of Ψ) of the covariance $\omega(\Psi_{t_i}(v_i)\Psi_{t_j}(v_j))$ for any $i,j=1,\ldots,2k$. This properties of the expectation (14) follows from the fact that

$$\omega((\Psi_{t_1}(v_1)-\Psi_0)\cdots(\Psi_{t_{2k}}(v_n)-\Psi_0))=\omega(\Xi(h(t_1,v_1))\cdots\Xi(h(t_{2k},v_{2k})))=$$
$$=\sum_\sigma(-1)^\sigma\prod_{i=1}^k\int_0^{\min(t_i,t_j)}\langle e^{A(t_i-s)}v_{\sigma(2i-1)},Ce^{A(t_j-s)}v_{\sigma(2i)}\rangle_V\mathrm{d}s$$

where C is the covariance of Ξ. Using the fact that Ψ_0 is a Gaussian random variable independent of Ξ, and so $\Psi_0(e^{A(t-s)}v)$ is also a Gaussian random variable independent of Ξ, we obtain the Gaussian behavior of Ψ_t. The compatibility of Ψ_t with (Ψ_0,Ξ) follows from the fact that Ψ is a linear function of (Ψ_0,Ξ).

Let us now study the family of random variables

$$\Psi_t^s(v)=\Xi(e^{A(t-\cdot)}v),\qquad t\in\mathbb{R}.$$

In the next proposition, we will show that this represents the stationary solution to the SDE (13) (this motivates our putting the upper index s to Ψ_t^s).

Proposition 18 *Assume that all eigenvalues of A have strictly negative real part less or equal than $-\lambda_A$, where $\lambda_A>0$. Then, we have $\sup_{t\in\mathbb{R}}\|\Psi_t^s\|_{\mathcal{G}(V)}<+\infty$, furthermore for any $G\in\Lambda V$ and any $t\in\mathbb{R}$ we get*

$$\omega(G(\Psi_t^s))=\omega(G(X^A)),\tag{15}$$

where $X^A\in\mathcal{G}(V)$ is a Gaussian random variable with covariance C_A given by

$$C_A:=\int_0^\infty e^{A^{\mathrm{T}}s}Ce^{As}\mathrm{d}s.\tag{16}$$

where A^{T} denote the transpose matrix.

Remark 19 Expression (15) in particular show that the independence of t of the expectation $\omega(G(\Psi_t^s))$, expressing the stationarity of Ψ_t^s.

Proof of Proposition 18 We have easily by the definition of Ψ_t^s:

$$\|\Psi_t^s\|_{\mathcal{G}(V)}\lesssim\|\mathbb{I}_{(-\infty,t]}(\cdot)e^{A(t-\cdot)}\|_{L^2(\mathbb{R})}\lesssim\frac{1}{\lambda_A},$$

where $\lesssim$ stands for inequality modulo some appropriate positive constant. The random variable Ψ_t^s is Gaussian so it is completely characterized (in term of moments) by its covariance. Note that the covariance of Ψ_t^s can be easily computed as follows

$$\omega(\Psi_t^s(v)\Psi_t^s(w))=\omega(\Xi(\mathbb{I}_{(-\infty,t]}(\cdot)e^{A(t-\cdot)}v)\Xi(\mathbb{I}_{(-\infty,t]}(\cdot)e^{A(t-\cdot)}w))=$$

$$=\langle\mathbb{I}_{(-\infty,t]}(\cdot)e^{A(t-\cdot)}v,C\mathbb{I}_{(-\infty,t]}(\cdot)e^{A(t-\cdot)}w\rangle=\int_{-\infty}^0\langle e^{-A^{\mathrm{T}}\tau}v,Ce^{-A\tau}w\rangle\mathrm{d}\tau=\langle v,C_Aw\rangle$$

with C_A given by (16). The appearance of the transposition is due to the fact that V is a *real* pre-Hilbert space. □

Let us point out that

$$A^{\mathrm{T}}C_A+C_AA=-C\tag{17}$$

by a simple integration by parts applied to (16).

Remark 20 Suppose that A commutes with C, then we have that Ψ_t^s is for all $t\in\mathbb{R}$ a Grassmann Gaussian process with correlation $C_A=-(A+A^{\mathrm{T}})^{-1}C$ (the operator A is invertible since all its eigenvalues has strictly negative real part).

3.2 Existence and uniqueness for general drift

Theorem 21 *For any $\Psi_0 \in \mathcal{G}(V)$ compatible with B, there exists $T > 0$ such that equation* (12) *admits a unique solution $\Psi \in C^0([0,T];\mathcal{G}(V))$. Moreover Ψ is compatible with $\Psi_0 \oplus B$.*

Proof We are going to construct a solution via Picard's iteration. Let $\Psi_t^0 = \Psi_0$ for all $t \geqslant 0$ and define the $n+1$-th Picard's iteration by

$$\Psi_t^{n+1}(v_\alpha) = \Psi_0(v_\alpha) + \int_0^t \Psi_s^n(F(v_\alpha))\mathrm{d}s + B_t(v_\alpha), \qquad t \in [0,T],$$

for any basis element $v_\alpha \in V$. Then $\Psi^0 \in C^0([0,T];\mathcal{G}(V))$ and since Ψ_0 is compatible with B we have that, for all $t \geqslant 0$, Ψ_t^1 belong to the Grassmann algebra generated by Ψ_0 and B in $\mathcal{A}$ and there $\Psi_t^1(v_\alpha)$ is an odd element for all α. Therefore $\Psi_t^1 \in \mathcal{G}(V)$ for all $t \geqslant 0$ and also Ψ_t^1 is compatible with $\Psi_0 \oplus B$. Since it is true for all t, by approximation in the operator norm we see that $\Psi^1 \in C^0([0,T];\mathcal{G}(V))$ and that it is compatible with $\Psi_0 \oplus B$. By induction we prove then that $\Psi^n \in C^0([0,T];\mathcal{G}(V))$ for all $n \geqslant 0$ and that it is compatible with $\Psi_0 \oplus B$.

Now observe that $\Psi_s^n(F(v_\alpha))$ is a polynomial function of $(\Psi_s^{n,\alpha})_\alpha$ and therefore by a standard contraction argument in the Banach space $\mathcal{A}$ we obtain that $(\Psi^n)_n$ converges, as $n \to +\infty$, in $C^0([0,T];\mathcal{A})$ for some positive T which depends only on F, on $\sup_\alpha \|\Psi_0^\alpha\|_{\mathcal{A}}$ and on $\sup_{0\leqslant t\leqslant T,\alpha} \|B_t^\alpha\|_{\mathcal{A}}$. If we call Ψ the limit we have that $\Psi \in C([0,T];\mathcal{G}(V))$ by the first part of the proof and we deduce easily that Ψ is compatible with $\Psi_0 \oplus B$. □

Theorem 22 *The solution to equation* (12) *exists for all times.*

Proof We will have global existence as soon as we can rule out explosions, that is prove that for all $t \geqslant 0$ we have $\sup_\alpha \|\Psi_t^\alpha\|_{\mathcal{A}} < \infty$ Let $\Theta_t(v) = \Psi_t(v) - B_t(v)$ and extend this map to a homeomorphism of ΛV into $\mathcal{A}$. Observe that

$$\Theta_t(v) = \Theta_0(v) + \int_0^t m_{\mathcal{A}}[(\Theta_s \otimes B_s)(\Delta F(v_\alpha))]\mathrm{d}s.$$

In particular, if we consider a (linear) basis of ΛV denoted by $(e_A)_A$ and let $\Theta_t^A := \Theta_t(e_A)$ and $B_t^A := B_t(e_A)$ we have

$$\Theta_t^A = \Theta_0^A + \int_0^t \sum_B h_{A,B,C}\Theta_s^B B_s^C \mathrm{d}s,$$

for real coefficients $(h_{A,B,C})_{A,B,C}$. This is a finite-dimensional, linear system of non-autonomous ODEs in $\mathcal{L}(H)$ for $(\Theta_t^A)_A$ ($\mathcal{H}$ being the Hilbert space the element of $\mathcal{A}$ act, see Section 2), and

$$\sum_A \|\Theta_t^A\|_{\mathcal{A}} \leqslant \sum_A \|\Theta_0^A\|_{\mathcal{A}} + |h| \int_0^t \sum_B \|\Theta_s^B\|_{\mathcal{A}} \sum_C \|B_s^C\|_{\mathcal{A}} \mathrm{d}s,$$

with $|h| := \sup_{A,B,C} |h_{A,B,C}|$. Therefore by Gronwall inequality

$$\sum_A \|\Theta_t^A\|_{\mathcal{A}} \leqslant \sum_A \|\Theta_0^A\|_{\mathcal{A}} \exp\left(|h| \int_0^t \sum_C \|B_s^C\|_{\mathcal{A}} \mathrm{d}s\right) < \infty$$

for all $t \geqslant 0$. In particular $\sup_\alpha \|\Psi_t(v_\alpha)\|_{\mathcal{A}} \leqslant \sup_\alpha \|\Theta_t(v_\alpha)\|_{\mathcal{A}} + \sup_\alpha \|B_t(v_\alpha)\|_{\mathcal{A}} < \infty$ for all $t \geqslant 0$ and the solution to equation (12) exists for all times. □

3.3 An Itô formula for solutions of SDEs

We want to prove a kind of Itô formula for the solution to equation (12). In order to do that let $Q_C \in \mathcal{L}(\Lambda V \otimes V \otimes V, \Lambda V)$ be given by

$$Q_C(f \otimes v \otimes w) = \langle v, Cw\rangle f, \qquad f \in \Lambda V, v, w \in V. \tag{18}$$

For $F \in \mathcal{L}(V, \Lambda V)$ and $G \in \Lambda V \otimes V$ we define $G \cdot F \in \Lambda V$ by extending linearly

$$(f \otimes v) \cdot F = fF(v), \qquad f \in \Lambda V, v \in V,$$

where in both sides we have multiplication of elements in ΛV.

Theorem 23 *For the global solution to equation* (12) *we have*

$$\omega(\Psi_t(G)) = \omega(\Psi_0(G)) + \int_0^t \omega\left(\Psi_s\left(\partial_R G \cdot F + \frac{1}{2} Q_C(\partial_R^2 G)\right)\right) \mathrm{d}s \tag{19}$$

for all $G \in \Lambda V$ *and* $t \geqslant 0$.

Proof Fix any $T \geqslant 0$. Note that

$$\|B_t - B_s\|_{\mathcal{G}(V)} \lesssim |t-s|^{1/2},$$

for all $0 \leqslant s < t \leqslant T$. From that it follows that, if F in (12) is a polynomial of degree $\deg(F)$:

$$\|(\Psi_t - \Psi_s) - (B_t - B_s)\|_{\mathcal{G}(V)} = \left\| \int_s^t F(\Psi_s) \mathrm{d}s \right\|_{\mathcal{G}(V)} \lesssim (1 + \|\Psi\|_{[0,T]})^{\deg(F)} |t-s|, \tag{20}$$

where $\|\Psi\|_{[0,T]} := \sup_{t\in[0,T]} \|\Psi_s\|_{\mathcal{G}(V)}$ which, by Theorem 22 and the continuity of Ψ with respect to the time t, is finite for every $T \in \mathbb{R}_+$. By Taylor formula (see Lemma 8) applied to polynomial G of degree $\deg(G)$ we have

$$\begin{gathered} \|G(\Psi_r) - G(\Psi_s) - (\Psi_s \otimes (\Psi_r - \Psi_s))\partial_R G + \\ + \tfrac{1}{2}(\Psi_s \otimes (\Psi_r - \Psi_s) \otimes (\Psi_r - \Psi_s))(\partial_R \partial_R G)\|_{\mathcal{A}} \lesssim (1 + \|\Psi\|_{[0,T]})^{\deg(F)(\deg(G)-2)} |r-s|^{3/2} \end{gathered} \tag{21}$$

since by (20) $\|\Psi_t - \Psi_s\|_{\mathcal{G}(V)} \lesssim (1 + \|\Psi\|_{[0,T]})^{\deg(F)} |t-s|^{1/2}$. In a similar way it is possible to obtain

$$\|\langle \partial_R G(\Psi_u), F(\Psi_u)\rangle - \langle \partial_R G(\Psi_s), F(\Psi_u)\rangle\|_{\mathcal{A}} \lesssim (1 + \|\Psi\|_{[0,T]})^{\deg(F)(\deg(G)-2)} |u-s|^{1/2} \tag{22}$$

On the other hand, by (20), we have

$$\begin{aligned} \omega((\Psi_s \otimes (\Psi_r - \Psi_s))\partial_R G) &= \omega\left(\textstyle\int_s^r \langle \partial_R G(\Psi_s), F(\Psi_u)\rangle \mathrm{d}u\right) \\ &\quad + \omega((\Psi_s \otimes (B_r - B_s))(\partial_R G)) \\ &= \omega\left(\textstyle\int_s^r \langle \partial_R G(\Psi_u), F(\Psi_u)\rangle \mathrm{d}u\right) \\ &\quad + \omega(\langle \partial_R G(\Psi_s), (B_r - B_s)\rangle) \\ &\quad + O((1 + \|\Psi\|_{[0,T]})^{\deg(F)(\deg(G)-2)} |r-s|^{3/2}) \end{aligned} \tag{23}$$

In a similar way, using inequality (20) we obtain

$$\begin{gathered} \omega((\Psi_s \otimes (\Psi_r - \Psi_s) \otimes (\Psi_r - \Psi_s))(\partial_R^2 G)) = \omega((\Psi_s \otimes (B_r - B_s) \otimes (B_r - B_s))(\partial_R^2 G)) \\ + O((1 + \|\Psi\|_{[0,T]})^{\deg(F)(\deg(G)-2)} |r-s|^{3/2}). \end{gathered} \tag{24}$$

Furthermore using the fact that $\partial_R G(\Psi_s)$ is independent of $(B_r - B_s)$ and $((B_r - B_s) \otimes (B_r - B_s))$ is independent of $\partial_R^2 G(\Psi_s)$ (this is due to the fact that Ψ_s is a function of $\{\Psi_0, B_k | k \in [0,s]\}$ and B_t is independent of Ψ_0 and has independent increments, we have:

$$\omega(\langle \partial_R G(\Psi_s), (B_r - B_s)\rangle) = \langle \omega((B_r - B_s)), \omega(\partial_R G(\Psi_s))\rangle = 0$$

(where on the r.h.s. we understand that averages w.r.t. ω are taken componentwise) and

$$\begin{aligned}
&\omega[\langle \partial_R \partial_R G(\Psi_s), (B_r - B_s) \otimes (B_r - B_s)\rangle_{V \otimes V}] \\
&= \langle \omega[(\partial_R \partial_R G(\Psi_s))], \omega[((B_r - B_s) \otimes (B_r - B_s))]\rangle \\
&= (r-s)\omega[Q_C(\partial_R^2 G(\Psi_s))].
\end{aligned}$$

By taking a partition $\{t_i\}_{i \in \{0,\dots,n\}}$, of diameter $\rho_n \to 0$ as $n \to +\infty$, of $[0,t]$ and exploiting inequality (21) we obtain

$$\begin{aligned}
\omega(G(\Psi_t)) &= \omega(G(\Psi_0)) + \textstyle\sum_{i=0}^{n+1} \omega(G(\Psi_{t_i})) - \omega(G(\Psi_{t_{i-1}})) \\
&= \omega(G(\Psi_0)) + \textstyle\sum_{i=1}^{n+1} \omega((\Psi_{t_{i-1}} \otimes (\Psi_{t_i} - \Psi_{t_{i-1}}))\partial_R G) \\
&\quad + \tfrac{1}{2}\omega((\Psi_s \otimes (\Psi_r - \Psi_s) \otimes (\Psi_r - \Psi_s))(\partial_R^2 G)) + O(\rho_n^{1/2}) \\
&= \omega(G(\Psi_0)) + \omega\left(\textstyle\int_0^t \partial_R G(\Psi_u) F(\Psi_u) \mathrm{d}u\right) \\
&\quad + \tfrac{1}{2}\textstyle\sum_{i=1}^{n+1}(t_i - t_{i-1})\omega[Q_C(\partial_R^2 G(\Psi_{t_i}))] + O(\rho_n^{1/2}),
\end{aligned}$$

where the constants in O are proportional to $(1 + \|\Psi\|_{[0,T]})^{\deg(F)\deg(G)}$ and do not depend on the partition. Taking the limit $n \to +\infty$, and so $\rho_n \to 0$, and using the fact that $\omega[Q_C(\partial_R^2 G(\Psi_s))]$ is continuous in s ($G(\Psi_s)$ being a continuous function from $\mathbb{R}_+$ to $\mathcal{A}$ since $\Psi \in C^0([0,T], \mathcal{G}(V))$) we obtain the thesis. □

3.4 Invariant measures

Consider an equation of the following kind

$$\Psi_t(v) = \Psi_0(v) + \int_0^t (\Psi_s(Av) + \Psi_s(F(v)))\mathrm{d}s + B_t(v), \qquad t \geqslant 0, v \in V, \tag{25}$$

for $\Psi \in C^0([0,T]; \mathrm{Hom}(V, \mathcal{A}))$, $F \in \mathrm{Hom}(V, \Lambda V)$, $A \in \mathcal{L}(V)$, B a Grassmann Brownian motion with correlation C and with initial condition Ψ_0 distributed as a Grassmann Gaussian independent of B with correlation C_A (compare to equation (16)). Fix $U \in \Lambda V$ even, and consider the linear functional $\tilde{\omega} : \mathcal{A} \to \mathbb{C}$ given by

$$\tilde{\omega}(\cdot) := \omega(\cdot e^{-2U(\Psi_0)}), \tag{26}$$

where ω is, as before, is the positive state on the $*$-algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$.

Definition 24 *We say that the functional $\tilde{\omega}$ is an invariant measure for the equation* (25) *if for any $G \in \Lambda V$ and any $t \in \mathbb{R}_+$ we have*

$$\tilde{\omega}(G(\Psi_t)) = \tilde{\omega}(G(\Psi_0)). \tag{27}$$

Remark 25 The functional is invariant in the sense of Definition 24, only if the expectations of the form (27) are constant when expressions of the form $G(\Psi_t)$ are evaluated. This means in particular that only polynomials in Ψ_t are considered.

There is a simple condition, analogous to the Fokker–Planck equation for commutative SDEs, for checking the invariance of the functional $\tilde{\omega}$. Here it is:

Lemma 26 *If Ψ is a solution to (25) and*

$$\tilde{\omega}(\mathcal{L}G(\Psi_0)) = 0, \qquad G \in \Lambda V, \tag{28}$$

for

$$\mathcal{L}G := \partial_R G \cdot F + \frac{1}{2} Q_C(\partial_R^2 G),$$

then $\tilde{\omega}$ is an invariant measure in the sense of Definition 24.

Proof Consider $G = G_A = v_A$ where $(v_A)_A$ is a (finite) linear basis of ΛV. Then we have $\mathcal{L}(G_A) = \sum_B \kappa_A^B v_B$ for a suitable family $(\kappa_A^B)_{A,B}$ of constants in $\mathbb{R}$. Write $\mathfrak{P}_A(t) := \tilde{\omega}(G_A(\Psi_t)) = \omega(G_A(\Psi_t)e^{-2U(\Psi_0)})$, due to definition (26). By Itô formula (19) we have that $\mathfrak{P}_A$ is the unique solution of the system of ODEs

$$\begin{aligned} \partial_t \mathfrak{P}_A(t) &= \partial_t \omega(G_A(\Psi_t)e^{-2U(\Psi_0)}) = \omega((\mathcal{L}G_A)(\Psi_t)e^{-2U(\Psi_0)}) \\ &= \sum_B \kappa_A^B \omega(G_B(\Psi_t)e^{-2U(\Psi_0)}) = \sum_B \kappa_A^B \mathfrak{P}_B(t), \end{aligned} \tag{29}$$

with initial condition $\mathfrak{P}_A(0) = \omega(G_A(\Psi_0)e^{-2U(\Psi_0)})$. On the other hand condition (28) implies that

$$\sum_B \kappa_A^B \mathfrak{P}_B(0) = \sum_B \kappa_A^B \omega(G_B(\Psi_0)e^{-2U(\Psi_0)}) = \omega(\mathcal{L}G(\Psi_0)e^{-2U(\Psi_0)}) = 0$$

which means that $(\mathfrak{P}_A(t))_A$ is constantly zero by uniqueness of solution to the ODEs system (29). Since the expectations of the form (27) are linear combinations of $(\mathfrak{P}_A(t))_A$ the proof is complete. □

The following theorem gives a sufficient condition on the SDE (25) to have the invariant measure (26).

Theorem 27 *For any even $U \in \Lambda_{\text{even}} V$, the SDE*

$$\Psi_t(v) = \Psi_0(v) + \int_0^t (\Psi_s(Av) + \Psi_s(\langle C\partial_R U, v\rangle)\mathrm{d}s + B_t(v) \qquad t \geqslant 0, v \in V, \tag{30}$$

with B a Brownian motion with correlation C and Ψ_0 an independent Gaussian initial condition with correlation C_A (defined in equation (16)) has $\tilde{\omega}$ as invariant measure provided

$$A^{\mathrm{T}} C_A - C_A A = 0. \tag{31}$$

Proof The proof is a consequence of Lemma 26. Denote $\omega^{\Psi_0}(G) := \omega(G(\Psi_0))$ the law of Ψ_0, and observe that for $H \in \Lambda V \otimes V$ we have the integration by parts formula

$$\omega((\Psi_0 \otimes \Psi_0)(H)) = \omega(\Psi_0(Q_{C_A}(\partial_R H))),$$

where Q_C is defined in (18). Then for all $G \in \Lambda V$ and $U \in \Lambda_{\text{even}} V$ we have

$$\omega((\Psi_0 \otimes \Psi_0)(A(\partial_R G)e^{-2U(\Psi_0)})) = \omega^{\Psi_0}(Q_{C_A}(\partial_R(A\partial_R G e^{-2U(\Psi_0)})))$$

$$= \omega^{\Psi_0}(Q_{C_A}((A \otimes 1)\partial_R\partial_R G)e^{-2U(\Psi_0)}) - 2\omega^{\Psi_0}(Q_{C_A}((A\partial_R G) \otimes \partial_R U)e^{-2U(\Psi_0)}).$$

Observe that using the equality (17) we have

$$Q_{C_A}((A\otimes 1)\partial_R\partial_R G) = \mathrm{Tr}((A\otimes C_A)\partial_R\partial_R G)$$

$$= \frac{1}{2}\mathrm{Tr}((C_A A + A^{\mathrm{T}}C_A)\partial_R\partial_R G) = -\frac{1}{2}Q_C(\partial_R\partial_R G),$$

where we have used the trace Tr on V given by $\mathrm{Tr}(v\otimes w) = \langle v, w\rangle$. Moreover, from (31) we have also

$$Q_{C_A}((A\partial_R G)\otimes\partial_R U) = \mathrm{Tr}((A\partial_R G)\otimes(C_A\partial_R U)) = \mathrm{Tr}(\partial_R G\otimes(A^{\mathrm{T}}C_A\partial_R U))$$

$$= \mathrm{Tr}\left(\partial_R G\otimes\left(\frac{A^{\mathrm{T}}C_A + C_A A}{2}\partial_R U\right)\right) + \mathrm{Tr}\left(\partial_R G\otimes\left(\frac{A^{\mathrm{T}}C_A - C_A A}{2}\partial_R U\right)\right)$$

$$= -\frac{1}{2}\mathrm{Tr}(\partial_R G\otimes(C\partial_R U)) = -\frac{1}{2}\langle\partial_R G, C\partial_R U\rangle.$$

since we assumed that $A^{\mathrm{T}}C_A - C_A A = 0$ and used (17) again. Therefore

$$\omega((\Psi_0\otimes\Psi_0)(A(\partial_R G)e^{-2U(\Psi_0)})) = -\frac{1}{2}\omega^{\Psi_0}(Q_C(\partial_R^2 G)e^{-2U(\Psi_0)}) + \omega^{\Psi_0}(\langle\partial_R G, C\partial_R U\rangle e^{-2U(\Psi_0)}).$$

Now observing that

$$\mathcal{L}G(\Psi_0) = (\Psi_0\otimes\Psi_0)(A\partial_R G) + \Psi_0\left(\langle\partial_R G, C\partial_R U\rangle + \frac{1}{2}Q_C(\partial_R\partial_R G)\right)$$

we can deduce

$$\omega(\mathcal{L}G(\Psi_0)e^{-2U(\Psi_0)}) = \omega((\Psi_0\otimes\Psi_0)(A\partial_R G)e^{-2U(\Psi_0)}) +$$

$$+\omega^{\Psi_0}\left(\left(\langle\partial_R G, C\partial_R U\rangle + \frac{1}{2}Q_C(\partial_R\partial_R G)\right)e^{-2U(\Psi_0)}\right) = 0.$$

□

3.5 Long-time behavior for small non-linearity

We will now investigate the existence of solutions which are globally bounded in time. In the commutative setting, non-linear equations can have globally bounded solutions only if the non-linearity stays uniformly small or if it shows some coercivity. Like all notions of positivity, also coercivity however does not apply well in the Grassmann setting. The only kind of coercive term we have identified is a linear drift with a negative sign. This is a very mild coercive term, but it turns out to be enough provided the non-linearity is small enough.

Consider the equation

$$\Psi_t(v) = \Psi_0(v) + \int_0^t(\Psi_s(Av) + \lambda\Psi_s(F(v)))\mathrm{d}s + B_t(v), \qquad t\geqslant 0, v\in V, \tag{32}$$

for $\Psi\in C([0,T];\mathrm{Hom}(V,\mathcal{A}))$, with $\lambda\geqslant 0$ and $F\in\mathrm{Hom}(V,\Lambda V)$ where A is an operator with eigenvalues having strictly negative real part less then $-\lambda_A < 0$. In this setting we introduce the notion of stationary solution to equation (32).

Definition 28 *We say that $\Psi_t^s\in C^0(\mathbb{R},\mathcal{G}(V))$ is a stationary solution to equation* (32) *of norm at most $K\in\mathbb{R}_+$ if the following two conditions hold:*

1. $\sup_{t\in\mathbb{R}} \|\Psi_t^s\| < K$,

2. $(\Psi_t^s)_{t\in\mathbb{R}}$ *solves the following integral equation*

$$\Psi_t^s = \lambda \int_{-\infty}^t \Psi_\tau^s(F(e^{A\tau}v))\mathrm{d}\tau + B_t^A \qquad t \in \mathbb{R} \tag{33}$$

where $B_t^A(v) = \Xi(e^{A(t-\cdot)}(v))$.

Hereafter we write

$$L(A,C) = \sup_{t\in\mathbb{R}} \|B_t^A\|_{\mathcal{G}(V)},$$

and we denote by $\mathfrak{K}_K \subset C^0(\mathbb{R},\mathcal{G}(V))$ the set of $\Psi \in C^0(\mathbb{R},\mathcal{G}(V))$ such that

$$\sup_{t\in\mathbb{R}} \|\Psi_t\|_{\mathcal{G}(V)} \leqslant K,$$

for some constant $K > 0$.

Theorem 29 *Assume* $K \geqslant 3L(A,C)$ *and suppose that all the eigenvalues of* A *have negative real part that are less or equal than* $-\lambda_A$*, then there exists* $\lambda_0(K,L,F)$ *depending on* K*,* $L(A,C)$ *and* F *such that if* $|\lambda| \leqslant 2\lambda_0(K,L,F)$ *there exists a unique stationary solution of norm at most* K *to equation* (32)*.*

Proof It is a simple application of the Banach fix point theorem by noting that the map

$$\mathcal{K}_\lambda(\Psi)_t := \lambda \int_{-\infty}^t \Psi_\tau(F(e^{A(t-\tau)}v))\mathrm{d}\tau + B_t^A, \qquad t \in \mathbb{R},$$

is a contraction in $\mathfrak{K}_K$. Indeed, if $\Psi \in \mathfrak{K}_K$ we have

$$\|\mathcal{K}_\lambda(\Psi)_t\|_{\mathcal{G}(V)} \leqslant |\lambda|\|F\|(1+K)^{\deg(F)} \int_{-\infty}^t e^{-\lambda_A(t-\tau)}\mathrm{d}\tau + L(A,C),$$

where we use the notation $\|F\|$ for the smallest constant such that

$$\sup_{v:\|v\|\leqslant 1} \|\psi(F(v))\|_{\mathcal{A}} \leqslant \|F\| \deg(F)(1+\|\psi\|_{\mathcal{G}(V)})^{\deg(F)} \qquad \psi \in \mathcal{G}(V).$$

If we request that

$$2\lambda_0 \frac{\|F\|(1+K)^{\deg(F)}}{\lambda_A} + 2L < K, \tag{34}$$

(where $\lambda_0 := \lambda_0(K,L,F)$) for $|\lambda| \leqslant 2\lambda_0$ small enough we have that $\mathcal{K}_\lambda$ maps $\mathfrak{K}_K$ into itself. Furthermore

$$\|\mathcal{K}_\lambda(\Psi_1)_t - \mathcal{K}_\lambda(\Psi_2)_t\|_{\mathcal{G}(V)} \leqslant \frac{|\lambda|}{\lambda_A}\|\partial_R F\|(1+K)^{\deg(F)-1}\|\Psi_{1,t} - \Psi_{2,t}\|_{\mathcal{G}(V)}.$$

Therefore, if we take

$$\frac{2\lambda_0}{\lambda_A}\|\partial_R F\|(1+K)^{\deg(F)-1} < 1 \tag{35}$$

the map $\mathcal{K}_\lambda$ is a strict contraction. The fact that $\mathfrak{K}_K \neq \emptyset$ will be a consequence of the Lemma (31) below. □

Remark 30 The function λ_0 in Theorem 29 can be taken to be a decreasing function of the parameters L and K.

We introduce an approximation for the solution to equation (33). Let $X \in \mathcal{G}(V)$ independent of B_t^A and consider the element $\Psi^X_{-T,t} \in C^0(\mathbb{R}, \mathcal{G}(V))$ which is $\Psi^X_{-T,t} = X$ if $t \leqslant -T$, for some $T > 0$, and is a solution to the SDE

$$\Psi^X_{-T,t} = X(e^{A(t+T)}v) + \lambda \int_{-T}^{t} F(\Psi^X_{-T,t}(e^{A(t-\tau)}v)) + B_t^A - B_{-T}^A(e^{A(t+T)}v),$$

when $t > -T$.

Lemma 31 *Under the assumptions of Theorem 29, take $\|X\|_{\mathcal{G}(V)} < \frac{1}{6}K$, then, for any $|\lambda| \leqslant \frac{1}{2}\lambda_0(K, L, F)$, we have that $\Psi^X_{-T,t} \in \mathfrak{K}_K$.*

Proof Let $\tau \in \mathbb{R}$ then we have the bound

$$\begin{aligned}\sup_{t\in(-\infty,\tau]} \|\Psi^X_{-T,t}\|_{\mathcal{G}(V)} &\leqslant e^{-\lambda_A \tau}(\|X\|_{\mathcal{G}(V)} + \|B^A_{-T}\|_{\mathcal{G}(V)}) + \\ &\quad + \|F\|\frac{|\lambda|}{\lambda_A}(1 + \sup_{t\in(-\infty,\tau]} \|\Psi^X_{-T,t}\|_{\mathcal{G}(V)})^{\deg(F)} + L(A, C).\end{aligned}$$

On the other hand by definition of λ_0 we have, for any $\tau > 0$

$$e^{-\lambda_A \tau}\left(\frac{1}{3}K + \frac{1}{6}K\right) + \|F\|\frac{|\lambda|}{\lambda_A}(1 + K)^{\deg(F)} + L(A, C) < K.$$

Let $R_\tau := \sup_{t\in(-\infty,\tau]} \|\Psi^X_{-T,t}\|_{\mathcal{G}(V)}$ and consider the set $Z = \left\{\tau \in \mathbb{R} : R_\tau \leqslant \frac{2K}{3}\right\}$. The set Z is non-empty, since $R_\tau < \frac{1}{2}K$ when $\tau \leqslant -T$, open since $\tau \mapsto R_\tau$ is an increasing continuous function of τ and closed since $\tau \to R_\tau$ is continuous. Then we must have $Z = \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \|\Psi^X_{-T,t}\|_{\mathcal{G}(V)} \leqslant \frac{2}{3}K,$$

for $|\lambda| \leqslant \lambda_0$. □

Theorem 32 *Under the hypotheses of Lemma 31 we have that $\Psi^X_{-T,t}$ converges to Ψ^s_t in $\mathcal{G}(V)$ as $T \to +\infty$, uniformly on compact subsets of $\mathbb{R}$.*

Proof If $t \geqslant -T$ we have that

$$\begin{aligned}\|\Psi^X_{-T,t} - \Psi^s_t\|_{\mathcal{G}(V)} &\lesssim e^{-\lambda_A(t+T)}(\|X\|_{\mathcal{G}(V)} + \|B^A_{-T}\|_{\mathcal{G}(V)}) + |\lambda|\|F\|(1 + K)^{\deg(F)} \times \\ &\quad \times \int_{-\infty}^{-T} e^{-\lambda_A(t-\tau)}\mathrm{d}\tau + |\lambda|\|\partial_R F\|(1 + K)^{\deg(F)-1} \times \\ &\quad \times \int_{-T}^{t} e^{-\lambda_A(t-s)}\|\Psi^X_{-T,\tau} - \Psi^s_\tau\|\mathrm{d}\tau.\end{aligned}$$

By Gronwall lemma we obtain

$$\begin{aligned}\|\Psi^X_{-T,t} - \Psi^s_t\|_{\mathcal{G}(V)} &\leqslant e^{-\lambda_A T}\left(\|X\|_{\mathcal{G}(V)} + \frac{K}{3} + \frac{|\lambda|\|F\|(1 + K)^{\deg(F)}}{\lambda_A}\right) \times \\ &\quad \times(1 + \exp(-(\lambda_A - |\lambda|(\|\partial_R F\|K)^{\deg(F)-1})(t + T)).\end{aligned}$$

Since $\lambda_A - |\lambda|(\|\partial_R F\|K)^{\deg(F)-1} > 0$, by the condition (35), we have that

$$\|\Psi^X_{-T,t} - \Psi^s_t\|_{\mathcal{G}(V)} \lesssim e^{-\lambda_A T}$$

uniformly in T when $t > P > -T$ for any such fixed P. This proves that $\Psi^X_{-T,t}$ converges to Ψ^s_t uniformly on compact sets. □

An important consequence of Theorem 32 is the following one.

Theorem 33 *Under the hypotheses and notations of Theorem 29 and Theorem 32, if* $\|X\|_{\mathcal{G}(V)} < \frac{1}{3}K$, $|\lambda| \leqslant \lambda_0(K, L, F)$ *and if* $\omega(\cdot e^{-2U(X)})$ *is an invariant measure for equation* (32) *(in the sense of Definition 24), for any* $t \in \mathbb{R}$ *and for any* $G \in \Lambda V$*, we have that*

$$\frac{\omega(G(X)e^{-2U(X)})}{\omega(e^{-2U(X)})} = \omega(G(\Psi^s_t)).$$

Proof Under the previous notation by Definition 24, we have that, for any $t \geqslant -T$,

$$\omega(G(X)e^{-2U(X)}) = \omega(G(\Psi^X_{-T,-T})e^{-2U(X)}) = \omega(G(\Psi^X_{-T,t})e^{-2U(X)}).$$

By Theorem 32 we can take the limit at the right hand side as $T \to +\infty$, obtaining

$$\omega(G(X)e^{-2U(X)}) = \omega(G(\Psi^s_t)e^{-2U(X)}).$$

Since X is independent of B^A_t and since Ψ^s_t is a function only of B^A we get

$$\omega(G(X)e^{-2U(X)}) = \omega(G(\Psi^s_t))\omega(e^{-2U(X)}),$$

from which the thesis follows. □

Remark 34 By Theorem 27, Theorem 33 applies to the case $\lambda F(v) = \lambda\langle C\partial_R U, v\rangle$ for λ small enough.

4 Infinite dimensional SDEs

We want to study SDEs of the form

$$\mathrm{d}\Psi_t = (A\Psi_t + \lambda F(\Psi_t))\mathrm{d}t + \mathrm{d}B_t \tag{36}$$

in the case where Ψ takes values in $\mathcal{G}(V)$ and $V = \mathcal{S}(\mathbb{R}^d) \otimes \mathbb{R}^n$, or $\mathcal{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$, or $C^\infty(\mathbb{T}^d) \otimes \mathbb{R}^n$ for some $n \in \mathbb{N}_0$ and with A a (possibly unbounded) linear operator on V. We discuss in detail the case $V = \mathcal{S}(\mathbb{R}^d)$, but every reasoning can be easily generalized to the other cases or more generally to any nuclear space of smooth functions.

The main difference between the finite and infinite dimensional case for Grassmann SDEs is that in general there is no natural topology on $\mathcal{G}(V)$. The space $V = \mathcal{S}(\mathbb{R}^d)$ is a Fréchet space. Since not every element of $\mathcal{G}(V)$ is continuous with respect to this topology we consider here the space $\mathcal{G}^{-\infty}(V) \subset \mathcal{G}(V)$, the subset of $\mathcal{G}(V)$ for which $X \in \mathcal{G}^{-\infty}(V)$ if and only if X restricted to $V \subset \Lambda V$ is a continuous linear map from V to $\mathcal{A}$. Hereafter we always consider $X \in \mathcal{G}^{-\infty}(V)$ both as a homomorphism from the algebra ΛV into $\mathcal{A}$, and as a continuous linear map from V into $\mathcal{A}$.

Since V is a nuclear space whose dual is nuclear the set of continuous linear maps from V into $\mathcal{A}$ can be identified with the tensor product $V^* \otimes \mathcal{A} = \mathcal{S}'(\mathbb{R}^n) \otimes \mathcal{A}$ (where V^* the topological dual of V, the Schwartz space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$). In other words we can look upon $\mathcal{G}^{-\infty}(V)$ as the subset of elements of $\mathcal{S}'(V) \otimes \mathcal{A}$ that can be extended to a homomorphism from ΛV into $\mathcal{A}$.

Using a reasoning similar to the one exploited in Lemma 4, it is possible to prove that $\mathcal{G}^{-\infty}(V)$ is a closed subset of $V^* \otimes \mathcal{A}$ with respect to the strong topology (of both $V^* = \mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{A}$), i.e. if $(X_\nu)_\nu$ is a net in $\mathcal{G}^{-\infty}(V)$ converging to X in the topology of $V^* \otimes \mathcal{A}$, then X can be extended to an homomorphism from ΛV into $\mathcal{A}$. Working on $\mathcal{G}^{-\infty}(V)$ is not so easy since $\mathcal{S}'(V) \otimes \mathcal{A}$ is not even a Fréchet space. For this reason we introduce now some subsets of $\mathcal{G}^{-\infty}(V)$ (which are analogous to Besov spaces) which admit a stronger metric topology.

We recall here only the definition of the Besov norm for functions taking values in a Banach space, further references and details are given in Appendix A (see also [6, 7]). Let $K_i \in \mathcal{S}(\mathbb{R}^d)$, with $i \geqslant -1$, be the functions corresponding to the Littlewood–Paley blocks used in the definition of Besov spaces $B^s_{p,q} := B^s_{p,q}(\mathbb{R}^d, \mathbb{R})$, where $s \in \mathbb{R}$ and $p, q \in [1, +\infty]$. If $X \in \mathcal{G}^{-\infty}(V)$ we define

$$\Delta_i X(x) = X(K_i(x - \cdot)), \qquad x \in \mathbb{R}^d, i \geqslant -1.$$

The function $\Delta_i X : \mathbb{R}^d \to \mathcal{A}$ can be identified with an element $\Delta_i X \in \mathcal{S}(\mathbb{R}^d) \otimes \mathcal{A} = \mathcal{S}(\mathbb{R}^d, \mathcal{A})$ and we define

$$\|X\|_{B^s_{p,q}(\mathbb{R}^d,\mathcal{A})} := \left[\sum_{i \geqslant -1} 2^{qis} \left(\int_{\mathbb{R}^d} \|\Delta_i X(x)\|^p_{\mathcal{A}} \mathrm{d}x \right)^{q/p} \right]^{1/q}.$$

Given $s \in \mathbb{R}$, we say that $X \in \mathcal{G}^s(V)$ when $X \in \mathcal{G}^{-\infty}(V)$ and $\|X\|_{B^s_{\infty,\infty}(\mathbb{R}^d,\mathcal{A})} < +\infty$. Thus we introduce the natural metric $d_{\mathcal{G}^s(V)}(X, Y) = \|X - Y\|_{B^s_{\infty,\infty}(\mathbb{R}^d,\mathcal{A})}$. Hereafter we always use the notation $\mathcal{C}^s(\mathbb{R}^d, \mathcal{A}) = B^s_{\infty,\infty}(\mathbb{R}^d, \mathcal{A})$ and $\mathcal{C}^s = \mathcal{C}^s(\mathbb{R}^d, \mathbb{R})$. Moreover $\mathcal{C}^{s*}$ will denote the topological dual of $\mathcal{C}^s$.

Lemma 35 *The set $\mathcal{G}^s(V)$ with the metric $d_{\mathcal{G}^s(V)}$ is a complete metric space.*

Proof The proof is analogous to that of Lemma 4. □

The case $s > 0$ can be understood more easily. In this case the elements $X \in \mathcal{G}^s(V)$ can be identified with some continuous function $\dot{X} : \mathbb{R}^d \to \mathcal{A}$. Under this identification if $f \in \mathcal{S}(\mathbb{R}^d)$ we have that

$$X(f) = \int_{\mathbb{R}^d} f(x) \dot{X}(x) \mathrm{d}x,$$

where the integral is taken in Bochner sense. If $X \in \mathcal{G}^s(V)$, with $s > 0$, we can obtain the function $\dot{X}$ in the following way. Let $\mathfrak{a} : \mathbb{R}^d \to \mathbb{R}$ be a smooth mollifier and define $\mathfrak{a}_{x,\varepsilon}(y) = \varepsilon^{-d} \mathfrak{a}\left(\frac{y-x}{\varepsilon}\right)$, where $x, y \in \mathbb{R}^d$ and $\varepsilon > 0$. Then we have that

$$\dot{X}(x) = \lim_{\varepsilon \to 0} X(\mathfrak{a}_{\varepsilon,x}) = X(\delta_x),$$

where the limit is taken with respect to the strong topology of $\mathcal{A}$, and where δ_x is the Dirac delta with unit mass at the point $x \in \mathbb{R}^d$. More generally, if $X \in \mathcal{G}^s(V)$ and $g \in \mathcal{C}^{s*}$, for any

sequence $(g_n)_n \subset \mathcal{S}(\mathbb{R}^d)$ converging to g with respect to the natural topology of $\mathcal{C}^{s*}$ we have that $(X(g_n))_n$ converges in $\mathcal{A}$ to a unique element

$$X(g) := \lim_{n \to +\infty} X(g_n).$$

The map X now defined on all of $\mathcal{C}^{s*}$ is continuous. In other words, if $X \in \mathcal{G}^s(V)$ then it can be identified with an element of $\mathcal{L}(\mathcal{C}^{s*}, \mathcal{A})$. Furthermore it is simple to prove, using suitable $\mathcal{S}(\mathbb{R}^d)$ approximations, that if $g, h \in \mathcal{C}^{s*}$, we have

$$X(g)X(h) = -X(h)X(g). \tag{37}$$

In particular we have

$$\dot{X}(x_1)\dot{X}(x_2) = -\dot{X}(x_2)\dot{X}(x_1), \qquad x_1, x_2 \in \mathbb{R}^d.$$

The relation (37) has important consequences. For any $n \geqslant 1$, let

$$\Lambda^n_\pi \mathcal{C}^{s*} \subset \underbrace{\mathcal{C}^{s*} \otimes_\pi \mathcal{C}^{s*} \otimes_\pi \cdots \otimes_\pi \mathcal{C}^{s*}}_{n \text{ times}}$$

be the completion of $\Lambda^n \mathcal{C}^{s*}$ in $\otimes^n_\pi \mathcal{C}^{s*}$. Here $\otimes_\pi$ is the projective tensor product of Banach spaces whose norm is defined as follows: if $f \in \otimes^n \mathcal{C}^{s*}$ then

$$\|f\|_{\otimes^n_\pi \mathcal{C}^{s*}} := \inf \left\{ \sum_{i=1}^k \prod_{j=1}^n \|f^i_j\|_{\mathcal{C}^{s*}}, \text{where } f = \sum_{i=1}^k f^i_1 \otimes \cdots \otimes f^i_n \right\}.$$

Lemma 36 *If $X \in \mathcal{G}^s(V)$ then it can be extended to a continuous homomorphism from $\bigoplus_{i=1}^n \Lambda^n_\pi \mathcal{C}^{s*}$ into $\mathcal{A}$ for any $n > 0$.*

Proof The proof is based on the fact that X can be identified with an element of $\mathcal{L}(\mathcal{C}^{s*}, \mathcal{A})$ and on the relation (37). □

In the case $s > 0$, we have for example that

$$X(\delta_{x_1} \wedge \cdots \wedge \delta_{x_n}) = \dot{X}(x_1) \cdots \dot{X}(x_n), \qquad x_1, \ldots, x_n \in \mathbb{R}^d$$

We want now to define a natural notion of function F of an element $X \in \mathcal{G}^s(V)$. For example, if $s > 0$, and we write $Y = F_n(X)$, we want to be able to take F_n of the form

$$\dot{Y}(x) = \sum_{(k_1, \ldots, k_n) \in D} K_{(k_1, \ldots, k_n)} \dot{X}(x + k_1)\dot{X}(x + k_2) \cdots \dot{X}(x + k_n), \tag{38}$$

where $D \subset \mathbb{R}^{dn}$ is some finite set and $K_{(k_1, \ldots, k_n)} \in \mathbb{R}$. When n is odd, the important property of the function F_n defined before is that $Y = F(X) \in \mathcal{C}^s(\mathbb{R}^d, \mathcal{A})$ and furthermore for any $g, h \in \mathcal{C}^{s*}$ we have

$$\begin{aligned} Y(g)Y(h) &= -Y(h)Y(g) \\ Y(g)X(h) &= -X(h)Y(g). \end{aligned} \tag{39}$$

We want to consider some more general setting than the one considered in equation (38) that maintains the important property (39).

Consider $F_n \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \Lambda^n_\pi \mathcal{C}^{s*})$, then F_n can be seen as an element of $\mathcal{S}'(\mathbb{R}^d) \otimes \Lambda^n_\pi \mathcal{C}^{s*}$. Using the same technique as before we can introduce the Banach space $\mathcal{C}^s(\mathbb{R}^d, \Lambda^n_\pi \mathcal{C}^{s*}) \subset \mathcal{S}'(\mathbb{R}^d) \otimes \Lambda^n_\pi \mathcal{C}^{s*}$ equipped with its natural norm. In this case we will we use the notation

$$F_n(X) := X \circ F_n, \tag{40}$$

where $(X \circ F_n)(v) = X(F_n(v)) \in \mathcal{A}$ for any $v \in \mathcal{S}(\mathbb{R}^d)$.

Definition 37 *We say that the linear function $F \in \mathcal{C}^s\left(\mathbb{R}^d, \bigoplus_{j=0}^k \Lambda^{2j+1}_\pi \mathcal{C}^{s*}\right)$ is a standard odd function of degree $2k+1$ if there is a sequence of functions $F^n \in \mathcal{C}^s\left(\mathbb{R}^d, \bigoplus_{j=0}^k \Lambda^{2j+1} \mathcal{C}^{s*}\right)$ (where $\Lambda^{2j+1}\mathcal{C}^{s*} = \underbrace{\mathcal{C}^{s*} \otimes \mathcal{C}^{s*} \otimes \cdots \otimes \mathcal{C}^{s*}}_{2j+1 \text{ times}}$ and $\otimes$ is the algebraic tensor product) such that for any $g \in V$ we have $F^n(g) \to F(g)$, as $n \to +\infty$, in $\bigoplus_{j=0}^k \Lambda^{2j+1}_\pi \mathcal{C}^{s*}$ equipped with its standard topology.*

Remark 38 When $s > 0$, an example of a function F satisfying Definition 37 is

$$F(X)(\delta_x) = \dot{X}(x) \int_{\mathbb{R}^d} \mathcal{G}_1(x-y)\dot{X}(y)\mathrm{d}y \int_{\mathbb{R}^d} \mathcal{G}_2(x-y)\dot{X}(y)\mathrm{d}y,$$

where $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{C}^{s+\epsilon}$ are functions that decrease faster than any polynomial at infinity. We note that a finite dimensional approximation of $F(X)(\delta_x)$ is given by

$$F^n(X)(\delta_x) = \sum_{D_i \in \mathcal{P}_n} \sum_{D_j \in \mathcal{P}_n} \dot{X}(x)\dot{X}(x-x_i)\dot{X}(x-x_j) \int_{D_i} \mathcal{G}_1(y)\mathrm{d}y \int_{D_j} \mathcal{G}_2(y)\mathrm{d}y,$$

where $\mathcal{P}_n$ is sequence of increasing partitions of $\mathbb{R}^d$ for which $\sup_i \left|\int_{D_i} \mathcal{G}_1(y)\mathrm{d}y\right| \to 0$ and $\sup_j \left|\int_{D_j} \mathcal{G}_2(y)\mathrm{d}y\right| \to 0$.

Theorem 39 *Let $s \in \mathbb{R}$, let $F \in \mathcal{C}^s\left(\mathbb{R}^d, \bigoplus_{j=0}^k \Lambda^{2j+1}_\pi \mathcal{C}^{s*}\right)$ be a standard odd function of degree $2k+1$ (see Definition 37), and consider $X \in \mathcal{G}^s(V)$. Then $F(X) \in \mathcal{C}^s(\mathbb{R}^d, \mathcal{A})$, furthermore for any $v_1, v_2 \in \mathcal{C}^{s*}$ we have*

$$\begin{aligned} F(X)(v_1)F(X)(v_2) &= -F(X)(v_2)F(X)(v_1) \\ F(X)(v_1)X(v_2) &= -X(v_2)F(X)(v_1). \end{aligned} \tag{41}$$

Proof The fact that $F(X) \in \mathcal{C}^s(\mathbb{R}^d, \mathcal{A})$ follows from Remark 36. The relations (41) are obvious when $F \in \mathcal{C}^s\left(\mathbb{R}^d, \bigoplus_{j=0}^k \Lambda^{2j+1}\mathcal{C}^{s*}\right)$. Using a finite dimensional approximation F^n converging to F the relations (41) are extended to the more general case. □

Remark 40 An important consequence of Theorem 39 is that if $X \in \mathcal{G}^s(V)$ and F satisfies the hypotheses of Theorem 39, $F(X)$ can be extended to a Grassmann random variable (which in the following we denote by the same symbol $F(X)$) in $\mathcal{G}^s(V)$. Furthermore $F(X)$ and X are compatible (in the sense of the corresponding definition of Section 2).

We are now in the position to give a precise meaning to equation (36) and to the various hypotheses on its data. Let the operator A be the (generally unbounded) generator of a strongly

continuous and exponentially contractive semigroup $(e^{At})_{t\geqslant 0}$ on $L^2(\mathbb{R}^d)$. Moreover consider the stationary Ornstein-Uhlenbeck motion

$$B_t^A(v) = \Xi(\mathbb{1}_{(-\infty,t]} \otimes e^{A(t-\cdot)}v), \qquad t \in \mathbb{R}, v \in \mathcal{S}(\mathbb{R}^d), \tag{42}$$

where Ξ is a Grassmann Gaussian noise on $\mathcal{G}(C^\infty(\mathbb{R}_+) \otimes \mathcal{S}(\mathbb{R}^d))$ with covariance $\mathbb{1} \otimes C$ (we consider on $C_c^\infty(\mathbb{R}_+) \otimes \mathcal{S}(\mathbb{R}^d)$ the pre-Hilbert space structure given by the natural inclusion of $C_c^\infty(\mathbb{R}_+) \otimes \mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$).

Definition 41 *We say that the process $\Psi \in C^0([0,T], \mathcal{G}^s(V))$ satisfies equation (36) with initial condition Ψ_0 if F is a standard odd function of finite degree, $\Psi_0 \in \mathcal{G}^s(V)$ is independent of Ξ, and for any $t \in [0,T]$ and any $v \in V$ we have*

$$\Psi_t(v) = \Psi_0(e^{At}v) + \lambda \int_0^t F(\Psi_s)(e^{A(t-s)}v)\mathrm{d}s + B_t^A(v) - B_0^A(e^{At}v), \tag{43}$$

where the integral in the variable s is taken in Bochner sense with respect the norm of $\mathcal{A}$.

We introduce some conditions on F, A and Ξ in order to prove a result on existence and uniqueness of solutions to (43).

Definition 42 *Let F be a standard odd function of degree $2k+1$. We say that F is a regular map on $\mathcal{C}^s(\mathbb{R}^d, \mathcal{A})$ if it is a locally bounded and Lipschitz function, i.e. it satisfies the following conditions:*

i. There is a continuous non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $X \in \mathcal{G}^s(V)$ we have

$$\|F(X)\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})} \leqslant f(\|X\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})}). \tag{44}$$

ii. The map F is locally Lipschitz continuous and there is a continuous non-decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $X, Y \in \mathcal{G}^s(V)$:

$$d_{\mathcal{G}^s(V)}(F(X), F(Y)) \leqslant g(\max(\|X\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})}, \|Y\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})}))d_{\mathcal{G}^s(V)}(X, Y). \tag{45}$$

Definition 43 *We say that the unbounded operator A on $L^2(\mathbb{R}^d)$, and the Gaussian noise Ξ with covariance C are adapted to the space $\mathcal{G}^s(V)$, if*

i. The operator A generates a contraction semigroup on $\mathcal{C}^s(\mathbb{R}^d)$ such that $\|e^{At}\|_{\mathcal{L}(\mathcal{C}^s(\mathbb{R}^d),\mathcal{C}^s(\mathbb{R}^d))} \lesssim e^{-\lambda_A t}$ for some strictly positive constant $\lambda_A > 0$.

ii. The Gaussian process B^A defined by (42) belongs to $C^0(\mathbb{R}, \mathcal{G}^s(V))$ and furthermore we have $\sup_{t\in\mathbb{R}} \|B_t^A\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})} < +\infty$.

Theorem 44 *Suppose that F satisfies Definition 42 and A, Ξ satisfy Definition 43 and let X be a variable in $\mathcal{G}^s(V)$ independent of the Grassmann Gaussian noise Ξ. Then there exists a $T > 0$ such that there is a unique solution $\Psi \in C^0([0,T], \mathcal{G}^s(V))$ to equation (43). Furthermore Ψ_t is compatible with (Ψ_0, Ξ, Ψ_s) for every $t, s \in [0,T]$.*

Proof The proof is a straightforward generalization of that of Theorem 21 for the case of our infinite dimensional setting. □

In order to obtain solutions for all times it is not possible to extend directly Theorem 22, since its proof was essentially based on the hypothesis that V is finite dimensional. On the other

hand the proofs of Theorem 29, Lemma 31 and Theorem 32 do not depend on the dimension. For this reason we introduce the space $\mathfrak{K}_K \subset C^0(\mathbb{R}, \mathcal{G}^s(V))$ defined as the set of functions $\Psi_t \in C^0(\mathbb{R}, \mathcal{G}^s(V))$ such that

$$\sup_{t\in\mathbb{R}} \|\Psi_t\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})} < K,$$

for some constant $K > 0$, the function L being given by

$$L(A, C) = \sup_{t\in\mathbb{R}} \|B^A_t\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})},$$

and the equation for stationary solutions corresponding to (43) as

$$\Psi^s_t = \lambda \int_{-\infty}^{t} F(\Psi^s_\tau(e^{A(t-\tau)}))\mathrm{d}\tau + B^A_t \qquad t \in \mathbb{R}. \tag{46}$$

We also denote by $\Psi^X_{-T,t}$ the Grassmann process defined by $\Psi^X_{-T,t} = X$ if $t \leqslant -T$ and such that $\Psi^X_{-T,t}$ is the solution to equation (43) with initial condition X at time $t = -T$.

Theorem 45 *Suppose that F satisfies Definition 42 and A, Ξ satisfy Definition 43 and let $K > 2L(A, C)$. Then there exists a function $\lambda_0(K, L, F)$ such that for any $|\lambda| \leqslant 2\lambda_0(K, L, F)$ there exists a unique (stationary) solution to equation (46) in $\mathfrak{K}_K$. Furthermore for any $|\lambda| \leqslant \lambda_0(K, L, F)$ and for any random variable $X \in \mathcal{G}^s(V)$, independent of B^A_t and such that $\|X\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})} < \frac{1}{3}K$, we have that $\Psi^X_{-T,t}$ converges, as $T \to +\infty$, in $\mathcal{G}^s(V)$ to Ψ^s_t uniformly on compact subsets of $\mathbb{R}$.*

Proof Due to our appropriate infinite dimensional setting the proof is a straightforward generalization of Theorem 29, Lemma 31 and Theorem 32. □

Remark 46 The function λ_0 defined in Theorem 45 has to satisfy the following inequalities

$$\frac{2\lambda_0(K, L, F) f_F(K)}{\lambda_A} + 2L < K, \tag{47}$$

$$\frac{2\lambda_0(K, L, F)}{\lambda_A} g_F(K) < 1 \tag{48}$$

where f_F, g_F and λ_A are the quantities introduced in Definition 42 and Definition 43 respectively.

5 The Yukawa$_2$ model

We are now in position to study the stochastic quantization for an Euclidean fermionic QFT. We consider the Euclidean counterpart of the Yukawa model describing the interaction of a scalar field with a spin 1/2 field in a $1 + 1$ Minkowski space-time. However we will not attempt here to remove the UV cutoff so the dimension of the space will not play any fundamental role. The Yukawa$_2$ model contains already the main features of a large class of Euclidean fermionic theories. It will be clear from the considerations below that more general results are easily derivable at the price of heavier notation. For the sake of readability we choose to stick to a definite model.

5.1 Osterwalder–Schrader fields

We define the Euclidean Yukawa$_2$ quantum field theory following Osterwalder and Schrader [66]. In the language of this paper, the Euclidean free Fermions introduced by Osterwalder and Schrader (compare with [66] page 282 and equation (3.3) page 284) are given by a quadruplet $\tilde{\psi} = (\psi, \bar{\psi}) \in \mathcal{G}(\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4)$ of complex Grassmann Gaussian fields on the complex pre-Hilbert space $\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4 \subseteq L^2(\mathbb{R}^2) \otimes \mathbb{C}^4$ with an Hermitian scalar product denoted $(\cdot, \cdot)$, a standard bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ (defined by $\langle f, g \rangle := (\bar{f}, g)$, for $f, g \in \mathcal{G}(\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4)$) and the correlations

$$\omega(\tilde{\psi}(f)\tilde{\psi}(g)) = \langle f, Kg \rangle_{\mathbb{C}}, \qquad f, g \in \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4, \tag{49}$$

where K is the bounded operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^4$ given by

$$K = (m_f^2 - \Delta)^{-1} \begin{pmatrix} 0 & (\not{\nabla} + m_f) \\ -(\not{\nabla} + m_f)^{\mathrm{T}} & 0 \end{pmatrix},$$

where Δ is the Laplacian on $\mathbb{R}^2$, $m_f > 0$ is a constant interpreted as the mass of the fermionic particle associated with the fermionic fields $(\psi, \bar{\psi})$. As before, $[\cdots]^{\mathrm{T}}$ denotes real transposition, that is $[\cdots]^{\mathrm{T}} = \overline{[\cdots]}^*$. Moreover $\not{\nabla} := \gamma_1 \nabla_1 + \gamma_2 \nabla_2$ with ∇_i the partial derivatives with respect to x_i, $i = 1, 2$, and with Euclidean Dirac matrices $\gamma_1, \gamma_2 : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

and such that

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}, \qquad a, b = 1, 2.$$

Note that K is complex anti-symmetric: $K^{\mathrm{T}} = -K$, but not Hermitian nor anti-Hermitian. Indeed $K = (m_f^2 - \Delta)^{-1/2} U$ with U an unitary operator in $L^2(\mathbb{R}^2) \otimes \mathbb{C}^4$. Hence the correlation matrix of the field $\tilde{\psi}$ can also be written as

$$\omega(\tilde{\psi}(f)\tilde{\psi}(g)) = \langle (m_f^2 - \Delta)^{-1/4} f, U (m_f^2 - \Delta)^{-1/4} g \rangle_{\mathbb{C}}.$$

Therefore we can realize $\tilde{\psi}$ as

$$\tilde{\psi}(f) := \zeta((m_f^2 - \Delta)^{-1/4} f), \qquad f \in \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4,$$

where $\tilde{\zeta} \in \mathcal{G}(\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4)$ is the complex Grassmann Gaussian field with the bounded covariance

$$\omega(\tilde{\zeta}(f)\tilde{\zeta}(g)) = \langle f, Ug \rangle_{\mathbb{C}}.$$

As a consequence, in this specific realization, we have

$$\|\tilde{\psi}(f)\| = \|\tilde{\zeta}((m_f^2 - \Delta)^{-1/4} f)\| \lesssim \|(m_f^2 - \Delta)^{-1/4} f\|_{L^2(\mathbb{R}^2) \otimes \mathbb{C}^4}, \tag{50}$$

where the constant, in the inequality, depends only on m_f.

In order to describe the Yukawa$_2$ model we need also to introduce an Euclidean boson field φ of mass $m_b > 0$: it is the centered Gaussian field on $\mathbb{R}^2$ with covariance $(m_b^2 - \Delta)^{-1}$ where $m_b > 0$ is looked upon as a fixed parameter and Δ is the Laplacian on $\mathbb{R}^2$. We can realize φ as the canonical multiplication operator on the Hilbert space $\mathcal{H}_\varphi = L^2\left(\mathcal{C}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^2), \nu\right)$ where ν is the centered Gaussian measure on $\mathcal{C}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^2)$ (the local version of the $\mathcal{C}^{-\kappa}$ Besov space) with

covariance $(m_b^2-\Delta)^{-1}$ and $\kappa>0$ is any small positive quantity. We shall denote by ω_φ the state associated with ν and $\mathcal{C}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^2)$.

In addition, large scale and small scale cutoffs are introduced as follows. Fix a smooth positive functions $h\in C^\infty(\mathbb{R}^2)$ with compact support and let $\mathfrak{a}$ be a regular enough (see below for the precise regularity) continuous function with compact support such that $\mathfrak{a}(0)=1$. Take $\varepsilon>0$ and let $\mathfrak{a}_\varepsilon(x)=\mathfrak{a}(x/\varepsilon)\varepsilon^{-2}$. Then we define new Grassmann variables

$$\tilde{\psi}_\varepsilon(x):=\tilde{\psi}(\mathfrak{a}_\varepsilon(\cdot-x)),\qquad x\in\mathbb{R}^2.$$

Then $\tilde{\psi}_\varepsilon\in\mathcal{G}^s(\mathcal{S}(\mathbb{R}^2)\otimes\mathbb{C}^4)$ for some $s>0$ (in general depending on the regularity of $\mathfrak{a}$ see below).

Following the argument of Section 6 of [66] we define the Euclidean Yukawa$_2$ model as follows.

Definition 47 *The approximate fermionic Schwinger functions $(\rho^n_{\varepsilon,h})_n$ of the Yukawa$_2$ model on $\mathbb{R}^2$ with infrared (IR) cutoff h and ultraviolet (UV) fermion cutoff ε are given by the Euclidean averages*

$$\rho^n_{\varepsilon,h}(f_1,\dots,f_n)=\frac{\omega\otimes\omega_\varphi(\tilde{\psi}_\varepsilon(f_1)\cdots\tilde{\psi}_\varepsilon(f_n)e^{-V_{\varepsilon,h}(\varphi)})}{\omega\otimes\omega_\varphi(e^{-V_\varepsilon})}\tag{51}$$

with potential given by the unbounded (non-self-adjoint) operator on $\mathcal{H}\otimes\mathcal{H}_\varphi$, where $\mathcal{H}$ is the fermionic Hilbert space,

$$V_{\varepsilon,h}(\varphi)=\lambda\int_{\mathbb{R}^2}\mathrm{d}x h(x)(\bar{\psi}_\varepsilon(x)\cdot\psi_\varepsilon(x))\varphi(x),\tag{52}$$

and $\lambda\in\mathbb{R}$ is a coupling constant.

The potential $V_{\varepsilon,h}(\varphi)$ can be expressed in terms of the fields $\tilde{\psi}_\varepsilon$ as follows

$$V_{\varepsilon,h}(\varphi)=\frac{\lambda}{2}\int_{\mathbb{R}^2}\mathrm{d}x h(x)(\tilde{\psi}_\varepsilon(x)\cdot J\widetilde{\psi}_\varepsilon(x))\varphi(x),$$

where

$$J:=\begin{pmatrix}0&-\mathbb{1}\\ \mathbb{1}&0\end{pmatrix}.$$

Note that there is no difficulty in making sense of (51), due to the following observations (see also [66]). The map $x\in\mathbb{R}^2\mapsto\tilde{\psi}_\varepsilon(x)\in\mathcal{A}=\mathcal{L}(\mathcal{H})$ is $\mathcal{C}^s$ due to the presence of the regularization $\varepsilon>0$ and the regularity of the Euclidean field $\tilde{\psi}$ given in (50). The massive Gaussian free field φ has (local) regularity $\mathcal{C}^{-\kappa}_{\mathrm{loc}}$ for arbitrarily small κ, therefore, taking $s>\kappa$, by a standard Besov estimates for Banach valued functions, we have, for any *fixed* $\varphi\in\mathcal{C}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^2)$

$$\|V_{\varepsilon,h}(\varphi)\|_{\mathcal{A}}\lesssim\lambda\|\tilde{\psi}_\varepsilon\|^2_{\mathcal{C}^\kappa(\mathcal{A})}\|h\varphi\|_{B^{-\kappa}_{1,1}(\mathbb{R}^2)}\lesssim\lambda\|h\varphi\|_{\mathcal{C}^{-\kappa}(\mathbb{R}^2)}.$$

It is also not difficult to see that the map $V_{\varepsilon,h}:\mathcal{C}^{-\kappa}\to\mathcal{A}$ given by $V_{\varepsilon,h}:\varphi\mapsto V_{\varepsilon,h}(\varphi)$ is linear and continuous in φ, therefore measurable and that

$$\omega\otimes\omega_\varphi(\tilde{\psi}(f_1)\cdots\tilde{\psi}(f_n)e^{-V_{\varepsilon,h}(\varphi)})=\omega(\tilde{\psi}(f_1)\cdots\tilde{\psi}(f_n)\omega_\varphi(e^{-V_{\varepsilon,h}(\varphi)}))$$

e.g. by proving this first for a piecewise approximations of φ and then taking the limit. Moreover we have (by the properties of the Bochner integral)

$$\|\omega_\varphi(e^{-V_{\varepsilon,h}(\varphi)})\|_{\mathcal{A}}\leqslant\omega_\varphi(\exp(C|\lambda|\|h\varphi\|_{\mathcal{C}^{-\kappa}}))<\infty,$$

for all $\lambda \in \mathbb{R}$. We conclude that the averages in (51) are well defined since

$$|(\omega \otimes \omega_\varphi)(\tilde{\psi}(f_1) \cdots \tilde{\psi}(f_n) e^{-V_{\varepsilon,h}})| \leqslant \|\tilde{\psi}(f_1) \cdots \tilde{\psi}(f_n) \omega_\varphi(e^{-V_{\varepsilon,h}(\varphi)})\|_{\mathcal{A}} < \infty.$$

By expanding the quantity $\omega_\varphi(e^{-V_{\varepsilon,h}(\varphi)}) \in \mathcal{A}$ in power series and integrating away the Gaussian field we obtain

$$\omega_\varphi(e^{-V_{\varepsilon,h}(\varphi)}) = e^{\mathcal{V}_{\varepsilon,h}},$$

with a purely fermionic potential given by

$$\mathcal{V}_{\varepsilon,h}(\tilde{\psi}_\varepsilon) = \lambda^2 \int_{(\mathbb{R}^2)^2} (h\bar{\psi}_\varepsilon \cdot \psi_\varepsilon)(x) \mathcal{G}(x-y) (h\bar{\psi}_\varepsilon \cdot \psi_\varepsilon)(y) \mathrm{d}x \mathrm{d}y.$$

Here $\mathcal{G} = (m_b^2 - \Delta)^{-1}$ is the Green function of the operator $(m_b^2 - \Delta)$ and m_b is the boson mass. This gives us the purely fermionic representation of the Schwinger functions (51) given by:

$$\rho^n_{\varepsilon,h}(f_1, \ldots, f_n) = \frac{\omega(\tilde{\psi}(f_1) \cdots \tilde{\psi}(f_n) e^{\mathcal{V}_{\varepsilon,h}})}{\omega(e^{\mathcal{V}_{\varepsilon,h}})}. \tag{53}$$

Perturbation theory suggests that the boson field requires a mass renormalization a_ε which depends on the fermion cutoff. In theory one would need to modify the Gaussian field covariance to include a mass renormalization and let φ have covariance $(m_b^2 - \Delta + a_\varepsilon)^{-1}$. However in this paper we will not discuss the removal of the UV cutoff and keep $\varepsilon > 0$ fixed. Therefore, for simplicity, we ignore this mass renormalization.

5.2 Stochastic quantization

The Grassmann SPDEs associated to the measure $\tilde{\omega}(\cdot) = \omega(\cdot e^{\mathcal{V}_{\varepsilon,h}})$ is formally

$$\mathrm{d}\tilde{\Psi}^h_{\varepsilon,t} = ((m_f^2 - \Delta)\tilde{\Psi}^h_{\epsilon,t} + \lambda^2 F_{\varepsilon,h,Y}(\tilde{\Psi}^h_{\varepsilon,t}))\mathrm{d}t + \mathrm{d}\tilde{B}_{\varepsilon,t} \tag{54}$$

where $\tilde{\Psi}^h_{\varepsilon,t} = (\Psi^h_{\varepsilon,t}, \bar{\Psi}^h_{\varepsilon,t})$,

$$F_{\varepsilon,h,Y}(\tilde{\Psi}^h_{\varepsilon,t}) = C_f \begin{pmatrix} \mathfrak{a}_\varepsilon^{*2} * \left(h\bar{\Psi}^h_{\varepsilon,t} \int \mathcal{G}(y-\cdot) h(y) \Psi^h_{\varepsilon,t}(y) \bar{\Psi}^h_{\varepsilon,t}(y) \mathrm{d}y\right) \\ \mathfrak{a}_\varepsilon^{*2} * \left(h\Psi^h_{\varepsilon,t} \int \mathcal{G}(y-\cdot) h(y) \Psi^h_{\varepsilon,t}(y) \bar{\Psi}^h_{\varepsilon,t}(y) \mathrm{d}y\right) \end{pmatrix},$$

with

$$C_f = \begin{pmatrix} 0 & (\not{\nabla} + m_f) \\ -(\not{\nabla} + m_f)^{\mathrm{T}} & 0 \end{pmatrix},$$

and where $\mathfrak{a}_\varepsilon(x) = \varepsilon^{-2}\mathfrak{a}\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$, $\mathfrak{a} : \mathbb{R}^2 \to \mathbb{R}^2$ is the regular enough function with compact support (see below for the more precise requirements) considered before, $\mathfrak{a}_\varepsilon^{*2} = \mathfrak{a}_\varepsilon * \mathfrak{a}_\varepsilon$, h is a smooth function (in the beginning with compact support, later on we will consider $h \equiv 1$).

The Brownian motion $\tilde{B}$ is described as follows. Let $\tilde{\Xi}$ be the white noise on the pre-Hilbert space $L^2(\mathbb{R}_+) \otimes \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4 \subseteq L^2(\mathbb{R}_+) \otimes B^{-1/2}_{2,2}(\mathbb{R}^2) \otimes \mathbb{C}^4$ equipped with the scalar product

$$((f,g)) = ((m_f^2 - \Delta)^{-1/4} f, (m_f^2 - \Delta)^{-1/4} g)_{L^2(\mathbb{R}_+ \times \mathbb{R}^2) \otimes \mathbb{C}^4} \qquad f, g \in L^2(\mathbb{R}_+) \otimes \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4, \tag{55}$$

and having bounded covariance

$$\tilde{C}_f = (m_f^2 - \Delta)^{-1/2} C_f.$$

The (two-sided) Brownian motion $\tilde{B}$ on $\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4$ is defined by

$$\tilde{B}_t(v) = \Xi((\mathbb{1}_{t>0} - \mathbb{1}_{t<0}) \otimes v), \qquad t \in \mathbb{R}, v \in \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4.$$

We set $\tilde{B}_\varepsilon = \mathfrak{a}_\varepsilon * \tilde{B}$.

According to Definition 41 the stationary version of equation (54) is the solution to the following integral problem

$$\tilde{\Psi}^{h,s}_{\varepsilon,t}(v) = \lambda^2 \int_{-\infty}^{t} F_{\varepsilon,h,Y}(\tilde{\Psi}^{h,s}_{\varepsilon,\tau})(e^{A(t-\tau)}v)\mathrm{d}\tau + \tilde{B}^A_t(v) - \tilde{B}^A_0(e^{At}v), \qquad t \in \mathbb{R}, v \in \mathbb{C}^4, \tag{56}$$

where we take $A = \Delta - m_f^2$ and

$$\tilde{B}^A_t(v) = \Xi(\mathbb{1}_{[0,t]} \otimes e^{At}v), \qquad t \in \mathbb{R}, v \in \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4.$$

From now on we will make the following assumption. We fix $\delta > 0$ and $\gamma > 0$, let $\alpha = \gamma + 1/2 + 2\delta$ and take $\mathfrak{a} \in B^\alpha_{1,\infty}(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$. Again we recall that $\varepsilon > 0$ will be fixed in all our study.

The main theorem of this section concerns the construction of a stochastic quantization for (53) and the removal of the IR cutoff h for λ small.

Theorem 48 *If $|\lambda|$ is small enough and h is a smooth function with compact support, then, for any $k \geqslant 1$, $f_1, \dots, f_k \in C^\infty_c(\mathbb{R}^2) \otimes \mathbb{C}^4$ and $t \in \mathbb{R}$, we have*

$$\frac{\omega(\tilde{\psi}(f_1) \cdots \tilde{\psi}(f_k) e^{\mathcal{V}_{\varepsilon,h}})}{\omega(e^{\mathcal{V}_{\varepsilon,h}})} = \omega(\tilde{\Psi}^{h,s}_{\varepsilon,t}(f_1) \cdots \tilde{\Psi}^{h,s}_{\varepsilon,t}(f_k)). \tag{57}$$

We are also able to obtain the following result on the removal of the IR cutoff.

Theorem 49 *If $|\lambda|$ is small enough and if we denote by $\tilde{\Psi}^s_{\varepsilon,t}$ the solution to equation (56) with $h \equiv 1$, we have that for any $f_1, \dots, f_k \in C^\infty_c(\mathbb{R}^2) \otimes \mathbb{C}^4$ and for any $t \in \mathbb{R}$*

$$\lim_{h_n \to 1} \frac{\omega\left(\tilde{\psi}(f_1) \cdots \tilde{\psi}(f_k) e^{\mathcal{V}_{\varepsilon,h_n}}\right)}{\omega\left(e^{\mathcal{V}_{\varepsilon,h_n}}\right)} = \omega(\tilde{\Psi}^s_{\varepsilon,t}(f_1) \cdots \tilde{\Psi}^s_{\varepsilon,t}(f_k)). \tag{58}$$

where $(h_n)_{n \geqslant 1}$ is any sequence of smooth functions with compact support converging to 1 uniformly in $\mathcal{C}^{\gamma+\delta}$ (see below the definition of δ) on compact subsets of $\mathbb{R}^2$.

The rest of the section focus on the proof of these two theorems. For the moment we record here two technical lemmas.

Lemma 50 *We have $\tilde{B}^A_\varepsilon \in \mathcal{C}^\delta(\mathbb{R}, \mathcal{C}^\gamma(\mathbb{R}^2, \mathcal{A}) \otimes \mathbb{C}^4)$ and*

$$\sup_{t \in \mathbb{R}} \|\tilde{B}^A_{\varepsilon,t}\|_{\mathcal{C}^{\gamma+2\delta}(\mathbb{R}^2,\mathcal{A}) \otimes \mathbb{C}^4} < +\infty.$$

Proof It is sufficient to prove the lemma for $\varepsilon = 0$ and $\gamma = -\frac{1}{2} - 2\delta$, since for $\varepsilon > 0$ we have $\mathfrak{a}_\varepsilon * : \mathcal{C}^s \to \mathcal{C}^{s+\alpha}$ by Theorem 63 in Appendix A. By definition of Grassmann Gaussian random variables, if $K_i \in \mathcal{S}(\mathbb{R}^2)$ is the function corresponding to the Littlewood–Paley block Δ_i (see Appendix A for the definition), we have

$$\sup_{x \in \mathbb{R}^2} \|\Delta_i \tilde{B}_{0,t} - \Delta_i \tilde{B}_{0,s}\|_{\mathcal{A} \otimes \mathbb{C}^4} \lesssim \left\| \int_{-\infty}^{t} e^{(\Delta - m_f^2)(t-\tau)} (-\Delta + m^2)^{1/4}(K_i)\mathrm{d}\tau + \right.$$

$$- \int_{-\infty}^{s} e^{(\Delta - m_f^2)(s-\tau)}(m_f^2 - \Delta)^{1/4}(K_i)\mathrm{d}\tau \Big\|_{L^2(\mathbb{R}^2)} \lesssim$$

$$\lesssim \left\| \int_s^t e^{(\Delta + m_f^2)(t-\tau)}(m_f^2 - \Delta)^{1/4}(K_i)\mathrm{d}\tau \right\| + \|(e^{(\Delta - m_f^2)(t-s)} - 1)(m_f^2 - \Delta)^{1/4}(K_i)\|_{L^2} \lesssim$$

$$\lesssim \left(\int_{\mathbb{R}^2} \left(\phi_i(p) \left(\int_0^{t-s} |p|^2 e^{-k(|p|^2 + m_f^2)} \mathrm{d}k \right) \right)^2 |p| \mathrm{d}p \right) \lesssim$$

$$\lesssim \left(\int_{2^{i-1} \lesssim |p| \lesssim 2^i} \left(\int_0^{t-s} \frac{\mathrm{d}k}{k^{(1-\delta)}} \right)^2 |p|^{1+4\delta} \mathrm{d}p \right)^{1/2} \lesssim 2^{i\left(\frac{1}{2} + 2\delta\right)} |t-s|^{\delta}.$$

From this the thesis follows from the definition of the norm of the Besov space $\mathcal{C}^\gamma(\mathbb{R}^2, \mathcal{A})$. □

Lemma 51 *For any $s > 0$, the function $F_{\varepsilon,h,Y}$ is well defined on $\mathcal{C}^s(\mathbb{R}^d, \mathcal{A})$ into itself, it satisfies Definition 37 and Definition 42. Furthermore the functions f_F, g_F in equations (44) and (45) (of Definition 42) can be chosen to be independent of h when $h \in \mathcal{C}^s$ has the norm bounded by some constant $C > 0$.*

Proof Since for $s > 0$ we have $\mathcal{C}^s(\mathbb{R}^d, \mathcal{A}) \subset C_b^0(\mathbb{R}^d, \mathcal{A})$ we have that the function

$$x \mapsto \int \mathcal{G}(y - x) h(y) \Psi_{\varepsilon,t}^h(y) \bar{\Psi}_{\varepsilon,t}^h(y) \mathrm{d}y$$

is well defined and continuous. Since $\mathcal{C}^s(\mathbb{R}^d, \mathcal{A})$ is an algebra, this proves that $F_{\varepsilon,h,Y}$ is well defined, continuous and locally bounded. This proves that $F_{\varepsilon,h,Y}$ satisfies Definition 42. The bounds required on $F_{\varepsilon,h,Y}$ are uniform in h indeed

$$\|F_{\varepsilon,h,Y}\|_{\mathcal{C}^s} \lesssim \|\tilde{\Psi}^h\|_{\mathcal{C}^s} \left\| \int \mathcal{G}(x - y) h(y) \Psi^h(y) \bar{\Psi}^h(y) \mathrm{d}y \right\|_{\mathcal{C}^s} \lesssim \|\tilde{\Psi}^h\|_{\mathcal{C}^s} \|\Psi^h \bar{\Psi}^h\|_{\mathcal{C}^s} \lesssim \|\tilde{\Psi}^h\|_{\mathcal{C}^s}^3.$$

Since $F_{\varepsilon,h,Y}$ is of third degree, we need only to prove that it can be approximated with finite dimensional operators. This can be done using the fact that when $s > 0$, $\tilde{\Psi}_{\varepsilon,t}$ is Hölder continuous in space and that $\mathcal{G}$ decreases exponentially at infinity. This means that if $\mathcal{P}_n$ is a sequence of increasing (finite) partition of the plane we can approximate the integral by the sum

$$S_n = \sum_{D_i \in \mathcal{P}_n} h(x - x_i) \Psi^h(x - x_i) \bar{\Psi}^h(x - x_i) \int_{D_i} \mathcal{G}(y) \mathrm{d}y,$$

where $x_i \in D_i$ is any point in D_i. If we choose the sequence of partitions $\mathcal{P}_n$ in such a way that $\lim_{n \to +\infty} \left\{ \sup_i \left(\int_{D_i} \mathcal{G}(y) \mathrm{d}y \right) \right\} = 0$, we have that

$$S_n \to \int \mathcal{G}(y - x) h(y) \Psi_{\varepsilon,t}^h(y) \bar{\Psi}_{\varepsilon,t}^h(y) \mathrm{d}y$$

in $\mathcal{C}^{s'}(\mathbb{R}^d, \mathcal{A})$, where $s' < s$. □

Using the previous two lemmas we can prove the following theorem.

Theorem 52 *Equation* (54) *admits a unique (local in time) solution, starting from* 0 *in* $\mathcal{G}^\gamma(V)$ *(*γ *being defined before Theorem 48 and Theorem 49). Furthermore if* $|\lambda|$ *is small enough, there exists a unique solution to the stationary equation* (56) *(associated with* (54)*) which is uniformly bounded in* $\mathcal{C}^\gamma(\mathbb{R}^d, \mathcal{A}) \otimes \mathbb{C}^4$ *by some constant* $K > 0$ *(that can be chosen independent of* h *when* $\|h\|_{\mathcal{C}^\gamma} < C$ *for some* $C > 0$*).*

Proof The proof is based on Theorem 44 and Theorem 45 whose hypotheses are satisfied in this setting because of Lemma 50 and Lemma 51. □

5.3 Finite dimensional approximations in finite volume

The first step in order to prove Theorem 48 and Theorem 49 is to introduce finite dimensional approximations of the Yukawa model. Consider the torus $\mathbb{T}^2_R = \mathbb{R}^2 \backslash (2\pi L\mathbb{Z})^2$ for $R > 0$ and the periodic version $\tilde{\psi}_R$ of the Fermion field $\tilde{\psi}$ defined by

$$\tilde{\psi}_R(f) := \sum_{n\in\mathbb{Z}^2} \tilde{\psi}(f(\cdot - 2\pi Rn)),$$

for all $f \in \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4$. Then letting $\tilde{\psi}_{R,\varepsilon}(x) := \tilde{\psi}_R(\mathfrak{a}_\varepsilon(\cdot - x))$ we have that the function $x \in \mathbb{T}^2_R \mapsto \tilde{\psi}_{R,\varepsilon}(x)$ is well defined and $\mathcal{C}^{\gamma+2\delta}(\mathbb{T}^2, \mathcal{A})$. Moreover $\tilde{\psi}_{R,\varepsilon}(x)$ is uniformly bounded in $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and

$$\|\tilde{\psi}_{R,\varepsilon}(x) - \tilde{\psi}_\varepsilon(x)\|_{\mathcal{C}^{\gamma+2\delta}(K,\mathcal{A}\otimes\mathbb{C}^4)} \to 0,$$

uniformly on compacts as $R \to \infty$. As a consequence we have

$$\frac{\omega(\tilde{\psi}(f_1)\cdots\tilde{\psi}(f_n)e^{\mathcal{V}_{\varepsilon,h}})}{\omega(e^{\mathcal{V}_{\varepsilon,h}})} = \lim_{R\to\infty} \frac{\omega(\tilde{\psi}_R(f_1)\cdots\tilde{\psi}_R(f_n)e^{\mathcal{V}_{R,\varepsilon,h}})}{\omega(e^{\mathcal{V}_{R,\varepsilon,h}})} \tag{59}$$

with

$$\mathcal{V}_{R,\varepsilon,h} := \lambda^2 \int_{\mathbb{R}^4} \mathrm{d}x (h\bar{\psi}_{R,\varepsilon} \cdot \psi_{R,\varepsilon})(x)\mathcal{G}(x-y)(h\bar{\psi}_{R,\varepsilon} \cdot \psi_{R,\varepsilon})(y)\mathrm{d}x\mathrm{d}y,$$

for all compactly supported $f_1, \ldots, f_n, h \in \mathcal{S}(\mathbb{R}^2)$.

Let now $(e_k)_{k\in\mathbb{Z}^2}$ be the orthonormal Fourier basis of (the complex) $L^2(\mathbb{T}^2_R)$ and $P_N : L^2(\mathbb{T}^2_R) \to L^2(\mathbb{T}^2_R)$ the orthogonal projection to the closed subspace generated by $\{e_k : k \in \mathbb{Z}^2, |k| \leqslant N\}$. Let $\tilde{\psi}_{N,R,\varepsilon} = P_N\tilde{\psi}_{R,\varepsilon}$ and observe again that $\|\tilde{\psi}_{N,R,\varepsilon}(x) - \tilde{\psi}_{R,\varepsilon}(x)\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \to 0$ as $N \to \infty$, for any $\alpha > 0$. Defining

$$\mathcal{V}_{N,R,\varepsilon,h} := \lambda^2 \int_{\mathbb{R}^4} (h\bar{\psi}_{N,R,\varepsilon} \cdot \psi_{N,R,\varepsilon})(x)\mathcal{G}_\varepsilon(x-y)(h\bar{\psi}_{N,R,\varepsilon} \cdot \psi_{N,R,\varepsilon})(y)\mathrm{d}x\mathrm{d}y$$

and we easily conclude that

$$\frac{\omega(\tilde{\psi}(f_1)\cdots\tilde{\psi}(f_n)e^{\mathcal{V}_{\varepsilon,h}})}{\omega(e^{\mathcal{V}_{\varepsilon,h}})} = \lim_{R\to\infty}\lim_{N\to\infty} \frac{\omega(\tilde{\psi}_{N,R}(f_1)\cdots\tilde{\psi}_{N,R}(f_n)e^{\mathcal{V}_{N,R,\varepsilon,h}})}{\omega(e^{\mathcal{V}_{N,R,\varepsilon,h}})}. \tag{60}$$

Now consider the finite set $\Lambda_N = \{k \in \mathbb{Z}^2 : |k| \leqslant N\}$ and let

$$\tilde{\theta}(k) := \int_{\mathbb{T}^2_R} \overline{e_k(x)}\tilde{\psi}_{N,R}(x)\mathrm{d}x, \qquad k \in \Lambda_N.$$

We can show easily that $(\tilde{\theta}(k))_{k\in\Lambda_N}$ is a Grassmann Gaussian vector with correlation given by

$$\omega(\tilde{\theta}_\alpha(k)\tilde{\theta}_\beta(\ell)) = \delta_{k+\ell=0}\begin{pmatrix} 0 & (i\not{k}+m_f)^{-1} \\ -(i\not{k}+m_f)^{-1\mathrm{T}} & 0 \end{pmatrix}_{\alpha,\beta} \tag{61}$$

where $\alpha,\beta = 1,\dots,4$ denote the canonical coordinates in $\mathbb{C}^4$ and $k,\ell\in\Lambda_N$. Moreover

$$\tilde{\psi}_{N,R}(x) = \sum_{k\in\Lambda_N} e_k(x)\tilde{\theta}(k), \qquad x\in\mathbb{T}^2_R$$

and $\tilde{\psi}_{N,R,\varepsilon} = \mathfrak{a}_\varepsilon *_{\mathbb{T}^2_R} \tilde{\psi}_{N,R}$ where the convolution is done on $\mathbb{T}^2_R$ as indicated.

At this point we have reduced via two subsequent approximations the number of degrees of freedom to a finite amount, described by the Grassmann random variable $\tilde{\theta}\in\mathcal{G}(\Lambda_N\otimes\mathbb{C}^4)$.

We introduce the following equation

$$\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t}(v) = \lambda^2\int_{-\infty}^t P_N(F_{\varepsilon,h_R,Y}(P_N(\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau})))(e^{(\Delta-m_f^2)(t-\tau)}v)\mathrm{d}\tau + \tilde{B}^A_{\varepsilon,R,t}(v) \tag{62}$$

where $(\tilde{B}^A_{\varepsilon,R,t})_{t\in\mathbb{R}}$ is the Ornstein–Uhlenbeck motion on $\mathbb{T}^2_R$ obtained from $(\tilde{B}^A_{\varepsilon,t})_{t\in\mathbb{R}}$ via the restriction from $\mathbb{R}^2$ to $\mathbb{T}^2$ of the periodicization

$$\tilde{B}^A_{\varepsilon,R,t}(f) := \sum_{n\in\mathbb{Z}^2}\tilde{B}^A_{\varepsilon,t}(f(\cdot-2\pi Rn)), \qquad f\in\mathcal{S}(\mathbb{R}^2),$$

and

$$h_R(z) := \sum_{n\in\mathbb{Z}^2} h(z-2\pi Rn).$$

Obviously we have that

$$(I-P_N)\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t} = (I-P_N)\tilde{B}^A_{\varepsilon,R,t}.$$

This implies that $P_N(\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t})$ solves a finite dimensional Grassmann SDE. Furthermore the drifts $P_N(F_{\varepsilon,h_R,Y}(P_N(\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t})))$ can be seen as the differentials of the function

$$U(P_N(\tilde{\Psi})) = \int_{\mathbb{T}^4_L} h_L(x)h_L(y)(P_N(\Psi)P_N(\bar{\Psi}))(x)\mathcal{G}(x-y)(P_N(\Psi)P_N(\bar{\Psi}))(y)\mathrm{d}x\mathrm{d}y.$$

times the matrix $P_N\mathfrak{a}^{*2}_\varepsilon * C_f$ (that is the covariance of the finite dimensional Brownian motion $P_N(\tilde{B}_{\varepsilon,R,t})$). This implies the following key lemma.

Lemma 53 *For any $k\geqslant 1$ and $f_1,\dots,f_k$ some linear combination of $e_1,\dots,e_N$ then, for $|\lambda|$ small enough (that can be chosen in a way independent of h) we have for any $t\in\mathbb{R}$*

$$\frac{\omega(\tilde{\psi}_{N,R}(f_1)\cdots\tilde{\psi}_{N,R}(f_k)e^{\mathcal{V}_{N,R,\varepsilon,h}})}{\omega(e^{\mathcal{V}_{N,R,\varepsilon,h}})} = \omega(\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t}(f_1)\cdots\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t}(f_k)).$$

Proof The proof is based on the fact that for any linear combination f_i of $e_1,\dots,e_N$ we have $\tilde{\Psi}^{h_L,s}_{\varepsilon,N,R,+\infty}(f_i) = P_N(\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,+\infty})(f_i)$. Thus the thesis follows from Theorem 33 and Remark 34 by noting that $\mathcal{V}_{N,R,\varepsilon,h}(\tilde{\psi}) = \lambda^2 U(\tilde{\psi})$. □

5.4 The infinite dimensional equation in finite volume

Consider the equation

$$\tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}(v) = \lambda^2 \int_{-\infty}^{t} F_{\varepsilon,h_R,Y}(\tilde{\Psi}^{h_R,s}_{\varepsilon,R,\tau})(e^{(\Delta - m_f^2)}v)\mathrm{d}t + \tilde{B}^A_{\varepsilon,R,t}. \tag{63}$$

This is a modification of (62) where the finite dimensional projections P_N are removed from the non-linearity $F_{\varepsilon,h_R,Y}$.

Remark 54 Lemma 51 holds for the drift of equation (63), and it is easy to generalize Lemma 50 to the OU motion $(\tilde{B}^A_{\varepsilon,R,t})_{t\in\mathbb{R}}$. This implies that it is possible to generalize Theorem 52 to equation (63) proving existence and uniqueness of its global solution (when $|\lambda|$ is small enough). Finally, it is important to note that the bound K on the solution $\tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}$ to equation (63), and the bounds on the constant $|\lambda|$ can be chosen independent of h when $\|h\|_{\mathcal{C}^\gamma} < C$ for some constant $C > 0$.

Equation (63) is infinite dimensional and in order to generalize Lemma 53 we cannot apply directly the finite dimensional results contained in Theorem 33 and Remark 34. For this reason we need to prove that the solution to equation (62) converges, as $N \to +\infty$, to the solution to equation (63).

Theorem 55 *For any $0 < q < \gamma$ we have that $\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t} \to \tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}$, in $\mathcal{G}^q(C^\infty(\mathbb{T}^d)\otimes\mathbb{C}^4)$ when $N \to +\infty$ and uniformly in t.*

Proof We have that the difference $\Psi^{h_R,s}_{\varepsilon,N,R,t} - \Psi^{h_R,s}_{\varepsilon,R,t}$ satisfies the following inequality

$$\begin{aligned}
\frac{\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t} - \tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}\|_{\mathcal{C}^q}}{\lambda^2} \lesssim\ & \int_{-\infty}^{t} e^{-\lambda_A(t-\tau)}\|(I-P_N)F_{\varepsilon,h_R,Y}(P_N(\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau}))\|_{\mathcal{C}^q}\mathrm{d}\tau + \\
& + \int_{-\infty}^{t} e^{-\lambda_A(t-\tau)} g_F(\max(\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau}\|_{\mathcal{C}^q}, \|P_N\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau}\|_{\mathcal{C}^q})) \times \\
& \times(\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau} - P_N\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau}\|_{\mathcal{C}^q})\mathrm{d}\tau + \\
& + \int_{-\infty}^{t} e^{-\lambda_A(t-\tau)} g_F(\max(\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau}\|_{\mathcal{C}^q}, \|\tilde{\Psi}^{h_R,s}_{\varepsilon,R,\tau}\|_{\mathcal{C}^q})) \times \\
& \times\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,\tau} - \tilde{\Psi}^{h_R,s}_{\varepsilon,R,\tau}\|_{\mathcal{C}^q}\mathrm{d}\tau.
\end{aligned}$$

Using Gronwall inequality we obtain that

$$\begin{aligned}
\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t} - \tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}\|_{\mathcal{C}^q} \lesssim (\|I-P_N\|_{\mathcal{L}(\mathcal{C}^\gamma,\mathcal{C}^q)})\times \\
\times\left(1 + \int_{-\infty}^{t}\exp(-(\lambda_A - \lambda^2 g_F(K))(t-\tau))\mathrm{d}\tau\right).
\end{aligned}$$

Since λ is small we have that $\lambda_A - \lambda^2 g_F(K) > 0$ (this is exactly the request (48) for $|\lambda| < \lambda_0(K,L,F)$). This means that

$$\|\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t} - \tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}\|_{\mathcal{C}^q} \lesssim (\|I-P_N\|_{\mathcal{L}(\mathcal{C}^\gamma,\mathcal{C}^q)}).$$

On the other hand, since, $\gamma > q$ by assumption, we have $\|I-P_N\|_{\mathcal{L}(\mathcal{C}^\gamma,\mathcal{C}^q)} \to 0$ as $N \to +\infty$. The thesis is proven noting that the convergence $\tilde{\Psi}^{h_R,s}_{\varepsilon,N,R,t} \to \tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}$ is uniform in t. □

An easy corollary of Theorem 55, Lemma 53, and the convergence (60) is the following.

Corollary 56 *For $|\lambda|$ small enough (that can be chosen independent of h when $\|h\|_{\mathcal{C}^\gamma} < C$ where $C > 0$), for any $k \geqslant 1$, $f_1, \ldots, f_k \in C^\infty(\mathbb{T}^d_L) \otimes \mathbb{C}^4$ and any $t \in \mathbb{R}$ we have*

$$\frac{\omega(\tilde{\psi}_R(f_1) \cdots \tilde{\psi}_R(f_k) e^{\mathcal{V}_{R,\varepsilon,h}})}{\omega(e^{\mathcal{V}_{R,\varepsilon,h}})} = \omega(\tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}(f_1) \cdots \tilde{\Psi}^{h_R,s}_{\varepsilon,R,t}(f_k)).$$

5.5 The infinite volume limit

In order to tackle the infinite volume limit and prove Theorem 48 and Theorem 49 we have to introduce weighted Besov spaces (see the Appendix A and [79] for the details). Let $\ell \in \mathbb{R}$ and consider $\rho_\ell : \mathbb{R}^d \to \mathbb{R}_+$ defined by $\rho_\ell(x) = (1 + |x|^2)^{(-\ell/2)}$. Consider the following norm on $\mathcal{S}'(\mathbb{R}^d) \otimes \mathcal{A}$ (where as before $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and $\mathcal{H}$ is the fermionic Hilbert space)

$$\|u\|_{\mathcal{C}^s_\ell(\mathbb{R}^d,\mathcal{A})} = \sup_{j \geqslant -1} \sup_{x \in \mathbb{R}^d} 2^{js} \|\Delta_j u(x) \rho_\ell(x)\|_{\mathcal{A}}.$$

Define the space $\mathcal{C}^s_\ell(\mathbb{R}^d, \mathcal{A})$ as the subspace of $\mathcal{S}'(\mathbb{R}^d) \otimes \mathcal{A}$ where $\| \cdot \|_{\mathcal{C}^s_\ell(\mathbb{R}^d,\mathcal{A})}$ is finite. We define also $\mathcal{G}^s_\ell(V) = \mathcal{C}^s_\ell(\mathbb{R}^d, \mathcal{A}) \cap \mathcal{G}^{-\infty}(V)$ (recall that $V = \mathcal{S}(\mathbb{R}^d) \otimes \mathbb{R}^n$ or $\mathcal{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ for some $n \geqslant 1$) and the distance $d_{\mathcal{G}^s_\ell(V)}(X - Y) = \|X - Y\|_{\mathcal{C}^s_\ell(\mathbb{R}^d,\mathcal{A})}$.

Remark 57 It is important to note that, by Theorem 62,

$$\sup_{x \in \mathbb{R}^d} \|\Delta_j u(x) \rho_\ell(x)\|_{\mathcal{A}} \sim \sup_{x \in \mathbb{R}^d} \|\Delta_j (\rho_\ell u)(x)\|_{\mathcal{A}},$$

where $\sim$ means that both quantities considered can be bounded from above and from below by some positive constant times the other quantity.

Lemma 58 *Let γ, δ be as stated just before Theorem 48, for any $s < \gamma + 2\delta$ and any $\ell > 0$ we have that*

$$\sup_{t \in \mathbb{R}} d_{\mathcal{G}^s_\ell(V)}(\tilde{B}^A_{\varepsilon,t}, \tilde{B}^A_{\varepsilon,R,t}) \to 0$$

when $R \to +\infty$.

Proof Let v be a function on $\mathbb{R}^2$ decreasing faster then any polynomial at infinity. Then we have

$$\tilde{B}^A_{\varepsilon,t}(v) - \tilde{B}^A_{\varepsilon,L,t}(v) = \tilde{\Xi}_\varepsilon(\mathbb{I}_{|x|_\infty > R}(\cdot) e^{A(t-\cdot)} v) - \tilde{\Xi}_{\varepsilon,L}(\mathbb{I}_{|x|_\infty > R}(\cdot) e^{A(t-\cdot)} v).$$

This means that $\tilde{B}^A_{\varepsilon,t}$ and $\tilde{B}^A_{\varepsilon,R,t}$ coincide on the set $\{|x|_\infty < R\} \subset \mathbb{R}^2$. Using Remark 57, this implies that

$$\begin{aligned} \|\Delta_i(\tilde{B}^A_{\varepsilon,t} - \tilde{B}^A_{\varepsilon,R,t}) \rho_\ell\|_{\mathcal{A} \otimes \mathbb{C}^4} &\lesssim \|\Delta_i(\rho_\ell(\tilde{B}^A_{\varepsilon,t} - \tilde{B}^A_{\varepsilon,R,t}))\|_{\mathcal{A} \otimes \mathbb{C}^4} \lesssim \\ &\lesssim R^{-\ell} \|\Delta_i(\tilde{B}^A_{\varepsilon,t} - \tilde{B}^A_{\varepsilon,R,t})\|_{\mathcal{A} \otimes \mathbb{C}^4} \lesssim R^{-\ell} 2^{j(-2\delta-\gamma)}, \end{aligned}$$

which implies that $\sup_{t \in \mathbb{R}} \|\tilde{B}^A_{\varepsilon,t} - \tilde{B}^A_{\varepsilon,R,t}\|_{\mathcal{C}^s_\ell(\mathbb{R}^2, \mathcal{A} \otimes \mathbb{C}^4)} \lesssim R^{-\ell} \to 0$, when $\ell > 0$ as $R \to +\infty$. □

We now prove the analogue of Theorem 55 in this case.

Theorem 59 *Let $0 < q < \gamma$, then $\tilde{\Psi}^{h_R,s}_{\varepsilon,R,t} \to \tilde{\Psi}^{h,s}_{\varepsilon,t}$, in $\mathcal{G}^q_\ell(\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4)$ uniformly in $t \in \mathbb{R}$, when $R \to +\infty$.*

Proof The proof is based on the fact that for any $\tilde{\Psi}_1, \tilde{\Psi}_2 \in \mathcal{G}_0^\gamma(\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4)$ (it is important to note that in these spaces we take the constant weight) we have, by Theorem 64 in Appendix A,

$$\|F_{\varepsilon,h_R,Y}(\tilde{\Psi}_1) - F_{\varepsilon,h,Y}(\tilde{\Psi}_2)\|_{\mathcal{C}_\ell^q(\mathbb{R}^2,\mathcal{A}\otimes\mathbb{C}^4)} \lesssim g_F(\max(\|h_R\tilde{\Psi}_1\|_{\mathcal{C}^q}, \|h\tilde{\Psi}_1\|_{\mathcal{C}^q}, \|h\tilde{\Psi}_2\|_{\mathcal{C}^q}))\times$$

$$\times (\|h\|_{\mathcal{C}^q}\|\tilde{\Psi}_1 - \tilde{\Psi}_2\|_{\mathcal{C}_\ell^q(\mathbb{R}^2,\mathcal{A}\otimes\mathbb{C}^4)} + \|\Psi_1\|_{\mathcal{C}_0^q(\mathbb{R}^2,\mathcal{A}\otimes\mathbb{C}^4)}\|h_R - h\|_{\mathcal{C}_\ell^q}), \tag{64}$$

where g_F is the function in Definition 42 and being h the IR cut-off in Definition 47. Using a reasoning similar to the one exploited in the proof of Theorem 55 we get

$$\|\tilde{\Psi}_{\varepsilon,R,t}^{h_R,s} - \tilde{\Psi}_{\varepsilon,t}^{h,s}\|_{\mathcal{C}_\ell^q(\mathbb{R}^2,\mathcal{A}\otimes\mathbb{C}^4)} \lesssim (\|h_R - h\|_{\mathcal{C}_\ell^q} + \sup_{\tau\in\mathbb{R}}\|\tilde{B}_{\varepsilon,\tau}^A - \tilde{B}_{\varepsilon,R,\tau}^A\|_{\mathcal{C}_\ell^q(\mathbb{R}^2,\mathcal{A}\otimes\mathbb{C}^4)})\times$$

$$\times \left(1 + \int_{-\infty}^t \exp(-(\lambda_A - \lambda^2 g_F(K))(t-\tau))\mathrm{d}\tau\right).$$

By $|\lambda|$ small enough, this converges to 0 as $R \to +\infty$ uniformly in t, since, by Lemma 58,

$$\lim_{R\to+\infty}\sup_{\tau\in\mathbb{R}}\|\tilde{B}_{\varepsilon,\tau}^A - \tilde{B}_{\varepsilon,R,\tau}^A\|_{\mathcal{C}_\ell^q(\mathbb{R}^2,\mathcal{A}\otimes\mathbb{C}^4)} \to 0$$

and $\|h_R - h\|_{\mathcal{C}_\ell^q} \to 0$, as $R \to +\infty$, by the definition of h_R. □

Hereafter we denote by $\tilde{\Psi}_{\varepsilon,t}^s$ the solution to equation (54) with $h \equiv 1$ (so with the spatial cut-off h removed).

Theorem 60 *Using the previous notations and hypotheses, let $(h_n : \mathbb{R}^2 \to \mathbb{R})_{n\geqslant 1}$ be a sequence of smooth functions with compact supports converging to 1 uniformly on compacts in the norm of $\mathcal{C}^\gamma$, $\gamma > 0$, and with uniformly bound in $\mathcal{C}^\gamma$. For any $0 < q < \gamma$ we have*

$$\tilde{\Psi}_{\varepsilon,t}^{h_n,s} \to \tilde{\Psi}_{\varepsilon,t}^s$$

in $\mathcal{G}_\ell^q(\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^4)$, for any $\ell > 0$ and uniformly in $t \in \mathbb{R}$ when $n \to +\infty$.

Proof The proof is analogous to the one of Theorem 59 where in inequality (64) we have $\|h_n - 1\|_{\mathcal{C}_\ell^q}$ (instead of $\|h_R - h\|_{\mathcal{C}_\ell^q}$) which converges to 0, as $n \to +\infty$, by hypothesis when $\ell > 0$. □

Proof of Theorem 48 and Theorem 49 The proof is analogous to the one of Corollary 56 where Theorem 55 is replaced by Theorem 59 and Theorem 60 respectively. □

A Besov spaces of Banach algebras

In this section we want to recall some results about Besov spaces of functions (or distributions) from $\mathbb{R}^d$ taking values in a Banach algebra $\mathcal{A}$. All the results of this section can be found in [6, 7] for the theory of Besov spaces taking values in a Banach space and [79] for weighted Besov spaces. We present here only the $\mathbb{R}^d$ case. The definition of Besov spaces on $\mathbb{T}^d$ is similar, and all the results announced here also hold on $\mathbb{T}^d$.

We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of smooth functions $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|f\|_{\ell,\alpha} := \|(1 + |x|)^\ell |D^\alpha f|\|_{L^\infty(\mathbb{R}^d)} < +\infty$$

where $\ell \in \mathbb{R}_+$ and $\alpha \in \mathbb{N}^d$. We denote by $\mathcal{S}'(\mathbb{R}^d)$ the strong dual of $\mathcal{S}(\mathbb{R}^d)$ (equipped with the topology of the semi-norms $\|\cdot\|_{\ell,\alpha}$) and by $\mathcal{O}_M(\mathbb{R}^d)$ the set of smooth functions $f \in \mathcal{O}_M(\mathbb{R}^d)$ such that for any $\alpha \in \mathbb{N}^d$ there exists $\ell_\alpha \in \mathbb{R}_+$ for which

$$\|(1+|x|)^{-\ell_\alpha}|D^\alpha f|\|_{L^\infty(\mathbb{R}^d)} < +\infty.$$

We use the notations

$$\mathcal{S}(\mathbb{R}^d,\mathcal{A}) = \mathcal{S}(\mathbb{R}^d)\otimes\mathcal{A}, \quad \mathcal{S}'(\mathbb{R}^d,\mathcal{A}) = \mathcal{S}'(\mathbb{R}^d)\otimes\mathcal{A}, \quad \mathcal{O}_M(\mathbb{R}^d,\mathcal{A}) = \mathcal{O}_M(\mathbb{R}^d)\otimes\mathcal{A},$$

where $\mathcal{A}$ is a Banach algebra. It is important to note that

$$\mathcal{S}(\mathbb{R}^d,\mathcal{A}) \overset{d}{\hookrightarrow} \mathcal{O}_M(\mathbb{R}^d,\mathcal{A}) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d,\mathcal{A})$$

where the arrows mean that each space is continuously embedded and dense in the following one.

Theorem 61 *It is possible to define uniquely* $\langle\cdot,\cdot\rangle : \mathcal{S}(\mathbb{R}^d,\mathcal{A})\times\mathcal{S}'(\mathbb{R}^d,\mathcal{A}) \to \mathcal{A}$, $\cdot : \mathcal{S}(\mathbb{R}^d,\mathcal{A})\times\mathcal{S}'(\mathbb{R}^d,\mathcal{A}) \to \mathcal{S}'(\mathbb{R}^d,\mathcal{A})$, $\cdot : \mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d,\mathcal{A}) \to \mathcal{S}'(\mathbb{R}^d,\mathcal{A})$, $* : \mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d,\mathcal{A}) \to \mathcal{O}_M(\mathbb{R}^d,\mathcal{A})$ *and* $D^\alpha : \mathcal{S}'(\mathbb{R}^d,\mathcal{A}) \to \mathcal{S}'(\mathbb{R}^d,\mathcal{A})$ *(where* $\alpha \in \mathbb{N}^d$*) which extend in a epicontinuous way the following operations: any* $f \in \mathcal{S}(\mathbb{R}^d)$, $u \in \mathcal{S}'(\mathbb{R}^d)$ *and* $a_1, a_2 \in \mathcal{A}$ *we have*

$$\begin{array}{rcl}
\langle f\otimes a_1, u\otimes a_2\rangle &=& \langle f,u\rangle a_1a_2\\
(f\otimes a_1)\cdot(u\otimes a_2) &=& (fu)\otimes(a_1a_2)\\
f\cdot(u\otimes a_2) &=& (fu)\otimes a_1\\
f{*}(u\otimes a_2) &=& (f{*}u)\otimes a_1\\
D^\alpha(u\otimes a_1) &=& (D^\alpha u)\otimes a_1
\end{array}$$

where $\langle f,u\rangle$ *is the normal pairing in* $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$, (fu) *is the product in* $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$, $(f{*}u)$ *is the convolution in* $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$ *and* D^α *is the* α *derivatives in* $\mathcal{S}'(\mathbb{R}^d)$.

Proof The proof of this theorem can be found in [7] Appendix 1. □

We recall the definition of Littlewood–Paley blocks: let χ,φ be smooth non-negative functions from $\mathbb{R}^n$ into $\mathbb{R}$ such that

- $\mathrm{supp}(\chi) \subset B_{\frac{4}{3}}(0)$ and $\mathrm{supp}(\varphi) \subset B_{\frac{8}{3}}(0)\setminus B_{\frac{3}{4}}(0)$,
- $\chi,\varphi \le 1$ and $\chi(y) + \sum_{j\ge 0}\varphi(2^{-j}y) = 1$ for any $y\in\mathbb{R}^n$,
- $\mathrm{supp}(\chi)\cap\mathrm{supp}(\varphi(2^{-i}\cdot)) = \emptyset$ for $i \geqslant 1$,
- $\mathrm{supp}(\varphi(2^{-j}\cdot))\cap\mathrm{supp}(\varphi(2^{-i}\cdot)) = \emptyset$ if $|i-j|>1$,

where by $B_r(x)$ we denote the ball of center $x\in\mathbb{R}^n$ and radius $r>0$. We introduce the following notations: $\varphi_{-1}=\chi$, $\varphi_j(\cdot)=\varphi(2^{-j}\cdot)$, $K_j = \hat{\varphi}_j$ and, for any $\ell>0$, we define $\rho_\ell(y) = (1+|x|^2)^{-\ell/2}$.

If $v\in\mathcal{S}'(\mathbb{R}^d,\mathcal{A})$ and if $i\in\mathbb{Z}$, $i\geqslant -1$ we define i-th Littlewood–Paley block as follows

$$\Delta_i v = K_i{*}v \in \mathcal{O}_M(\mathbb{R}^d,\mathcal{A}).$$

Then, if $s\in\mathbb{R}$, $p,q\in[1,+\infty]$ and $\ell\in\mathbb{R}$ we define the function

$$\|v\|_{B^s_{p,q,\ell}(\mathbb{R}^d,\mathcal{A})} = \left(\sum_{j=-1}^{+\infty} 2^{jsq}\|\Delta_j v\|^q_{L^p_\ell(\mathbb{R}^d,\mathcal{A})}\right)^{1/q},$$

when $q \in [1, +\infty)$ and $\|v\|_{B^s_{p,+\infty,\ell}(\mathbb{R}^d,\mathcal{A})} = \sup_j(2^{js}\|\Delta_j v\|_{L^p_\ell(\mathbb{R}^d,\mathcal{A})})$, where $\|\cdot\|_{L^p_\ell(\mathbb{R}^d,\mathcal{A})}$ is the norm in the space $L^p_\ell(\mathbb{R}^d,\mathcal{A})$ that is

$$\|f\|_{L^p_\ell(\mathbb{R}^d,\mathcal{A})} = \left(\int_{\mathbb{R}^n}(\|f(y)\|_{\mathcal{A}}\rho_\ell(y))^p \mathrm{d}y\right)^{1/p},$$

if $p \in [1, +\infty)$ and

$$\|f\|_{L^\infty_\ell(\mathbb{R}^d,\mathcal{A})} = \sup_{y\in\mathbb{R}^d}(\|f(y)\|_{\mathcal{A}}\rho_\ell(y)),$$

if $p = +\infty$. For any $v \in \mathcal{S}(\mathbb{R}^d,\mathcal{A})$ the norm $\|v\|_{B^s_{p,q,\ell}(\mathbb{R}^d,\mathcal{A})} < +\infty$ is finite. Then we look at $B^s_{p,q,\ell}(\mathbb{R}^d,\mathcal{A})$ as the closure of $\mathcal{S}(\mathbb{R}^d,\mathcal{A})$ in $\mathcal{S}'(\mathbb{R}^d,\mathcal{A})$ with respect to the norm $\|\cdot\|_{B^s_{p,q,\ell}(\mathbb{R}^d,\mathcal{A})}$. Hereafter, if $s \in \mathbb{R}$, $p, q \in [1, +\infty]$ and $\ell \in \mathbb{R}$, we use the following notations $\mathcal{C}^s_\ell(\mathbb{R}^d,\mathcal{A}) := B^s_{\infty,\infty,\ell}(\mathbb{R}^d,\mathcal{A})$, $B^s_{p,q}(\mathbb{R}^d,\mathcal{A}) = B^s_{p,q,0}(\mathbb{R}^d,\mathcal{A})$, $\mathcal{C}^s(\mathbb{R}^d,\mathcal{A}) := B^s_{\infty,\infty,0}(\mathbb{R}^d,\mathcal{A})$, $B^s_{p,q,\ell} := B^s_{p,q,\ell}(\mathbb{R}^d,\mathbb{R})$ etc.

In this paper we need only the next three results.

Theorem 62 *For any $s, \ell \in \mathbb{R}$ and $v \in \mathcal{C}^s_\ell(\mathbb{R}^d,\mathcal{A})$ we have that $\rho_\ell v \in \mathcal{C}^s(\mathbb{R}^d,\mathcal{A})$ and*

$$\|v\|_{\mathcal{C}^s_\ell(\mathbb{R}^d,\mathcal{A})} \sim \|\rho_\ell v\|_{\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})}.$$

Proof By Theorem 6.5 in [79] for weighted Besov spaces on $\mathbb{R}^d$ with polynomial-like weights, the theorem for a general Banach algebra $\mathcal{A}$ is a straightforward generalization. □

Theorem 63 *Let $\alpha > 0$ and consider $\mathfrak{a} \in B^\alpha_{1,\infty}$ then we have that the convolution $\mathfrak{a}*$ can be extended in a unique continuous way from an operator from $\mathcal{S}(\mathbb{R}^d,\mathcal{A})$ into itself, into an operator from $\mathcal{C}^s(\mathbb{R}^d,\mathcal{A})$ into $\mathcal{C}^{s+\alpha}(\mathbb{R}^d,\mathcal{A})$.*

Proof Using the notations of [6], if $\mathfrak{a} \in B^\alpha_{1,\infty}$ then $((-\Delta+1)^{\alpha/2}(\mathfrak{a})) \in \mathcal{F}L_1$, then the thesis follows from Corollary 6.4 in [6]. □

Theorem 64 *Let $s > 0$ and $\ell_1, \ell_2 \in \mathbb{R}$ we have that the product $\cdot : \mathcal{S}(\mathbb{R}^d,\mathcal{A}) \times \mathcal{S}(\mathbb{R}^d,\mathcal{A}) \to \mathcal{S}(\mathbb{R}^d,\mathcal{A})$ can be (uniquely) extended in a continuous way from $\mathcal{C}^s_{\ell_1}(\mathbb{R}^d,\mathcal{A}) \times \mathcal{C}^s(\mathbb{R}^d,\mathcal{A})$ into $\mathcal{C}^s_{\ell_1+\ell_2}(\mathbb{R}^d,\mathcal{A})$, furthermore for any $v_1 \in \mathcal{C}^s_{\ell_1}(\mathbb{R}^d,\mathcal{A})$ and $v_2 \in \mathcal{C}^s_{\ell_2}(\mathbb{R}^d,\mathcal{A})$ we have*

$$\|v_1 \cdot v_2\|_{\mathcal{C}^s_{\ell_1+\ell_2}(\mathbb{R}^d,\mathcal{A})} \lesssim \|v_1\|_{\mathcal{C}^s_{\ell_1}(\mathbb{R}^d,\mathcal{A})}\|v_2\|_{\mathcal{C}^s_{\ell_2}(\mathbb{R}^d,\mathcal{A})},$$

Proof The proof can be found in [7] in the unweighted case. The proof in the Besov spaces with polynomial weights ρ_ℓ is similar. □

B Series expansion of solutions

In this appendix we sketch some implications of the exclusion principle for the existence of global solutions to finite-dimensional Grassmann SDEs. In principle some of these considerations also apply to some more realistic models like the Yukawa$_2$ model however, to our surprise, stochastic quantization, at least in the perturbative regime, can be carried on without establishing them.

The solutions to the non-linear equation (32) can be represented via a series. We investigate now the properties of this series and its absolute convergence. The explicit control of the series is not needed for the development of the theory of stochastic quantization as we have shown. For simplicity let us assume that $(v_\alpha)_{\alpha=1,\dots,N}$ is a finite basis of V and that $F \in \mathrm{Hom}(V, \Lambda V)$ is given by a sum of cubic monomials

$$F(v_\alpha) = \sum_{\alpha_1,\alpha_2,\alpha_3} \lambda^\alpha_{\alpha_1\alpha_2\alpha_3} \psi(v_{\alpha_1})\psi(v_{\alpha_2})\psi(v_{\alpha_2})$$

where $\psi \in \mathrm{Hom}(V, \Lambda V)$ is the canonical injection of V into ΛV and $(\lambda^\alpha_{\alpha_1\alpha_2\alpha_3})$ a family of coefficients in $\mathbb{R}$. Moreover we take $A = -\mathbb{I}$. equation (32) has then the integral formulation given by

$$\Psi_t(v) = \Phi_t(v) + \int_0^t \Psi_s(e^{-(t-s)}F(v))\mathrm{d}s, \qquad t \geqslant 0, v \in V. \tag{65}$$

where

$$\Phi_t(v) := \Psi_0(e^{-t}v) + B^A_t(v) - B^A_0(e^{-t}v), \qquad t \geqslant 0, v \in V.$$

By iteratively expanding Ψ_s on the right hand side of (65) we obtain a series expansion for Ψ_t of the form

$$\Psi_t = \sum_\tau J_\tau(\Phi)(t)$$

$$= J_\bullet(\Phi)(t) + J_{[\bullet\bullet\bullet]}(\Phi)(t) + J_{[[\bullet\bullet\bullet]\bullet\bullet]}(\Phi)(t) + \cdots + J_{[[\bullet\bullet\bullet][\bullet[\bullet\bullet[\bullet\bullet\bullet]]\bullet]\bullet]}(\Phi)(t) + \cdots$$

The series is indexed by (planar) trees τ which have branches of order 3 and where J is a multilinear integral operator such that

$$J_\bullet(\Phi)(t)^\alpha = \Phi_t(v_\alpha)$$

$$J_{[\tau_1\tau_2\tau_3]}(\Phi)(t)^\alpha = \sum_{\alpha_1,\alpha_2,\alpha_3} \int_0^t e^{-(t-s)} \lambda^\alpha_{\alpha_1,\alpha_2,\alpha_3} J_{\tau_1}(\Phi)(s)^{\alpha_1} J_{\tau_2}(\Phi)(s)^{\alpha_2} J_{\tau_3}(\Phi)(s)^{\alpha_3} \mathrm{d}s$$

where $\bullet$ denotes the simple tree and $[\tau_1, \dots, \tau_3]$ the tree with branches $\tau_1, \dots, \tau_3$. Our goal is to prove that the above series converges for all times and derive estimates for the norm $\|\Psi_t\|$ of the solution Ψ. Expanding the expression for $J_\tau(\Phi)$ we have

$$J_\tau(\Phi) = \int \left(\prod_{p \in I(\tau)} \lambda^{\alpha_p}_{\alpha_{p_1}\alpha_{p_2}\alpha_{p_3}} \right) \left(\prod_{p \in L(\tau)} \Phi^{\alpha_p}(s_{p_-}) \right) \left(\prod_{p \in I(\tau)} \mathrm{d}s_p \right)$$

where $I(\tau)$ denotes the internal nodes, $L(\tau)$ the leaves, p the parent node and p_1, p_2, p_3 the three children of every internal node and $\Phi^\alpha(t) = \Phi_t(v_\alpha)$. We have $|\tau| = |L(\tau)| + |I(\tau)|$, $|\tau| = 1 + 3|I(\tau)|$, $|L(\tau)| = 1 + 3|I(\tau)| - |I(\tau)| = 1 + 2|I(\tau)|$.

Note that $(\Phi^\alpha_t)^2 = 0$ and that the increments $\Phi^\alpha(t,s) := \Phi^\alpha_t - \Phi^\alpha_s$ have the norm bound

$$\|\Phi^\alpha(t,s)\|^2 \leqslant \int_s^t e^{-(t-r)}\mathrm{d}r + |e^{-(t-s)} - 1|^2 \leqslant 2(1 - e^{-(t-s)}) =: H(t-s)$$

which can be estimated as

$$\|\Phi^\alpha(t,s)\| \leqslant |t-s|^{1/2}.$$

The following Lemma uses $(\Phi^\alpha_t)^2 = 0$ (Pauli exclusion principle) to derive good estimates for products of fields.

Lemma 65 *Assume $m \geqslant 1$. For any $n \geqslant 1$ and any $t_1, \dots, t_n \in [0,T]$, $\alpha_1, \dots, \alpha_n \in \{1, \dots, N\}$ we have*

$$\|\Phi_{t_1}^{\alpha_1} \cdots \Phi_{t_n}^{\alpha_n}\| \leqslant \frac{C^{n+1} T^{n/2}}{(n!)^{1/2}}$$

where C is a universal constant depending only on N.

Proof Partition $[0,T]$ in $r = n/(2N)$ equal intervals $(I_k)_k$ and let $(s_k)_k$ be the centers of those intervals. Now in the product replace each $\Phi_{t_1}^{\alpha_i}$ by $\Phi_{t_1}^{\alpha_i} = \Phi_{s_k}^{\alpha_i} + \Phi^{\alpha_i}(t_i, s_k)$ where s_k is the nearest to t_i of the centers. By doing so we rewrite $\Phi_{t_1}^{\alpha_1} \cdots \Phi_{t_n}^{\alpha_n}$ as a sum S of 2^n products of fields which are either $\Phi_{s_k}^{\alpha_i}$ or $\Phi^{\alpha_i}(t_i, s_k)$, moreover for a given I_k there cannot be more than N factors $\Phi_{s_k}^{\alpha_i}$ by the exclusion principle. Denote by n_k the number of $(t_i)_i$ in I_k. Then $n = \sum_k n_k$ and using the estimate $\|\Phi^{\alpha_i}(t_i, s_k)\| \leqslant |t_i - s_k| \leqslant |I_k| = (2NT)/n$ and the fact that in each cell we have at most $n_k - N$ increments, we have

$$\|S\| \leqslant 2^n \prod_{k=1}^{r} |I_k|^{n_k - N} \leqslant 2^n \left(\frac{2NT}{n}\right)^{n - Nr} = 2^n \left(\frac{2NT}{n}\right)^{n/2} \leqslant \frac{C^{n+1} T^{n/2}}{(n!)^{1/2}}.$$

□

Using this lemma above we can estimate

$$\|J_\tau(\Phi)(t)\| \leqslant \int \left(\prod_{p \in I(\tau)} \left| \lambda_{\alpha_{p_1} \alpha_{p_2} \alpha_{p_3}}^{\alpha_p} \right| \right) \left\| \prod_{p \in L(\tau)} \Phi_{s_{p_-}}^{\alpha_p} \right\| \left(\prod_{p \in I(\tau)} \mathrm{d}s_p \right)$$

$$\lesssim |\lambda|^{|I(\tau)|} N^{|L(\tau)|} \frac{C^{|L(\tau)|+1} t^{|L(\tau)|/2}}{(|L(\tau)|!)^{1/2}}$$

where we estimated all the integrals by constants uniformly in t due to the presence of the exponential factors. The number of trees τ with a given number of nodes $|\tau| = n$ is no more than D^n for some n and $D > 0$ and $|L(\tau)| = 1 + 2|I(\tau)| = 1 + (2/3)(|\tau| - 1) = 1/3 + 2|\tau|/3$ so we obtain at the end for the solution Ψ_t of (65)

$$\|\Psi_t\| \leqslant \sum_\tau \|J_\tau(\Phi)(t)\| \lesssim \sum_{n \geqslant 0} D^n |\lambda|^{(2/3)(n-1)} N^{2n/3} \frac{C^{2n/3} t^{n/3+1/6}}{(n/3 + 1/6)!}$$

which is a series which converges for all t (with some stretched exponential behavior).

Theorem 66 *There exists an increasing function $E(t)$ depending on $N, |\lambda|, m$ such that for all $m > 0$,*

$$\sup_{s \leqslant t} \|\Psi_s\| \leqslant E(t).$$

In particular this shows again that explosion is not possible for this kind of SDEs, unlike their bosonic analogs.

Our goal now is to remove the time dependence on our estimate for the solution exploiting the exponential decay. Let us partition the interval $[0, \infty]$ into intervals $(I_h)_{h \geqslant 0}$ of size 1. In each of these intervals we will use the finer partition of the previous lemma to estimate products, which will then not depend on T anymore. Of course now the problem is that we have a bound of the form

$$\|\Phi^{\alpha_1}(t_1) \cdots \Phi^{\alpha_n}(t_n)\| \leqslant \prod_{h \geqslant 0} \frac{C^{n_h+1}}{(n_h!)^{1/2}}$$

where n_h is the number of time variables in I_h and $\sum_{h\geqslant 0} n_h = n$, and so far there is no useful upper bound on this (since we observe that we can have the situation where $n_h = 1$ for all h and so we loose the factorial contribution). Let Q be the number of intervals with $n_h > 0$. In the expression for $J_\tau(\Phi)(t)$ there are at least Q factors of the form $e^{-(s-s')}$, $s' < s$, for which the sum of the quantities $s - s'$ is at least Q. So we have the bounds

$$\|J_\tau(\Phi)(t)\| \leqslant |\lambda|^{(2/3)(n-1)} N^n \sum_{Q\geqslant 1} e^{-Q/2} \prod_{h:n_h>0} \frac{C^{n_h+1}}{(n_h!)^{1/2}} \leqslant \frac{C^n |\lambda|^{(2/3)(n-1)}}{(n!)^{1/2}} \sum_{Q\geqslant 1} e^{-Q/2} Q^{n/2}$$

where we used that

$$\prod_{h:n_h>0} \frac{1}{n_h!} \leqslant \frac{1}{n!} \sum_{\substack{k_1,\dots,k_Q \geqslant 1 \\ k_1+\cdots+k_Q = n}} \frac{n!}{k_1!\cdots k_Q!} = \frac{1}{n!} Q^n.$$

Now, using the bound $\sum_{Q\geqslant 1} e^{-Q/2} Q^{n/2} \lesssim c^n (n!)^{1/2}$ we obtain that

$$\|J_\tau(\phi)(t)\| \lesssim |\lambda|^{(2/3)(n-1)} C^n$$

so provided $|\lambda|$ is small enough (depending on our choice of N) we conclude that the uniform bound $\sup_{t\geqslant 0} \|\Psi_t\| \lesssim 1$ holds.

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