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Hyperviscous stochastic Navier–Stokes equations with white noise invariant measure in two dimensions

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Abstract

We prove existence and uniqueness of martingale solutions to a (slightly) hyper-viscous stochastic Navier–Stokes equation in 2d with initial conditions absolutely continuous with respect to the Gibbs measure associated to the energy.

1 Introduction

Consider the following stochastic hyper-viscous Navier–Stokes equation on $\mathbb{R}_+ \times \mathbb{T}^2$

$$\begin{aligned} \partial_t u &= -A^\theta u - u \cdot \nabla u - \nabla p - \sqrt{2}\nabla^\perp A^{(\theta-1)/2}\xi, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1}$$

where $\mathbb{T}^2$ is the two dimensional torus, $A = -\Delta$ on $\mathbb{T}^2$, $\nabla^\perp := (\partial_2, -\partial_1)$, $\theta > 1$, and ξ denotes a space-time white noise. The initial condition for u will be taken distributed according to the white noise on $\mathbb{T}^2$ or an absolute continuous perturbation thereof with density in L^2. The white noise on $\mathbb{T}^2$ is formally invariant for the dynamics described by (1) and the existence theory for the corresponding stationary process has been addressed by Gubinelli and Jara in [12] using the concept of *energy solutions* for any $\theta > 1$. Uniqueness was left open in the aforementioned paper, and the main aim of the present work, which can be thought of as a continuation of [14], is to introduce a martingale problem formulation (1) for which we can prove uniqueness.

In order to properly formulate the martingale problem, we need to investigate the infinitesimal generator for eq. (1) and uniqueness will result from suitable solutions of the associated Kolmogorov backward equation.

The variable u appearing in eq. (1) represents physically the *velocity* of a fluid. Rewriting the equation for the *vorticity* $\omega := \nabla^\perp \cdot u$ yields

$$\partial_t \omega = -A^\theta \omega - u \cdot \nabla \omega + \sqrt{2} A^{(\theta+1)/2}\xi. \tag{2}$$

We also have the relation $u = K * \omega$, for the *Biot-Savart kernel* K on $\mathbb{T}^2$ given by

$$K(x) = \frac{1}{2\pi\iota} \sum_{k \in \mathbb{Z}_0^2} \frac{k^\perp}{|k|^2} e^{2\pi\iota k \cdot x} = - \sum_{k \in \mathbb{Z}_0^2} \frac{2\pi\iota k^\perp}{|2\pi k|^2} e^{2\pi\iota k \cdot x},$$

where $k^\perp = (k_2, -k_1)$ and $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash \{0\}$. It is more convenient to work with the scalar quantity ω and with eq. (2).

The standard stochastic Navier-Stokes equation corresponds to the case $\theta = 1$. However, this regime is quite singular for the white noise initial condition and no results are known, not even existence of a stationary solution, e.g. from limit of Galerkin approximations. While a bit unphysical, we will stick here to the *hyper-viscous regime*, namely $\theta > 1$. Note that the noise has to be coloured accordingly in order to preserve the white noise as invariant measure. Moreover, we call *energy measure* the law under which the velocity field is a (vector-valued, incompressible) white noise. In terms of vorticity ω, the kinetic energy of the fluid configuration u is

$$\|u\|_{L^2}^2 = \int_{\mathbb{T}^2} |K * \omega|^2(x) \mathrm{d}x = \sum_{k \in \mathbb{Z}_0^2} |\hat{K}(k)|^2 |\hat{\omega}(k)|^2 = \sum_{k \in \mathbb{Z}_0^2} \left| \frac{2\pi\iota k^\perp}{|2\pi k|^2} \right|^2 |\hat{\omega}(k)|^2 = \|(-\Delta)^{-1/2} \omega\|_{L^2}^2,$$

where $\hat{f} : \mathbb{Z}^2 \to \mathbb{C}$ denotes the Fourier transform of $f : \mathbb{T}^2 \to \mathbb{R}$ defined as to have $f(x) = \sum_{k \in \mathbb{Z}^2} e^{2\pi\iota k \cdot x} \hat{f}(k)$. The energy measure is thus formally given by

$$\mu(\mathrm{d}\omega) = \frac{1}{C} e^{-\frac{1}{2}\|A^{-1/2}\omega\|_{L^2}^2} \mathrm{d}\omega,$$

where $\mathrm{d}\omega$ denotes the "Lebesgue measure" on functions on $\mathbb{T}^2$. Rigorously, this of course means the product Gaussian measure

$$\mu(\mathrm{d}\omega) = \prod_{k \in \mathbb{Z}_0^2} \frac{1}{C_k} \exp\left(-\frac{|\hat{\omega}(k)|^2}{2|2\pi k|^2} \right) \mathrm{d}\hat{\omega}(k),$$

with the restriction that $\hat{\omega}(-k) = \overline{\hat{\omega}(k)}$. For $f, g \in C^\infty(\mathbb{T}^2)$, we have

$$\int \omega(f)\omega(g)\mu(\mathrm{d}\omega) = \sum_{k \in \mathbb{Z}_0^2} |2\pi k|^2 \overline{\hat{f}(k)} \hat{g}(k) = \langle A^{1/2} f, A^{1/2} g \rangle_{L^2(\mathbb{T}^2)} = \langle f, g \rangle_{H^1(\mathbb{T}^2)}.$$

We can use the right-hand side as the definition of the covariance of $(\omega(f))_{f \in C^\infty(\mathbb{T}^2)}$, which determines the law of ω as a centred Gaussian process indexed by $H^1(\mathbb{T}^2)$. If η is a white noise on $L^2(\mathbb{T}^2)$, then μ has the same distribution as $A^{1/2}\eta$ and it is only supported on $H^{-2-}(\mathbb{T}^2)$.

A different situation occurs if we consider initial conditions distributed according to the *enstrophy measure*, namely the Gaussian measure for which the initial vorticity is a white noise. This measure is more regular than the energy measure and more results are known, both for the Euler dynamics (i.e., without dissipation and noise) and for the stochastic Navier-Stokes dynamics, see e.g. [1, 2, 3].

As we already remarked, the present paper present results obtained using the technique introduced in [14] and strongly rooted in the notion of energy solution of Gonçalves and Jara [11], extended in [12]. With respect to [14] we give a slightly different formulation which simplifies certain technical estimates. The core of the argument however remains the same. The main point is to consider the well-posedness problem for (1) as a problem of *singular diffusion*, i.e. diffusions with distributional drift. The papers [9,

10, 7, 4] all follow a similar strategy in order to identify a domain for the formal infinitesimal generator $\mathcal{L} = \frac{1}{2}\Delta + b \cdot \nabla$ of a finite dimensional diffusion. Then they show existence and uniqueness of solutions for the corresponding martingale problem. The key difficulty is that for distributional b the domain does not contain any smooth functions and instead one has to identify a class of non-smooth test functions with a special structure, adapted to b. Roughly speaking they must be local perturbations of a linear functional constructed from b. Recently other results of regularization by noise for SPDEs [5, 6] have been obtained. An important difference is that our drift is unbounded and not even a function. The connection between energy solutions and regularisation by noise was first observed in [12].

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Plan of the paper. In Section 2 we introduce a Galerkin approximation for the nonlinearity $u \cdot \nabla \omega$ and study the infinitesimal generator of the approximating equation. The martingale problem for cylinder function related to eq. (2) is introduced in Section 3. In Section 4 we prove uniqueness for the martingale problem via existence of classical solutions to the backward Kolmogorov equation for the operator $\mathcal{L}$ involved in the martingale problem. The construction of a domain to such an operator is the core of the work and it will be tackled in Section 5, where we provide also existence and uniqueness for the associated Kolmogorov equation. Finally, in Section 6 we prove some crucial bounds on the drift. Appendix A contains some auxiliary results.

2 Galerkin approximations

In order to rigorously study the eq. (2), consider the solution $(\omega_t^m)_{t \geqslant 0}$ to its Galerkin approximation:

$$\partial_t \omega^m = -A^\theta \omega^m - B_m(\omega^m) + \sqrt{2} A^{(1+\theta)/2} \xi, \tag{3}$$

where

$$B_m(\omega) := \operatorname{div} \Pi_m((K * \Pi_m \omega)\Pi_m \omega),$$

and Π_m denotes the projection onto Fourier modes of size less than m, namely $\Pi_m f(x) = \sum_{|k| \leqslant m} e^{2\pi \iota k \cdot x} \hat{f}(k)$.

Proposition 1 *Eq.* (3) *has a unique strong solution* $\omega^m \in C(\mathbb{R}_+, H^{-2-}(\mathbb{T}^2))$ *for every deterministic initial condition in* $H^{-2-}(\mathbb{T}^2)$. *The solution is a strong Markov process and it is invariant under* μ.

Proof We can rewrite ω^m in Fourier variables as $\omega^m = w^m + W^m := \Pi_m \omega^m + (1 - \Pi_m)\omega^m$, in such a way that w^m and W^m solve respectively a finite-dimensional SDE with locally Lipschitz continuous coefficients and an infinite-dimensional linear SDE. Global existence and invariance of μ follow by Section 7 in [12]. Now, w^m has compact spectral support and therefore $w^m \in C(\mathbb{R}_+, C^\infty(\mathbb{T}))$, while it can be proved that W^m has trajectories in $C(\mathbb{R}_+, H^{-2-}(\mathbb{T}^2))$. Thus, ω^m has trajectories in $C(\mathbb{R}_+, H^{-2-}(\mathbb{T}^2))$. □

We define the semigroup of ω^m for all bounded and measurable functions φ as $T_t^m \varphi(\omega_0) := \mathbb{E}_{\omega_0}[\varphi(\omega_t^m)]$, where, under $\mathbb{P}_{\omega_0}$, the process ω^m solves (3) with initial condition $\omega_0 \in H^{-2-}(\mathbb{T}^2)$.

Lemma 1 *For all $p \in [1,\infty]$, the family of operators $(T^m_t)_{t\geqslant 0}$ can be uniquely extended to a contraction semigroup on $L^p(\mu)$ which is continuous for $p \in [1,\infty[$.*

Let $\mathcal{C}$ denote the set of cylinder functions on $H^{-2-}(\mathbb{T}^2)$, namely those functions $\varphi : H^{-2-}(\mathbb{T}^2) \to \mathbb{R}$ of the form $\varphi(\omega) = \Phi(\omega(f_1),\dots,\omega(f_n))$ for some $n \geqslant 1$ where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is smooth and $f_1,\dots,f_n \in C^\infty(\mathbb{T}^2)$. On such functions the generator of the semigroup T^m has an explicit representation: Itô's formula gives

$$\mathrm{d}\varphi(\omega^m_t) = \mathcal{L}^m\varphi(\omega^m_t)\mathrm{d}t + \sum_{i=1}^n \partial_i\Phi(\omega^m_t(f_1),\dots,\omega^m_t(f_n))\mathrm{d}M_t(f_i), \tag{4}$$

where $\mathcal{L}^m := \mathcal{L}_\theta + \mathcal{G}^m$ with

$$\mathcal{L}_\theta\varphi(\omega) = \sum_{i=1}^n \partial_i\Phi(\omega(f_1),\dots,\omega(f_n))\omega(-A^\theta f_i) + \frac{1}{2}\sum_{i=1}^n \partial^2_{i,j}\Phi(\omega(f_1),\dots,\omega(f_n))\langle A^{\theta+1}f_i, f_j\rangle,$$

and

$$\mathcal{G}^m\varphi(\omega) = -\sum_{i=1}^n \partial_i\Phi(\omega(f_1),\dots,\omega(f_n))\langle B_m(\omega), f_i\rangle.$$

Here, $(M_t(f_i))_{t\geqslant 0}$ is a continuous martingale with quadratic variation

$$\langle M(f_i)\rangle_t = 2t\|A^{(\theta+1)/2}f_i\|^2_{L^2(\mathbb{T}^2)},$$

and therefore $\int_0^\cdot \sum_{i=1}^n \partial_i\Phi(\omega^m_t(f_1),\dots,\omega^m_t(f_n))\mathrm{d}M_t(f_i)$ is a martingale. Consequently, we have

$$T^m_t\varphi(\omega) - \varphi(\omega) = \int_0^t T^m_s(\mathcal{L}^m\varphi)(\omega)\mathrm{d}s, \qquad \text{for all } \omega \in H^{-2-}.$$

To extend this to more general functions φ, we work via Fock space techniques. The Hilbert space $L^2(\mu)$ can be identified with the Fock space $\mathcal{H} = \Gamma H^1_0(\mathbb{T}^2) := \bigoplus_{n=0}^\infty (H^1_0(\mathbb{T}^2))^{\otimes n}$ with $H^1_0(\mathbb{T}^2) := \{\psi \in H^1(\mathbb{T}^2) : \hat\psi(0) = 0\}$ and norm

$$\|\varphi\|^2 = \sum_{n=0}^\infty n!\|\varphi_n\|^2_{(H^1_0(\mathbb{T}^2))^{\otimes n}} = \sum_{n=0}^\infty n! \sum_{k_{1:n}\in(\mathbb{Z}^2_0)^n} \left(\prod_{i=1}^n |2\pi k_i|^2\right)|\hat\varphi_n(k_{1:n})|^2,$$

by noting that any $\varphi \in L^2(\mu)$ can be written in chaos expansion $\varphi = \sum_{n\geq 0} W_n(\varphi_n)$, where W_n is the n-th order Wiener-Itô integral and $\varphi_n \in H^1_0(\mathbb{T}^2)^{\otimes n}$ for every $n \in \mathbb{N}$, see e.g. [15, 17] for details. We will use the convention that φ_n is symmetric in its n arguments, that is, we identify it with its symmetrization. Note that cylinder functions are dense in $\mathcal{H}$. We denote by $\mathcal{N}$ the *number operator*, i.e. the self–adjoint operator on $\mathcal{H}$ such that $(\mathcal{N}\varphi)_n := n\varphi_n$. It is well known that the semigroup generated by the number operator satisfies an hypercontractivity estimate, see Theorem 1.4.1 in [17]. We record it in the next lemma.

Lemma 2 *For $p \geqslant 2$, let $c_p = \sqrt{p-1}$. Then*

$$\||\varphi|^{p/2}\|^2 \leqslant \|c_p^{\mathcal{N}}\varphi\|^p, \qquad \textit{for every } \varphi \in \mathcal{H}.$$

With these preparations we are ready to give expressions for the operators $\mathcal{L}_\theta$ and $\mathcal{G}^m$ in terms of the Fock space representation of $\mathcal{H}$.

Lemma 3 *For sufficiently nice $\varphi \in \mathcal{H}$, the operator $\mathcal{L}_\theta$ is given by*

$$\mathcal{F}(\mathcal{L}_\theta \varphi)_n(k_{1:n}) = -(2\pi)^{2\theta} L_\theta(k_{1:n}) \hat{\varphi}_n(k_{1:n})$$

where $L_\theta(k_{1:n}) := |k_1|^{2\theta} + \cdots + |k_n|^{2\theta}$. Moreover, writing $\mathcal{G}^m = \mathcal{G}^m_+ + \mathcal{G}^m_-$ we have

$$\mathcal{F}(\mathcal{G}^m_+ \varphi)_n(k_{1:n}) = (n-1)\mathbb{1}_{|k_1|,|k_2|,|k_1+k_2| \leqslant m} \frac{(k_1^\perp \cdot (k_1+k_2))((k_1+k_2)\cdot k_2)}{|k_1|^2 |k_2|^2} \hat{\varphi}_{n-1}(k_1+k_2, k_{3:n}), \quad (5)$$

$$\mathcal{F}(\mathcal{G}^m_- \varphi)_n(k_{1:n}) = (2\pi)^2 (n+1) n \sum_{p+q=k_1} \mathbb{1}_{|k_1|,|p|,|q| \leqslant m} \frac{(k_1^\perp \cdot p)(k_1 \cdot q)}{|k_1|^2} \hat{\varphi}_{n+1}(p, q, k_{2:n}). \quad (6)$$

For all $\varphi_{n+1} \in (H^1_0(\mathbb{T}^2))^{\otimes(n+1)}$ and for all $\varphi_n \in (H^1_0(\mathbb{T}^2))^{\otimes n}$, we have

$$\langle \varphi_{n+1}, \mathcal{G}^m_+ \varphi_n \rangle = -\langle \mathcal{G}^m_- \varphi_{n+1}, \varphi_n \rangle. \quad (7)$$

Proof The computations are analogous to those of Lemma 3.7 of [13] for $\mathcal{L}_\theta$ and of Lemma 2.4 and Lemma 2.7 in [14]. □

Remark 1 $\mathcal{G}^m_+$ and $\mathcal{G}^m_-$ are (unbounded) operators which increase and decrease, respectively, the "number of particles" by one. Moreover, we know from (7) that they are formally the adjoint of the other (modulo a sign change).

A key result is given by the following bounds for $\mathcal{G}^m_\pm$ acting on weighted subspaces of $\mathcal{H}$.

Lemma 4 *Let $w : \mathbb{N}_0 \to \mathbb{R}_+$ and $\varphi \in \mathcal{H}$. The following m-dependent bound holds:*

$$\|w(\mathcal{N}) \mathcal{G}^m \varphi\| \lesssim m \|(w(\mathcal{N}+1) + w(\mathcal{N}-1))(1+\mathcal{N})(1-\mathcal{L}_\theta)^{1/2} \varphi\|. \quad (8)$$

Moreover, uniformly in m, we have

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma} \mathcal{G}^m_+ \varphi\| \lesssim \|w(\mathcal{N}+1)(1+\mathcal{N})(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma} \varphi\|, \qquad \text{for all } \gamma > \frac{1}{2\theta}, \quad (9)$$

and

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma} \mathcal{G}^m_- \varphi\| \lesssim \|w(\mathcal{N}-1)\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma} \varphi\|, \qquad \text{for all } \gamma < \frac{1}{2}. \quad (10)$$

These bounds will be proven later on in Section 6. In view of eq. (8), it is natural to identify a dense domain $\mathcal{D}(\mathcal{L}^m)$ for $\mathcal{L}^m$ as

$$\mathcal{D}(\mathcal{L}^m) := \{\varphi \in \mathcal{H} : \|(1+\mathcal{N})(1-\mathcal{L}_\theta)\varphi\| < \infty\} = (1+\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H}/$$

Note that $\langle \psi, (\mathcal{L}_\theta + \mathcal{G}^m)\varphi \rangle = \langle (\mathcal{L}_\theta - \mathcal{G}^m)\psi, \varphi \rangle$ for $\psi, \varphi \in \mathcal{D}(\mathcal{L}^m)$ and in particular that $\mathcal{L}_\theta$ is dissipative since for all $\varphi \in \mathcal{D}(\mathcal{L}^m)$ we have

$$\langle \varphi, (\mathcal{L}_\theta + \mathcal{G}^m)\varphi \rangle = \langle \mathcal{L}_\theta \varphi, \varphi \rangle = -\|(-\mathcal{L}_\theta)^{1/2} \varphi\|^2 \leqslant 0.$$

A priori $\mathcal{L}^m$ is only the restriction to $\mathcal{D}(\mathcal{L}^m)$ of the generator $\hat{\mathcal{L}}^m$ of the semigroup $(T^m_t)_t$. However, we will also prove in Lemma 11 below that the operator $\mathcal{L}^m$ is closable and that its closure is indeed the generator $\hat{\mathcal{L}}^m$.

In order to exploit these pieces of information, we have to work with solutions of Galerkin approximations having "near-stationary" fixed-time marginal.

Definition 1 *We say that a stochastic process $(\omega_t)_{t\geqslant 0}$ with values in $\mathcal{S}'(\mathbb{T}^2)$ is* (L^2–)incompressible *if, for all $T>0$, there exists a constant $C(T)$ such that we have*

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}|\varphi(\omega_t)|\leqslant C(T)\|\varphi\|,\qquad \varphi\in\mathcal{C}.$$

For an incompressible process $(\omega_t)_{t\geqslant 0}$ it makes sense, using a density argument involving cylinder functions, to define $s\mapsto\varphi(\omega_s)$ for all $\varphi\in\mathcal{H}$ as a stochastic process continuous in L^1.

Lemma 5 *Let $\mathbb{E}_{\eta\mathrm{d}\mu}$ be the law of the solution ω^m to the Galerkin approximation* (3) *starting from an initial condition $\omega_0^m\sim\eta\mathrm{d}\mu$ with $\eta\in L^2(\mu)$. Then, for any $\Psi: C(\mathbb{R}_+;\mathcal{S}')\to\mathbb{R}$,*

$$\mathbb{E}_{\eta\mathrm{d}\mu}|\Psi(\omega^m)|\leqslant\|\eta\|\mathbb{E}_\mu(\Psi(\omega^m)^2)^{1/2}.$$

In particular, any such process is incompressible uniformly in m.

Proof We get

$$\mathbb{E}_{\eta\mathrm{d}\mu}|\Psi(\omega^m)|=\mathbb{E}_\mu[\eta(w_0)|\Psi(\omega^m)|]\leqslant\|\eta\|(\mathbb{E}_\mu\Psi(\omega^m)^2)^{1/2}.$$

Incompressibility easily follows from the fact that μ is an invariant measure for the Galerkin approximations independently of m. □

Definition 2 *A* weight *is a measurable increasing map $w:\mathbb{R}_+\to(0,\infty)$ such that there exists $C>0$ with $w(x)\leqslant Cw(x+y)$, for all $x\geqslant 1$ and for $|y|\leqslant 1$. We write as $|w|$ the smallest such constant C. We denote $w(\mathcal{N})$ the self-adjoint operator on $\mathcal{H}$ defined as spectral multiplier.*

Lemma 6 *Let $\eta\in L^2(\mu)$ and let ω^m be a solution to* (3) *with* $\mathrm{Law}(\omega_0^m)\sim\eta\mathrm{d}\mu$. *Then this solution is incompressible and, for any $\varphi\in\mathcal{D}(\mathcal{L}^m)$, the process*

$$M_t^{m,\varphi}=\varphi(\omega_t^m)-\varphi(\omega_0^m)-\int_0^t\mathcal{L}^m\varphi(\omega_s^m)\mathrm{d}s,\qquad t\geqslant 0,$$

is a continuous martingale with quadratic variation

$$\langle M^{m,\varphi}\rangle_t=\int_0^t\mathcal{E}(\varphi)(\omega_s^m)\mathrm{d}s,\qquad \textit{with}\qquad \mathcal{E}(\varphi)=2\int_{\mathbb{T}^2}|A_x^{\frac{\theta+1}{2}}D_x\varphi|^2\mathrm{d}x.\tag{11}$$

For any weight w, we have

$$\|w(\mathcal{N})(\mathcal{E}(\varphi))^{1/2}\|\lesssim\|w(\mathcal{N}-1)(1-\mathcal{L}_\theta)^{1/2}\varphi\|.\tag{12}$$

Moreover, for all $p\geqslant 1$, it holds

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\varphi(\omega_s^m)\mathrm{d}s\right|^p\lesssim(T^{p/2}\vee T^p)\|c_{2p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p,\tag{13}$$

uniformly in m.

Proof If φ is a cylinder function, then we have eq. (4) and in that case Doob's inequality and Lemma 5 yield, for all $T>0$,

$$\mathbb{E}\sup_{t\in[0,T]}|M_t^{m,\varphi}|\lesssim\mathbb{E}(\langle M^{m,\varphi}\rangle_T^{1/2})\leqslant\|\eta\|\mathbb{E}_\mu(\langle M^{m,\varphi}\rangle_T)^{1/2}\lesssim\|\eta\|T^{1/2}\|(\mathcal{E}(\varphi))^{1/2}\|_{L^2(\mu)}.$$

The norm appearing on the right-hand side can be estimated as follows:

$$\begin{aligned}
\|w(\mathcal{N})(\mathcal{E}(\varphi))^{1/2}\|^2 &= 2\int_x \left\|w(\mathcal{N})A_x^{\frac{\theta+1}{2}}D_x\varphi\right\|^2 \mathrm{d}x \\
&= 2\int_x \left(\sum_{n\geq 0}(n-1)!w(n-1)^2n^2\left\|A_x^{\frac{\theta+1}{2}}\varphi_n(x,\cdot)\right\|^2_{H_0^1(\mathbb{T}^2)^{\otimes(n-1)}}\right) \\
&\simeq 2\sum_{n\geq 1} n!w(n-1)^2n\sum_{k_{1:n}}\left(\prod_{i=2}^n|2\pi k_i|^2\right)|k_1|^{2(\theta+1)}|\hat{\varphi}_n(k_{1:n})|^2 \\
&= 2\sum_{n\geq 1} n!w(n-1)^2n\sum_{k_{1:n}}\left(\prod_{i=1}^n|2\pi k_i|^2\right)|k_1|^{2\theta}|\hat{\varphi}_n(k_{1:n})|^2 \\
&= 2\sum_{n\geq 1} n!w(n-1)^2\sum_{k_{1:n}}\left(\prod_{i=1}^n|2\pi k_i|^2\right)L_\theta(k_{1:n})|\hat{\varphi}_n(k_{1:n})|^2 \\
&\lesssim 2\|w(\mathcal{N}-1)(1-\mathcal{L}_\theta)^{1/2}\varphi\|^2,
\end{aligned}$$

where we used a symmetrisation in the arguments of $\hat{\varphi}_n$ in the 5th line. Using the bounds (8) and (12), one can extend formula (6) to all functions in $\mathcal{D}(\mathcal{L}^m)$ by a density argument.

As far as (13) is concerned, let us remark that, provided the process ω^m is started from its stationary measure μ, then the *reversed process* $(\tilde{\omega}_t = \omega_{T-t})_{t\geqslant 0}$ is also stationary and with (martingale) generator $\tilde{\mathcal{L}}^m = \mathcal{L}_\theta - \mathcal{G}^m$. The forward–backward Itô trick used in [12] allows us to represent additive functionals of the form $\int_0^t \mathcal{L}_\theta\psi(\omega_s^m)\mathrm{d}s$ as a sum of forward and backward martingales whose quadratic variations satisfy (11). Therefore,

$$\begin{aligned}
\mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t \mathcal{L}_\theta\varphi(\omega_s^m)\mathrm{d}s\right|^p\right] &\lesssim T^{p/2}\|(\mathcal{E}(\varphi))^{p/4}\|^2 \\
&\lesssim T^{p/2}\|c_p^{\mathcal{N}}\mathcal{E}(\varphi)^{1/2}\|^p \lesssim T^{p/2}\|c_p^{\mathcal{N}}(1-\mathcal{L}_\theta)^{1/2}\varphi\|^p.
\end{aligned} \tag{14}$$

Let $\psi = (1-\mathcal{L}_\theta)^{-1}\varphi$ and exploit (14) to compute

$$\begin{aligned}
\mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t \varphi(\omega_s^m)\mathrm{d}s\right|^p\right] &= \mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t(1-\mathcal{L}_\theta)\psi(\omega_s^m)\mathrm{d}s\right|^p\right] \\
&\lesssim \mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t(-\mathcal{L}_\theta)\psi(\omega_s^m)\mathrm{d}s\right|^p\right] + \mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t\psi(\omega_s^m)\mathrm{d}s\right|^p\right] \\
&\lesssim T^{p/2}\|c_p^{\mathcal{N}}(1-\mathcal{L}_\theta)^{1/2}\psi\|^p + T^p\|c_p^{\mathcal{N}}\psi\|^p \\
&\lesssim (T^{p/2}\vee T^p)(\|c_p^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p + \|c_p^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1}\varphi\|^p) \\
&\lesssim (T^{p/2}\vee T^p)\|c_p^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p,
\end{aligned}$$

which is uniform in m. □

3 The cylinder martingale problem

We want now to take limits of Galerkin approximations and have a characterisation of the limiting dynamics. The main problem is that the formal limiting (martingale) generator $\mathcal{L}$ does not send cylinder functions to $\mathcal{H}$, therefore we cannot properly formulate a martingale problem for incompressible solutions. However, estimate (13) suggests that it is reasonable to ask that any limit process $(\omega_t)_{t\geqslant 0}$ satisfies

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\varphi(\omega_s)\mathrm{d}s\right|^p \lesssim (T^{p/2}\vee T^p)\|c_{2p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p, \tag{15}$$

for all $p \geqslant 1$ and all cylinder functions $\varphi \in \mathcal{C}$. The proof of the next lemma is almost immediate.

Lemma 7 *Assume that a process $(\omega_t)_t$ satisfies* (15) *and let $I_t(\varphi) = \int_0^t \varphi(\omega_s)\mathrm{d}s$ for all $\varphi \in \mathcal{C}$. Then the map $\varphi \mapsto (I_t(\varphi))_{t\geqslant 0}$ can be extended to all $\varphi \in (1-\mathcal{L}_\theta)^{1/2}\mathcal{H}$. The process $(I_t(\varphi))_{t\geqslant 0}$ is almost surely continuous.*

Proof Take $(\varphi_n)_n \subseteq \mathcal{C}$ such that $\sum_n \|(1-\mathcal{L}_\theta)^{1/2}\varphi_n - \varphi\| < \infty$, then it is easy to see that $(I(\varphi_n))_n$ is a Cauchy sequence in $C([0,T];\mathbb{R})$ a.s. with limit $I(\varphi) \in C([0,T];\mathbb{R})$. It satisfies (15) by Fatou's lemma and, therefore, depends only on φ and not on the particular approximating sequence. □

From this we deduce that for such processes we have

$$\lim_{m\to\infty} \int_0^t (\mathcal{L}^m\varphi)(\omega_s)\mathrm{d}s = \int_0^t (\mathcal{L}\varphi)(\omega_s)\mathrm{d}s,$$

in probability and in L^p for cylinder functions $\varphi \in \mathcal{C}$. Here, on the right-hand side the quantity $\mathcal{L}\varphi$ is defined as $\mathcal{L}\varphi = \mathcal{L}_\theta\varphi + \lim_{m\to\infty} \mathcal{G}^m\varphi$, that is an element of the space of distributions $(1-\mathcal{L}_\theta)^{1/2}\mathcal{H}$. The limit exists and is unique thanks to the uniform estimates on $\mathcal{G}^m$ in Lemma 4. As a consequence, we have also a notion of martingale problem w.r.t. the operator $\mathcal{L}$ involving only cylinder functions.

Definition 3 *A process $(\omega_t)_{t\geqslant 0}$ with trajectories in $C(\mathbb{R}_+;\mathcal{S}')$ solves the* cylinder martingale problem for $\mathcal{L}$ with initial distribution ν *if $\omega_0 \sim \nu$ and if the following conditions are satisfied:*

i. $(\omega_t)_t$ is incompressible,

ii. the Itô trick works: for all cylinder functions φ and all $p \geqslant 1$, we have eq. (15)*.*

iii. for any $\varphi \in \mathcal{C}$, the process

$$M_t^\varphi = \varphi(\omega_t) - \varphi(\omega_0) - \int_0^t \mathcal{L}\varphi(\omega_s)\mathrm{d}s, \qquad t \geqslant 0, \tag{16}$$

is a continuous martingale with quadratic variation $\langle M^\varphi\rangle_t = \int_0^t \mathcal{E}(\varphi)(\omega_s)\mathrm{d}s$. The integral on the right-hand side of eq. (16) *is defined according to Lemma 7.*

Theorem 1 *Let $\eta \in L^2(\mu)$ and, for each $m \geqslant 1$, let (ω^m) be the solution to* (3) *with $\omega_0^m \sim \eta\mathrm{d}\mu$. Then the family $(\omega^m)_{m\in\mathbb{N}}$ is tight in $C(\mathbb{R}_+;\mathcal{S}')$ and any weak limit ω solves the cylinder martingale problem for $\mathcal{L}$ with initial distribution $\eta\mathrm{d}\mu$ according to Definition 3 and we have*

$$\mathbb{E}[|\varphi(\omega_t) - \varphi(\omega_s)|^p] \lesssim (|t-s|^{p/2} \vee |t-s|^p)\|c_{4p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p \tag{17}$$

for any $p \geqslant 2$ and $\varphi \in \mathcal{C}$.

Proof The proof follows the one for Theorem 4.6 in [14].

Step 1. Consider $p \geqslant 2$ and $\varphi \in \mathcal{C}$. We want to derive an estimate for $\mathbb{E}[|\varphi(\omega_t^m) - \varphi(\omega_s^m)|^p]$. We write then $\varphi(\omega_t^m) - \varphi(\omega_s^m) = \int_s^t \mathcal{L}^m\varphi(\omega_r^m)\mathrm{d}r + M_t^{m,\varphi} - M_s^{m,\varphi}$, and get from Lemma 5 and eq. (13) the following bound

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t \mathcal{L}^m\varphi(\omega_r^m)\mathrm{d}r\right|^p\right] &\lesssim \left[\mathbb{E}_\mu\left|\int_s^t \mathcal{L}^m\varphi(\omega_r^m)\mathrm{d}r\right|^{2p}\right]^{1/2} \\ &\lesssim (|t-s|^{p/2} \vee |t-s|^p)\|c_{4p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p.\end{aligned}$$

The martingale term can be bounded by means of the Burkholder-Davis-Gundy inequality and (12) as follows:

$$\begin{aligned}\mathbb{E}[|M_t^{m,\varphi} - M_s^{m,\varphi}|^p] &\lesssim \mathbb{E}\left[\left(\int_s^t \mathcal{E}(\varphi)(\omega_r^m)\mathrm{d}r\right)^{p/2}\right] \lesssim \left[\mathbb{E}_\mu\left(\int_s^t \mathcal{E}(\varphi)(\omega_r^m)\mathrm{d}r\right)^p\right]^{1/2} \\ &\lesssim |t-s|^{p/2}\|(\mathcal{E}(\varphi))^{p/2}\| \lesssim |t-s|^{p/2}\|c_{2p}^{\mathcal{N}}(\mathcal{E}(\varphi))^{1/2}\|^p \\ &\lesssim |t-s|^{p/2}\|c_{2p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{1/2}\varphi\|^p.\end{aligned}$$

Therefore,

$$\mathbb{E}[|\varphi(\omega_t^m) - \varphi(\omega_s^m)|^p] \lesssim (|t-s|^{p/2} \vee |t-s|^p)\|c_{4p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|^p.$$

The law of the initial condition $\varphi(\omega_0^m)$ is independent of m, and by Kolmogorov's continuity criterion the sequence of real-valued processes $(\varphi(\omega^m))_m$ is tight in $C(\mathbb{R}_+;\mathbb{R})$ whenever $p \geqslant 4$ and $\varphi \in \mathcal{C}$ is such that $\|c_{4p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\| < \infty$. Note that this space contains in particular all the functions of the form $\varphi(\omega) = \omega(f)$ with $f \in C^\infty(\mathbb{T}^2)$, where by $u(f)$ we mean the application of the distribution $\omega \in \mathcal{S}'$ to the test function f. Hence, we can apply Mitoma's criterion [16] to get the tightness of the sequence $(\omega^m)_m$ in $C(\mathbb{R}_+;\mathcal{S}')$.

Step 2. Since $\omega_0^m \sim \eta\mathrm{d}\mu$, any weak limit has initial distribution $\eta\mathrm{d}\mu$. Incompressibility is also clear since, for any $\varphi \in \mathcal{H}$, we have

$$\mathbb{E}[|\varphi(\omega_t)|] \leqslant \liminf_{m\to\infty}\mathbb{E}[|\varphi(\omega_t^m)|] \leqslant \|\eta\|\|\varphi\|.$$

Using cylinder functions, we can pass to the limit in eq. (13) and prove that any accumulation point $(\omega_t)_t$ satisfies eq. (15). It remains to check the martingale characterisation (16). Fix $\varphi \in \mathcal{C}$ and let $(\psi_n)_n \subseteq \mathcal{C}$ be such that $\psi_n \to \mathcal{L}\varphi$ in $(1+\mathcal{L}_\theta)^{1/2}\mathcal{H}$. By convergence in law, incompressibility, eq. (13) and eq. (15), we have that

$$\begin{aligned}&\mathbb{E}\left[\left(\varphi(\omega_t) - \varphi(\omega_s) - \textstyle\int_s^t \mathcal{L}\varphi(\omega_r)\mathrm{d}r\right) G((\omega_r)_{r\in[0,s]})\right] \\ =\ &\lim_{n\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t) - \varphi(\omega_s) - \textstyle\int_s^t \psi_n(\omega_r)\mathrm{d}r\right) G((\omega_r)_{r\in[0,s]})\right] \\ =\ &\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t^m) - \varphi(\omega_s^m) - \textstyle\int_s^t \psi_n(\omega_r^m)\mathrm{d}r\right) G((\omega_r^m)_{r\in[0,s]})\right] \\ =\ &\lim_{m\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t^m) - \varphi(\omega_s^m) - \textstyle\int_s^t \mathcal{L}\varphi(\omega_r^m)\mathrm{d}r\right) G((\omega_r^m)_{r\in[0,s]})\right],\end{aligned}$$

where the exchange of limits in the last line is justified by the uniformity in m of the bound in eq. (13). By dominated convergence in the estimates leading to Lemma 6 one has

$$\|(1-\mathcal{L}_\theta)^{-1/2}(\mathcal{L}\varphi - \mathcal{L}^m\varphi)\| = \|(1-\mathcal{L}_\theta)^{-1/2}(\mathcal{G}\varphi - \mathcal{G}^m\varphi)\| \lesssim o(1)\|(1+\mathcal{N})^{3/2}(1-\mathcal{L}_\theta)^{1/2}\varphi\|$$

as $m \to \infty$. This is enough to conclude (again using eq. (13)) that

$$\begin{aligned}&\lim_{m\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t^m) - \varphi(\omega_s^m) - \textstyle\int_s^t \mathcal{L}\varphi(\omega_r^m)\mathrm{d}r\right) G((\omega_r^m)_{r\in[0,s]})\right] \\ =\ &\lim_{m\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t^m) - \varphi(\omega_s^m) - \textstyle\int_s^t \mathcal{L}^m\varphi(\omega_r^m)\mathrm{d}r\right) G((\omega_r^m)_{r\in[0,s]})\right] = 0,\end{aligned}$$

since $(\omega_t^m)_t$ solves indeed the martingale problem for $\mathcal{L}^m$. This establishes that any accumulation point $(\omega_t)_t$ is a solution to the cylinder martingale problem for $\mathcal{L}$. Similarly, one can pass to the limit on the martingales $(M_t^{m,\varphi})_t$ to show that the limiting quadratic variation is as claimed. □

4 Uniqueness of solutions

Uniqueness of solutions to the cylinder martingale problem depends on the control of the associated Kolmogorov equation.

The following standard fact on generators of semigroups that will be useful in our further considerations. For the sake of the reader we provide also a proof to illustrate the relation between the Kolmogorov equation for a concrete operator and abstract semigroup theory.

Lemma 8 *Let $\mathcal{A}$ be a densely defined, dissipative operator on $\mathcal{H}$ and assume that we can solve the Kolmogorov equation $\partial_t\varphi(t) = \mathcal{A}\varphi(t)$ in $C(\mathbb{R}_+;\mathcal{D}(\mathcal{A}))\cap C^1(\mathbb{R}_+;\mathcal{H})$ with initial condition $\varphi(0)=\varphi_0$ in a dense set $\mathcal{U}_\mathcal{A}\subseteq\mathcal{D}(\mathcal{A})$. Then $\mathcal{A}$ is closable and its closure $\mathcal{B}$ is the unique extension of $\mathcal{A}$ which generates a strongly continuous semigroup of contractions $(T_t)_{t\geqslant 0}$. Moreover, we have*

$$\mathcal{A}T_t\varphi_0 = T_t\mathcal{A}\varphi_0, \tag{18}$$

for all $\varphi_0\in\mathcal{U}_\mathcal{A}$.

Proof Since $\mathcal{A}$ is dissipative, the solution to the Kolmogorov equation is unique and $\|\varphi(t)\|\leqslant\|\varphi_0\|$. Then, if we let $T_t\varphi_0=\varphi(t)$ for $\varphi_0\in\mathcal{U}_\mathcal{A}$ we can extend T_t by continuity to the whole space $\mathcal{H}$ as a contraction. By uniqueness, we have then $T_{t+s}\varphi_0 = T_tT_s\varphi_0$, since $t\mapsto T_{t+s}\varphi_0$ solves the equation with initial condition $T_s\varphi_0$. Moreover, for $\varphi_0\in\mathcal{U}_\mathcal{A}$, we have that

$$T_t\varphi_0-\varphi_0=\int_0^t\mathcal{A}T_s\varphi_0\mathrm{d}s, \tag{19}$$

which implies that $t\mapsto T_t\varphi_0$ is strongly continuous. Again by density, we deduce that $(T_t)_{t\geqslant0}$ is a strongly continuous semigroup. Let now $\mathcal{B}$ be its Hille–Yosida generator. Then (19) implies that $\mathcal{B}\varphi_0 = \partial_tT_t\varphi_0|_{t=0}=\mathcal{A}\varphi_0$ for all $\varphi_0\in\mathcal{U}_\mathcal{A}$, and therefore for all $\varphi_0\in\mathcal{D}(\mathcal{A})$ since $\mathcal{B}$ is closed. So $\mathcal{B}$ is an extension of $\mathcal{A}$ and therefore $\mathcal{A}$ is closable. Assume now that there exists another extension $\tilde{\mathcal{B}}$ which is the generator of another strongly continuous semigroup $(S_t)_{t\geqslant0}$ of contractions. Now, for all $\varphi_0\in\mathcal{U}_\mathcal{A}\subseteq\mathcal{D}(\mathcal{A})\subseteq\mathcal{D}(\tilde{\mathcal{B}})$ we have $\partial_tS_t\varphi_0=\tilde{\mathcal{B}}S_t\varphi_0$, but also $\partial_tT_t\varphi_0=\mathcal{A}T_t\varphi_0=\tilde{\mathcal{B}}T_t\varphi_0$. Since $\tilde{\mathcal{B}}$ is dissipative (due to the fact that its semigroup is contractive), the associated Kolmogorov equation must have a unique solution and, as a consequence, $T_t\varphi_0=S_t\varphi_0$, which by density implies that $T=S$ and that $\mathcal{B}=\tilde{\mathcal{B}}$. Now observe that, if $\varphi_0\in\mathcal{U}_\mathcal{A}$, then $T_t\varphi_0\in\mathcal{D}(\mathcal{A})$ and by standard results on contraction semigroups (see e.g. Proposition 1.1.5 in [8]) we have $\mathcal{A}T_t\varphi_0=\mathcal{B}T_t\varphi_0=T_t\mathcal{B}\varphi_0=T_t\mathcal{A}\varphi_0$. □

Theorem 3 below tells us that we can find a dense domain $\mathcal{D}(\mathcal{L})\subseteq\mathcal{H}$ for $\mathcal{L}$ such that the Kolmogorov equation

$$\partial_t\varphi(t)=\mathcal{L}\varphi(t),\qquad t\geqslant0, \tag{20}$$

has a unique solution in $C(\mathbb{R}_+;\mathcal{D}(\mathcal{L}))\cap C^1(\mathbb{R}_+;\mathcal{H})$ for any initial condition in a dense set $\mathcal{U}\subseteq\mathcal{H}$. As a first consequence, Lemma 8 tells us that $\mathcal{L}$ is closable and its closure $\mathcal{L}^\natural$ is the generator of a strongly continuous semigroup $(T_t)_{t\geqslant0}$ and $\varphi(t)=T_t\varphi$ for all $\varphi\in\mathcal{U}$.

Lemma 9 *Let $\varphi\in C(\mathbb{R}_+;\mathcal{D}(\mathcal{L}))\cap C^1(\mathbb{R}_+;\mathcal{H})$ and let ω be a solution to the cylinder martingale problem for $\mathcal{L}$. Then*

$$\varphi(t,\omega_t)-\varphi(0,\omega_0)-\int_0^t(\partial_s+\mathcal{L})\varphi(s,\omega_s)\mathrm{d}s,\qquad t\geqslant0,$$

is a martingale.

Proof By an approximation argument, is easy to see that for any $\varphi \in \mathcal{D}(\mathcal{L})$ the process

$$\varphi(\omega_t) - \varphi(\omega_0) - \int_0^t \mathcal{L}\varphi(\omega_s)\mathrm{d}s, \qquad t \geqslant 0,$$

is a martingale, where the integral on the right-hand side is now understood as a standard Lebesgue integral of the continuous process $s \mapsto (\mathcal{L}\varphi)(\omega_s)$ (which is well defined a.s.). The proof of the extension to time-dependent functions follows the same lines as that of Lemma A.3 in [14]. □

For an incompressible process we have that, for all $t \geqslant 0$,

$$\int_0^s (\partial_r + \mathcal{L})T_{t-r}\varphi(\omega_r)\mathrm{d}r = 0, \qquad s \in [0,t]$$

for all $\varphi \in \mathcal{D}(\mathcal{L}^\natural)$, and therefore also that $(T_{t-s}\varphi(\omega_s))_{s\in[0,t]}$ is a martingale for any solution of the cylinder martingale problem for $\mathcal{L}$. This easily implies the main result of the paper.

Theorem 2 *There exists a unique solution ω to the cylinder martingale problem for $\mathcal{L}$ with initial distribution $\omega_0 \sim \eta\mathrm{d}\mu$ with $\eta \in L^2(\mu)$. Moreover, ω is a homogeneous Markov process with transition kernel $(T_t)_{t\geqslant 0}$ and with invariant measure μ.*

Proof Let us first prove that $(\omega_t)_{t\geqslant 0}$ is Markov. Let $0 \leqslant t < s$, let X be an $\mathcal{F}_t$-measurable bounded random variable, where $\mathcal{F}_t = \sigma(\omega_r : r \in [0,t])$, and let $\varphi \in \mathcal{D}(\mathcal{L}^\natural)$, then $(T_{s-t}\varphi(\omega_t))_{t\in[0,s]}$ is a martingale and

$$\mathbb{E}[X\varphi(\omega_s)] \quad = \quad \mathbb{E}[XT_{s-s}\varphi(\omega_s)] = \mathbb{E}[XT_{s-t}\varphi(\omega_t)]$$

i.e., $\mathbb{E}[\varphi_0(\omega_s)|\mathcal{F}_t] = T_{s-t}\varphi(\omega_t) = \mathbb{E}[\varphi_0(\omega_s)|\omega_t]$, and the Markov property is a consequence of another density argument. Moreover, its transition kernel is given by the semigroup $(T_t)_{t\geqslant 0}$. By an induction argument, it is clear that any finite-dimensional marginal is determined by T and by the law of $\omega_0 \sim \eta\mathrm{d}\mu$. As a consequence, the law of the process is unique. If $\omega_0 \sim \mu$, then the process is stationary. □

Remark 2 As a by-product note that the formula $(T_t\varphi)(\omega_0) = \mathbb{E}[\varphi(\omega_t)|\omega_0]$ allows to extend the semigroup T to a bounded semigroup in L^p for all $p \in [1,\infty]$ since

$$|\langle\psi, T_t\varphi\rangle| = |\mathbb{E}_\mu[\psi(\omega_0)(T_t\varphi)(\omega_0)]| = |\mathbb{E}_\mu[\psi(\omega_0)\varphi(\omega_t)]| \leqslant \|\psi\|_{L^p}\|\varphi\|_{L^q}$$

for all $\psi, \varphi \in L^\infty(\mu)$ and all $p, q \in [1,\infty]$ with $1/p + 1/q = 1$. Therefore $\|T_t\varphi\|_{L^q} \leqslant \|\varphi\|_{L^q}$. Moreover for all $\varphi \in \mathcal{C}$ such that $\|c_{4p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\| < \infty$ we have

$$\|T_t\varphi - \varphi\|_{L^p} = \sup_{\psi:\|\psi\|_{L^q}\leqslant 1} \mathbb{E}_\mu[\psi(\omega_0)(\varphi(\omega_t) - \varphi(\omega_0))] \leqslant (\mathbb{E}_\mu[|\varphi(\omega_t) - \varphi(\omega_0)|^p])^{1/p} \to 0$$

as $t \to 0$ by eq. (17). An approximation argument gives that $(T_t)_{t\geqslant 0}$ is strongly continuous in L^p for all $1 \leqslant p < \infty$.

5 The Kolmogorov equation

It remains to determine a suitable domain for $\mathcal{L}$ and solve the Kolmogorov backward equation

$$\partial_t\varphi(t) = \mathcal{L}\varphi(t),$$

for a sufficiently large class of initial data. In order to do so, we consider the backward equation for the Galerkin approximation with generator $\mathcal{L}^m$ and derive uniform estimates. By compactness, this yields the existence of strong solutions to the backward equation after removing the cutoff. Uniqueness follows by the dissipativity of $\mathcal{L}$.

5.1 A priori estimates

Lemma 10 *For any $\varphi_0 \in \mathcal{V} := (1+\mathcal{N})^{-2}(1-\mathcal{L}_\theta)^{-1}\mathcal{H}$, there exists a solution*

$$\varphi^m \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L}^m)) \cap C^1(\mathbb{R}_+; \mathcal{H})$$

to the backward Kolmogorov equation

$$\partial_t \varphi^m(t) = \mathcal{L}^m \varphi^m(t)$$

with $\varphi^m(0) = \varphi_0$ and which satisfies the estimates

$$\|(1+\mathcal{N})^p \varphi^m(t)\|^2 + \int_0^t e^{-C(t-s)} \|(1+\mathcal{N})^p (1-\mathcal{L}_\theta)^{1/2} \varphi^m(s)\|^2 \mathrm{d}s \lesssim_p e^{Ct} \|(1+\mathcal{N})^p \varphi_0\|^2,$$

$$\|(1+\mathcal{N})^p (1-\mathcal{L}_\theta) \varphi^m(t)\| \lesssim_{t,m,p} \|(1+\mathcal{N})^{p+1} (1-\mathcal{L}_\theta)\varphi_0\|,$$

for all $t \geqslant 0$ and $p \geqslant 1$.

Proof Take $h > 0$ and let $\mathcal{G}^{m,h} = J_h \mathcal{G}^m J_h$, where $J_h = e^{-h(\mathcal{N}-\mathcal{L}_\theta)}$. The operator $\mathcal{G}^{m,h}$ is bounded on $\mathcal{H}$ by the estimates in Lemma 4. Consider $\varphi_0^m \in \mathcal{D}(\mathcal{L}^m)$. Using the fact that $\mathcal{L}_\theta$ is the generator of a contraction semigroup, we take $(\varphi^m(t))_{t\geqslant 0}$ to be the solution to the integral equation

$$\varphi^{m,h}(t) = e^{\mathcal{L}_\theta t}\varphi_0 + \int_0^t e^{\mathcal{L}_\theta (t-s)} \mathcal{G}^{m,h} \varphi^{m,h}(s) \mathrm{d}s \tag{21}$$

in $C(\mathbb{R}_+; (1-\mathcal{L}_\theta)\mathcal{H})$ and deduce easily that $\varphi^{m,h}$ solves the equation $\partial_t \varphi^{m,h}(t) = (\mathcal{L}_\theta + \mathcal{G}^{m,h})\varphi^{m,h}(t)$. Moreover

$$\|(1-\mathcal{L}_\theta)(1+\mathcal{N})^{2p}\varphi^{m,h}(t)\| \leqslant C_{t,h,m} \|(1-\mathcal{L}_\theta)(1+\mathcal{N})^{2p}\varphi_0\|,$$

for any finite $t \geqslant 0$ and $p > 0$ but not uniformly in h and m. Now

$$\begin{aligned}
&\langle (1+\mathcal{N})^{2p}\varphi^{m,h}(t), \mathcal{G}^{m,h}\varphi^{m,h}(t)\rangle \\
&= \langle (1+\mathcal{N})^{2p}\varphi^{m,h}(t), \mathcal{G}_+^{m,h}\varphi^{m,h}(t)\rangle - \langle \mathcal{G}_+^{m,h}(1+\mathcal{N})^{2p}\varphi^{m,h}(t), \varphi^{m,h}(t)\rangle \\
&= \langle (1+\mathcal{N})^{2p}\varphi^{m,h}(t), \mathcal{G}_+^{m,h}\varphi^{m,h}(t)\rangle - \langle \mathcal{N}^{2p}\mathcal{G}_+^{m,h}\varphi^{m,h}(t), \varphi^{m,h}(t)\rangle \\
&= \langle ((1+\mathcal{N})^{2p} - \mathcal{N}^{2p})\varphi^{m,h}(t), \mathcal{G}_+^{m,h}\varphi^{m,h}(t)\rangle.
\end{aligned}$$

Using $|(1+\mathcal{N})^{2p} - \mathcal{N}^{2p}| \lesssim (1+\mathcal{N})^{2p-1}$ and the uniform estimates in Lemma 4 we have that, for some $\sigma \in (0, 1/2)$,

$$\begin{aligned}
&|\langle ((1+\mathcal{N})^{2p} - \mathcal{N}^{2p})\varphi^{m,h}(t), \mathcal{G}_+^{m,h}\varphi^{m,h}(t)\rangle| \\
&\leqslant \|(1+\mathcal{N})^p (1-\mathcal{L}_\theta)^\sigma \varphi^{m,h}(t)\| \|(1+\mathcal{N})^{p-1}(1-\mathcal{L}_\theta)^{-\sigma}\mathcal{G}_+^{m,h}\varphi^{m,h}(t)\| \\
&\leqslant \|(1+\mathcal{N})^p (1-\mathcal{L}_\theta)^\sigma \varphi^{m,h}(t)\|^2.
\end{aligned}$$

Therefore,

$$|\langle (1+\mathcal{N})^{2p}\varphi^{m,h}(t), \mathcal{G}^{m,h}\varphi^{m,h}(t)\rangle| \lesssim \|(1+\mathcal{N})^p (1-\mathcal{L}_\theta)^\sigma \varphi^{m,h}(t)\|^2$$

and by interpolation we can bound this by

$$|\langle (1+\mathcal{N})^{2p}\varphi^{m,h}(t), \mathcal{G}^{m,h}\varphi^{m,h}(t)\rangle| \leqslant C_\delta \|(1+\mathcal{N})^p \varphi^{m,h}(t)\|^2 + \delta \|(1+\mathcal{N})^p (1-\mathcal{L}_\theta)^{1/2}\varphi^{m,h}(t)\|^2,$$

for some small $\delta > 0$. Therefore, we have

$$\partial_t \frac{1}{2}\|(1+\mathcal{N})^p \varphi^{m,h}(t)\|^2 = \langle (1+\mathcal{N})^{2p}\varphi^{m,h}(t), (\mathcal{L}_\theta + \mathcal{G}^{m,h})\varphi^{m,h}(t)\rangle$$

$$= -\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,h}(t)\|^2 + \|(1+\mathcal{N})^p\varphi^{m,h}(t)\|^2 + \langle(1+\mathcal{N})^{2p}\varphi^{m,h}(t), \mathcal{G}^{m,h}\varphi^{m,h}(t)\rangle$$
$$\leqslant -(1-\delta)\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,h}(t)\|^2 + C'_\delta\|(1+\mathcal{N})^p\varphi^{m,h}(t)\|^2$$

uniformly in m and h. Integrating this inequality gives

$$\|(1+\mathcal{N})^p\varphi^{m,h}(t)\|^2 + \int_0^t e^{-C(t-s)}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,h}(s)\|^2\mathrm{d}s \lesssim e^{Ct}\|(1+\mathcal{N})^p\varphi_0\|^2$$

for all $p \geqslant 1$ where the constants are uniform in m and h. Inserting this a priori bound in the mild formulation in eq. (21) we obtain

$$\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi^{m,h}(t)\| \leqslant \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi^{m,h}(0)\| + \int_0^t \|(1+\mathcal{N})^{p+1}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,h}(s)\|\mathrm{d}s$$
$$\lesssim_t \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi_0\| + \|(1+\mathcal{N})^{p+1}\varphi_0\|,$$

where we also used that

$$\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\mathcal{G}^{m,h}\varphi^{m,h}(s)\| \leqslant C(m)\|(1+\mathcal{N})^{p+1}\mathcal{G}^{m,h}\varphi^{m,h}(s)\|$$
$$\lesssim_m \|(1+\mathcal{N})^{p+1}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,h}(s)\|$$

by the presence of the Galerkin projectors and our (non-uniform) bounds. Indeed, note that

$$(1-\mathcal{L}_\theta)\Pi_m \lesssim |m|^{2\theta}(1+\mathcal{N})\Pi_m.$$

We conclude that

$$\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi^{m,h}(t)\| \lesssim_{t,m} \|(1+\mathcal{N})^{p+1}(1-\mathcal{L}_\theta)\varphi_0\|,$$

uniformly in h. We can then pass to the limit (by subsequence) as $h \to 0$ and obtain a function $\varphi^m \in C(\mathbb{R}_+, (1+\mathcal{N})^{-p}(1-\mathcal{L}_\theta)^{-1}\mathcal{H})$ satisfying the estimates

$$\|(1+\mathcal{N})^p\varphi^m(t)\|^2 + \int_0^t e^{-C(t-s)}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^m(s)\|^2\mathrm{d}s \lesssim e^{Ct}\|(1+\mathcal{N})^p\varphi_0\|^2$$

and

$$\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi^m(t)\| \lesssim_{t,m} \|(1+\mathcal{N})^{p+1}(1-\mathcal{L}_\theta)\varphi_0\|,$$

for all $t \geqslant 0$ and $p \geqslant 1$. As a consequence, $\varphi^m \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L}^m))$ for all $t \geqslant 0$ as soon as $\|(1+\mathcal{N})^2(1-\mathcal{L}_\theta)\varphi_0\| < \infty$. By passing to the limit in the equation, φ^m also satisfies

$$\partial_t\varphi^m(t) = (\mathcal{L}_\theta + \mathcal{G}^m)\varphi^m(t) = \mathcal{L}^m\varphi^m(t).$$

□

Recall that we write T^m to indicate the semigroup generated by the Galerkin approximation ω^m. Moreover, if we denote by $\hat{\mathcal{L}}^m$ its Hille–Yosida generator, we have the following result.

Lemma 11 *$(\mathcal{L}^m, \mathcal{D}(\mathcal{L}^m))$ is closable and its closure is the generator $\hat{\mathcal{L}}^m$. In particular, if $\varphi \in \mathcal{V}$, then $\varphi^m(t) = T^m_t\varphi$ solves*

$$\partial_t\varphi^m(t) = \mathcal{L}^m\varphi^m(t),$$

and we have

$$\mathcal{L}^m T^m_t\varphi = T^m_t\mathcal{L}^m\varphi.$$

Proof Let $(\omega_t^m)_{t\geqslant 0}$ be a solution to the Galerkin approximation (3) with initial condition ω_0. If $\varphi \in \mathcal{C}$ is a cylinder function, then we have

$$T_t^m\varphi(\omega_0) - \varphi(\omega_0) = \mathbb{E}_{\omega_0}\left[\int_0^t \mathcal{L}^m\varphi(\omega_s^m)\mathrm{d}s\right] = \int_0^t T_s^m(\mathcal{L}^m\varphi)(\omega_0)\mathrm{d}s.$$

By approximation (using a Bochner integral in $\mathcal{H}$ on the right-hand side), we can extend this point-wise formula to all $\varphi \in \mathcal{D}(\mathcal{L}^m)$ obtaining for them that $T_t^m\varphi - \varphi = \int_0^t T_s^m\mathcal{L}^m\varphi\mathrm{d}s$ in $\mathcal{H}$. For every $\varphi \in \mathcal{D}(\mathcal{L}^m)$, Lemma 1 implies that the map $s \mapsto T_s^m\mathcal{L}^m\varphi \in \mathcal{H}$ is continuous, and therefore

$$\frac{T_t^m\varphi - \varphi}{t} \to \mathcal{L}^m\varphi, \qquad \text{as } t \to 0, \qquad \varphi \in \mathcal{D}(\mathcal{L}^m),$$

with convergence in $\mathcal{H}$. As a consequence, $\varphi \in \mathcal{D}(\hat{\mathcal{L}}^m)$ and we conclude that $\hat{\mathcal{L}}^m$ is an extension of $(\mathcal{L}^m, \mathcal{D}(\mathcal{L}^m))$. By Lemma 8, we have that the closure of $\mathcal{L}^m$ is $\hat{\mathcal{L}}^m$ and that $\mathcal{L}^m T_t^m\varphi = T_t^m\mathcal{L}^m\varphi$ for all $\varphi \in \mathcal{V}$. □

Using the commutation $\mathcal{L}^m T_t^m\varphi = T_t^m\mathcal{L}^m\varphi$, we are able to get better estimates, uniform in m.

Corollary 1 *For all $\varphi_0 \in \mathcal{V}$ and for all $\alpha \geqslant 1$, we have*

$$\|(1+\mathcal{N})^\alpha\partial_t\varphi^m(t)\|^2 = \|(1+\mathcal{N})^\alpha\mathcal{L}^m\varphi^m(t)\|^2 \lesssim e^{tC}\|(1+\mathcal{N})^\alpha\mathcal{L}^m\varphi_0\|^2, \tag{22}$$

and

$$\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\varphi^m(t)\|^2 \lesssim te^{tC}\|(1+\mathcal{N})^\alpha\mathcal{L}^m\varphi_0^m\|^2 + \|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\varphi_0\|^2. \tag{23}$$

Proof Recall $T_t^m\varphi_0^m = \varphi^m(t)$. We already know

$$e^{-tC}\|(1+\mathcal{N})^\alpha T_t^m\varphi_0^m\|^2 + \int_0^\infty e^{-sC}\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}T_s^m\varphi_0^m\|^2\mathrm{d}s \lesssim \|(1+\mathcal{N})^\alpha\varphi_0\|^2,$$

which yields

$$\begin{aligned}\int_0^\infty e^{-tC}\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\partial_t T_t^m\varphi_0^m\|^2\mathrm{d}t &= \int_0^\infty e^{-tC}\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}T_t^m\mathcal{L}^m\varphi_0\|^2\mathrm{d}t \\ &\lesssim \|(1+\mathcal{N})^\alpha\mathcal{L}^m\varphi_0\|^2,\end{aligned}$$

and

$$\begin{aligned}&\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}T_t^m\varphi_0\|^2 \\ \lesssim\ &\left\|\textstyle\int_0^t(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\partial_s T_s^m\varphi_0^m\mathrm{d}s\right\|^2 + \|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\varphi_0\|^2 \\ \leqslant\ &t\textstyle\int_0^t\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\partial_s T_s^m\varphi_0\|^2\mathrm{d}s + \|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\varphi_0\|^2 \\ \leqslant\ &te^{tC}\textstyle\int_0^t e^{-sC}\|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\partial_s T_s^m\varphi_0\|^2\mathrm{d}s + \|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\varphi_0\|^2 \\ \lesssim\ &te^{tC}\|(1+\mathcal{N})^\alpha\mathcal{L}^m\varphi_0\|^2 + \|(1+\mathcal{N})^\alpha(1-\mathcal{L}_\theta)^{1/2}\varphi_0\|^2,\end{aligned}$$

which is what claimed. □

5.2 Controlled structures

The a priori bounds (22) and (23) bring us in position to control $\|\varphi^m(t)\|$, $\|\partial_t\varphi^m(t)\|$, and $\|\mathcal{L}^m\varphi^m(t)\|$ uniformly in m and locally uniformly in t, but in order to study the limiting Kolmogorov backward equation we have first to deal with the limiting operator $\mathcal{L}$ and to define a domain $\mathcal{D}(\mathcal{L})$.

To take care of the term $\mathcal{G}$ in the limiting operator $\mathcal{L}$, we decompose it by means of a cut-off function $\mathcal{M} = M(\mathcal{N})$ as follows

$$\mathcal{G}^m = \mathbb{1}_{|\mathcal{L}_\theta| \geqslant \mathcal{M}} \mathcal{G}^m + \mathbb{1}_{|\mathcal{L}_\theta| < \mathcal{M}} \mathcal{G}^m =: \mathcal{G}^{m,\succ} + \mathcal{G}^{m,\prec}.$$

We then set

$$\varphi^{m,\sharp} := \varphi^m - (1-\mathcal{L}_\theta)^{-1} \mathcal{G}^{m,\succ} \varphi^m, \tag{24}$$

so that

$$(1-\mathcal{L}^m)\varphi^m = (1-\mathcal{L}_\theta)\varphi^{m,\sharp} + \mathcal{G}^{m,\prec}\varphi^m. \tag{25}$$

Lemma 12 *Let w be a weight, $L \geqslant 1$, $\bar{\varepsilon} \in]0, (\theta-1)/(2\theta)[$ and $M(n) = L(n+1)^{3\theta/(\theta-1-2\theta\varepsilon)}$. Then we have*

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}^{m,\succ}\psi\| \lesssim |w| L^{-\frac{(\theta-1)}{2\theta}+\bar{\varepsilon}} \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\|. \tag{26}$$

Consequently, there exists $L_0 = L_0(|w|)$ such that, for all $L \geqslant L_0$ and all $\varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$, there is a unique $\varphi^m = \mathcal{K}^m\varphi^\sharp$ such that

$$\varphi^m = (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}\varphi^m + \varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H},$$

which satisfies the bound

$$\begin{array}{rl} & \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\mathcal{K}^m\varphi^\sharp\| + |w|^{-1}L^{(\theta-1)/(2\theta)-\bar{\varepsilon}}\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}(\mathcal{K}^m\varphi^\sharp - \varphi^\sharp)\| \\ \lesssim & \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|. \end{array} \tag{27}$$

All the estimates are uniform in m and true in the limit $m \to \infty$. We denote $\mathcal{K} = \mathcal{K}^\infty$.

Proof We start with the estimate on $\mathcal{G}_+^{m,\succ}$. We have, for $\varepsilon \in]0, 1/2 - 1/(2\theta)[$, using Lemma 4,

$$\begin{array}{rcl} \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}_+^{m,\succ}\psi\| & \lesssim & \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathbb{1}_{|\mathcal{L}_\theta|\geqslant M(\mathcal{N})}\mathcal{G}_+^m\psi\| \\ & \lesssim & \|w(\mathcal{N})M(\mathcal{N})^{-1/2+1/(2\theta)+\varepsilon}(1-\mathcal{L}_\theta)^{-1/(2\theta)-\varepsilon}\mathcal{G}_+^m\psi\| \\ & \lesssim & \|w(\mathcal{N}+1)M(\mathcal{N}+1)^{-1/2+1/(2\theta)+\varepsilon}(\mathcal{N}+1)(1-\mathcal{L}_\theta)^{1/2-\varepsilon}\psi\|. \end{array}$$

The bound on $\mathcal{G}_-^{m,\succ}$ can be obtained using again Lemma 4:

$$\begin{array}{rcl} \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}_-^{m,\succ}\psi\| & \lesssim & \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathbb{1}_{|\mathcal{L}_\theta|\geqslant M(\mathcal{N})}\mathcal{G}_-^m\psi\| \\ & \lesssim & \|w(\mathcal{N})M(\mathcal{N})^{-1/2+1/(2\theta)}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-^m\psi\| \\ & \lesssim & \|w(\mathcal{N}-1)M(\mathcal{N}-1)^{-1/2+1/(2\theta)}\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{1/2}\psi\|. \end{array}$$

In conclusion, for $\varepsilon \in]0, (\theta-1)/(2\theta)[$, choosing $M(n) = L(n+1)^{3\theta/(\theta-1-2\theta\varepsilon)}$, for $L \geqslant 1$,

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}^{m,\succ}\psi\| \lesssim L^{-1/2+1/(2\theta)+\varepsilon}|w|\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\|.$$

Now let $\varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$, the map

$$\begin{array}{c} \Psi^m : w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H} \to w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}, \\ \psi \mapsto \Psi^m(\psi) := (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}\psi + \varphi^\sharp, \end{array}$$

satisfies, for some positive constant C,

$$\begin{array}{rcl} \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\Psi^m(\psi)\| & \leqslant & \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}^{m,\succ}\psi\| + \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\| \\ & \leqslant & CL^{-1/2+1/(2\theta)+\varepsilon}|w|\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\| + \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|. \end{array}$$

Namely, Ψ^m is well-defined and, choosing L large enough, it is a contraction leaving the ball of radius $2\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|$ invariant. Therefore, it has a unique fixed point $\mathcal{K}^m\varphi^\sharp$ satisfying the claimed inequalities. □

Remark 3 In the previous lemma, the cut-off $M(n)$ depends via $|w|$ on the weight w. In the following we will only use *polynomial weights* of the form $w(n) = (1+n)^\alpha$ with $|\alpha| \leqslant K$ for a fixed K. In this case $|w|$ is uniformly bounded and it is possible to select a cut-off which is adapted to all those weights. This will be fixed once and for all and not discussed further.

Proposition 2 *Let w be a polynomial weight, $\gamma \geqslant 0$, $\bar\varepsilon$ as in Lemma 12,*

$$\alpha(\gamma) = \frac{\theta(6\gamma+5)-2}{2(\theta-1)}.$$

Let

$$\varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(1+\mathcal{N})^{-\alpha(\gamma)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H},$$

and set $\varphi^m := \mathcal{K}^m\varphi^\sharp$. Then $\mathcal{L}^m\varphi^m$ is a well-defined operator and we have the bound

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma \mathcal{G}^{m,\prec}\varphi^m\| \lesssim \|w(\mathcal{N})(1+\mathcal{N})^{\alpha(\gamma)}(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|. \tag{28}$$

Proof By eq. (25) we need only to estimate $\mathcal{G}^{m,\prec}\varphi^m$. We first deal with $\mathcal{G}_+^{m,\prec}$: we have by (9), for $\delta < 1/2 - 1/(2\theta)$,

$$\begin{aligned}
\|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma \mathcal{G}_+^{m,\prec}\varphi^m\| &= \|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma \mathbb{1}_{|\mathcal{L}_\theta|<M(\mathcal{N})}\mathcal{G}_+^m\varphi^m\| \\
&\lesssim \|w(\mathcal{N})M(\mathcal{N})^{\gamma+1/2-\delta}(1-\mathcal{L}_\theta)^{-1/2+\delta}\mathcal{G}_+^m\varphi^m\| \\
&\lesssim \|w(\mathcal{N}+1)M(\mathcal{N}+1)^{\gamma+1/2}(\mathcal{N}+1)(1-\mathcal{L}_\theta)^{1/(2\theta)+\delta}\varphi^m\|.
\end{aligned}$$

For $\mathcal{G}_-^{m,\prec}$, it follows in a similar way from estimate (10) that, for every $\delta \in]0, 1/(2\theta)]$,

$$\begin{aligned}
\|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma \mathcal{G}_-^{m,\prec}\varphi^m\| &= \|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma \mathbb{1}_{|\mathcal{L}_\theta|<M(\mathcal{N})}\mathcal{G}_-^m\varphi^m\| \\
&\lesssim \|w(\mathcal{N})M(\mathcal{N})^{\gamma+1/(2\theta)}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-^m\varphi^m\| \\
&\lesssim \|w(\mathcal{N}-1)M(\mathcal{N}-1)^{\gamma+1/(2\theta)}\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{1/2}\varphi^m\|.
\end{aligned}$$

These bounds and the definition of $M(n)$ give the claimed bound on $\mathcal{G}^{m,\prec}$. □

5.3 Limiting generator and its domain

Lemma 13 *Let w be a weight and take a cut-off function as in Proposition 2 with $\gamma = 0$. Set*

$$\mathcal{D}_w(\mathcal{L}) := \{\mathcal{K}\varphi^\sharp : \varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}\}.$$

Then $\mathcal{D}_w(\mathcal{L})$ is dense in $w(\mathcal{N})^{-1}\mathcal{H}$. If $w \equiv 1$ we simply write $\mathcal{D}(\mathcal{L})$.

Proof Note that $w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$ is dense in $w(\mathcal{N})^{-1}\mathcal{H}$, therefore, in order to prove Lemma 13, it suffices to show that, for any $\psi \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$ and for all $\nu \geqslant 1$, there exists $\varphi^\nu \in \mathcal{D}_w(\mathcal{L})$ such that

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}(\varphi^\nu - \psi)\| \lesssim \nu^{-(\theta-1)/(2\theta)+\bar\varepsilon}\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\|, \tag{29}$$

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\nu\| \lesssim \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\|, \tag{30}$$

$$\begin{aligned}
\|w(\mathcal{N})(1-\mathcal{L})\varphi^\nu\| \lesssim{}& \nu^{1/(2\theta)+\delta}(\|w(\mathcal{N})(1-\mathcal{L}_\theta)\psi\| \\
&+\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi\|),
\end{aligned} \tag{31}$$

for some $\delta > 0$. By Lemma 12, there exists $\varphi^\nu \in w(\mathcal{N})^{-1}\mathcal{H}$ such that

$$\varphi^\nu = \mathbb{1}_{\nu M(\mathcal{N}) \leqslant |\mathcal{L}_\theta|}(1-\mathcal{L}_\theta)^{-1}\mathcal{G}\varphi^\nu + \psi$$

and satisfying estimates (29)–(30). We are left to show that $\varphi^\nu \in \mathcal{D}_w(\mathcal{L})$ and (31). Note that

$$\varphi^\nu = (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^\succ\varphi^\nu + \varphi^{\nu,\sharp},$$

where

$$\varphi^{\nu,\sharp} = \psi - \mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}(1-\mathcal{L}_\theta)^{-1}\mathcal{G}\varphi^\nu.$$

In particular, we have $\mathcal{L}\varphi^\nu = \varphi^\nu + \mathcal{G}^\prec\varphi^\nu - (1-\mathcal{L}_\theta)\varphi^{\nu,\sharp}$, and, by Proposition 2, it suffices to estimate $\varphi^{\nu,\sharp}$ in $w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H}\cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$. The first contribution, ψ, satisfies the required bounds by assumption, so it is enough to show that the second contribution, which we denote by ψ^ν, satisfies

$$\begin{aligned}
\|w(\mathcal{N})(1-\mathcal{L}_\theta)\psi^\nu\| &\lesssim \nu^{1/(2\theta)+\delta}\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi\|, &(32)\\
\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi^\nu\| &\lesssim \|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi\|. &(33)
\end{aligned}$$

Notice that $(1-\mathcal{L}_\theta)\psi^\nu = -\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}\mathcal{G}\varphi^\nu$, hence estimate (32) can be obtained from the uniform bounds in Lemma 4 as follows (note that those bounds are valid also when $m=+\infty$). We have, for $\mathcal{G}_+$,

$$\begin{aligned}
&\|w(\mathcal{N})\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}\mathcal{G}_+\varphi\|\\
\lesssim\ &\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/(2\theta)+\delta}\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}(1-\mathcal{L}_\theta)^{-1/(2\theta)-\delta}\mathcal{G}_+\varphi\|\\
\lesssim\ &\nu^{1/(2\theta)+\delta}\|w(\mathcal{N}+1)M(\mathcal{N}+1)^{1/(2\theta)+\delta}(\mathcal{N}+1)(1-\mathcal{L}_\theta)^{1/2-\delta}\varphi\|.
\end{aligned}$$

For $\mathcal{G}_-$ we have, instead

$$\begin{aligned}
&\|w(\mathcal{N})\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}\mathcal{G}_-\varphi\|\\
\lesssim\ &\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/(2\theta)}\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-\varphi\|\\
\lesssim\ &\nu^{1/(2\theta)}\|w(\mathcal{N}-1)M(\mathcal{N}-1)^{1/(2\theta)}\mathcal{N}^{3/2}\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-\varphi\|,
\end{aligned}$$

which gives estimate (32) if we choose $\bar\varepsilon$ small enough. In order to obtain estimate (33), note that, for $\kappa\in]0,(\theta-1)/(2\theta)[$,

$$\begin{aligned}
\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi^\nu\| &= \|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathbb{1}_{M(\mathcal{N})\leqslant|\mathcal{L}_\theta|<\nu M(\mathcal{N})}\mathcal{G}\varphi^\nu\|\\
&\lesssim M(n)^{-(\theta-1)/\theta+2\kappa}\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{-1/(2\theta)-\kappa}\mathcal{G}_+\varphi^\nu\|\\
&\quad + M(n)^{-(\theta-1)/\theta+2\kappa}\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-\varphi^\nu\|
\end{aligned}$$

Now recall that $M(n)\simeq(n+1)^{3\theta/(\theta-1-2\theta\bar\varepsilon)}$ and get by (9)–(10) the inequality

$$\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi^\nu\| \lesssim \|w(\mathcal{N})(1+\mathcal{N})^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi^\nu\|.$$

Applying (27) yields the result. □

Lemma 14 *For any $\varphi\in\mathcal{D}(\mathcal{L})$, we have*

$$\langle\varphi,\mathcal{L}\varphi\rangle \leqslant 0.$$

In particular, the operator $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ is dissipative.

Proof Notice that $\varphi\in\mathcal{D}(\mathcal{L})$ implies $\mathcal{L}_\theta\varphi,\mathcal{G}\varphi\in(1-\mathcal{L}_\theta)^{1/2}\mathcal{H}$ and $\varphi\in(1-\mathcal{L}_\theta)^{-1/2}(1+\mathcal{N})^{-1}\mathcal{H}$. These regularities are enough to proceed by approximation and establish that

$$\langle\varphi,\mathcal{L}\varphi\rangle = -\langle\varphi,(-\mathcal{L}_\theta)\varphi\rangle + \langle\varphi,\mathcal{G}\varphi\rangle = -\langle\varphi,(-\mathcal{L}_\theta)\varphi\rangle = -\|(-\mathcal{L}_\theta)^{1/2}\varphi\|^2 \leqslant 0,$$

where we used the anti-symmetry of the form associated to $\mathcal{G}$, i.e. $\langle\varphi,\mathcal{G}\varphi\rangle = 0$. □

5.4 Existence and uniqueness for the Kolmogorov equation

Having defined a domain for $\mathcal{L}$ it remains to study the Kolmogorov equation $\partial_t \varphi = \mathcal{L}\varphi$. In particular, we consider the equation for $\varphi^{m,\sharp}$, which was defined in (24),

$$\begin{aligned}\partial_t \varphi^{m,\sharp} + (1-\mathcal{L}_\theta)\varphi^{m,\sharp} &= \mathcal{L}^m \varphi^m + (1-\mathcal{L}_\theta)\varphi^{m,\sharp} - (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}\partial_t \varphi^m \\ &= \varphi^m + \mathcal{G}^{m,\prec}\varphi^m - (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}\partial_t \varphi^m \\ &= \varphi^m + \mathcal{G}^{m,\prec}\varphi^m - (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}(\varphi^m + \mathcal{G}^{m,\prec}\varphi^m - (1-\mathcal{L}_\theta)\varphi^{m,\sharp}) \\ &=: \Phi^{m,\sharp}.\end{aligned}$$

We want to get a suitable bound in terms of $\varphi_0^{m,\sharp}$ for each term of $\Phi^{m,\sharp}$. The Schauder estimate in Lemma 17 will be crucial. We will also need the following result.

Lemma 15 *We have*

$$\begin{aligned}&\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(t)\| \\ \lesssim\ & (te^{tC}+1)^{1/2}(\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\| + \|(1+\mathcal{N})^{p+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\|).\end{aligned} \tag{34}$$

Proof By (23) and Lemma 12 it follows that

$$\begin{aligned}&\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(t)\| \\ \lesssim\ & \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^m(t)\| + \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}^{m,\succ}\varphi^m(t)\| \\ \lesssim\ & te^{tC}\|(1+\mathcal{N})^p\mathcal{L}^m\varphi_0^m\| + \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi_0^m\| \\ \lesssim\ & (te^{tC}+1)^{1/2}(\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\| + \|(1+\mathcal{N})^{p+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\|),\end{aligned}$$

where in the last step we exploited Proposition 2. □

For $\gamma \in]1/2, 1-1/(2\theta)[$, we have that, by the estimates (9) and (10),

$$\begin{aligned}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma-1}\mathcal{G}^{m,\succ}(1-\mathcal{L}_\theta)\varphi^{m,\sharp}(s)\| &\lesssim \|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma+1/2+1/(2\theta)}\varphi^{m,\sharp}(s)\| \\ &\lesssim \|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma+1/2+1/(2\theta)}\varphi^{m,\sharp}(s)\|.\end{aligned}$$

By interpolation for products, there exists $q>0$ such that, for all $\varepsilon \in]0,1[$,

$$\begin{aligned}\|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma+1/2+1/(2\theta)}\varphi^{m,\sharp}(s)\| \lesssim\ & C_\varepsilon\|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\| \\ & +\varepsilon\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma+1}\varphi^{m,\sharp}(s)\|,\end{aligned}$$

where the first term on the right-hand side can be controlled via the a priori estimate (34), while the second term can be absorbed on the left-hand side. Moreover, we have by (28) and by estimate (34),

$$\begin{aligned}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma}\mathcal{G}^{m,\prec}\varphi^m(s)\| &\lesssim \|(1+\mathcal{N})^{p+\alpha(\gamma)}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\| \\ &\lesssim \|(1+\mathcal{N})^{p+\alpha(\gamma)}(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\| \\ &\quad +\|(1+\mathcal{N})^{p+\alpha(\gamma)+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\|.\end{aligned}$$

Recalling $\gamma \in]1/2, 1-1/(2\theta)[$ and exploiting estimates (9)–(10), we get

$$\begin{aligned}&\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma-1}\mathcal{G}^{m,\succ}\mathcal{G}^{m,\prec}\varphi^m(s)\| \\ \lesssim\ & \|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma-1/2+1/(2\theta)}\mathcal{G}^{m,\prec}\varphi^m(s)\| \\ \lesssim\ & \|(1+\mathcal{N})^{p+3/2+\alpha(\gamma-1/2+1/(2\theta))}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\| \\ \lesssim\ & \|(1+\mathcal{N})^{p+\alpha(\gamma)}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\|,\end{aligned}$$

where we used $3/2+\alpha(\gamma-1/2+1/(2\theta))<\alpha(\gamma)$ whenever $\bar\varepsilon<1/3-1/(3\theta)$. This bound can be controlled via (34) as above. As a consequence, we established that, after renaming $q=q(p,\gamma)>0$,

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^\gamma\Phi^{m,\sharp}(t)\| &\lesssim_T \|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\|\\ &\quad+\|(1+\mathcal{N})^{q+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\|\\ &\quad+\varepsilon\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma+1}\varphi^{m,\sharp}(t)\|,\end{aligned}$$

and hence, for $\gamma\in]1/2,1-1/(2\theta)[$,

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma}\varphi^{m,\sharp}(t)\| &\lesssim_T \|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\|+\|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\|\\ &\quad+\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\|\\ &\lesssim_T \|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\|.\end{aligned}$$

Recall $\partial_t\varphi^{m,\sharp}=(1-\mathcal{L}_\theta)\varphi^{m,\sharp}+\Phi^{m,\sharp}(t)$, so that

$$\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^\gamma\partial_t\varphi^{m,\sharp}(t)\|\lesssim\|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\|.$$

By interpolation, this gives

$$\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma/2}(\varphi^{m,\sharp}(t)-\varphi^{m,\sharp}(s))\|\leqslant|t-s|^\kappa\|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\|.$$

Introduce now, for $p>0$, the sets

$$\mathcal{U}_p=\bigcup_{\gamma\in\left]\frac12,1-\frac{1}{2\theta}\right[}\mathcal{K}(1+\mathcal{N})^{q(p,\gamma)}(1-\mathcal{L}_\theta)^{-1-\gamma}\mathcal{H}\subset\mathcal{H},$$

and $\mathcal{U}=\bigcup_{p>\alpha(0)}\mathcal{U}_p$.

Theorem 3 *Let $p>0$ and $\varphi_0\in\mathcal{U}_p$. Then there exists a solution*

$$\varphi\in\bigcup_{\delta>0}C(\mathbb{R}_+;(1+\mathcal{N})^{-p+\delta}(1-\mathcal{L}_\theta)^{-1}\mathcal{H})$$

to the Kolmogorov backward equation $\partial_t\varphi=\mathcal{L}\varphi$ with initial condition $\varphi(0)=\varphi_0$. For $p>\alpha(0)$, we have $\varphi\in C(\mathbb{R}_+,\mathcal{D}(\mathcal{L}))\cap C^1(\mathbb{R},\mathcal{H})$ and, by dissipativity of $\mathcal{L}$, this solution is unique.

Proof Let $\varphi_0\in\mathcal{U}_p$ and set $\varphi_0^\sharp:=\mathcal{K}^{-1}\varphi_0\in(1+\mathcal{N})^{-q}(1-\mathcal{L}_\theta)^{-1-\gamma}\mathcal{H}$ for $\gamma\in]1/2,1-1/(2\theta)[$ and $p>0$. For $m\in\mathbb{N}$, let φ^m be the solution to $\partial_t\varphi^m=\mathcal{L}^m\varphi^m$ with initial condition $\varphi^m(0)=\mathcal{K}^m\varphi_0^\sharp$. A diagonal argument yields the relative compactness of bounded sets of $(1+\mathcal{N})^{-p}(1-\mathcal{L}_\theta)^{-1-\gamma/2}\mathcal{H}$ in the space $(1+\mathcal{N})^{-p+\delta}(1-\mathcal{L}_\theta)^{-1}\mathcal{H}$ for $\delta>0$, with the consequence that, by Ascoli-Arzelà the sequence $(\varphi^{m,\sharp})_m$ is relatively compact in $C(\mathbb{R}_+;(1+\mathcal{N})^{-p+\delta}(1-\mathcal{L}_\theta)^{-1}\mathcal{H})$ equipped with the topology of uniform convergence on compact sets. We denote $\varphi^\sharp$ a limit point of such a sequence and let $\varphi=\mathcal{K}\varphi^\sharp$. Then, along the convergent subsequence,

$$\begin{aligned}\varphi(t)-\varphi(0) &= \lim_{m\to\infty}(\varphi^m(t)-\varphi^m(0))\\ &= \lim_{m\to\infty}\int_0^t\mathcal{L}^m\varphi^m(s)\mathrm{d}s\\ &= \lim_{m\to\infty}\int_0^t(\mathcal{L}_\theta\varphi^{m,\sharp}(s)+\mathcal{G}^{m,\prec}\mathcal{K}^m\varphi^{m,\sharp}(s))\mathrm{d}s\\ &= \lim_{m\to\infty}\int_0^t(\mathcal{L}_\theta\varphi^\sharp(s)+\mathcal{G}^{m,\prec}\mathcal{K}^m\varphi^{m,\sharp}(s))\mathrm{d}s\\ &= \int_0^t(\mathcal{L}_\theta\varphi^\sharp(s)+\mathcal{G}^\prec\mathcal{K}\varphi^\sharp(s))\mathrm{d}s,\end{aligned}$$

where we exploited our uniform bounds on $\mathcal{L}_\theta$, $\mathcal{G}^{m,\succ}$, $\mathcal{K}^m$ and the convergence of $\varphi^{m,\sharp}$ to $\varphi^\sharp$ as $m \to \infty$ to get the 4th equality, while the last step follows from our bounds for $\mathcal{G}^\prec$ and $\mathcal{K}$, together with the dominated convergence theorem.

If we take $p > \alpha(0)$, then by definition (cfr. Lemma 13) $\varphi \in \mathcal{D}(\mathcal{L})$. Furthermore, $\mathcal{L}\varphi \in C(\mathbb{R}_+;\mathcal{H})$ and we have $\varphi \in C^1(\mathbb{R}_+;\mathcal{H})$ because of the relation $\varphi(t) - \varphi(s) = \int_s^t \mathcal{L}\varphi(\tau)\mathrm{d}\tau$. We can hence compute,

$$\partial_t \|\varphi(t)\|^2 = 2\langle \varphi(t), \mathcal{L}\varphi(t)\rangle \leqslant 0,$$

by the dissipativity of the operator $\mathcal{L}$ given by Lemma 14. Therefore, for any solution we have $\|\varphi(t)\| \leqslant \|\varphi_0\|$, which together with the linearity of the equation yields the uniqueness. □

6 Bounds on the drift

We prove there the key bounds on the drift $\mathcal{G}^m$.

Proof of Lemma 4 We start by estimating $\mathcal{G}^m_+$. We have, by Lemma 16 and since $\gamma > 1/(2\theta)$,

$$\begin{aligned}
&\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}^m_+\varphi\|^2 = \textstyle\sum_{n\geqslant 0} n!w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) |\mathcal{F}((1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}^m_+\varphi)_n(k_{1:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 2} n!n^2w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) \frac{\mathbb{1}_{|k_1|,|k_2|,|k_1+k_2|\leqslant m}|k_1+k_2|^4}{(1+L_\theta(k_{1:n}))^{2\gamma}|k_1|^2|k_2|^2} |\hat\varphi_{n-1}(k_1+k_2,k_{3:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 2} n!n^2w(n)^2 \sum_{\ell,k_{3:n}} \left(\prod_{i=3}^n |2\pi k_i|^2\right) |\ell|^4 |\hat\varphi_{n-1}(\ell,k_{3:n})|^2 \sum_{k_1+k_2=\ell} (1+L_\theta(k_{1:n}))^{-2\gamma} \\
\lesssim\ & \textstyle\sum_{n\geqslant 2} n!n^2w(n)^2 \sum_{\ell,k_{3:n}} \left(\prod_{i=3}^n |2\pi k_i|^2\right) |\ell|^4 (1+L_\theta(\ell,k_{3:n}))^{-2\gamma+1/\theta} |\hat\varphi_{n-1}(\ell,k_{3:n})|^2.
\end{aligned}$$

Introducing the notation $\ell_1 = \ell = k_1 + k_2$ and $\ell_i = k_{i+1}$ for $i \geqslant 2$, we get

$$\lesssim\ \textstyle\sum_{n\geqslant 2} n!n^2w(n)^2 \sum_{\ell_{1:n-1}} \left(\prod_{i=1}^{n-1} |2\pi\ell_i|^2\right) |\ell_1|^2 (1+L_\theta(\ell_{1:n-1}))^{-2\gamma+1/\theta} |\hat\varphi_{n-1}(\ell_{1:n-1})|^2.$$

then using the symmetry of $\hat\varphi_{n-1}$ we reduce this to

$$\lesssim\ \textstyle\sum_{n\geqslant 2} n!nw(n)^2 \sum_{\ell_{1:n-1}} \left(\prod_{i=1}^{n-1} |2\pi\ell_i|^2\right) \frac{|\ell_1|^2+\cdots+|\ell_n|^2}{1+L_\theta(\ell_{1:n-1})} (1+L_\theta(\ell_{1:n-1}))^{-2\gamma+1/\theta+1} |\hat\varphi_{n-1}(\ell_{1:n-1})|^2.$$

from which we obtain

$$\begin{aligned}
&\lesssim \sum_{n\geqslant 1} n!(n+1)^2w(n+1)^2 \sum_{\ell_{1:n}} \left(\prod_{i=1}^n |2\pi\ell_i|^2\right) (1+L_\theta(\ell_{1:n}))^{-2\gamma+1/\theta+1} |\hat\varphi_n(\ell_{1:n})|^2 \\
&\lesssim \|w(\mathcal{N}+1)(1+\mathcal{N})(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma}\varphi\|^2.
\end{aligned}$$

For $\mathcal{G}^m_-$, note first that, by the Cauchy-Schwarz inequality and by Lemma 16 (since $\gamma < 1/2$),

$$\begin{aligned}
&\left|\textstyle\sum_{p+q=k_1} (k_1^\perp\cdot p)(k_1\cdot q)\hat\varphi_{n+1}(p,q,k_{2:n})\right|^2 \\
\lesssim\ & \textstyle\sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{2\gamma-1-1/\theta} \\
& \times \textstyle\sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma} |k_1^\perp\cdot p|^2 |k_1\cdot q|^2 |\hat\varphi_{n+1}(p,q,k_{2:n})|^2 \\
\lesssim\ & (1+|k_1|^{2\theta})^{2\gamma-1} \textstyle\sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma} |k_1^\perp\cdot p|^2 |k_1\cdot q|^2 |\hat\varphi_{n+1}(p,q,k_{2:n})|^2,
\end{aligned}$$

therefore,

$$\begin{aligned}
& \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}_-^m\varphi\|^2 = \textstyle\sum_{n\geqslant 0} n!w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) |\mathcal{F}((1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}_-^m\varphi)_n(k_{1:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) \frac{1}{|k_1|^4(1+L_\theta(k_{1:n}))^{2\gamma}} \\
& \times \left|\textstyle\sum_{p+q=k_1} (k_1^\perp\cdot p)(k_1\cdot q)\hat\varphi_{n+1}(p,q,k_{2:n})\right|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) \frac{1}{|k_1|^4(1+L_\theta(k_{1:n}))^{2\gamma}} (1+|k_1|^{2\theta})^{2\gamma-1} \\
& \times \textstyle\sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|k_1|^4|p|^2|q|^2|\hat\varphi_{n+1}(p,q,k_{2:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \sum_{k_{1:n}} \sum_{p+q=k_1} \left(\prod_{i=2}^n |2\pi k_i|^2\right) |2\pi p|^2|2\pi q|^2 \\
& \times (1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|\hat\varphi_{n+1}(p,q,k_{2:n})|^2,
\end{aligned}$$

we now let $\ell_1 = p$, $\ell_2 = q$, and $\ell_i = k_{i-1}$ for $3 \leqslant i \leqslant n+1$, so that

$$\begin{aligned}
& \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}_-^m\varphi\|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \sum_{\ell_{1:n+1}} \left(\prod_{i=1}^{n+1} |2\pi \ell_i|^2\right) (1+|\ell_1|^{2\theta}+|\ell_2|^{2\theta})^{1+1/\theta-2\gamma}|\hat\varphi_{n+1}(\ell_{1:n+1})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \sum_{\ell_{1:n+1}} \left(\prod_{i=1}^{n+1} |2\pi \ell_i|^2\right) (1+|\ell_1|^{2\theta}+\cdots+|\ell_{n+1}|^{2\theta})^{1+1/\theta-2\gamma}|\hat\varphi_{n+1}(\ell_{1:n+1})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 1} n!w(n-1)^2n^3 \sum_{\ell_{1:n}} \left(\prod_{i=1}^{n} |2\pi \ell_i|^2\right) (1+|\ell_1|^{2\theta}+\cdots+|\ell_{n}|^{2\theta})^{1+1/\theta-2\gamma}|\hat\varphi_{n}(\ell_{1:n})|^2 \\
\lesssim\ & \|w(\mathcal{N}-1)\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma}\varphi\|^2
\end{aligned}$$

which gives the uniform bound.

Let us now discuss the m-dependent estimates, we have for $\mathcal{G}_+^m$

$$\begin{aligned}
& \|w(\mathcal{N})\mathcal{G}_+^m\varphi\|^2 = \textstyle\sum_{n\geqslant 0} n!w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) |\mathcal{F}(\mathcal{G}_+^m\varphi)_n(k_{1:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 2} n!w(n)^2n^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) \mathbb{1}_{|k_1|,|k_2|,|k_1+k_2|\leqslant m} \frac{|k_1+k_2|^4}{|k_1|^2|k_2|^2}|\hat\varphi_{n-1}(k_1+k_2,k_{3:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 2} n!w(n)^2n^2 \\
& \times \textstyle\sum_{k_{1:n}} \left(\prod_{i=3}^n |2\pi k_i|^2\right) |2\pi(k_1+k_2)|^2 \mathbb{1}_{|k_1|,|k_2|,|k_1+k_2|\leqslant m}|k_1+k_2|^{2\theta}|\hat\varphi_{n-1}(k_1+k_2,k_{3:n})|^2 \\
\lesssim\ & m^2 \textstyle\sum_{n\geqslant 2} n!w(n)^2n^2 \sum_{\ell_{1:n-1}} \left(\prod_{i=1}^{n-1} |2\pi \ell_i|^2\right) |\ell_1|^{2\theta}|\hat\varphi_{n-1}(\ell_{1:n-1})|^2 \\
\lesssim\ & m^2 \textstyle\sum_{n\geqslant 2} n!w(n)^2n \sum_{\ell_{1:n-1}} \left(\prod_{i=1}^{n-1} |2\pi \ell_i|^2\right) L_\theta(\ell_{1:n-1})|\hat\varphi_{n-1}(\ell_{1:n-1})|^2 \\
\lesssim\ & m^2 \textstyle\sum_{n\geqslant 1} n!w(n+1)^2(n+1)^2 \sum_{\ell_{1:n}} \left(\prod_{i=1}^{n} |2\pi \ell_i|^2\right) (1+L_\theta(\ell_{1:n}))|\hat\varphi_{n}(\ell_{1:n})|^2 \\
\lesssim\ & m^2\|w(\mathcal{N}+1)(1+\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi\|^2.
\end{aligned}$$

Finally, for $\mathcal{G}_-^m$ we have,

$$\begin{aligned}
& \|w(\mathcal{N})\mathcal{G}_-^m\varphi\|^2 = \textstyle\sum_{n\geqslant 0} n!w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) |\mathcal{F}(\mathcal{G}_-^m\varphi)_n(k_{1:n})|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) \frac{\mathbb{1}_{|k_1|,|p|,|q|\leqslant m}}{|k_1|^4} \\
& \times \left|\textstyle\sum_{p+q=k_1} (k_1^\perp\cdot p)(k_1\cdot q)\hat\varphi_{n+1}(p,q,k_{2:n})\right|^2 \\
\lesssim\ & \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^4 \\
& \times \textstyle\sum_{k_{1:n}} \sum_{p+q=k_1} \left(\prod_{i=2}^n |2\pi k_i|^2\right) |2\pi p|^2|2\pi q|^2 \mathbb{1}_{|k_1|,|p|,|q|\leqslant m}|k_1|^2|\hat\varphi_{n+1}(p,q,k_{2:n})|^2 \\
\lesssim\ & m^2 \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^3 \sum_{p,q,k_{2:n}} \left(\prod_{i=2}^n |2\pi k_i|^2\right) \\
& \times |2\pi p|^2|2\pi q|^2(|p|^{2\theta}+|q|^{2\theta}+|k_2|^{2\theta}+\cdots+|k_n|^{2\theta})|\hat\varphi_{n+1}(p,q,k_{2:n})|^2 \\
\lesssim\ & m^2 \textstyle\sum_{n\geqslant 0} n!w(n)^2(n+1)^3 \sum_{\ell_{1:n+1}} \left(\prod_{i=1}^{n+1} |2\pi \ell_i|^2\right) L_\theta(\ell_{1:n+1})|\hat\varphi_{n+1}(\ell_{1:n+1})|^2 \\
\lesssim\ & m^2 \textstyle\sum_{n\geqslant 1} n!w(n-1)^2n^2 \sum_{\ell_{1:n}} \left(\prod_{i=1}^{n} |2\pi \ell_i|^2\right) L_\theta(\ell_{1:n})|\hat\varphi_{n}(\ell_{1:n})|^2 \\
\lesssim\ & m^2\|w(\mathcal{N}-1)\mathcal{N}(1-\mathcal{L}_\theta)^{1/2}\varphi\|^2.
\end{aligned}$$

□

A Some auxiliary results

Lemma 16 *Let $C, \beta \geqslant 0$, $\alpha > (d+\beta)/(2\theta)$. Then*

$$\sum_{p\in\mathbb{Z}^d} \frac{|p|^\beta}{(|p|^{2\theta}+|k-p|^{2\theta}+C)^\alpha} \lesssim (|k|^{2\theta}+C)^{(\beta+d)/(2\theta)-\alpha}, \qquad k\in\mathbb{Z}^d.$$

Proof Since $|p|^{2\theta}+|k-q|^{2\theta} \gtrsim |p|^{2\theta}+|k|^{2\theta}$, we have

$$\textstyle\sum_p \frac{|p|^\beta}{(|p|^{2\theta}+|k-p|^{2\theta}+C)^\alpha} \quad\lesssim\quad \int_{\mathbb{R}^d} \frac{|y|^\beta}{(|y|^{2\theta}+|k|^{2\theta}+C)^\alpha}\mathrm{d}y$$

By scaling

$$\int_{\mathbb{R}^d} \frac{|y|^\beta}{(|y|^{2\theta}+|k|^{2\theta}+C)^\alpha}\mathrm{d}y = (|k|^{2\theta}+C)^{(\beta+d)/2\theta-\alpha}\int_{\mathbb{R}^d}\frac{|y|^\beta}{(|y|^{2\theta}+1)^\alpha}\mathrm{d}y$$

and the integral is finite if $\beta - 2\theta\alpha < -d$. □

Lemma 17 *We have, for any $T>0$, $\gamma>0$,*

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma}\psi(t)\| \quaternion{\leqslant}{} &\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma}\psi(0)\| \\ &+\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma}(\partial_t-(1-\mathcal{L}_\theta))\psi(t)\|.\end{aligned}$$

Proof The proof is standard and proceeds by spectral calculus. Write $\Psi(t) := (\partial_t - (1-\mathcal{L}_\theta))\psi(t)$

$$\Psi_i(s) = \mathbb{1}_{|1-\mathcal{L}_\theta|\sim 2^i}\Psi(s),$$

where $\mathbb{1}_{|1-\mathcal{L}_\theta|\sim 2^i}$ denotes a dyadic partition of unity such that $\|\varphi\|^2 \approx \sum_i \|\mathbb{1}_{|1-\mathcal{L}_\theta|\sim 2^i}\varphi\|^2$ for any φ.

Let $S_t = e^{-t(1-\mathcal{L}_\theta)}$, so that

$$\psi(t) = S_t\psi(0) + \int_0^t S_{t-s}\Psi(s)\mathrm{d}s.$$

Then, using $\|(1-\mathcal{L}_\theta)^{1+\gamma}S_{t-s}\psi\| \lesssim ((t-s)^{-1-\gamma}\vee 1)\|\psi\|$ and $\|(1-\mathcal{L}_\theta)^{1+\gamma}\mathbb{1}_{|1-\mathcal{L}_\theta|\sim 2^i}\| \lesssim 2^{(1+\gamma)i}$, and letting $\delta = 2^{-i}$, we have

$$\begin{aligned}
&\left\|(1-\mathcal{L}_\theta)^{1+\gamma}\textstyle\int_0^t S_{t-s}\Psi_i(s)\mathrm{d}s\right\| \\
\leqslant\ &\left\|(1-\mathcal{L}_\theta)^{1+\gamma}\textstyle\int_0^{t-\delta} S_{t-s}\Psi_i(s)\mathrm{d}s\right\| + \left\|(1-\mathcal{L}_\theta)^{1+\gamma}\textstyle\int_{t-\delta}^t S_{t-s}\Psi_i(s)\mathrm{d}s\right\| \\
\lesssim\ &\textstyle\int_0^{t-\delta}((t-s)^{-1-\gamma}\vee 1)\|\Psi_i(s)\|\mathrm{d}s + 2^{(1+\gamma)i}\int_{t-\delta}^t\|S_{t-s}\Psi_i(s)\|\mathrm{d}s \\
\lesssim\ &(\delta^{-\gamma}+2^{i(1+\gamma)}\delta)\sup_{0\leqslant s\leqslant T}\|\Psi_i(s)\| \\
\lesssim\ &2^{i\gamma}\sup_{0\leqslant s\leqslant T}\|\Psi_i(s)\| \\
\lesssim\ &\sup_{0\leqslant s\leqslant T}\|(1-\mathcal{L}_\theta)^\gamma\Psi_i(s)\|,
\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\left\|(1-\mathcal{L}_\theta)^{1+\gamma}\textstyle\int_0^t S_{t-s}\Psi(s)\mathrm{d}s\right\|^2 \quad&\lesssim\quad \textstyle\sum_i\left\|(1-\mathcal{L}_\theta)^{1+\gamma}\int_0^t S_{t-s}\Psi_i(s)\mathrm{d}s\right\|^2 \\ &\lesssim\ \sup_{0\leqslant s\leqslant T}\textstyle\sum_i\|(1-\mathcal{L}_\theta)^\gamma\Psi_i(s)\|^2 \\ &\lesssim\ \sup_{0\leqslant s\leqslant T}\|(1-\mathcal{L}_\theta)^\gamma\Psi(s)\|^2.\end{aligned}$$

Therefore, since $\mathcal{N}$ commutes with $\mathcal{L}_\theta$, we also have

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma}\psi(t)\| \quaternion{\leqslant}{} &\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1+\gamma}\psi(0)\| \\ &+\sup_{0\leqslant t\leqslant T}\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma}\Psi(s)\|,\end{aligned}$$

that is the claimed estimate. □

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