Energy solutions of KPZ are unique

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Abstract

Energy solutions is a weak notion of solution to the KPZ equation which was introduced by Gonçalves and Jara [ARMA 212 (2013)] to describe the large scale fluctuations of a wide class of weakly asymmetric particle systems. In this paper we prove that energy solutions of the periodic, stationary, stochastic Burgers equation are unique and in particular they coincide with the Hopf–Cole solution. To obtain uniqueness we use the reformulation of the energy solution concept given by Gubinelli and Jara [SPDE–AC 1 (2013)] and an observation by Funaki and Quastel [arXiv:1407.7310].

1 Introduction

Energy solutions [3, 4, 5] are a tool to give meaning to (stationary) solutions to the periodic stochastic Burgers equation (SBE)

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi \tag{1}$$

defined on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with initial condition $u_0 \sim \mu$ where μ is the white noise on $L^2_0(\mathbb{T})$ (the space of square integrable functions on $\mathbb{T}$ with zero mean), Δ is the periodic Laplacian and ∂_t, ∂_x denote the time and space derivative respectively. The term ξ is a gaussian centered space–time white noise with covariance $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$.

Energy solutions were introduced by Gonçalves and Jara [3] in order to describe the fluctuations of a large class of weakly asymmetric interacting particle system. Subsequently Gubinelli and Jara [4] used the language of controlled paths to give a more abstract framework where energy solutions are seen as some kind of "weak solutions" to the SBE. An introductory exposition is present in the lecture notes [5] of Gubinelli and Perkowski.

The main difficulty in eq. (1) is due to the interpretation of the nonlinearity $\partial_x u^2$. Indeed any reasonable candidate solution to (1) features trajectories which, at fixed time and as functions of the space variable, look like white noise on $\mathbb{T}$ so that the term $\partial_x u^2$ does not have an obvious definition.

Bertini and Giacomin [1] determined the fluctuations of the weakly asymmetric exclusion process (WASEP) by heavily exploiting the fact that it is expected that any reasonable solution to (1) should be obtained as the Cole–Hopf transform $u = \partial_x \log \psi$ where ψ is the solution of the stochastic (linear, multiplicative) heat equation (SHE):

$$\mathrm{d}\psi = \Delta\psi + \psi\xi \tag{2}$$

where ξ is the space–time white noise and where the product $\psi\xi$ has to be understood as an Ito product between ψ and a the time–differential $\mathrm{d}W$ of a cylindrical Brownian motion on $L^2(\mathbb{T})$ [10]. This last equation can be solved via standard stochastic techniques giving a unique strong solution ψ which is space–time Hölder continuous and stricly positive for positive times. In this way the relation $u = \partial \log \psi$ identifies a unique generalized process which is called the Cole–Hopf solution of

the SBE (1). Using a microscopic version of the Cole–Hopf transform due to Gärtner, Bertini and Giacomin [1] were able to prove that after rescaling and recentering the fluctuation field of the WASEP converged to a process having the law of the Cole–Hopf solution. This result relies on the precise structure of the WASEP and on the availability of a Cole–Hopf correspondence at the microscopic level. However it is expected (as *the weak KPZ conjecture* goes) that the fluctuation field of a large class of microscopic non–equilibrium one–dimensional particle system should likewise converge to the same object Hopf–Cole solution. Gonçalves and Jara [3] were able to prove the tightness of the fluctuation field of a large class of weakly asymmetric interacting particle systems which do not enjoy a microscopic Hopf–Cole transformation. The difficulty they encountered is that caracterizing the limiting fluctuation field u without the Hopf–Cole correspondence hits the difficulty of describing the dynamics of u in term of (1) which is not a well–posed equation as it stands, even interpreted as a weak solution in the PDE sense. Their way out of this impasse has been to give a new notion of solution which however, up to know, were not know of enjoy uniqueness. In this way they were able to give a partial caracterization of the accumulation points of the fluctuation field u without being able to describe exactly the limit.

Note that in the meanwhile Hairer [7] has found a notion of solution to the KPZ equation

$$\partial_t h = \Delta h + (\partial_x h)^2 - \infty + \xi \tag{3}$$

which enjoy a good existence and uniqueness theory and from which solution of the SBE can be reobtained as $u = \partial_x h$, solving a long standing problem of mathematical physics. The problem with Hairer's solution theory for KPZ is that it is not simple to check that the limiting fluctuation fields of the microscopic models satisfy the conditions of the theory. This difficulty persists also in other recent (but, in this case, mostly equivalent) methods of giving a meaning to the SBE: Hairer's regularity structures [8] or Gubinelli, Imkeller and Perkowski's paracontrolled distributions [6, 5].

The aim of the present work is to show that stationary energy solutions of (1), as defined by Gubinelli and Jara in [4], are unique and coincide with the Hopf–Cole solution. Existence was already obtained by standard Galerkin methods [4]. In this way we fill a gap which could lead to possibly estabilish the weak KPZ universality conjecture in a quite large extent, at least in the stationary situation.

Our proof was motivated by the recent paper of Funaki and Quastel [2] in which they estabilish the invariance of the white noise for the Hopf–Cole solution. In our framework invariance comes for free from Galerkin approximations but their strategy can be adapted to show that energy solution can be transformed via Hopf–Cole to give solutions of the SHE. Uniqueness for the SHE then implies easily uniqueness for energy solutions. In this way the full power of the weak formulation comes into play since even if Hopf–Cole is not available at the microscopic level (or for other kind of approximations, like Galerkin) the limiting energy solution does allow the Hopf–Cole transformation. Note that our results does not seems to apply to the original formulation of Gonçalves and Jara since it is not clear how to perform in their setting the computations leading to the control of the remainder in the Hopf–Cole transformation.

Extension of the uniqueness result in $\mathbb{R}$ is in progress and we do not expect additional difficulties. Applications to the convergence toward KPZ of the stationary version of the model considered by Hairer and Quastel [9] are under investigation.

Notation. Let $\mathscr{S}$ the space of C^∞ functions on $\mathbb{T}$ and by $\mathscr{S}'$ the dual space of distributions. For $k \in \mathbb{Z}$ we denote by $e_k(x) = \exp(-ikx)/(2\pi)^{1/2}$ the Fourier basis. For any $f \in \mathscr{S}'$ let $\hat{f} = \mathscr{F}f$ the Fourier transform defined by $\mathscr{F}f(k) = \langle e_k, f\rangle$. We have also $f(x) = \sum_k \langle e_k, f\rangle e_k(x)$ and Plancherel formula $\langle f, g\rangle_{L^2} = \sum_{k\in\mathbb{Z}} (\mathscr{F}f(k))^* \mathscr{F}g(k)$. Let Π_0 the projection from $L^2(\mathbb{T})$ to $L^2_0(\mathbb{T})$ and Π^N_0 the projection on the span of $(e_k)_{0<|k|\leqslant N}$ and by Π^N the projection on the the span of $(e_k)_{0\leqslant|k|\leqslant N}$. Let H^α the standard Sobolev spaces on $\mathbb{T}$ of index $\alpha \in \mathbb{R}$ and $H^\alpha_0 = \Pi_0 H^\alpha$ their mean–zero counterparts.

The white noise μ can be realized on $H_0^{-\alpha}$ for some $\alpha > 1/2$. If $u_0 \sim \mu$ then for all $k \in \mathbb{Z}\backslash\{0\}$ $\mathscr{F}u_0(k)$ is a complex Gaussian variable such that $\mathscr{F}u_0(k)^* = \mathscr{F}u_0(-k)$ and $\mathbb{E}\left|\mathscr{F}u_0(k)\right|^2 = 2$.

Denote $\mathscr{C}_T V = C([0,T], V)$ the space of continuous functions from $[0,T]$ to the Banach space V endowed with the supremum norm and with $\mathscr{C}_T^\gamma V = C^\gamma([0,T], V)$ the subspace of γ-Hölder continuous functions in $\mathscr{C}_T V$ with the γ-Hölder norm.

We introduce also a fixed family of smoothing operators indexed by $L \geqslant 1$. Let $q : \mathbb{T} \to \mathbb{R}_+$ be a even smooth function of compact support around 0 and such that $\int_{\mathbb{T}} q(x)\mathrm{d}x = 1$. Let $\delta^L(y) = Lq(Ly)$, $\delta_x^L(y) = \delta^L(x-y)$, $\rho_x^L = \delta_x^L - 1/(2\pi)$ and $\mathcal{J}_0^L f = \rho^L * f$. We let also $\rho_x^\infty(y) = \delta_x(y) - 1/(2\pi)$ and then $\mathcal{J}_0 f = \rho^\infty * f = \Pi_0 f$.

2 Controlled processes and energy solutions

In this section we define stationary energy solutions of (1), following [4]. We introduce first a class of processes u which at fixed time are distributed like the (zero–mean) white noise on $\mathbb{T}$ but for which the nonlinear term $\partial_x u^2$ is defined as a space–time distribution. In this class of processes it makes then sense to look for solutions of the SBE (1).

Definition 1 (Controlled process) *Denote with $\mathcal{Q}$ the space of pairs $(u, \mathcal{A})_{0\leqslant t\leqslant T}$ of generalized stochastic processes with continuous paths in $\mathscr{S}'$ such that*

i) the law of u_t is the white noise μ for all $t \in [0,T]$;

ii) For any test function $\varphi \in \mathscr{S}$ the process $t \mapsto \mathcal{A}_t(\varphi)$ is a.s. of zero quadratic variation, $\mathcal{A}_0(\varphi) = 0$ and the pair $(u(\varphi), \mathcal{A}(\varphi))_{0\leqslant t\leqslant T}$ satisfies the equation

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\Delta\varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) + \partial_x M_t(\varphi) \tag{4}$$

where $(M_t(\varphi))_{0\leqslant t\leqslant T}$ is a martingale with respect to the filtration generated by $(u, \mathcal{A})_{0\leqslant t\leqslant T}$ with quadratic variation $[M(\varphi)]_t = 2t\|\varphi\|^2_{L^2(\mathbb{T})}$;

iii) the reversed processes $\hat{u}_t = u_{T-t}$, $\hat{\mathcal{A}}_t = -(\hat{\mathcal{A}}_T - \mathcal{A}_{T-t})$ satisfies the same equation with respect to its own filtration.

When $\mathcal{A} = 0$ the process $(X, 0) \in \mathcal{Q}$ is the stationary Ornstein–Uhlenbeck (OU) process with invariant measure μ. It is the unique solution of the SPDE

$$\mathrm{d}X = \Delta X \mathrm{d}t + \partial_x \mathrm{d}M$$

with initial condition $u_0 \sim \mu$. Allowing $\mathcal{A} \neq 0$ has the intuitive meaning of considering perturbation of the OU process with zero quadratic variation antisymmetric drifts. In this sense we say that a couple $(u, \mathcal{A}) \in \mathcal{Q}$ is a process *controlled* by the Ornstein–Uhlenbeck process.

Controlled processes allow the definition of some interesting non–linear functionals. We do not insists on generality here, all we need is that the Burgers drift makes sense as a space–time distribution:

Theorem 1 *Assume that $(u, \mathcal{A}) \in \mathcal{Q}$ and for any $N \geqslant 1$ and $0 \leqslant t \leqslant T$. Let*

$$\mathcal{B}_t^N(\varphi) = \int_0^t \partial_x \left(\mathcal{J}_0^N u_s\right)^2 (\varphi)\mathrm{d}s$$

then $(\mathcal{B}_t^N)_{N\geqslant 1}$ converges in probability in $C^{1/2+}H^{-1-} \cap C^{0+}H^{0-}$, we denote the limit by

$$\int_0^t \partial_x u_s^2(\varphi)\mathrm{d}s.$$

Proof The proof can be found in [4, 5] for approximations given by $\tilde{\mathcal{B}}_t^N(\varphi) = \int_0^t \partial_x(\Pi_0^N u_s)^2(\varphi)\mathrm{d}s$. The estimates required to go from $\tilde{\mathcal{B}}_t^N$ to $\mathcal{B}_t^N$ simply uses the fact that $(s,x) \mapsto \mathcal{J}_0^N u_s(x) - \Pi_0^N u_s(x)$ is a smooth Gaussian random field which converges to zero as a continuous space-time function. □

We are ready to give a rigorous meaning to the SBE (1) in the class of controlled processes:

Definition 2 *A controlled process $(u, \mathcal{A}) \in \mathcal{Q}$ is an energy solution of SBE iff*

$$\mathcal{A}_t(\varphi) = \int_0^t \partial_x u_s^2(\varphi)\mathrm{d}s.$$

almost surely for all $t \in [0,T]$ and $\varphi \in \mathscr{S}$.

Existence of energy solutions of SBE can be proved via Galerkin approximations or approximation of the non–linear term as done in [4, 5]. The aim of this note is to show that energy solutions of SBE are in fact adapted to the noise M and pathwise unique. In order to state the result let Iu_0 the unique centered Gaussian random field such that $\partial_x Iu_0 = u_0$ and $\int_{\mathbb{T}} Iu_0(x)\mathrm{d}x = 0$.

Theorem 2 *There exists only one energy solution u of SBE and it is given by the Cole–Hopf transformation*

$$u_t(x) = \partial_x \log \psi_t(x)$$

where ψ is the (Itô) solution of SHE :

$$\mathrm{d}\psi_t = \Delta\psi_t + \psi_t \mathrm{d}M_t$$

with the inital condition given by $\psi_0(x) = \exp(Iu_0(x))$.

The rest of the paper is devoted to the proof of this theorem.

3 Regularization by noise for controlled processes

In this section u will always denote a generic controlled process in $\mathcal{Q}$, not necessarily an energy solution to SBE. Controlled processes have regularization properties coming from the fast decorrelation in time of the OU process associated to the Laplacian Δ. Using an Itô formula and following the classic approach of Kipnis–Varadhan, we can replace time–averages of some functional with a sum of forward and backward martingales whose quadratic variation is controlled in terms of a more regular (or smaller) functional.

For any test function $\varphi \in \mathscr{S}$ the processes $(u_t(\varphi))_{t\in[0,T]}$ and $(\hat{u}_t(\varphi))_{t\in[0,T]}$ are Dirichlet processes: the sum of a martingale and a zero quadratic variation process. This is compatible with the regularity of our solutions and there is no clue that solutions of SBE are distributional semimartingales. Dirichlet processes (wheter Markov or not) can be handled via the stochastic calculus via regularization developed by Russo and Vallois [11]. In this approach the Itô formula holds also for Dirichlet processes and if $X = (X^i)_{i=1,\dots,k}$ is an $\mathbb{R}^k$ valued Dirichlet process and g is a $C^2(\mathbb{R}^k;\mathbb{R})$ function then we have

$$g(X_t) = g(X_0) + \sum_{i=1}^k \int_0^t \partial_i g(X_s)\mathrm{d}^- X_s^i + \frac{1}{2}\sum_{i,j=1}^k \int_0^t \partial_{i,j}^2 g(X_s)\mathrm{d}^-[X^i, X^j]_s \tag{5}$$

where d^- denotes the forward integral and $[X,X]$ the quadratic covariation of the vector process X. Decomposing $X = M + N$ as the sum of a martingale M and a zero quadratic variation process N we have $[X,X] = [M,M]$ and

$$g(X_t) = g(X_0) + \sum_{i=1}^k \int_0^t \partial_i g(X_s)\mathrm{d}^- M_s^i + \sum_{i=1}^k \int_0^t \partial_i g(X_s)\mathrm{d}^- N_s^i$$

$$+\sum_{i,j=1}^{k}\frac{1}{2}\int_0^t \partial^2_{i,j}g(X_s)\mathrm{d}^-[M^i,M^j]_s$$

where now $\mathrm{d}^- M$ coincide with the usual Itô integral and $[M,M]$ is the usual quadratic variation of the martingale M. The integral $\int_0^t \partial_i g(X_s)\mathrm{d}^- N^i_s$ is well-defined due to the fact that all the other terms in this formula are well defined. The case the function g depends explicitly on time can be handled by the above formula by considering time as an additional (0-th) component of the process X and using the fact that $[X^i, X^0] = 0$ for all $i = 1, .., k$. In the computations which follows we will only need to apply the Itô formula to smooth functions.

On $H = L^2_0(\mathbb{T})$ we consider the space of smooth cylinder functions $\mathcal{C}\,\mathrm{yl}$ which depends only on finitely many coordinates on the basis $\{e_k\}_{k\in\mathbb{Z}_0}$ and we consider the Malliavin derivative $D : L^2(\mu) \to L^2(\mu)\otimes H$ defined as $D\varphi(x) = \sum_{k\in\mathbb{Z}_0}\partial_{x_k}\varphi(x)e_k$ for $\varphi \in \mathcal{C}\,\mathrm{yl}$ where $x_k = \langle e_k, x\rangle$ are the coordinates of x. We let $D_k\varphi = \langle e_k, D\varphi\rangle$.

With this notations we can write the generator L_0 of the Ornstein-Uhlenbeck process associated to the operator Δ as

$$L_0\varphi(x) = \sum_{|k|>0}|k|^2\left(-x_k D_k\varphi(x) + \tfrac{1}{2}D_{-k}D_k\varphi(x)\right). \tag{6}$$

For any smooth cylinder function $h : [0,T]\times \mathcal{J}_N H \to \mathbb{R}$ the Itô formula for the finite quadratic variation process $(u^N_t = \Pi^N_0 u_t)_t$ gives

$$h(t,u^N_t) = h(0,u^N_0) + \int_0^t(\partial_s + L^N_0)h(s,u^N_s)\mathrm{d}s + \int_0^t Dh(s,u^N_s)\mathrm{d}\mathcal{J}^N\mathcal{A}_s + M^+_t$$

where

$$L^N_0 h(s,x) = \sum_{0<|k|\le N}|k|^2(-x_k D_k h(s,x) + \frac{1}{2}D_k D_{-k}h(s,x))$$

is the restriction of the operator L_0 to $\Pi^N H$ and where the martingale part denoted M^+ has quadratic variation given by $[M^+]_t = \int_0^t \mathcal{E}^N(h(s,\cdot))(u^N_s)\mathrm{d}s$, where

$$\mathcal{E}^N(\varphi)(x) = \frac{1}{2}\sum_{0<|k|\le N}|k|^2|D_k\varphi(x)|^2 = \left\|\mathcal{J}^N_0 D\varphi\right\|^2_{H^1_0},$$

is the density of the Dirichlet form associated to the generator L^N_0. The Itô formula on the backward process reads

$$h(T-t,u^N_{T-t}) = h(T,u^N_T) + \int_0^t(-\partial_s + L^N_0)h(T-s,u^N_{T-s})\mathrm{d}s$$
$$-\int_0^t Dh(T-s,u^N_{T-s})\mathrm{d}\mathcal{J}^N\mathcal{A}_{T-s} + M^-_t$$

with $[M^-]_t = \int_0^t \mathcal{E}^N(h(T-s,\cdot))(u^N_{T-s})\mathrm{d}s$. Adding the two Itô formulas we have the key equality

$$\int_0^t 2L^N_0 h(s,u^N_s)\mathrm{d}s = -M^+_t + M^-_{T-t} - M^-_T. \tag{7}$$

which allows us to represent the time integral of $2L^N_0 h$ as a sum of martingales which allows better control since their quadratic variation depends only on $\mathcal{E}^N(h)$. From this we can prove easily (see [4]) the following lemma.

Lemma 1 (Itô trick) *Let $h : [0,T] \times \Pi_N H \to \mathbb{R}$ be a cylinder function. Then for any $p \geq 1$,*

$$\left\| \sup_{t\in[0,T]} \left| \int_0^t L_0 h(s,u_s)\mathrm{d}s \right| \right\|_{L^p(\mathbb{P}_\mu)} \lesssim_p T^{1/2} \sup_{s\in[0,T]} \|\mathcal{E}(h(s,\cdot))\|^{1/2}_{L^{p/2}(\mu)} \tag{8}$$

where $\mathcal{E}(\varphi)(x) = \frac{1}{2}\sum_{|k|>0} |k|^2 |D_k\varphi(x)|^2$. In the particular case $h(s,x) = e^{a(T-s)}\tilde{h}(x)$ for some $a \in \mathbb{R}$ we have the improved estimate

$$\left\| \int_0^T e^{a(T-s)} L_0 \tilde{h}(u_s)\mathrm{d}s \right\|_{L^p(\mathbb{P}_\mu)} \lesssim_p \left(\frac{1-e^{2aT}}{2a} \right)^{1/2} \|\mathcal{E}(\tilde{h})\|^{1/2}_{L^{p/2}(\mu)}. \tag{9}$$

If we take $p = 2$ in Lemma 1 we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t L_0 h(u_s)\mathrm{d}s \right|^2 \right] \lesssim T\|h\|_1^2$$

where

$$\|h\|_1^2 = \mathbb{E}\mathcal{E}(h(u_0)) = \mathbb{E}\|\mathrm{D}h(u_0)\|^2_{H^1}$$

is the Dirichlet form of the OU process.

We denote by $\mathscr{H}^1$ the completion of the space of smooth cylinder functions with respect to the norm $\| * \|_1$ and denote with $\mathscr{H}^{-1}$ the dual of $\mathscr{H}^1$ with norm

$$\|f\|_{-1} = \sup_{g\in L^2(\mu)} (2\langle f, g\rangle - \|g\|_1^2)$$

(see [12], Chap. 2 for details).

In order to exploit the Itô trick of Lemma 1 we need to be able to solve the Poisson equation $L_0 h_f = f$ for given function f. Sometimes we can do it explicitly and estimate directly the Dirichlet energy of the solution. In other situations however it is preferable to have a bound involving directly f without need to find h_f explicitly. (In order not to bother us with domain considerations the reader can think of cylindrical and smooth f, h since this will be enough for our purposes.) Fix an arbitrary h and consider the decomposition

$$f = L_0 h + L_0 h - f$$

then

$$\begin{aligned}
\left\| \sup_{t\in[0,T]} \left| \int_0^t f(u_s)\mathrm{d}s \right| \right\|_{L^2(\mathbb{P})} &\leqslant \left\| \sup_{t\in[0,T]} \left| \int_0^t L_0 h(u_s)\mathrm{d}s \right| \right\|_{L^2(\mathbb{P})} + \left\| \sup_{t\in[0,T]} \left| \int_0^t (L_0 h - f)(u_s)\mathrm{d}s \right| \right\|_{L^2(\mathbb{P})} \\
&\leqslant \left\| \sup_{t\in[0,T]} \left| \int_0^t L_0 h(u_s)\mathrm{d}s \right| \right\|_{L^2(\mathbb{P})} + \int_0^T \|(L_0 h - f)(u_s)\|_{L^2(\mathbb{P})}\mathrm{d}s \\
&\leqslant CT^{1/2}\|h\|_1 + T\|L_0 h - f\|_{L^2(\mu)}
\end{aligned}$$

Now note that, for fixed $\lambda > 0$ a solution h_λ to $(\lambda - L_0)h_\lambda = -f$ is given explicitly by

$$h_\lambda(x) = -\int_0^\infty e^{-\lambda s}(P_s^{\mathrm{OU}} f)(x)\mathrm{d}s$$

where P^{OU} is the contraction semigroup generated by L_0 on $L^2(\mu)$. Then

$$\langle h_\lambda, -f\rangle = \langle h_\lambda, (\lambda - L_0)h_\lambda, \rangle = \lambda\|h_\lambda\|^2_{L^2} + \|h_\lambda\|_1^2$$

and if $f \in \mathscr{H}^{-1}$ we have

$$\lambda \|h_\lambda\|_{L^2}^2 + \|h_\lambda\|_1^2 = |\langle h_\lambda, f\rangle| \leqslant \|f\|_{-1}\|h_\lambda\|_1$$

which implies $\|h_\lambda\|_1 \leqslant \|f\|_{-1}$ and $\|h_\lambda\|_{L^2} \leqslant \lambda^{-1/2}\|f\|_{-1}$. So noting that $L_0 h_\lambda - f = \lambda h_\lambda$ we have

$$CT^{1/2}\|h_\lambda\|_1 + T\|L_0 h_\lambda - f\|_{L^2(\mu)} = CT^{1/2}\|h_\lambda\|_1 + T\|\lambda h_\lambda\|_{L^2(\mu)} \leqslant CT^{1/2}\|f\|_{-1} + T\lambda^{1/2}\|f\|_{-1}.$$

Taking $\lambda \to 0$ we end up with the following lemma which extends the Kipnis–Varadhan lemma (cfr. Lemma 2.4 in [12]) to controlled processes :

Lemma 2 *Assume that $f \in \mathscr{H}^{-1}$ then for every controlled process u we have*

$$\left\| \sup_{t\in[0,T]} \left| \int_0^t f(u_s)\mathrm{d}s \right| \right\|_{L^2(\mathbb{P})} \leqslant CT^{1/2}\|f\|_{-1}.$$

with some constant C which does not depends on u.

4 Mapping to the SHE

The strategy to prove Theorem 2 is the following. Fix $L \geqslant 1$ and consider the process $\phi_t^L(x) = \exp(Iu_t^L(x))$ where $u_t^L = \mathcal{J}_0^L u_t$ and $I = \partial_x^{-1}$ is the antiderivative defined on $L_0^2(\mathbb{T})$ by $\partial_x I\varphi = \varphi$ for all $\varphi \in L_0^2(\mathbb{T})$. Let $I^* = -I$ the adjoint of I and $\rho_x^L = \mathcal{J}_0^L\delta_x$ is a smooth test function of zero mean. Now $I\rho_x^L(y) = (\delta^L * \Theta)(y-x)$ where $\Theta : \mathbb{T} \to R$ is the odd periodic function given by $\Theta(y) = \frac{1}{2}\operatorname{sgn}(y) - y/(2\pi)$ for $y \in (-\pi, \pi]$ with $\Theta(0) = 0$.

The process $(\phi_t^L(x))_{t,x}$ is smooth in x and of finite quadratic variation in t. Note that $\phi_t^L(x) = \exp(Iu_t^L(x)) = \exp(u_t(I^*\rho_x^L))$ is not a cylindrical function but by a direct approximation procedure we can pretend it is and since $(u_t, \mathcal{A})$ is controlled we have

$$\mathrm{d}(Iu_t^L)(x) = \mathrm{d}u_t(I^*\rho_x^L) = u_t(\Delta I^*\rho_x^L)\mathrm{d}t + \mathrm{d}\mathcal{A}_t(I^*\rho_x^L) - \mathrm{d}M_t(\partial_x I^*\rho_x^L).$$

The Itô formula for Dirichlet processes (5) gives

$$\begin{aligned}\mathrm{d}\phi_t^L(x) &= \phi_t^L(x)(u_t(\Delta I^*\rho_x^L)\mathrm{d}t + \mathrm{d}\mathcal{A}_t(I^*\rho_x^L) - \mathrm{d}M_t(\partial_x I^*\rho_x^L)) + \phi_t(x)\langle \partial_x I^*\rho_x^L, \partial_x I^*\rho_x^L\rangle \mathrm{d}t \\ &= \phi_t^L(x)(u_t(\Delta I^*\rho_x^L)\mathrm{d}t + \mathrm{d}\mathcal{A}_t(I^*\rho_x^L) + \mathrm{d}M_t(\rho_x^L)) + \phi_t(x)\langle \rho_x^L, \rho_x^L\rangle \mathrm{d}t\end{aligned}$$

Moreover since $\rho_x^L(y) = \rho_0^L(y-x)$ we have $\Delta(I^*\rho_x^L) = \Delta_x(I^*\rho_x^L)$ and

$$\begin{aligned}\phi_t^L(x)u_t(\Delta I^*\rho_x^L) &= \phi_t^L(x)\Delta_x u_t(I^*\rho_x^L) = \Delta_x\phi_t^L(x) - \phi_t^L(x)(\partial_x u_t(I^*\rho_x^L))^2 \\ &= \Delta_x\phi_t^L(x) - \phi_t^L(x)(u_t(\partial_x I^*\rho_x^L))^2 = \Delta_x\phi_t^L(x) - \phi_t^L(x)(u_t(\rho_x^L))^2.\end{aligned}$$

Being ϕ_t^L is smooth in space we can rewrite this as an approximate stochastic heat equation (SHE)

$$\mathrm{d}\phi_t^L(x) = \Delta_x\phi_t^L(x)\mathrm{d}t + \phi_t^L(x)(K^L - Q_t^L)\mathrm{d}t + \phi_t^L(x)\mathrm{d}M_t(\rho_x^L) + \mathrm{d}R_t^L(x)$$

where the remainder R^L is given by

$$R_t^L(x) = \int_0^t \phi_s^L(x)(\mathrm{d}\mathcal{A}_s(I^*\rho_x^L) - \mathcal{J}_0(u_s(\rho_*^L))^2(x)\mathrm{d}s - K^L\mathrm{d}s)$$

and where we introduced a constant K^L and the one–dimensional process

$$Q_t^L = \frac{1}{2\pi}\int_{\mathbb{T}}[(u_t(\rho_x^L))^2 - \langle \rho_x^L, \rho_x^L\rangle]\mathrm{d}x$$

By definition of energy solution of SBE we have also

$$\mathcal{A}_t(I^*\rho_x^L) = -\lim_{N\to 0}\int_0^t \left(\mathcal{J}_0^N u_s\right)^2 (\partial_x I^*\rho_x^L)\mathrm{d}s = \lim_{N\to 0}\int_0^t \left(\mathcal{J}_0^N u_s\right)^2 (\rho_x^L)\mathrm{d}s \tag{10}$$

Thanks to eq. (10) we can define for any $N \geqslant 1$ the process

$$R_t^{L,N}(x) = \int_0^t \phi_s^L(x)(\left(\mathcal{J}_0^N u_s\right)^2 (\rho_x^L) - \mathcal{J}_0\left((\mathcal{J}_0^L u_s)^2\right)(x) - K^{L,N})\mathrm{d}s$$

$$= \int_0^t \phi_s^L(x)(\mathcal{J}_0^L\left(\mathcal{J}_0^N u_s\right)^2 (x) - \mathcal{J}_0\left((\mathcal{J}_0^L u_s)^2\right)(x) - K^{L,N})\mathrm{d}s$$

and observe that $\lim_{N\to\infty} R^{L,N} = R^L$ if we choose the sequence of constants $K^{L,N}$ so that $K^{L,N} \to K^L$. In Sec. 5 below we prove the following lemma which is the key for the convergence

Lemma 3 *There exists a choice of $K^{L,N}$ such that for any $\varphi \in \mathscr{S}$ we have $\mathbb{E}[R_t^{L,N}(\varphi)] = 0$ and*

$$\lim_{L\to\infty}\lim_{N\to\infty}\mathbb{E}[\sup_{0\leqslant t\leqslant T}(R_t^{L,N}(\varphi))^2] = 0$$

where $R_t^{L,N}(\varphi) = \langle R_t^{L,N}, \varphi\rangle_{L^2(\mathbb{T})}$. Moreover $K^{L,N} \to K^L$ as $N \to +\infty$ and $K^L \to K = -1/12$ as $L \to +\infty$.

We are now ready to prove our main result.

Proof (of Theorem 2) We have

$$R_t^L(\varphi) = \phi_t^L(\varphi) - \phi_0^L(\varphi) - \int_0^t \phi_s^L(\Delta\varphi)\mathrm{d}s - \int_0^t \phi_s^L(\varphi)(K^L - Q_s^L)\mathrm{d}s - \int_0^t\int_{\mathbb{T}} \phi_s^L(x)\varphi(x)\mathrm{d}M_s(\rho_x^L)\mathrm{d}x$$

We already know that the family $(\phi_t^L(x))_{t,x}$ convergences in the space of continuous paths on $[0,T]\times\mathbb{T}$ to $\phi_t(x) = \exp((Iu_t)(x))$. It is not difficult also to show that the process $(Q_s^L)_s$ converges in $L^2([0,T])$ to a limit which we call Q. The quadratic variation of the martingale part converges also, so we have that, for fixed $\varphi \in \mathscr{S}$

$$R_t(\varphi) = \lim_{L\to\infty} R_t^L(\varphi)$$

$$= \phi_t(\varphi) - \phi_0(\varphi) - \int_0^t \phi_s(\Delta\varphi)\mathrm{d}s - \int_0^t \phi_s(\varphi)(K - Q_s)\mathrm{d}s - \int_0^t\int_{\mathbb{T}} \phi_s(x)\varphi(x)\mathrm{d}\left(\mathcal{J}_0 M_s\right)(x)\mathrm{d}x$$

as a continuous process in $t \in [0,T]$. But from Lemma 3 we have also by Fatou

$$\mathbb{E}[\sup_{0\leqslant t\leqslant T}(R_t(\varphi))^2] \leqslant \liminf_{L\to\infty}\mathbb{E}[\sup_{0\leqslant t\leqslant T}(R_t^L(\varphi))^2] \leqslant \liminf_{L\to\infty}\liminf_{N\to\infty}\mathbb{E}[\sup_{0\leqslant t\leqslant T}(R_t^{L,N}(\varphi))^2] = 0$$

so we conclude that ϕ satisfy

$$\phi_t(\varphi) - \phi_0(\varphi) - \int_0^t \phi_s(\Delta\varphi)\mathrm{d}s - \int_0^t \phi_s(\varphi)(K - Q_s)\mathrm{d}s - \int_0^t\int_{\mathbb{T}} \phi_s(x)\varphi(x)\mathrm{d}\left(\mathcal{J}_0 M_s\right)(x)\mathrm{d}x = 0$$

almost surely for all $\varphi \in \mathscr{S}$ and $t \in [0,T]$. Note also that the process

$$\tilde{\phi}_t(x) = \phi_t(x)\exp\left(M_t(1) - \frac{t}{2} + Kt - \int_0^t Q_s\mathrm{d}s\right) = \exp\left((Iu_t)(x) + M_t(1) - \frac{t}{2} + Kt - \int_0^t Q_s\mathrm{d}s\right)$$

satisfies

$$\tilde{\phi}_t(\varphi) - \tilde{\phi}_0(\varphi) - \int_0^t \tilde{\phi}_s(\Delta\varphi)\mathrm{d}s - \int_0^t\int_{\mathbb{T}} \tilde{\phi}_s(x)\varphi(x)\mathrm{d}M_s(x)\mathrm{d}x = 0.$$

However we also know that there exists a unique strictly positive solution ψ for this equation which is adapted to the filtration generated by M, so we must have $\tilde{\phi}_t(x) = \psi_t(x)$ and in particular

$$(Iu_t)(x) + M_t(1) - \frac{t}{2} + Kt - \int_0^t Q_s \mathrm{d}s = \log \tilde{\phi}_t(x) = \log \psi_t(x)$$

By differentiating this equality (in the sense of distributions) and projecting away the constants we get

$$u_t(x) = \partial_x \log \psi_t(x)$$

which shows that the only energy solution of the SBE is the Cole–Hopf solution obtained from solving the SHE associated to the noise M. □

5 Convergence of the remainder

Let

$$r^{L,N}(u_s, x) = \phi_s^L(x)(\mathcal{J}_0^L \left(\mathcal{J}_0^N u_s\right)^2 (x) - \mathcal{J}_0(\mathcal{J}_0^L u_s)^2(x) - K^{L,N})$$

so that $R_t^{L,N}(x) = \int_0^t r^{L,N}(u_s, x)\mathrm{d}s$. Using Lemma 2 we can estimate

$$\mathbb{E}[\sup_{0 \leqslant t \leqslant T} (R_t^{L,N}(\varphi))^2] \lesssim T \|r^{L,N}(\cdot, \varphi)\|_{-1}^2$$

where

$$\|r^{L,N}(\cdot, \varphi)\|_{-1}^2 = \sup_{\Phi \in L^2(\mu)} 2\mathbb{E}[r^{L,N}(u_0, \varphi)\Phi(u_0)] - \|\Phi\|_1^2$$

with $\|\Phi\|_1^2 = \mathbb{E}\|\mathrm{D}\Phi\|_{H^1}^2$ in terms of the Malliavin derivative D associated to the measure μ. We prove below that we can choose $K^{L,N}$ so that $\mathbb{E}[r^{L,N}(u_t, x)] = \mathbb{E}[r^{L,N}(u_0, x)] = 0$ for all $x \in \mathbb{T}$. This is necessary in order for $\|r^{L,N}(\cdot, \varphi)\|_{-1}$ to be finite for all φ. At this point everything boils down to control

$$\mathbb{E}[r^{L,N}(u_0, \varphi)\Phi(u_0)]$$

and show that it goes to zero as $N \to +\infty$ and $L \to +\infty$ where φ is a smooth test function and $\Phi \in L^2(\mu)$ is such that $\|\Phi\|_1 < +\infty$.

For any $n \geqslant 1$ we denote by

$$W_n(h) = \int_{\mathbb{T}^n} h(z_1, \dots, z_n) W(\mathrm{d}z_1 \cdots \mathrm{d}z_n)$$

the n-th chaos of the white noise u_0. It is also possible to check that when Ψ is in the domain of the Malliavin derivative D we can partially integrate by parts in the expression $\mathbb{E}[W_n(h)\Psi]$ to obtain the equality

$$\mathbb{E}[W_n(h)\Psi] = \int_{\mathbb{T}} \mathbb{E}[W_{n-1}(h(z_1, \cdot, \dots, \cdot)) D_{z_1}\Psi]\mathrm{d}z_1$$

this will be used repeatedly below.

Observe that the random variable $(\mathcal{J}_0^L \left(\mathcal{J}_0^N u_0\right)^2 (x) - \mathcal{J}_0 \left((\mathcal{J}_0^L u_0)^2\right)(x))$ is an element of the second chaos of u_0. Let us compute its kernel:

$$\mathcal{J}_0^L \left(\mathcal{J}_0^N u_0\right)^2 (x) = \int_{\mathbb{T}^2} \left[\int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) \rho_y^N(z_1) \rho_y^N(z_2)\right] W(\mathrm{d}z_1 \mathrm{d}z_2)$$

and

$$\mathcal{J}_0\left((\mathcal{J}_0^L u_0)^2\right)(x) = \int_{\mathbb{T}^2}\left[\int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y)\rho_y^L(z_1)\rho_y^L(z_2)\right] W(\mathrm{d}z_1\mathrm{d}z_2)$$

where

$$\rho_x^\infty(y) = \lim_{K\to\infty}\rho_y^K(y) = \delta_x(y) - \frac{1}{2\pi}$$

is the distributional kernel of $\mathcal{J}_0$. Let

$$g_x^{L,N}(z_1,z_2) = \int_{\mathbb{T}} \mathrm{d}y(\rho_x^L(y)\rho_y^N(z_1)\rho_y^N(z_2) - \rho_x^\infty(y)\rho_y^L(z_1)\rho_y^L(z_2))$$

so that

$$(\mathcal{J}_0^L\left(\mathcal{J}_0^N u_0\right)^2(x) - \mathcal{J}_0\left((\mathcal{J}_0^L u_0)^2\right)(x)) = W_2(g_x^{L,N}) = \int_{\mathbb{T}^2} g_x^{L,N}(z_1,z_2)W(\mathrm{d}z_1\mathrm{d}z_2).$$

We let also

$$W_1(g_x^{L,N}(z_1,\cdot)) = \int_{\mathbb{T}} g_x^{L,N}(z_1,z_2)W(\mathrm{d}z_2).$$

Consider now

$$\mathbb{E}[r^\varepsilon(u_0,\varphi)\Phi(u_0)] = \int_{\mathbb{T}}\varphi(x)\mathbb{E}[(W_2(g_x^{L,N}) - K^{L,N})\phi_0^L(x)\Phi(u_0)]\mathrm{d}x$$

Partially integrating by parts W_2 we have

$$\mathbb{E}[W_2(g_x^{L,N})\phi_0^L(x)\Phi(u_0)] = \int_{\mathbb{T}}\mathbb{E}[W_1(g_x^{L,N}(z_1,\cdot))\mathrm{D}_{z_1}[\phi_0^L(x)\Phi(u_0)]]\mathrm{d}z_1$$

$$= \int_{\mathbb{T}}\mathbb{E}[W_1(g_x^{L,N}(z_1,\cdot))\phi_0^L(x)\mathrm{D}_{z_1}\Phi(u_0)]\mathrm{d}z_1 + \int_{\mathbb{T}}\mathbb{E}[W_1(g_x^{L,N}(z_1,\cdot))\mathrm{D}_{z_1}\phi_0^L(x)\Phi(u_0)]\mathrm{d}z_1$$

The second term can be again integrated by parts to obtain

$$\int_{\mathbb{T}}\mathbb{E}[W_1(g_x^{L,N}(z_1,\cdot))\mathrm{D}_{z_1}\phi_0^L(x)\Phi(u_0)]\mathrm{d}z_1 = \int_{\mathbb{T}^2} g_x^{L,N}(z_1,z_2)\mathbb{E}[\mathrm{D}^2_{z_1,z_2}\phi_0^L(x)\Phi(u_0)]\mathrm{d}z_1\mathrm{d}z_2$$

$$+\int_{\mathbb{T}^2} g_x^{L,N}(z_1,z_2)\mathbb{E}[\mathrm{D}_{z_1}\phi_0^L(x)\mathrm{D}_{z_2}\Phi(u_0)]\mathrm{d}z_1\mathrm{d}z_2.$$

while the first can be written as

$$\int_{\mathbb{T}}\mathbb{E}[W_1(g_x^{L,N}(z_1,\cdot))\phi_0^L(x)\mathrm{D}_{z_1}\Phi(u_0)]\mathrm{d}z_1 = \int_{\mathbb{T}}\mathbb{E}[(W_1(g_x^{L,N}(z_1,\cdot))\diamond\phi_0^L(x))\mathrm{D}_{z_1}\Phi(u_0)]\mathrm{d}z_1$$

$$+\int_{\mathbb{T}^2} g_x^{L,N}(z_1,z_2)\mathbb{E}[\mathrm{D}_{z_2}\phi_0^L(x)\mathrm{D}_{z_1}\Phi(u_0)]\mathrm{d}z_1\mathrm{d}z_2.$$

where

$$(W_1(g_x^{L,N}(z_1,\cdot))\diamond\phi_0^L(x)) = W_1(g_x^{L,N}(z_1,\cdot))\phi_0^L(x) - \int_{\mathbb{T}} g_x^{L,N}(z_1,z_2)\mathrm{D}_{z_2}\phi_0^L(x)\mathrm{d}z_2$$

is a (partial) Wick contraction of the two terms. In the end we have the decomposition

$$\mathbb{E}[r^{L,N}(u_0,\varphi)\Phi(u_0)] = A^{L,N} + B^{L,N} + C^{L,N}$$

where

$$A^{L,N} = \int_{\mathbb{T}}\varphi(x)\int_{\mathbb{T}}\mathbb{E}[(W_1(g_x^{L,N}(z_1,\cdot))\diamond\phi_0^L(x))\mathrm{D}_{z_1}\Phi(u_0)]\mathrm{d}z_1\mathrm{d}x$$

$$B^{L,N} = \int_{\mathbb{T}} \varphi(x) \left[\int_{\mathbb{T}^2} g_x^{L,N}(z_1, z_2) \mathbb{E}\left[(\mathrm{D}^2_{z_1,z_2} \phi_0^L(x)) \Phi(u_0) \right] \mathrm{d}z_1 \mathrm{d}z_2 - K^{L,N} \mathbb{E}[\phi_0^L(x) \Phi(u_0)] \right] \mathrm{d}x$$

and

$$C^{L,N} = 2 \int_{\mathbb{T}} \varphi(x) \int_{\mathbb{T}^2} g_x^{L,N}(z_1, z_2) \mathbb{E}[(\mathrm{D}_{z_1} \phi_0^L(x)) \mathrm{D}_{z_2} \Phi(u_0)] \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}x.$$

So we can bound these three terms independently. In order to proceed, observe that

$$\mathrm{D}_{z_1} \phi_0^L(x) = \phi_0^L(x) (I^* \rho_x^L)(z_1)$$

and

$$\mathrm{D}^2_{z_1,z_2} \phi_0^L(x) = \phi_0^L(x) (I^* \rho_x^L)(z_1) (I^* \rho_x^L)(z_2)$$

so

$$B^{L,N} = \int_{\mathbb{T}} \varphi(x) \left[\int_{\mathbb{T}^2} g_x^{L,N}(z_1, z_2) (I^* \rho_x^L)(z_1) (I^* \rho_x^L)(z_2) \mathrm{d}z_1 \mathrm{d}z_2 - K^{L,N} \right] \mathbb{E}[\phi_0^L(x) \Phi(u_0)] \mathrm{d}x$$

and

$$C^{L,N} = 2 \int_{\mathbb{T}} \varphi(x) \int_{\mathbb{T}^2} g_x^{L,N}(z_1, z_2) (I^* \rho_x^L)(z_1) \mathbb{E}[\phi_0^L(x) \mathrm{D}_{z_2} \Phi(u_0)] \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}x.$$

Let

$$K^{L,N} = \int_{\mathbb{T}^2} g_x^{L,N}(z_1, z_2) (I^* \rho_x^L)(z_1) (I^* \rho_x^L)(z_2) \mathrm{d}z_1 \mathrm{d}z_2$$

to have $B^{L,N} = 0$. From the expression of $A^{L,N}$ and $C^{L,N}$ we deduce that with this choice

$$\mathbb{E}[r^{L,N}(u_0, \varphi)] = 0$$

as required. By Cauchy–Schwarz we have

$$(A^{L,N})^2 \leqslant \mathbb{E} \left\| \int_{\mathbb{T}} \varphi(x) (W_1(g_x^{L,N}(z_1, \cdot)) \diamond \phi_0^L(x)) \mathrm{d}x \right\|^2_{H^{-1}_{z_1}} \mathbb{E} \| \mathrm{D}_{z_1} \Phi(u_0) \|^2_{H^1_{z_1}} = (A_1^{L,N})^2 \|\Phi\|_1^2$$

where

$$(A_1^{L,N})^2 = \mathbb{E} \left\| \int_{\mathbb{T}} \varphi(x) W_1(g_x^{L,N}(z_1, \cdot)) \diamond \phi_0^L(x) \mathrm{d}x \right\|^2_{H^{-1}_{z_1}}$$

$$= \int_{\mathbb{T}^2} \varphi(x) \varphi(x') \mathbb{E} \left[\langle \phi_0^L(x) \diamond W_1(g_x^{L,N}(z_1, \cdot)), \phi_0^L(x') \diamond W_1(g_{x'}^{L,N}(z_1, \cdot)) \rangle_{H^{-1}_{z_1}} \right] \mathrm{d}x \mathrm{d}x'$$

and

$$(C^{L,N})^2 \lesssim \mathbb{E} \left\| \int_{\mathbb{T}} \varphi(x) \phi_0^L(x) \int_{\mathbb{T}} g_x^{L,N}(z_1, z_2) (I^* \rho_x^L)(z_1) \mathrm{d}z_1 \mathrm{d}x \right\|^2_{H^{-1}_{z_2}} \mathbb{E} \| \mathrm{D}_{z_2} \Phi(u_0) \|^2_{H^1_{z_2}}$$

$$\lesssim (C_1^{L,N})^2 \|\Phi\|_1^2$$

with

$$(C_1^{L,N})^2 = \mathbb{E} \left\| \int_{\mathbb{T}} \varphi(x) \phi_0^L(x) \int_{\mathbb{T}} g_x^{L,N}(z_1, z_2) (I^* \rho_x^L)(z_1) \mathrm{d}z_1 \mathrm{d}x \right\|^2_{H^{-1}_{z_2}}$$

So we have proven

Lemma 4 *We have*

$$|\mathbb{E}[r^{L,N}(u_0, \varphi) \Phi(u_0)]| \lesssim \|\Phi\|_1 (A_1^{L,N} + C_1^{L,N})$$

and in particular

$$\|r^{L,N}(u_0, \varphi)\|_{-1} \lesssim A_1^{L,N} + C_1^{L,N}.$$

Then it remains to bound each of these constants to show that they vanish in the limit $N, L \to +\infty$ and to study the limit of $K^{L,N}$. This is the aim of the following lemmas.

Lemma 5 *We have* $\lim_{N\to+\infty} K^{L,N} = K^L$ *and* $\lim_{L\to+\infty} K^L = K = -1/12$.

Proof Since

$$\int_{\mathbb{T}} (I^*\rho_x^L)(z_1)\rho_y^N(z_1)\mathrm{d}z_1 = \left\langle I^*\mathcal{J}_0^L\delta_x, \mathcal{J}_0^N\delta_y\right\rangle_{L^2} = \left\langle \delta_x, I\mathcal{J}_0^L\mathcal{J}_0^N\delta_y\right\rangle_{L^2} = \left(I\mathcal{J}_0^L\mathcal{J}_0^N\delta_y\right)(x)$$

we have

$$K^{L,N} = \int_{\mathbb{T}^2} g_x^{L,N}(z_1,z_2)(I^*\rho_x^L)(z_1)(I^*\rho_x^L)(z_2)\mathrm{d}z_1\mathrm{d}z_2$$

$$= \int_{\mathbb{T}} \mathrm{d}y(\rho_x^L(y)\left(I\mathcal{J}_0^{L\wedge N}\delta_y\right)(x)^2 - \rho_x^\infty(y)\left(I\mathcal{J}_0^L\delta_y\right)(x)^2).$$

Taking $N \to \infty$ we get

$$K^{L,N} \to K^L = \int_{\mathbb{T}} \mathrm{d}y(\rho_x^L(y)\left(I\mathcal{J}_0^L\delta_y\right)(x)^2 - \rho_x^\infty(y)\left(I\mathcal{J}_0^L\delta_y\right)(x)^2).$$

Now note that the integral of the first term in the expression of K^L vanishes, indeed:

$$\int_{\mathbb{T}} \mathrm{d}y\rho_x^L(y)\left(I\mathcal{J}_0^L\delta_y\right)(x)^2 = \int_{\mathbb{T}} \mathrm{d}y\rho_x^L(y)\left(I\mathcal{J}_0^L\delta_x\right)(y)^2$$

$$= \int_{\mathbb{T}} \mathrm{d}y\left(\partial_y I\mathcal{J}_0^L\delta_x\right)(y)\left(I\mathcal{J}_0^L\delta_x\right)(y)^2 = \frac{1}{3}\int_{\mathbb{T}} \mathrm{d}y\partial_y\left(I\mathcal{J}_0^L\delta_x\right)(y)^3 = 0.$$

Moreover the second term satisfies

$$\int_{\mathbb{T}} \mathrm{d}y\rho_x^\infty(y)\left(I\mathcal{J}_0^L\delta_y\right)(x)^2 = \left(I\mathcal{J}_0^L\delta_x\right)(x)^2 - \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{d}y\left(I\mathcal{J}_0^L\delta_y\right)(x)^2$$

since $\rho_x^\infty = \delta_x - 1/2\pi$. By symmetry also the $\left(I\mathcal{J}_0^L\delta_x\right)(x)^2$ contribution vanishes, since

$$\left(I\mathcal{J}_0^L\delta_x\right)(x) = \left(I^*\mathcal{J}_0^L\delta_x\right)(x) = -\left(I\mathcal{J}_0^L\delta_x\right)(x) = 0.$$

So we ends up with

$$K^L = -\frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{d}y\left(I\mathcal{J}_0^L\delta_x\right)(y)^2 = -\frac{1}{2\pi}\left\langle I\mathcal{J}_0^L\delta_x, I\mathcal{J}_0^L\delta_x\right\rangle = -\frac{1}{4\pi^2}\sum_{0<|k|}\frac{|\hat{\rho}^L(k)|^2}{|k|^2}$$

since

$$\mathscr{F}\left(I\mathcal{J}_0^L\delta_x\right)(k) = \frac{\hat{\rho}^L(k)}{ik}e_k(x)^*.$$

And as $L \to \infty$ we obtain $\hat{\rho}^L(k) \to 1$ for all $k \neq 0$ so

$$K^L \to -\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{12} = K.$$

□

Lemma 6 *We have* $A_1^{L,N} \to 0$ *as* $N \to \infty$ *and* $L \to \infty$.

Proof

$$(A_1^{L,N})^2 = \mathbb{E}\left\| \int_{\mathbb{T}} \varphi(x) W_1(g_x^{L,N}(z_1,\cdot)) \diamond \phi_0^L(x) \mathrm{d}x \right\|^2_{H_{z_1}^{-1}}$$

$$= \int_{\mathbb{T}^2} \varphi(x)\varphi(x') \mathbb{E}\left[\langle \phi_0^L(x) \diamond W_1(g_x^{L,N}(z_1,\cdot)), \phi_0^L(x') \diamond W_1(g_{x'}^{L,N}(z_1,\cdot)) \rangle_{H_{z_1}^{-1}} \right] \mathrm{d}x \mathrm{d}x'$$

Integrating by parts the W_1 terms and taking into account the cancellations due to the partial $\diamond$ contractions we get

$$\mathbb{E}\left[\langle \phi_0^L(x) \diamond W_1(g_x^{L,N}(z_1,\cdot)), \phi_0^L(x') \diamond W_1(g_{x'}^{L,N}(z_1,\cdot)) \rangle_{H_{z_1}^{-1}} \right]$$

$$= \mathbb{E}[\phi_0^L(x)\phi_0^L(x')] \int_{\mathbb{T}^2} \langle g_x^{L,N}(z_1,z_2), g_{x'}^{L,N}(z_1,z_2) \rangle_{H_{z_1}^{-1}} \mathrm{d}z_2$$

$$+ \int_{\mathbb{T}^2} \mathbb{E}[(\mathrm{D}_{z_3}\phi_0^L(x))(\mathrm{D}_{z_2}\phi_0^L(x'))] \langle g_x^{L,N}(z_1,z_2), g_{x'}^{L,N}(z_1,z_3) \rangle_{H_{z_1}^{-1}} \mathrm{d}z_2 \mathrm{d}z_3$$

The second term can be written as

$$\mathbb{E}[\phi_0^L(x)\phi_0^L(x')] \int_{\mathbb{T}^2} (I^*\rho_x^L)(z_3)(I^*\rho_{x'}^L)(z_2) \langle g_x^{L,N}(z_1,z_2), g_{x'}^{L,N}(z_1,z_3) \rangle_{H_{z_1}^{-1}} \mathrm{d}z_2 \mathrm{d}z_3$$

so letting

$$V^L(x,x') = \varphi(x)\varphi(x')\mathbb{E}[\phi_0^L(x)\phi_0^L(x')]$$

we have

$$(A_1^{L,N})^2 = \int_{\mathbb{T}^2} V^L(x,x') \int_{\mathbb{T}^2} (I^*\rho_x^L)(z_3)(I^*\rho_{x'}^L)(z_2) \langle g_x^{L,N}(z_1,z_2), g_{x'}^{L,N}(z_1,z_3) \rangle_{H_{z_1}^{-1}} \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}x \mathrm{d}x'$$

$$+ \int_{\mathbb{T}^2} V^L(x,x') \int_{\mathbb{T}^2} \langle g_x^{L,N}(z_1,z_2), g_{x'}^{L,N}(z_1,z_2) \rangle_{H_{z_1}^{-1}} \mathrm{d}z_2 \mathrm{d}x \mathrm{d}x' = A_{1,1} + A_{1,2}$$

Let us consider $A_{1,1}$ first :

$$A_{1,1} = \int_{\mathbb{T}^4} V^L(x,x') \langle G_{x,x'}, G_{x',x} \rangle_{H^{-1}} \mathrm{d}x \mathrm{d}x'$$

where

$$G_{x,x'}(z_1) = \int_{\mathbb{T}} (I^*\rho_{x'}^L)(z_2) g_x^{L,N}(z_1,z_2) \mathrm{d}z_2.$$

Now observe that

$$G_{x,x'}(z_1) = \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) \rho_y^N(z_1) (\mathcal{J}_0^N I^* \rho_{x'}^L)(y) - \int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y) \rho_y^L(z_1) (I^*\rho_{x'}^L)(y)$$

so the H^{-1} scalar product in $A_{1,1}$ can be expanded as

$$\langle G_{x,x'}, G_{x',x} \rangle_{H^{-1}} = \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) (\mathcal{J}_0^N I^* \rho_{x'}^L)(y) \int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^L(y') (\mathcal{J}_0^N I^* \rho_x^L)(y') \langle \rho_y^N(z_1), \rho_{y'}^N(z_1) \rangle_{H_{z_1}^{-1}}$$

$$- \int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y) (I^*\rho_{x'}^L)(y) \int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^L(y') (\mathcal{J}_0^N I^* \rho_x^L)(y') \langle \rho_y^L(z_1), \rho_{y'}^N(z_1) \rangle_{H_{z_1}^{-1}}$$

$$- \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) (\mathcal{J}_0^N I^* \rho_{x'}^L)(y) \int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^\infty(y') (I^* \rho_x^L)(y') \langle \rho_y^N(z_1), \rho_{y'}^L(z_1) \rangle_{H_{z_1}^{-1}}$$

$$+\int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y)(I^*\rho_{x'}^L)(y)\int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^\infty(y')(I^*\rho_x^L)(y')\langle \rho_y^L(z_1),\rho_{y'}^L(z_1)\rangle_{H_{z_1}^{-1}}$$

Taking $N\to\infty$ we get

$$\begin{aligned}\langle G_{x,x'},G_{x',x}\rangle_{H^{-1}} \to &\int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y)(I^*\rho_{x'}^L)(y)\int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^L(y')(I^*\rho_x^L)(y')\langle \rho_y^\infty(z_1),\rho_{y'}^\infty(z_1)\rangle_{H_{z_1}^{-1}}\\
&-\int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y)(I^*\rho_{x'}^L)(y)\int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^L(y')(I^*\rho_x^L)(y')\langle \rho_y^L(z_1),\rho_{y'}^\infty(z_1)\rangle_{H_{z_1}^{-1}}\\
&-\int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y)(I^*\rho_{x'}^L)(y)\int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^\infty(y')(I^*\rho_x^L)(y')\langle \rho_y^\infty(z_1),\rho_{y'}^L(z_1)\rangle_{H_{z_1}^{-1}}\\
&+\int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y)(I^*\rho_{x'}^L)(y)\int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^\infty(y')(I^*\rho_x^L)(y')\langle \rho_y^L(z_1),\rho_{y'}^L(z_1)\rangle_{H_{z_1}^{-1}}\\
=&\left\langle I\mathcal{J}_0(\rho_x^L(I^*\rho_{x'}^L)),I\mathcal{J}_0(\rho_{x'}^L(I^*\rho_x^L))\right\rangle - \left\langle I\mathcal{J}_0^L(\rho_x^\infty(I^*\rho_{x'}^L)),I\mathcal{J}_0(\rho_{x'}^L(I^*\rho_x^L))\right\rangle\\
-&\left\langle I\mathcal{J}_0(\rho_x^L(I^*\rho_{x'}^L)),I\mathcal{J}_0^L(\rho_{x'}^\infty(I^*\rho_x^L))\right\rangle + \left\langle I\mathcal{J}_0^L(\rho_x^\infty(I^*\rho_{x'}^L)),I\mathcal{J}_0^L(\rho_{x'}^\infty(I^*\rho_x^L))\right\rangle.\end{aligned}$$

The point is to show that, as $L\to+\infty$,

$$\left\langle I\mathcal{J}_0(\rho_x^L(I^*\rho_{x'}^L)),I\mathcal{J}_0(\rho_{x'}^L(I^*\rho_x^L))\right\rangle \to \langle I\mathcal{J}_0(\rho_x^\infty(I^*\rho_{x'}^\infty)),I\mathcal{J}_0(\rho_{x'}^\infty(I^*\rho_x^\infty))\rangle$$

and the same limit for the other three quantities. Writing things in Fourier space we have

$$\begin{aligned}J_L(x-x') &= \left\langle I\mathcal{J}_0(\rho_x^L(I^*\rho_{x'}^L)),I\mathcal{J}_0(\rho_{x'}^L(I^*\rho_x^L))\right\rangle\\
&= \frac{1}{2\pi}\sum_{\substack{0<|k_1|,|k_2|,|k_3|,|k_4|\\ k_1+k_2+k_3+k_4=0}} \frac{\mathbb{I}_{k_1+k_2\neq 0}}{(k_1+k_2)^2}e^{ik_1x}\frac{e^{ik_2x'}}{-ik_2}e^{ik_3x'}\frac{e^{ik_4x}}{-ik_4}\hat\rho^L(k_1)\hat\rho^L(k_2)\hat\rho^L(k_3)\hat\rho^L(k_4)\\
&= \frac{1}{2\pi}\sum_{\substack{0<|k_1|,|k_2|,|k_3|,|k_4|\\ k_1+k_2+k_3+k_4=0}} \frac{\mathbb{I}_{k_1+k_2\neq 0}}{(k_1+k_2)^2}\frac{e^{i(k_2+k_3)(x'-x)}}{-ik_2}\frac{1}{-ik_4}\hat\rho^L(k_1)\hat\rho^L(k_2)\hat\rho^L(k_3)\hat\rho^L(k_4)\end{aligned}$$

Now taking Littlewood–Paley blocks $(\Delta_q)_{q\geqslant -1}$ of this quantity we get

$$\Delta_q J_L(x) = \frac{1}{2\pi}\sum_{\substack{0<|k_1|,|k_2|,|k_3|,|k_4|\\ k_1+k_2+k_3+k_4=0}} \frac{\mathbb{I}_{k_1+k_2\neq 0}}{(k_1+k_2)^2}\frac{e^{i(k_2+k_3)(x'-x)}}{-ik_2}\frac{\theta_q(k_2+k_3)}{-ik_4}\hat\rho^L(k_1)\hat\rho^L(k_2)\hat\rho^L(k_3)\hat\rho^L(k_4).$$

Where θ_q is the Fourier multiplier associated to Δ_q by $\Delta_q=\theta_q(\mathrm{D})$. Taking into accout that $\sup_k|\hat\rho^L(k)|<+\infty$ if we show that the quantity

$$S_q = \frac{1}{2\pi}\sum_{\substack{0<|k_1|,|k_2|,|k_3|,|k_4|\\ k_1+k_2+k_3+k_4=0}} \mathbb{I}_{k_1+k_2\neq 0}\frac{\theta_q(k_2+k_3)}{(k_1+k_2)^2|k_2||k_3|}$$

is finite then we can conclude by dominated convergence. Note that

$$S_q \lesssim 2^{q\varepsilon}\sum_{0<|k_1|,|k_2|,|k_3|}\frac{\mathbb{I}_{k_1+k_2\neq 0,k_2+k_3\neq 0}}{(k_1+k_2)^2|k_2||k_3||k_2+k_3|^\varepsilon}$$

for any $\varepsilon>0$ since

$$|\theta_q(k)| = |\theta(2^{-q}k)| \lesssim (1+|2^{-q}k|)^{-\varepsilon} \lesssim 2^{q\varepsilon}|k|^{-\varepsilon}$$

due to the fact that the function θ is supported in a ball of finite radius. We can now perform the sum over k_3 and get

$$S_q \lesssim 2^{q\varepsilon} \sum_{0<|k_1|,|k_2|} \frac{\mathbb{I}_{k_1+k_2\neq 0}}{(k_1+k_2)^2|k_2|}$$

since

$$\sum_{0<|k_3|} \frac{\mathbb{I}_{k_2+k_3\neq 0}}{|k_3||k_2+k_3|^\varepsilon} \lesssim \frac{1}{|k_2|^\varepsilon}$$

uniformly in $k_2 \neq 0$ and L. Now performing the other two sums we show simply that

$$S_q \lesssim 2^{q\varepsilon} \sum_{0<|k_2|} \frac{1}{|k_2|^{1+\varepsilon}} \lesssim 2^{q\varepsilon}$$

where we used the fact that

$$\sum_{0<|k_1|} \frac{\mathbb{I}_{k_1+k_2\neq 0}}{(k_1+k_2)^2} \leqslant \sum_{k_1\in\mathbb{Z}} \frac{\mathbb{I}_{k_1+k_2\neq 0}}{(k_1+k_2)^2} = \sum_{k_1\in\mathbb{Z}} \frac{\mathbb{I}_{k_1\neq 0}}{(k_1)^2} \lesssim 1.$$

By dominated convergence we can conclude that

$$\left\langle I\mathcal{J}_0(\rho^L_x(I^*\rho^L_{x'})), I\mathcal{J}_0(\rho^L_{x'}(I^*\rho^L_x))\right\rangle \to \langle I\mathcal{J}_0(\rho^\infty_x(I^*\rho^\infty_{x'})), I\mathcal{J}_0(\rho^\infty_{x'}(I^*\rho^\infty_x))\rangle$$

as distributions of arbitrarily small negative order of the variable $x-x'$. This is ok since we test this quantity against the function V^L which belongs to $C^{1/2-}(\mathbb{T}^2)$ uniformly in L. The other terms can be handled similarly and thus we conclude that $A_{1,1} \to 0$.

Let us turn now to $A_{1,2}$. The relevant computation is the following:

$$\int_{\mathbb{T}^2} \langle g^{L,N}_x(z_1,z_2), g^{L,N}_{x'}(z_1,z_2)\rangle_{H^{-1}_{z_1}} \mathrm{d}z_2 =$$

$$\begin{aligned}
&\int_{\mathbb{T}} \mathrm{d}y \rho^L_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^L_{x'}(y') \langle \rho^N_y(z_1), \rho^N_{y'}(z_1)\rangle_{H^{-1}_{z_1}} \langle \rho^N_y(z_2), \rho^N_{y'}(z_2)\rangle_{L^2_{z_2}} \\
&- \int_{\mathbb{T}} \mathrm{d}y \rho^L_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^\infty_{x'}(y') \langle \rho^N_y(z_1), \rho^L_{y'}(z_1)\rangle_{H^{-1}_{z_1}} \langle \rho^N_y(z_2), \rho^L_{y'}(z_2)\rangle_{L^2_{z_2}} \\
&- \int_{\mathbb{T}} \mathrm{d}y \rho^\infty_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^L_{x'}(y') \langle \rho^L_y(z_1), \rho^N_{y'}(z_1)\rangle_{H^{-1}_{z_1}} \langle \rho^L_y(z_2), \rho^N_{y'}(z_2)\rangle_{L^2_{z_2}} \\
&+ \int_{\mathbb{T}} \mathrm{d}y \rho^\infty_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^\infty_{x'}(y') \langle \rho^L_y(z_1), \rho^L_{y'}(z_1)\rangle_{H^{-1}_{z_1}} \langle \rho^L_y(z_2), \rho^L_{y'}(z_2)\rangle_{L^2_{z_2}}
\end{aligned}$$

Now consider one of the terms:

$$\begin{aligned}
&\int_{\mathbb{T}} \mathrm{d}y \rho^L_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^L_{x'}(y') \langle \rho^N_y(z_1), \rho^N_{y'}(z_1)\rangle_{H^{-1}_{z_1}} \langle \rho^N_y(z_2), \rho^N_{y'}(z_2)\rangle_{L^2_{z_2}} \\
&= \int_{\mathbb{T}} \mathrm{d}y \rho^L_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^L_{x'}(y') \rho^N_y(y') (I^* I \rho^N_y)(y')
\end{aligned}$$

since

$$\langle \rho^N_y(z_1), \rho^N_{y'}(z_1)\rangle_{H^{-1}_{z_1}} = \langle I\rho^N_y, I\rho^N_{y'}\rangle_{L^2} = \langle I^*I\rho^N_y, \delta_{y'}\rangle_{L^2} = (I^*I\rho^N_y)(y')$$

As $N\to\infty$ this quantity becomes

$$\to \int_{\mathbb{T}} \mathrm{d}y \rho^L_x(y) \int_{\mathbb{T}} \mathrm{d}y' \rho^L_{x'}(y') \rho^\infty_y(y') (I^* I \rho^\infty_y)(y')$$

$$= \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) \rho_{x'}^L(y) (I^* I \rho_y^\infty)(y) - \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) \int_{\mathbb{T}} \mathrm{d}y' \rho_{x'}^L(y') (I^* I \rho_y^\infty)(y')$$

$$= (I^* I \rho_0^\infty)(0) \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) \rho_{x'}^L(y) - \frac{1}{2\pi} (I^* I \rho_x^L)(x')$$

$$= (I^* I \rho_0^\infty)(0) \rho_x^L(x') - \frac{1}{2\pi} (I^* I \rho_x^L)(x')$$

since $(I^* I \rho_y^\infty)(y) = (I^* I \rho_0^\infty)(0)$. As $L \to \infty$ we obtain

$$= (I^* I \rho_0^\infty)(0) \rho_x^\infty(x') - \frac{1}{2\pi} (I^* I \rho_x^\infty)(x')$$

For the other terms the discussion of the convergence is similar and the limit is the same so we obtain that

$$\int_{\mathbb{T}^2} \langle g_x^{L,N}(z_1, z_2), g_{x'}^{L,N}(z_1, z_2) \rangle_{H_{z_1}^{-1}} \mathrm{d}z_2 \to 0$$

as a distribution in the x, x' variables and we conclude that $A_{1,2} \to 0$. □

Lemma 7 *We have $C_1^{L,N} \to 0$ as $N \to \infty$ and $L \to \infty$.*

Proof Let

$$G_x(z_2) = \int_{\mathbb{T}} g_x^{L,N}(z_1, z_2) (I^* \rho_x^L)(z_1) \mathrm{d}z_1 = G_{x,x}(z_2)$$

and recall that

$$(C_1^{L,N})^2 = \mathbb{E} \left\| \int_{\mathbb{T}} \varphi(x) \phi_0^L(x) G_x(z_2) \mathrm{d}x \right\|_{H_{z_2}^{-1}}^2 = \int_{\mathbb{T}^2} V^L(x, x') \langle G_x, G_{x'} \rangle_{H^{-1}} \mathrm{d}x \mathrm{d}x'$$

In order to estimate the scalar product can take $f \in H^1$ and consider $\int G_{x,x}(z) f(z) \mathrm{d}z$ instead. Recall that

$$\int G_{x,x}(z) f(z) \mathrm{d}z = \int_{\mathbb{T}} \mathrm{d}y \rho_x^L(y) \mathcal{J}_0^N f(y) (\mathcal{J}_0^N I^* \rho_x^L)(y) - \int_{\mathbb{T}} \mathrm{d}y \rho_x^\infty(y) \mathcal{J}_0^L f(y) (I^* \rho_x^L)(y)$$

replace $\rho_x^\infty(y)$ by $\delta_x(y) - 1/2\pi$ and $\rho_x^L(y)$ by $\delta_x^L(y) - 1/2\pi$ to get

$$= \int_{\mathbb{T}} \mathrm{d}y \delta_x^L(y) \left(\mathcal{J}_0^N f(y) \right) (\mathcal{J}_0^N I^* \rho_x^L)(y) - \left(\mathcal{J}_0^L f(x) \right) (I^* \rho_x^L)(x)$$

$$- \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{d}y \mathcal{J}_0^N f(y) (\mathcal{J}_0^N I^* \rho_x^L)(y) + \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{d}y \mathcal{J}_0^L f(y) (I^* \rho_x^L)(y)$$

$$= \int_{\mathbb{T}} \mathrm{d}y \delta_x^L(y) \left(\mathcal{J}_0^N f(y) - \mathcal{J}_0^N f(x) \right) (\mathcal{J}_0^N I^* \rho_x^L)(y)$$

$$+ \left(\mathcal{J}_0^N f(x) \right) (I^* \left(\delta^L * \mathcal{J}_0^N \rho_x^L \right))(x) - \left(\mathcal{J}_0^L f(x) \right) (I^* \rho_x^L)(x)$$

$$- \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{d}y \mathcal{J}_0^N f(y) (\mathcal{J}_0^N I^* \rho_x^L)(y) + \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{d}y \mathcal{J}_0^L f(y) (I^* \rho_x^L)(y).$$

We note that, by symmetry,

$$\left(\mathcal{J}_0^N f(x) \right) (I^* \left(\delta^L * \mathcal{J}_0^N \rho_x^L \right))(x) = \left(\mathcal{J}_0^L f(x) \right) (I^* \rho_x^L)(x) = 0$$

and moreover that in this expression we have that

$$\sup_x \left| \left\langle \mathcal{J}_0^L f, I^* \rho_x^L \right\rangle - \langle f, \Theta_x \rangle \right| + \sup_x \left| \left\langle \mathcal{J}_0^N f, \mathcal{J}_0^N I^* \rho_x^L \right\rangle - \langle f, \Theta_x \rangle \right| \lesssim \|f\|_{H^1} o(1)$$

as $L \wedge N \to \infty$. Finally it remains to bound

$$H = \int_{\mathbb{T}} \mathrm{d}y \delta_x^L(y) \left(\mathcal{J}_0^N f(y) - \mathcal{J}_0^N f(x) \right) (\mathcal{J}_0^N I^* \rho_x^L)(y)$$

However using $H^1 \subseteq C^{1/2}$ compactly and that $\left\| \mathcal{J}_0^N I^* \rho_x^L \right\|_\infty < +\infty$ uniformy in x, L, N we have

$$|H| \lesssim \left\| \mathcal{J}_0^N f \right\|_{H^1} \int_{\mathbb{T}} \mathrm{d}y \delta_x^L(y) |y - x|^{1/2} \lesssim \|f\|_{H^1} L^{-1/2} \to 0.$$

In conclusion

$$\sup_{x \in \mathbb{T}} \left| \int G_{x,x'}(z) f(z) \mathrm{d}z \right| \lesssim \|f\|_{H^1} o(1)$$

which implies $\sup_{x \in \mathbb{T}} \langle G_{x,x}, G_{x,x} \rangle_{H^{-1}} = o(1)$ and thus we can conclude that $C_1^{L,N} \to 0$. □

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