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NONLINEAR PDES WITH MODULATED DISPERSION II: KORTEWEG-DE VRIES EQUATION

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Abstract. We study dispersive equations with a time non-homogeneous modulation acting on the linear dispersion term. As primary models, we consider the Korteweg-de Vries equation (KdV) and related equations such as the Benjamin-Ono equation (BO) and the intermediate long wave equation (ILW), imposing certain irregularity conditions on the time non-homogeneous modulation. In this work, we establish phenomena called *regularization by noise* in three-folds: (i) When the modulation is sufficiently irregular, we show that the modulated KdV on both the circle and the real line is locally well-posed in the regime where the (unmodulated) KdV equation is known to be ill-posed. In particular, given *any* $s \in \mathbb{R}$, we show that the modulated KdV on the circle with a sufficiently irregular modulation is locally well-posed in $H^s(\mathbb{T})$. For example, this result implies that if the modulation is given by a fractional Brownian motion with Hurst index $0 < H < \frac{2}{3}$, the modulated KdV on the circle is locally well-posed in $H^{-\frac{1}{2H}+\varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$. Moreover, by adapting the I-method to the current modulated setting, we prove global well-posedness of the modulated KdV in negative Sobolev spaces. (ii) It is known that (semilinear) dispersive equations such as KdV and BO exhibit quasilinear nature below certain regularity thresholds. We show that sufficiently irregular modulations make the modulated versions of these equations semilinear by establishing their local well-posedness by a contraction argument, providing local Lipschitz continuity of the solution map. (iii) We also prove nonlinear smoothing for these modulated equations, where we show that a gain of regularity of the nonlinear part becomes (arbitrarily) larger for more irregular modulations.

As applications of our approach, we also include the following examples: (iv) deep-water and shallow-water convergence of the modulated (scaled) ILW in arbitrary negative Sobolev spaces. (v) the modulated KdV on the circle with white noise initial data. (vi) stochastic modulated KdV on the circle with an additive space-time white noise and singular additive noises of arbitrarily low regularity.

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1. Introduction

1.1. **Modulated dispersive equations.** In this paper, we study a modulated dispersive equation of the following form:

$$\begin{cases} \partial_t u + Lu \cdot \partial_t w = \mathcal{N}(u) \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{M}, \tag{1.1}$$

where $\mathcal{M} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$,[1] $u : \mathbb{R}_+ \times \mathcal{M} \to \mathbb{R}$ or $\mathbb{C}$ is the unknown, the operator L denotes an (unbounded) linear dispersive operator, and $\mathcal{N}(u)$ denotes the nonlinearity. Here, $w : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary continuous function of time, called a *modulation*, which will play a central role in our analysis. Previously, de Bouard and Debussche [48] and Debussche and Tsutsumi [51] used stochastic calculus to study the modulated nonlinear Schrödinger equations (NLS) with the modulation w given by a Brownian motion; see also [56, 147]. For a general modulation w, such an approach based on stochastic calculus, interpreting (1.1) as an Ito or Stratonovich stochastic PDE is not available, and our goal in this paper is to develop a *pathwise* approach to study the modulated dispersive equation (1.1), exploiting *irregularity* of the modulation function w; see Definition 1.1. In the previous work [34], the first two authors developed a pathwise approach to study well-posedness of the modulated NLS. In this paper, we study the following modulated Korteweg-de Vries equation (KdV) on both the circle and the real line:

$$\partial_t u + \partial_x^3 u \cdot \partial_t w = \partial_x u^2 \tag{1.2}$$

as our primary model example. The (unmodulated) KdV equation:

$$\partial_t u + \partial_x^3 u = \partial_x u^2, \tag{1.3}$$

derived as a model for uni-directional shallow water waves, is of fundamental importance in mathematics and has been studied extensively from both the applied and theoretical viewpoints. The modulated KdV (1.2) naturally appears as a model for weakly nonlinear long waves in an inhomogeneous waveguide; see [37, 89], where the modulation w is taken to be periodic but not differentiable. There is, however, no analytical study on the modulated KdV (1.2) and we hope to make an important first step in this direction. We also study the related modulated dispersive equations, appearing in the study of fluid motion, such as

- modulated mKdV equation (1.13),
- modulated Benjamin-Ono equation (1.15),
- modulated intermediate long wave equation (1.16),
- stochastic modulated KdV equation with an additive forcing (1.29).

Our main motivation to study the modulated dispersive equation of form (1.1) is to underline the regularization effect of a non-homogeneous time modulation, in the spirit of the work of Flandoli, Priola, and the second author on the stochastic transport equation; see [59]. The so-called *regularization by noise* phenomenon has been observed for different models:

- many results in the SDE case since the 70's [153, 151, 103, 45, 42, 20, 43],

[1] By convention, we endow $\mathbb{T}$ with the normalized Lebesgue measure $dx_{\mathbb{T}} = (2\pi)^{-1}dx$ such that we do not need to carry factors involving 2π.

- stochastic parabolic equations [84, 79, 55, 18], linear transport equations with rough drifts [59], scalar conservation laws [33], Hamilton-Jacobi equations [108],
- regularization by non-random noises: [66, 60, 144].

See [144] for further references. We refer interested readers to survey works [58, 71].

We point out, however, that rather few results on regularization by noise are known for dispersive equations, mainly limited to random initial data / additive noises of super-critical regularity; see Remark 1.9. In this paper, we carry out pathwise analysis to establish several forms of regularization by noise for canonical dispersive equations such as the KdV and BO equations. See the subsequent subsections in this introduction for concrete examples.

The presence of a modulation w in (1.1) prevents a straightforward use of the Fourier restriction norm method due to Bourgain [6], thus forcing us to develop an alternative approach based on nonlinear Young integration as in [34]. See Section 3, where we go over a general theory of nonlinear Young integrals and well-posedness of a Young differential equation. We point out that the nonlinear Young integral approach we develop here has some similarity with the refinement of the Fourier restriction norm method due to Koch and Tataru [101], where an 'endpoint Young integral' via the U^2-V^2 duality plays a crucial role. Lastly, we mention recent works [149, 140, 141, 142, 143], on pathwise well-posedness of various modulated dispersive equations. See Remark 1.15 for a further discussion.

1.2. **Irregularity of the modulation.** At this stage, the modulated dispersive equation (1.1) is merely formal, since the time derivative of the modulation w does not exist in general. One of our primary interests is to consider the case where w is a sample path of a stochastic process such as a Brownian motion and, more generally, a fractional Brownian motion.

To bypass this problem, we will interpret the equation (1.1) in terms of the Duhamel formulation (= mild formulation):

$$u(t) = U^w(t)u_0 + U^w(t)\int_0^t U^w(t')^{-1}\mathcal{N}(u(t'))dt', \tag{1.4}$$

where $U^w(t) = e^{-w(t)L}$ denotes the modulated linear propagator and we impose $w(0) = 0$ such that $U^w(0) = \mathrm{Id}$.[2] Note that the Duhamel formulation (1.4) only involves the modulation w but not its derivative, allowing us to study the problem even when w is only continuous. The aim of this paper is to study the Duhamel formulation (1.4) under the assumption on the *irregularity* of the modulation w introduced in [20, 34]; see Definition 1.1. In particular, for the modulated KdV (1.2), we will show that if w is sufficiently irregular, then (1.4) is well-posed even in Sobolev spaces of very low regularities.

The next definition concerns a particular notion of irregularity of the modulation w, introduced in [20, 34], which will play a fundamental role in our analysis.

Definition 1.1. Let $\rho > 0$ and $0 < \gamma < 1$. Given $T > 0$, we say that a function $w \in C([0,T];\mathbb{R})$ is (ρ,γ)-irregular on the time interval $[0,T]$ if we have

$$\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} := \sup_{a\in\mathbb{R}}\sup_{0\le r<t\le T}\langle a\rangle^{\rho}\frac{|\Phi^w_{t,r}(a)|}{|t-r|^{\gamma}} < \infty, \tag{1.5}$$

[2]The normalization $w(0) = 0$ is not an additional restriction since only the time derivative $\partial_t w$ appears in the modulated equation (1.1).

where

$$\Phi^w_{t,r}(a) = \int_r^t e^{iaw(t')}dt'. \tag{1.6}$$

We say that w is (ρ,γ)-irregular on $\mathbb{R}_+$ if it is (ρ,γ)-irregular on $[0,T]$ for each finite $T>0$.

For further study on the notion of irregularity, we refer readers to recent works [67, 144] which appeared after the first version of the current paper.

In order to study (1.4) with an irregular modulation w, we need to exploit the temporal regularity of a solution u. More precisely, we will exploit the temporal regularity of the *modulated interaction representation* $\mathbf{u}$ of the unknown u defined by

$$\mathbf{u}(t) = U^w(t)^{-1}u(t). \tag{1.7}$$

When $w(t)=t$ (i.e. for usual (unmodulated) dispersive PDEs), the (standard) interaction representation $e^{tL}u(t)$ has played a fundamental role in the Fourier restriction norm method [6, 101] (see also [72], where an analogy to the interaction representation in quantum mechanics was made) and the normal form method in both the deterministic and probabilistic settings [2, 104, 83, 132, 131, 135]. In general, there are two advantages in working with (modulated) interaction representations: (i) it allows us to exploit temporal regularity of the interaction representation, while, at the level of the original unknown, positive temporal regularity normally comes with a loss of spatial derivative (precisely due to the presence of the modulated linear propagator $U^w(t)$ in (1.4)) and (ii) it allows us to show multilinear dispersive smoothing in a more direct manner.

In terms of the modulated interaction representation $\mathbf{u}$, the Duhamel formulation (1.4) is written as

$$\mathbf{u}(t) = u_0 + \int_0^t U^w(t')^{-1}\mathcal{N}(U^w(t')\mathbf{u}(t'))dt'. \tag{1.8}$$

In this paper, we study the fixed point problem (1.8) by assuming a certain regularity condition on the modulated interaction representation $\mathbf{u}$. A particular subspace of $C([0,T];H^s(\mathcal{M}))$ will play a special role in our analysis, which we introduce in the next definition.

Definition 1.2. Let $\mathcal{M}=\mathbb{R}$ or $\mathbb{T}$ and $s\in\mathbb{R}$. Suppose that $w\in C(I;\mathbb{R})$ for an interval $I\subset\mathbb{R}_+$ is (ρ,γ)-irregular for some $\rho>0$ and $\frac{1}{2}<\gamma<1$. Then, we denote by

$$\mathscr{D}^s_w(I\times\mathcal{M}) \subset C(I;H^s(\mathcal{M}))$$

the space of *paths controlled by* w, which consists of paths $u\in C(I;H^s(\mathcal{M}))$ such that their modulated interaction representations $\mathbf{u}(t)=U^w(t)^{-1}u(t)$ belong to $\mathcal{C}^\gamma(I;H^s(\mathcal{M}))$; see (2.3) for the definition of the $\mathcal{C}^\gamma$-norm.

We also define the class $\dot{\mathscr{D}}^s_w(I\times\mathcal{M})$ in a similar manner. Namely, we say that $u\in C(I;\dot{H}^s(\mathcal{M}))$ belongs to $\dot{\mathscr{D}}^s_w(I\times\mathcal{M})$ if its modulated interaction representation $\mathbf{u}$ belongs to $\mathcal{C}^\gamma(I;\dot{H}^s(\mathcal{M}))$.

We conclude this subsection by presenting an example of irregular modulations. In [20], the authors proved that a fractional Brownian motion is a (ρ,γ)-irregular function.

Theorem A. *Let $\{W_t\}_{t\in\mathbb{R}_+}$ be a fractional Brownian motion of Hurst index $H\in(0,1)$. Then, for any $\rho<\frac{1}{2H}$, there exists $\frac{1}{2}<\gamma<1$ such that, with probability one, the sample paths of W are (ρ,γ)-irregular on $\mathbb{R}_+$.*

Theorem A shows that there exist continuous paths which are (ρ, γ)-irregular for arbitrarily large ρ. In a later work [67], it was shown that (ρ, γ)-irregularity is a generic property of Hölder functions of sufficiently low regularity; see [67, p. 2418 and Theorem 3.1].

Theorem B. *Let $d \geq 1$. Given any $\delta \in (0, 1)$, almost every δ-Hölder continuous function $w \in C^\delta([0, 1]; \mathbb{R}^d)$ is (ρ, γ)-irregular for any $\rho < \frac{1}{2\delta}$ with some $\gamma = \gamma(\rho) \in (\frac{1}{2}, 1)$.*

In this statement, "almost every" is to be understood according to the notion of *prevalence*; see [67] for details and the references therein.

As observed in [20, 34], the irregularity of the path w allows us to obtain a regularizing effect in the ODE context and to prove pathwise well-posedness for the modulated NLS. In the present paper, we show that irregular modulations induce strong regularization effects in the case of KdV and related equations. In order to treat irregular modulations in the sense of Definition 1.1, we will use a nonlinear version of Young integration theory introduced in [34] to give a proper meaning to the integral term in (1.8); see Section 3, where we provide an extensive review on the nonlinear Young integration theory for non-experts. These ideas, essentially coming from the controlled path theory [74], were already used in the context of the (unmodulated) KdV (1.3) on the circle in the work [75] by the second author, where the equation has been studied in Sobolev spaces (and more general Fourier–Lebesgue spaces) of negative regularity, without relying on the Fourier restriction norm method. We also point out that the refinement of the Fourier restriction norm method due to Koch and Tataru [101] is also based on interpreting the Duhamel integral (at the level of the interaction representation) as an 'endpoint Young integral' via the U^2-V^2 duality. We refer interested readers to the lecture notes [100]; see also [85]. Indeed, this point of view was used in recent works [141, 142] to study modulated dispersive PDEs; see Remark 1.15. Lastly, we mention recent works [30, 31, 29] on pathwise well-posedness of stochastic dispersive PDEs with multiplicative noises, where the Young / rough integration theory (via the sewing lemma) are combined with the Fourier restriction norm method, where the random tensor estimate, originally introduced in the context of random variables by Deng, Nahmod, and Yue [53] and later adapted to multiple stochastic integrals with respect to (fractional) Brownian motions [136, 30], plays a crucial role in establishing a pathwise bound on the effect of multiplicative stochastic forcing; see also [24, 146].

1.3. **Well-posedness results.** In this subsection, we state our main results on well-posedness of the modulated dispersive equations mentioned above. For simplicity of the presentation, we impose the mean-zero assumption[3] on solutions to these modulated dispersive equations on the circle in the remaining part of this paper.

Our primary model example is the modulated KdV equation (1.2) on both the circle and the real line. Before stating our main result on the modulated KdV (1.2), let us go over the known well-posedness (and ill-posedness) results on the (unmodulated) KdV (1.3). KdV has attracted wide attention from both the applied and theoretical communities and its well-posedness has been studied extensively on both the circle and the real line. In [6], Bourgain

[3]For the modulated KdV, BO, and ILW equations, if initial data has non-zero mean α_0, then the following Galilean transformation: $u(t, x) \mapsto u(t, x - 2\alpha_0 t) - \alpha_0$, along with the conservation of the spatial mean (which can be verified by arguing as in Section 6), transforms these equations with a non-zero mean into the mean-zero versions. Note that, while the mean-zero assumption is not necessary for the modulated mKdV (1.13), we need to apply a gauge transform and consider its renormalized version (1.14).

introduced the so-called Fourier restriction norm method and proved local and global well-posedness of (1.3) in $L^2(\mathcal{M})$, $\mathcal{M} = \mathbb{T}$ and $\mathbb{R}$, which was improved by the subsequent works [94, 38, 98, 82] (see also [13]) and now KdV (1.3) is known to be globally well-posed in $H^s(\mathbb{T})$ for $s \geq -\frac{1}{2}$ and in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{4}$. In particular, in these works, local well-posedness was shown by a contraction argument, exhibiting semilinear nature of the equation; in particular, the solution map: $u_0 \in H^s(\mathcal{M}) \mapsto u \in C([-T, T]; H^s(\mathcal{M}))$ is smooth (in fact, real analytic).[4] In the following, we refer local well-posedness by a contraction argument as *semilinear local well-posedness*. On the other hand, it is known that the solution map fails to be locally uniformly continuous below the aforementioned regularity thresholds [35]; see also [9]. Namely, the KdV equation (1.3) exhibits *quasilinear* nature in the low regularity setting (i.e. below $s = -\frac{1}{2}$ on $\mathbb{T}$ and $s = -\frac{3}{4}$ on $\mathbb{R}$).

There are also well-posedness results of KdV based on its complete integrability; see [93, 97] for the sharp global well-posedness of KdV in $H^{-1}(\mathcal{M})$, $\mathcal{M} = \mathbb{T}$ and $\mathbb{R}$ (see [113, 114] for ill-posedness below $s = -1$). We note that the time modulation effect breaks the integrability of the KdV equation, and thus we aim to develop a Fourier analytic approach adapted to the current modulated setting.

We now state our main well-posedness results for the modulated KdV (1.2) on the circle.

Theorem 1.3. *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1.*

(i) *Suppose that $\rho \geq \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy one of the following conditions*:

$$\begin{aligned} &\text{(i.a)} \quad \tfrac{1}{2} \leq \rho \leq \tfrac{3}{4} \quad \text{and} \quad s > \tfrac{3}{2} - 3\rho, \\ &\text{(i.b)} \quad \rho > \tfrac{3}{4} \quad \text{and} \quad s \geq -\rho. \end{aligned} \tag{1.9}$$

Then, the modulated KdV equation (1.2) *on $\mathbb{T}$ is semilinearly locally well-posed in $H^s(\mathbb{T})$.*

(ii) *In addition, suppose that w is (ρ, γ)-irregular on $\mathbb{R}_+$ and that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy one of the following conditions*:

$$\begin{aligned} &\text{(ii.a)} \quad \tfrac{1}{2} < \rho \leq \tfrac{3}{4\gamma} \quad \text{and} \quad s > \tfrac{3-6\rho}{6-4\gamma}, \\ &\text{(ii.b)} \quad \tfrac{3}{4\gamma} < \rho < \tfrac{3}{2} \quad \text{and} \quad s \geq -\rho, \\ &\text{(ii.c)} \quad \rho \geq \tfrac{3}{2} \quad \text{and} \quad s > -\tfrac{3}{2}. \end{aligned} \tag{1.10}$$

Then, the modulated KdV equation (1.2) *on $\mathbb{T}$ is globally well-posed in $H^s(\mathbb{T})$.*

Remark 1.4. (i) Uniqueness of a solution u constructed in Theorem 1.3 (i) holds in the class $\mathscr{D}^s_w([0, \tau] \times \mathbb{T})$ defined in Definition 1.2, where $0 < \tau \leq T$ denotes the local existence time. A similar comment applies to the other modulated equations considered in this paper.

(ii) The solution map:

$$(u_0, w) \in H^s(\mathbb{T}) \times \mathcal{W}^{\rho,\gamma}_T \longrightarrow \mathbf{u} \in \mathcal{C}^\gamma([0, \tau]; H^s(\mathbb{T}))$$

is locally Lipschitz continuous; see Remarks 3.9 (iii) and 4.2. A similar comment applies to the other modulated equations considered in this paper.

[4] On the circle, the smoothness of the solution map holds only after restricting our attention to the subspace $H^s_\alpha(\mathbb{T})$ of $H^s(\mathbb{T})$, consisting of functions with spatial mean $\alpha \in \mathbb{R}$. Recall that, in the periodic setting, we restrict our attention to $H^s_0(\mathbb{T})$ in this paper.

(iii) All of our local well-posedness results in this paper come with *persistence of regularity*. Suppose that a modulated equation is locally well-posed in $H^s(\mathcal{M})$ for some $s \in \mathbb{R}$ with local existence time[5] $\tau = \tau(\|u_0\|_{H^s}) \in (0, T]$. If, in addition, we have $u_0 \in H^{s_0}(\mathcal{M})$ for some $s_0 > s$, then the corresponding solution exists with the same local existence time τ, depending only on the H^s-norm of the initial data u_0.

As a corollary to Theorem 1.3 (i) with Theorem A, we have the following well-posedness result for the modulated KdV (1.2) on $\mathbb{T}$, where a modulation is given by a fractional Brownian motion. For simplicity of the presentation, we only state its local well-posedness.

Corollary 1.5. *Let $\{W_t\}_{t\in\mathbb{R}_+}$ be a fractional Brownian motion of Hurst index $H \in (0,1)$. Suppose that $s \in \mathbb{R}$ satisfies*

$$\begin{aligned} &\text{(a)}\ \ \tfrac{2}{3} \le H < 1 \quad \text{and} \quad s > \tfrac{3}{2} - \tfrac{3}{2H}, \\ &\text{(b)}\ \ 0 < H < \tfrac{2}{3} \quad \text{and} \quad s > -\tfrac{1}{2H}. \end{aligned}$$

Then, with probability one, the modulated KdV equation (1.2) *on $\mathbb{T}$ with the modulation given by a sample path of the fractional Brownian motion $\{W_t\}_{t\in\mathbb{R}_+}$ is semilinearly locally well-posed in $H^s(\mathbb{T})$.*

Remark 1.6. Similar results hold for each of the dispersive equations considered in this paper, with the appropriate choice of exponents, when the modulation function is given by a fractional Brownian motion, or, thanks to Theorem B, when the modulation function is given by a generic (in the sense of prevalence) Hölder function.

Next, we state our main well-posedness results for the modulated KdV (1.2) on the real line.

Theorem 1.7. *Given $\rho > \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1.*

(i) *Suppose that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy one of the following conditions*:

$$\begin{aligned} &\text{(i.a)}\ \ \tfrac{1}{2} < \rho \le \tfrac{3}{4} \quad \text{and} \quad s > \tfrac{3}{2} - 3\rho, \\ &\text{(i.b)}\ \ \tfrac{3}{4} < \rho < \tfrac{3}{2} \quad \text{and} \quad s \ge -\rho, \\ &\text{(i.c)}\ \ \rho \ge \tfrac{3}{2} \quad \text{and} \quad s > -\tfrac{3}{2}. \end{aligned} \tag{1.11}$$

Then, the modulated KdV equation (1.2) *on $\mathbb{R}$ is semilinearly locally well-posed in $H^s(\mathbb{R})$.*

(ii) *In addition, suppose that w is (ρ,γ)-irregular on $\mathbb{R}_+$ and that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy* (1.10)*. Then, the modulated KdV equation* (1.2) *on $\mathbb{R}$ is globally well-posed in $H^s(\mathbb{R})$.*

Theorems 1.3 and 1.7 show that an irregular modulation provides a *regularization effect* on the KdV equation, allowing us to prove well-posedness in the regime, where the unmodulated KdV equation (1.3) is ill-posed (namely, $s < -1$). We also mention that our construction of a solution is based on a contraction argument, yielding local Lipschitz continuous dependence. Such a semilinear property fails for the unmodulated KdV equation (1.3) in $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$ and in $H^s(\mathbb{R})$ for $s < -\frac{3}{4}$, thus exhibiting a *semilinearization phenomenon*. Last but not least, in the periodic case, Theorem 1.3 shows a very strong regularization effect; given *any* $s \in \mathbb{R}$,

[5]Clearly, τ also depends on the modulation w but we suppress its dependence on w in this discussion.

the modulated KdV (1.2) on the circle with a sufficiently irregular modulation is locally well-posed in $H^s(\mathbb{T})$; see also Theorem 1.12 for an analogous result on the modulated BO and ILW equations. Theorems 1.3 and 1.7 present the first results on regularization by noise in the study of nonlinear dispersive equations with rough (deterministic) initial data. See also Remark 1.9. In the next subsection, we also present another kind of regularization-by-noise phenomenon in the context of *nonlinear smoothing*; see Theorem 1.16.

Remark 1.8. Contrary to the periodic case, the local well-posedness result of the modulated KdV (1.2) on the real line (Theorem 1.7 (i)) holds only in the scaling-subcritical regime $s > s_{\text{crit}} = -\frac{3}{2}$, even if we take ρ to be large. This is due to the fact that frequencies can be arbitrarily small on the real line case, which prevents us from exploiting the irregularity of the modulation function w in a certain frequency configuration. See Remark 4.5, where we show the essential sharpness of the condition $s > -\frac{3}{2}$ for $\rho \geq \frac{3}{2}$ in (1.11) (modulo the endpoint $s = -\frac{3}{2}$) by establishing "mild" ill-posedness of the modulated KdV (1.2) on the real line for $s < -\frac{3}{2}$.

Let us point out that our results are limited to sufficiently irregular modulations and they do not provide any sharp information about the dependence of solutions on modulations, especially about the transition between smooth and irregular modulations. Indeed, a bit surprisingly, in the current modulated context, the application of controlled path techniques (including nonlinear Young integration theory via the sewing lemma) is easier if the modulation is very irregular. In fact, by considering sufficiently irregular modulations, we have avoided the need of a second-order controlled expansion which was necessary in the work [75], where the second author developed a controlled path approach to study the (unmodulated) KdV equation (1.3). It remains an open problem to fill the gap between regular and irregular modulations.

Our main strategy for proving local well-posedness is to make sense of the integral term in (1.8) as a *nonlinear Young integral* and prove local well-posedness of the equation (1.8) via a contraction argument, just as in the previous work [34] by the first two authors. While our approach is equivalent to that presented in [34], we first construct the integral in (1.8) as a nonlinear Young integral $\mathcal{I}^X(\mathbf{u})$ with a given driver X in an abstract setting via the sewing lemma (Lemma 3.3) from the theory of controlled rough paths [74] developed by the second author; see Lemma 3.4. We made a particular effort to present this part in a pedestrian manner, accessible to non-experts. We then study well-posedness and related properties of the following Young differential equation (YDE):

$$\mathbf{u} = u_0 + \mathcal{I}^X(\mathbf{u}) \tag{1.12}$$

in our functional framework adapted to Sobolev spaces, under appropriate assumptions on the driver X; see Proposition 3.8. In Sections 4 and 5, we verify that such assumptions are indeed satisfied for each of the modulated dispersive equations we consider in this paper.

The controlled rough path theory, introduced as an alternative formulation of the rough path theory due to Lyons [109, 110], has been applied and extended further in a variety of contexts; for example,

- well-posedness of stochastic parabolic equations [81, 55],
- Hairer's work on well-posedness for the KPZ equation [86],

- Hairer's theory of regularity structures, dealing with very singular parabolic PDEs such as the parabolic Φ^4_3-model [87];
- paracontrolled calculus [76, 19, 117] which has also been applied to study nonlinear wave equations (NLW) in recent years [77, 125, 10, 126, 11].

Our approach to study modulated dispersive equations, exhibiting various regularization-by-noise effects is an interesting and useful addition to this list.

All our techniques are deterministic and they provide novel results even in the stochastic context; for example, we can take the modulation w to be the sample path of a fractional Brownian motion as we have seen in Corollary 1.5. Even for a Brownian motion, our results on KdV, mKdV, BO, and ILW are novel to our knowledge. See Subsections 1.5 and 1.6 for applications with random initial data and stochastic forcing. We also note that YDEs allow for a straightforward Euler approximation scheme to discretize in time; see, for example, [34, Subsection 2.3].

In Section 7, we prove the global well-posedness results in Theorems 1.3 (ii) and 1.7 (ii) by the so-called I-method (= method of almost conservation laws) introduced in [38]. In particular, we prove these results by combining a key crucial commutator estimate (Proposition 7.6), a common tool in an application of the I-method, with the sewing lemma (Lemma 3.3). Such a combination of tools and ideas from dispersive PDEs and the controlled rough path theory is novel in the literature. We also mention a recent work [78] for a highly non-trivial application of the I-method in the context of the stochastic NLW. The I-method relies on the scaling property of a given equation (see (7.4)) and thus is limited to the scaling-subcritical regime $s > s_{\text{crit}} = -\frac{3}{2}$ in the KdV case. This is the reason why the restriction $s > -\frac{3}{2}$ appears in Theorems 1.3 (ii) and 1.7 (ii).[6] On the other hand, it is intriguing to see the same scaling-subcritical restriction appears in local well-posedness on the real line (Theorem 1.7 (i)). Our implementation of the I-method in the current modulated context is of the first order. It would be of interest to see if we can improve the range of regularity by adding a "correction" term and proceed with the second order I-method; see, for example, [38] for a higher order implementation of the I-method in the unmodulated setting.

Remark 1.9. In [7, 8], Bourgain initiated the study of nonlinear dispersive equations with random initial data of low regularity. Over the last fifteen years, this study has been particularly active [8, 16, 41, 17, 4, 137, 127]; see [5] for a (slightly outdated) survey on this topic. See also more recent works [77, 54, 125, 10, 126, 11], including those on stochastic dispersive PDEs with additive noises (which can be easily adapted to random initial data). In these works, non-deterministic viewpoints allowed us to go beyond the limit of deterministic analysis, and thus, broadly speaking, we can regard them as regularization-by-noise phenomena (by considering random initial data).

We also note that a noise can sometimes worsen the behavior of an equation; see, for example, [46, 47] for the construction of finite-time blowup for the stochastic NLS.

[6] In this paper, we apply the classical KdV scaling (7.4) to u, while we apply (7.7) to the modulation w. In fact, by adjusting the scaling for w, we can have the modulated KdV (on $\mathbb{R}$) invariant under a non-KdV scaling applied to u. In a recent work [73], the second and fifth authors with D. Greco and S. Liu made use of this observation and proved that, given any $s \in \mathbb{R}$, the modulated KdV on the circle is globally well-posed in $H^s(\mathbb{T})$, provided that the modulation is sufficiently irregular.

Let us now consider other modulated dispersive equations. For simplicity of presentation, we restrict our attention to the periodic case. We first consider the following modulated modified KdV equation (mKdV) on $\mathbb{T}$:

$$\partial_t u + \partial_x^3 u \cdot \partial_t w = \partial_x u^3. \tag{1.13}$$

As in the unmodulated setting [6], the equation (1.13) on the circle is not suitable for well-posedness studies. Using the L^2-conservation,[7] the gauge transform: $x \mapsto x - 3\|u_0\|_{L^2}^2 t$ converts (1.13) into the following renormalized modulated mKdV:

$$\partial_t u + \partial_x^3 u \cdot \partial_t w = 3\big(u^2 - \|u\|_{L^2}^2\big)\partial_x u, \tag{1.14}$$

where certain resonant interactions have been removed; see (4.16) and (4.17).

Theorem 1.10. *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1. Then, for $s \geq \frac{1}{2}$, the modulated (renormalized) mKdV equation* (1.14) *on $\mathbb{T}$ is semilinearly locally well-posed in $H^s(\mathbb{T})$.*

At the level of the original modulated mKdV (1.13), local well-posedness holds in a class $H^s(\mathbb{T}) \cap \{\|f\|_{L^2} = \mu\}$ for any prescribed value of $\mu \in \mathbb{R}_+$.

Remark 1.11. In the unmodulated setting, mKdV on the circle is known to be semilinearly locally well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$ [6, 104], while the solution map fails to be locally uniformly continuous below $s = \frac{1}{2}$ [35, 148]; see also [9]. See [92, 61] for well-posedness results via the complete integrability of the equation.

Unlike the KdV case, Theorem 1.10 does not provide any improvement as compared to the unmodulated setting. This is due to the fact that the regularity restriction $s \geq \frac{1}{2}$ in Theorem 1.10 comes from the resonant part of the nonlinearity, where the modulation plays no role; see the proof of Proposition 4.3. By adapting the approach in [148, 120] (see also [133, 12]) to weaken the resonant interaction, one may be able to improve Theorem 1.10 for some $s < \frac{1}{2}$, but we do not pursue this issue here. See also Remarks 1.15 and 1.18. We also mention the recent works [21, 22] for a further discussion on the renormalization of the complex-valued mKdV.

Next, we consider the following modulated Benjamin-Ono equation (BO) on the circle:

$$\partial_t u - \mathcal{H}\partial_x^2 u \cdot \partial_t w = \partial_x u^2, \tag{1.15}$$

where $\mathcal{H}$ denotes the Hilbert transform with the Fourier multiplier $-\mathbf{1}_{n\neq 0} \cdot i\,\mathrm{sgn}(n)$. Given $0 < \delta < \infty$, we also consider the modulated intermediate long wave equation (ILW) on the circle:

$$\partial_t u - \mathcal{G}_\delta \partial_x^2 u \cdot \partial_t w = \partial_x u^2, \tag{1.16}$$

where the operator $\mathcal{G}_\delta$ is given as the following Fourier multiplier operator:

$$\widehat{\mathcal{G}_\delta f}(n) = \widehat{\mathcal{G}_\delta}(n)\widehat{f}(n) := -\mathbf{1}_{n\neq 0} \cdot i\bigg(\coth(\delta n) - \frac{1}{\delta n}\bigg)\widehat{f}(n), \quad n \in \mathbb{Z}. \tag{1.17}$$

In the unmodulated setting (i.e. $w(t) = t$), the ILW equation models the propagation of an internal wave at the interface of a stratified fluid of finite depth $\delta > 0$, and the unknown u denotes the amplitude of the internal wave at the interface. Moreover, it serves as an "intermediate" equation of finite depth $0 < \delta < \infty$, providing a natural connection between the BO

[7] In the current modulated setting, we need to justify the L^2-conservation; see Section 6.

equation, modeling fluid of infinite depth ($\delta = \infty$), and the KdV equation (1.3), modeling shallow water ($\delta = 0$). See [145, 99] for an overview of the subject and the references therein.

In considering the shallow-water limit ($\delta \to 0$), we need to magnify the amplitude u of the internal wave at the interface to observe any non-trivial limiting behavior. For this purpose, we introduce the following scaling transform [1]:

$$v(t,x) = \tfrac{3}{\delta} u\big(\tfrac{3}{\delta} t, x\big), \tag{1.18}$$

which converts (1.16) into the following modulated scaled ILW equation, where the rescaled modulation $\widetilde{w}_\delta(t) = \frac{\delta}{3} w\big(\frac{3}{\delta} t\big)$ is denoted again by w:

$$\partial_t v - \widetilde{\mathcal{G}_\delta} \partial_x^2 v \cdot \partial_t w = \partial_x v^2, \tag{1.19}$$

where

$$\widetilde{\mathcal{G}_\delta} = \frac{3}{\delta} \mathcal{G}_\delta. \tag{1.20}$$

In studying shallow-water convergence, we fix the modulation w in the modulated scaled ILW (1.19), independent of δ.

We first state well-posedness results of these modulated equations.

Theorem 1.12. *Given $\rho \geq 1$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1.*

(i) *Suppose that $\rho \geq 1$ and $s \in \mathbb{R}$ satisfy one of the following conditions*:

$$\begin{aligned} &\text{(i.a)} \quad \rho = 1 \quad \text{and} \quad s > -\tfrac{1}{2}, \\ &\text{(i.b)} \quad \rho > 1 \quad \text{and} \quad s \geq -\tfrac{1}{2}\rho. \end{aligned} \tag{1.21}$$

Then, the modulated BO equation (1.15) *on $\mathbb{T}$ is semilinearly locally well-posed in $H^s(\mathbb{T})$. Given $0 < \delta < \infty$, the modulated ILW* (1.16) *and the modulated scaled ILW* (1.19) *on $\mathbb{T}$ are also semilinearly locally well-posed in $H^s(\mathbb{T})$.*

(ii) *In addition, suppose that w is (ρ, γ)-irregular on $\mathbb{R}_+$. Then, the modulated BO equation* (1.15) *on $\mathbb{T}$ is globally well-posed in $H^s(\mathbb{T})$ for any $s \geq 0$. Similarly, given $0 < \delta < \infty$, the modulated ILW* (1.16) *and the modulated scaled ILW* (1.19) *on $\mathbb{T}$ are globally well-posed in $H^s(\mathbb{T})$ for any $s \geq 0$.*

In the unmodulated setting, Fourier analytic approaches allow us to show well-posedness of BO and the (scaled) ILW in $L^2(\mathbb{T})$ [111, 112, 115, 27]; see also [52]. Furthermore, by the complete integrability of the equation, the (unmodulated) BO equation is known to be globally well-posed in the entire scaling-subcritical regime, namely in $H^s(\mathbb{T})$ for any $s > -\frac{1}{2}$, while it is ill-posed for $s \leq -\frac{1}{2}$ [69, 95].[8] See also [23] for the study of ILW in negative Sobolev spaces. In the current modulated setting, Theorem 1.12 shows that, given any $s \in \mathbb{R}$, the modulated BO, ILW, and scaled ILW on the circle with sufficiently irregular modulations are locally well-posed in $H^s(\mathbb{T})$, exhibiting a strong regularization-by-noise phenomenon.

Remark 1.13. In the unmodulated setting, BO and ILW on the circle are known to behave quasilinearly on $H^s(\mathbb{T})$ for $s < 0$. More precisely, semilinear local well-posedness of these equations on $H^s(\mathbb{T})$ fails for $s < 0$ (even if we restrict our attention to the subclass $H^s_0(\mathbb{T})$, consisting of mean-zero functions); see [112, Theorem 1.2]. See also [112, second paragraph

[8] This global well-posedness for $s > -\frac{1}{2}$ was recently extended to ILW via a perturbative argument; see [68].

on p. 640]. Thus, Theorem 1.12 establishes a semilinearization phenomenon on the modulated BO and ILW equations in negative Sobolev spaces.

Given $\alpha \in \mathbb{R}$, let $H^s_\alpha(\mathbb{T})$ denote the subspace of $H^s(\mathbb{T})$ consisting of functions with spatial mean α. It is worthwhile to note that while Molinet [112, Theorem 1.2] showed that the solution map for the unmodulated BO on $\mathbb{T}$ is analytic in $H^s_\alpha(\mathbb{T})$ for any $s \geq 0$ and $\alpha \in \mathbb{R}$, there is *no* known well-posedness result for the unmodulated BO in $H^s_\alpha(\mathbb{T})$, $s \geq 0$, via a contraction argument at this point, which is in sharp contrast to the semilinear local well-posedness of the modulated BO on $\mathbb{T}$, stated in Theorem 1.12. We also mention a recent work [70], where certain quasilinear nature of the unmodulated BO on $\mathbb{T}$ was manifested.

Lastly, we point out that semilinear local well-posedness of the unmodulated BO and ILW on the real line fails in $H^s(\mathbb{R})$ for *any* $s \in \mathbb{R}$; see [116, 102, 25]. Namely, on the real line, these (unmodulated) equations manifest their quasilinear nature in a stronger manner. Thus, it would be of interest to investigate if an analogue of Theorem 1.12 holds on the real line (which would then show that, given any $s \in \mathbb{R}$, irregular modulations semilinearize these equations). We, however, do not pursue this issue further in this paper.

Lastly, we consider deep-water convergence ($\delta \to \infty$) and shallow-water convergence ($\delta \to 0$) of the modulated (scaled) ILW. In the unmodulated setting, the convergence issue for the (scaled) ILW has been studied in [1, 106, 107, 27, 62, 26, 68, 88, 28, 25] from both deterministic and probabilistic viewpoints. The following theorem establishes deep-water and shallow-water convergence results for the modulated (scaled) ILW.

Theorem 1.14. *Given $\rho > 1$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1.*

(i) (deep-water convergence). *Let $s > -\frac{1}{2}\rho$. Given $u_0 \in H^s(\mathbb{T})$, let u^δ, $0 < \delta < \infty$, and u denote the local-in-time solutions to the modulated ILW equation* (1.16) *and the modulated BO equation* (1.15) *with $u^\delta|_{t=0} = u|_{t=0} = u_0$, respectively. Then, as $\delta \to \infty$, u^δ converges to u in $C([0, \tau]; H^s(\mathbb{T}))$, where $\tau \in (0, T]$ denotes the local existence time of the modulated BO equation* (1.15). *Moreover, if w is (ρ, γ)-irregular on $\mathbb{R}_+$ and $s \geq 0$, then the deep-water convergence holds globally in time.*

(ii) (shallow-water convergence). *Let $s > -\frac{1}{2}\rho$. Given $u_0 \in H^s(\mathbb{T})$, let v^δ, $0 < \delta < \infty$, and u denote the local-in-time solutions to the modulated scaled ILW equation* (1.19) *and the modulated KdV equation* (1.2) *with $v^\delta|_{t=0} = u|_{t=0} = u_0$, respectively. Then, as $\delta \to 0$, v^δ converges to u in $C([0, \tau]; H^s(\mathbb{T}))$, where $\tau \in (0, T]$ denotes the local existence time of the modulated KdV equation* (1.2). *Moreover, if w is (ρ, γ)-irregular on $\mathbb{R}_+$ and $s \geq 0$, then the shallow-water convergence holds globally in time.*

It follows from the discussion in Subsections 5.3 and 5.4 together with Lemma 3.1 (iv), Remark 3.2, and Proposition 3.8 (iv) that, at the level of the modulated interaction representations, convergence holds in $\mathcal{C}^\gamma([0, \tau]; H^s(\mathcal{M}))$. For example, $\mathbf{u}^\delta(t) = e^{-w(t)\mathcal{G}_\delta \partial_x^2} u^\delta(t)$ converges to $\mathbf{u}(t) = e^{-w(t)\mathcal{H}\partial_x^2} u(t)$ in $\mathcal{C}^\gamma([0, \tau]; H^s(\mathcal{M}))$ as $\delta \to \infty$.

Remark 1.15. (i) After the appearance of the first version of this paper, there have been interesting well-posedness results on modulated dispersive equations [56, 64, 147, 149, 140, 141, 142, 143]. In particular, in recent preprints [141, 142], Robert studied pathwise well-posedness of the modulated dispersive equation (1.1) such as the modulated (fractional) NLS

and modulated (fractional) KdV, by working within the framework of the refined Fourier restriction norm method involving the U^2- and V^2-spaces (namely, without the use of the sewing lemma). In [142], he proved local well-posedness of the following modulated fractional KdV on $\mathbb{T}$:

$$\partial_t u + \partial_x |\partial_x|^\beta u \cdot \partial_t w = \partial_x u^2,$$

which essentially agrees with Theorem 1.3 (i) for the modulated KdV (1.2) on $\mathbb{T}$ (corresponding to the $\beta = 2$ case) and Theorem 1.12 (i) for the modulated BO (1.15), the modulated ILW (1.16), and the modulated scaled ILW (1.19) on $\mathbb{T}$ (corresponding to the $\beta = 1$ case).

In [142], Robert employed the so-called short-time Fourier restriction norm method (introduced in [101]; see also [36, 91, 134]), which is often used to weaken resonant interactions, and proved local well-posedness of the (real-valued) renormalized modulated mKdV (1.14) in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$, provided that $\rho = \rho(s) \gg 1$ is sufficiently large, thus improving Theorem 1.10 when ρ is large. We point out that the solution map constructed in [142] is merely continuous, which is consistent with the failure of semilinear local well-posedness in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$ discussed in Remark 1.11.

As mentioned above, our strategy in this paper is to study the YDE (1.12) by constructing $\mathcal{I}^X(\mathbf{u})$ as a nonlinear Young integral via the sewing lemma (without relying on the (refined) Fourier restriction norm method). One of the benefits of our approach is that we can allow the temporal regularity $\alpha > 0$ of a solution $\mathbf{u}$ to (1.12) to be less than $\frac{1}{2}$ as long as $\alpha + \gamma > 1$; see Lemma 3.4.[9] This in particular allows us to study well-posedness of stochastic modulated dispersive PDEs with additive forcing. See Subsection 1.6 and Section 9 for a further discussion.

(ii) In a recent preprint [24], with A. Chapouto, we studied pathwise well-posedness of stochastic modulated dispersive equations with multiplicative noises and established a new regularization-by-noise phenomenon by exploiting the nonlinear interaction between the unknown and the noise. For example, for the stochastic modulated KdV with a multiplicative fractional-in-time noise in the Young regime, we showed that irregularity of the modulation induces smoothing on the stochastic convolution in a pathwise manner, where a gain of spatial regularity becomes (arbitrarily) larger for more irregular modulations.

In a recent preprint [80], we implemented a normal form method to study modulated dispersive PDEs, which provides an alternative way to give a meaning to the integral term in (1.8) (and $\mathcal{I}^X(\mathbf{u})$ in (1.12)) without relying on the sewing lemma (or the Fourier restriction norm method as in [141, 142]). Furthermore, our normal form approach in [80] does not require any positive temporal regularity of the modulated interaction representation $\mathbf{u}$ defined in (1.7), allowing us to prove unconditional uniqueness of solutions to modulated dispersive PDEs; see [80] for further details.

1.4. **Nonlinear smoothing & Galerkin approximation.** Our approach to modulated dispersive equations on the circle also yields (i) nonlinear smoothing and (ii) convergence of the Galerkin approximation.

Given $N \in \mathbb{N}$, we consider the following truncated modulated KdV on the circle:

$$\partial_t u^N + \partial_x^3 u^N \cdot \partial_t w = \partial_x \mathbf{P}_N\big((\mathbf{P}_N u^N)^2\big), \tag{1.22}$$

[9] In a recent preprint [80], we implemented a normal form approach to the modulated KdV (1.2) and the modulated BO (1.15), where we constructed a solution with temporal regularity $\alpha = 0$.

where $\mathbf{P}_N$ denotes the Dirichlet projector onto the (spatial) frequencies $\{|n| \leq N\}$; see (2.1). Then, a slight modification of the proof of Theorem 1.3 (i) yields the following results for the modulated KdV (1.2) on the circle.

Theorem 1.16. *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1.*

(i) (nonlinear smoothing). *Suppose that $\rho \geq \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy (1.9). In addition, suppose that $s_0 > s$ satisfies one of the following conditions*:

$$\begin{aligned} &\text{(i.a)} \quad 0 \leq s + \rho \leq \tfrac{1}{2} \quad \text{and} \quad s_0 < 2s + 3\rho - \tfrac{3}{2}, \\ &\text{(i.b)} \quad s + \rho > \tfrac{1}{2} \quad \text{and} \quad s_0 \leq s + 2\rho - 1. \end{aligned} \tag{1.23}$$

Given $u_0 \in H^s(\mathbb{T})$, let $u \in C([0, \tau]; H^s(\mathbb{T}))$ be the solution to the modulated KdV equation (1.2) on $\mathbb{T}$ with $u|_{t=0} = u_0$, where $\tau \in (0, T]$ denotes the local existence time. Then, we have

$$u - e^{-w(t)\partial_x^3} u_0 \in C([0, \tau]; H^{s_0}(\mathbb{T})).$$

(ii) (Galerkin approximation). *Suppose that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy one of the following conditions*:

$$\begin{aligned} &\text{(ii.a)} \quad \tfrac{1}{2} < \rho \leq \tfrac{3}{4} \quad \text{and} \quad s > \tfrac{3}{2} - 3\rho, \\ &\text{(ii.b)} \quad \rho > \tfrac{3}{4} \quad \text{and} \quad s > -\rho. \end{aligned} \tag{1.24}$$

Then, given $u_0 \in H^s(\mathbb{T})$, the solution u^N to the truncated modulated KdV equation (1.22) on $\mathbb{T}$ with $u^N|_{t=0} = u_0$ converges to the solution u to the modulated KdV equation (1.2) on $\mathbb{T}$ with $u|_{t=0} = u_0$ in $C([0, \tau]; H^s(\mathbb{T}))$, where $\tau \in (0, T]$ denotes the local existence time of the modulated KdV equation (1.2). Moreover, given any $r > 0$, the rate of convergence of u^N to u is uniform in $u_0 \in B_r$ where B_r denotes the ball in $H^s(\mathbb{T})$ of radius $r > 0$ centered at the origin.

We point out that our argument does not yield nonlinear smoothing or convergence of the Galerkin approximation for the modulated KdV on the real line; see Remark 4.6.

In the case of the (unmodulated) KdV (1.3) on $\mathbb{T}$, nonlinear smoothing was shown for $s > -\frac{1}{2}$ and $s_0 < \min(3s+1, s+1)$ by the normal form method [57]. We prove Theorem 1.16 (i) by a slight modification of the contraction argument needed to prove local well-posedness and thus we do not need to rely on the normal form method. Furthermore, the regularity gain on the nonlinear part $u - e^{-w(t)\partial_x^3} u_0$ can be (arbitrarily) large for irregular modulations, which is in sharp contrast to the unmodulated case. We also point out that the uniformity claim in Theorem 1.16 (ii) fails for the (unmodulated) KdV (1.3) on $\mathbb{T}$ (see [40, Theorem 1.1]), which shows another example of regularization by noise for the modulated KdV (1.2) on the circle.

Next, we state analogous results for the modulated mKdV (1.13) on the circle.

Theorem 1.17. *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1.*

(i) (nonlinear smoothing). *Suppose that $s_0 > s \geq \frac{1}{2}$ satisfies*

$$s_0 \leq \min(s + 2\min(\rho, s) - 1, 3s - 1). \tag{1.25}$$

Given $u_0 \in H^s(\mathbb{T})$, let $u \in C([0,\tau]; H^s(\mathbb{T}))$ be the solution to the modulated mKdV equation (1.13) *on $\mathbb{T}$ with $u|_{t=0} = u_0$, where $\tau \in (0,T]$ denotes the local existence time. Then, we have*

$$u - e^{-w(t)\partial_x^3 + 3t\|u_0\|_{L^2}^2 \partial_x} u_0 \in C([0,\tau]; H^{s_0}(\mathbb{T})).$$

(ii) (Galerkin approximation). *Let $\rho > \frac{1}{2}$ and $s > \frac{1}{2}$. Then, given $u_0 \in H^s(\mathbb{T})$, the solution u^N to the truncated modulated mKdV equation on $\mathbb{T}$:*

$$\begin{cases} \partial_t u^N + \partial_x^3 u^N \cdot \partial_t w = \partial_x \mathbf{P}_N \big((\mathbf{P}_N u^N)^3 \big) \\ u^N|_{t=0} = u_0 \end{cases}$$

converges to the solution u to the modulated mKdV equation (1.13) *on $\mathbb{T}$ with $u|_{t=0} = u_0$ in $C([0,\tau]; H^s(\mathbb{T}))$, where $\tau \in (0,T]$ denotes the local existence time of the modulated mKdV equation* (1.13).

Remark 1.18. Contrary to Theorem 1.16 (i) for the modulated KdV (1.2) (and also Theorem 1.19 (i) for the modulated BO (1.15)), the gain of regularity in nonlinear smoothing for the modulated mKdV (1.13) is bounded by $3s - 1$ even if we take $\rho \gg 1$. This is (once again) due to the resonant part of the nonlinearity, where the modulation plays no role; see the proof of Proposition 4.3. See also Remark 1.11.

Lastly, we state the corresponding results for the modulated BO and the modulated (scaled) ILW on the circle.

Theorem 1.19. *Given $\rho > 1$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1.*

(i) (nonlinear smoothing). *Suppose that $\rho > 1$ and $s \in \mathbb{R}$ satisfy* (1.21). *In addition, suppose that $s_0 > s$ satisfies*

$$\begin{aligned} &\text{(i.a)} \quad 0 \le s < s_0 \le \rho - 1, \\ &\text{(i.b)} \quad s < 0 \quad \text{and} \quad s_0 \le s + \rho - 1. \end{aligned} \tag{1.26}$$

Given $u_0 \in H^s(\mathbb{T})$, let $u \in C([0,\tau]; H^s(\mathbb{T}))$ be the solution to the modulated BO equation (1.15) *on $\mathbb{T}$ with $u|_{t=0} = u_0$, where $\tau \in (0,T]$ denotes the local existence time. Then, we have*

$$u - e^{w(t)\mathcal{H}\partial_x^2} u_0 \in C([0,\tau]; H^{s_0}(\mathbb{T})).$$

(ii) (Galerkin approximation). *Suppose that $\rho > 1$ and $s > -\frac{1}{2}\rho$. Then, given $u_0 \in H^s(\mathbb{T})$, the solution u^N to the truncated modulated BO equation on $\mathbb{T}$:*

$$\begin{cases} \partial_t u^N - \mathcal{H}\partial_x^2 u^N \cdot \partial_t w = \partial_x \mathbf{P}_N \big((\mathbf{P}_N u^N)^2 \big) \\ u^N|_{t=0} = u_0 \end{cases}$$

converges to the solution u to the modulated BO equation (1.15) *on $\mathbb{T}$ with $u|_{t=0} = u_0$ in $C([0,\tau]; H^s(\mathbb{T}))$, where $\tau \in (0,T]$ denotes the local existence time of the modulated BO equation* (1.15). *Moreover, given any $r > 0$, the rate of convergence of u^N to u is uniform in $u_0 \in B_r \subset H^s(\mathbb{T})$.*

(iii) *Let $0 < \delta < \infty$. Then, the conclusions in Part (i) and (ii) also hold for the modulated ILW equation* (1.16) *and the modulated scaled ILW equation* (1.19) *on $\mathbb{T}$.*

1.5. **Invariance of the white noise under the modulated KdV.** In Theorem 1.3 (ii), we established global well-posedness of the periodic modulated KdV (1.2) via a deterministic tool (the I-method). In this subsection, we use a probabilistic tool to globalize solutions to the periodic modulated KdV (1.2), which allows us to construct global-in-time solutions in a regime not covered by Theorem 1.3 (ii).

Let μ be a white noise[10] on the circle $\mathbb{T}$ with formal density:

$$d\mu = Z^{-1} \exp\bigg(-\frac{1}{2}\int_{\mathbb{T}} u^2 dx_{\mathbb{T}}\bigg) du. \tag{1.27}$$

More precisely, μ is the induced probability measure under the map:[11]

$$\omega \in \Omega \longmapsto u_0^\omega = \sum_{n\in\mathbb{Z}^*} g_n(\omega) e_n, \tag{1.28}$$

where $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$, $e_n(x) = e^{inx}$, and $\{g_n\}_{n\in\mathbb{Z}^*}$ is a family of independent standard complex-valued Gaussian random variables[12] on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned that $g_{-n} = \overline{g_n}$, $n\in\mathbb{Z}^*$. It is easy to see that u_0^ω in (1.28) belongs almost surely to $H^s(\mathbb{T})\setminus H^{-\frac{1}{2}}(\mathbb{T})$, $s<-\frac{1}{2}$.

In particular, Theorem 1.3 (i) states that, for $\rho>\frac{2}{3}$, the modulated KdV (1.2) on $\mathbb{T}$ is locally well-posed almost surely with respect to the white noise initial data. On the other hand, the regularity threshold in (1.10) for (deterministic) global well-posedness tends to $-\frac{1}{4}$ as $\rho\to\frac{2}{3}$ and $\gamma\to\frac{1}{2}$. More precisely, when $\rho<1-\frac{1}{3}\gamma$, the regularity threshold in (1.10) is above $-\frac{1}{2}$ and thus we can not apply Theorem 1.3 (ii) to globalize local-in-time solutions with the white noise initial data.

By using a probabilistic argument, we establish the following global-in-time result for the periodic modulated KdV equation (1.2) with the white noise initial data, which complements the deterministic global well-posedness result in Theorem 1.3 (ii).

Theorem 1.20. *Given $\rho>\frac{2}{3}$ and $\frac{1}{2}<\gamma<1$, let w be (ρ,γ)-irregular on $\mathbb{R}_+$ in the sense of Definition 1.1. Then, the modulated KdV equation* (1.2) *on the circle $\mathbb{T}$ is almost surely globally well-posed with the white noise initial data*[13] *u_0^ω in* (1.28)*, and the white noise μ is invariant under the resulting dynamics.*

Theorem 1.20 extends invariance of the white noise under the KdV dynamics from the unmodulated setting (see [138, 121, 122, 123, 130, 96]) to the current modulated setting, thus establishing the first result on the construction of invariant measures for modulated dispersive PDEs. We also mention that Theorem 1.20 plays a key role in the globalization argument for the stochastic modulated KdV (1.29) with an additive space-time white noise (Theorem 1.23 (ii)) considered in the next subsection.

[10] As it is customary in the literature, with a slight abuse of notation, we use the term 'white noise' to refer to both the distribution-valued random variable u_0^ω in (1.28) and its law $\mu = \mathrm{Law}(u_0^\omega)$, when there is no confusion. Here, $\mathrm{Law}(X)$ denotes the law of a random variable X. For clarity, we may refer to $\mu = \mathrm{Law}(u_0^\omega)$ as the white noise measure.

[11] For simplicity of the presentation, we assume that the white noise has spatial mean zero on $\mathbb{T}$.

[12] Note that $\mathrm{Var}(g_n) = 1$.

[13] As it is common in the study of dispersive PDEs with random initial data, we use u_0^ω to denote both the white noise and a sample path of the white noise but the meaning is clear from the context.

In view of Theorem 1.16 on the convergence of the truncated modulated KdV dynamics (1.22) to the modulated KdV dynamics (1.2), Theorem 1.20 follows from a standard application of Bourgain's invariant measure argument [7, 8] (whose details we omit; see [15] for details of Bourgain's invariant measure argument), once we prove invariance of the white noise for the truncated modulated KdV (1.22); see Proposition 8.1.

Theorem 1.12 states that for $\rho > 1$, the modulated BO (1.15), the modulated ILW (1.16), and the modulated scaled ILW (1.19) are locally well-posed in $H^s(\mathbb{T})$, $s > -\frac{1}{2}\rho$, containing the support of the white noise. Moreover, for the same range of ρ and s, Theorem 1.19 (ii) guarantees convergence of the Galerkin approximation. Hence, a slight modification of a proof of Theorem 1.20 yields the following result on almost sure global well-posedness of these equations with the white noise initial data and invariance of the white noise. We omit details.

Theorem 1.21. *Given $\rho > 1$ and $\frac{1}{2} < \gamma < 1$, let w be (ρ, γ)-irregular on $\mathbb{R}_+$ in the sense of Definition 1.1.*

(i) *The modulated BO equation* (1.15) *on the circle $\mathbb{T}$ is almost surely globally well-posed with the white noise initial data u_0^ω in* (1.28)*, and the white noise μ is invariant under the resulting dynamics. The same result also holds for the modulated ILW equation* (1.16) *and the modulated scaled ILW equation* (1.19) *on the circle.*

(ii) *Let $\{\delta_j\}_{j\in\mathbb{N}}$ be an increasing sequence of positive numbers, tending to ∞ as $j \to \infty$. Let u^{δ_j}, $j \in \mathbb{N}$, and u denote the solutions to the modulated ILW equation* (1.16) *and the modulated BO equation* (1.15) *with the white noise initial data u_0^ω in* (1.28)*, respectively. Then, as $j \to \infty$ (and hence $\delta_j \to \infty$), u^{δ_j} converges to u in $C(\mathbb{R}_+; H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}))$, $\varepsilon > 0$, almost surely. Here, $C(\mathbb{R}_+; H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}))$ is endowed with the compact-open topology.*

(iii) *Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a decreasing sequence of positive numbers, tending to 0 as $j \to \infty$. Let v^{δ_j}, $j \in \mathbb{N}$, and u denote the solutions to the modulated scaled ILW equation* (1.19) *and the modulated KdV equation* (1.2) *with the white noise initial data u_0^ω in* (1.28)*, respectively. Then, as $j \to \infty$ (and hence $\delta_j \to 0$), v^{δ_j} converges to u in $C(\mathbb{R}_+; H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}))$, $\varepsilon > 0$, almost surely.*

Remark 1.22. In Theorem 1.21 (ii), we took a discrete sequence of depth parameters δ_j, $j \in \mathbb{N}$, such that, in view of Theorem 1.21 (i), there exists a set of full probability, guaranteeing global existence of u^{δ_j} for any $j \in \mathbb{N}$.

In the case of a continuous parameter $\delta \in (0, \infty)$, we have almost sure asymptotic global-in-time deep-water convergence in the following sense. It follows from Theorem 1.21 (i) that there exists $\Sigma \subset \Omega$ with $\mathbb{P}(\Sigma) = 1$ such that for each $\omega \in \Sigma$, the solution u to the modulated BO (1.15) with $u|_{t=0} = u_0^\omega$ exists globally in time. Given any $T \gg 1$, by iterating the local-in-time argument (as in the proof of Theorem 1.14 (i)), we see that there exists $\delta_0 = \delta_0(\omega, T) \gg 1$ such that, for any $\delta > \delta_0$, the solution u^δ to the modulated ILW (1.16) with $u^\delta|_{t=0} = u_0^\omega$ exists on the time interval $[0, T]$ and, moreover, u^δ converges to u on $[0, T]$ as $\delta \to \infty$.

By making use of a bi-parameter Kolmogorov continuity criterion [3, Lemma 2.1], it may be possible to prove full deep-water convergence along a continuous parameter $\delta \in (0, \infty)$. We, however, do not pursue this issue here. See [152] for such an argument in the context of the Smoluchowski-Kramers approximation for singular stochastic wave equations.

Similar comments apply to the shallow-water convergence considered in Theorem 1.21 (iii).

1.6. **Stochastic modulated KdV.** As the last example, we consider the following stochastic modulated KdV equation on the circle with an additive white-in-time noise:

$$\partial_t u + \partial_x^3 u \cdot \partial_t w = \partial_x u^2 + \phi\xi, \tag{1.29}$$

where ξ denotes a (Gaussian) space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$ whose space-time covariance is (formally) given by

$$\mathbb{E}[\xi(t_1, x_1)\xi(t_2, x_2)] = \delta(t_1 - t_2)\delta(x_1 - x_2)$$

for $t_1, t_2 \in \mathbb{R}_+$ and $x_1, x_2 \in \mathbb{T}$ with δ denoting the Dirac delta function. Here, ϕ is a Hilbert-Schmidt operator from $L^2_0(\mathbb{T}) = \mathbf{P}_{\neq 0}L^2(\mathbb{T})$ to $H^s_0(\mathbb{T}) = \mathbf{P}_{\neq 0}H^s(\mathbb{T})$ for some $s \in \mathbb{R}$, where $\mathbf{P}_{\neq 0}$ denotes the projection onto non-zero frequencies. Namely, $L^2_0(\mathbb{T})$ and $H^s_0(\mathbb{T})$ are subspaces of $L^2(\mathbb{T})$ and $H^s(\mathbb{T})$, consisting of mean-zero functions.

Given $\sigma \geq 0$,[14] we say that a distribution-valued random variable X_σ on $\mathbb{T}$ (and its law, denoted by μ_σ) is a (spatial) white noise on $\mathbb{T}$ with variance σ if

$$\mu_\sigma = \mathrm{Law}(X_\sigma) = \mathrm{Law}(\sqrt{\sigma}\, u_0^\omega), \tag{1.30}$$

where u_0^ω is the white noise (with variance 1) in (1.28).

Let us recall the notion of an *evolution system of measures* [44, 48] in a somewhat formal manner. Let $\Phi_{t_1,t_2} = \Phi^\omega_{t_1,t_2}$, $t_1 \geq t_2 \geq 0$, be a solution map for a given (random) dynamical system, sending the data φ at time t_2 to the solution $\Phi_{t_1,t_2}\varphi$ at time t_1. Then, we define the transition semigroup P_{t_1,t_2} by

$$P_{t_1,t_2}F(\varphi) = \mathbb{E}[F(\Phi^\omega_{t_1,t_2}\varphi)] \tag{1.31}$$

for a bounded measurable function F on the phase space $\mathcal{X}$. Then, we say that[15] a family $\{\rho_t\}_{t\in\mathbb{R}_+}$ of probability measures on $\mathcal{X}$ is an evolution system of measures indexed by $\mathbb{R}_+$ if

$$\int_{\mathcal{X}} F(\varphi)\rho_{t_1}(d\varphi) = \int_{\mathcal{X}} P_{t_1,t_2}F(\varphi)\rho_{t_2}(d\varphi) \tag{1.32}$$

for any bounded continuous function F on $\mathcal{X}$ and $t_1 \geq t_2 \geq 0$. Note that (1.32) is equivalent to

$$\rho_{t_1} = P^*_{t_1,t_2}\rho_{t_2} \tag{1.33}$$

for any $t_1 \geq t_2 \geq 0$. If there exists an invariant measure ρ, then by setting $\rho_t = \rho$, $t \in \mathbb{R}_+$, the family $\{\rho_t\}_{t\in\mathbb{R}_+}$ is obviously an evolution system of measures. It is in this sense that the notion of an evolution system of measures is a generalization of the notion of an invariant measure.

We now state our main result on the stochastic modulated KdV (1.29).

Theorem 1.23. (i) *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1. Suppose that $s \in \mathbb{R}$ satisfies* (1.9) *and that $\phi \in \mathrm{HS}(L^2_0(\mathbb{T}); H^s_0(\mathbb{T}))$. Then, the stochastic modulated KdV equation* (1.29) *is locally well-posed in $H^s(\mathbb{T})$ and the modulated interaction representation $\mathbf{u}(t) = U^w(t)^{-1}u(t)$ of the solution u belongs to*

[14]By convention, we have $X \equiv 0$ when $\sigma = 0$. Namely, $\mu_0 = \delta_0$, where δ_0 is the Dirac delta distribution at the trivial function.

[15]Strictly speaking, an evolution system of measures is the mapping $t \in \mathbb{R}_+ \mapsto \rho_t \in \mathcal{P}(\mathcal{M})$, where $\mathcal{P}(\mathcal{M})$ denotes the family of probability measures on $\mathcal{M}$. However, we simply refer to the family $\{\rho_t\}_{t\in\mathbb{R}_+}$ of measures as an evolution system of measures.

$\mathcal{C}^\alpha([0,\tau];H^s(\mathbb{T}))$ *for any* $0<\alpha<\frac{1}{2}$, *where* $\tau>0$ *denotes the almost surely positive local existence time.*

In particular, when $\rho>\frac{2}{3}$, *the stochastic modulated KdV equation* (1.29) *with an additive space-time white noise* (*namely,* $\phi=\mathrm{Id}$) *is locally well-posed in* $H^s(\mathbb{T})$.

(ii) *Given* $\rho>\frac{2}{3}$ *and* $\frac{1}{2}<\gamma<1$, *let* w *be* (ρ,γ)*-irregular on* $\mathbb{R}_+$. *Then, the stochastic modulated KdV equation* (1.29) *on* $\mathbb{T}$ *with an additive space-time white noise* (*namely,* $\phi=\mathrm{Id}$) *is globally well-posed with the* (*spatial*) *white noise initial data* u_0^ω *in* (1.28), *independent of the space-time white noise forcing* ξ. *Moreover, for any* $t\geq 0$, *we have*

$$\mathrm{Law}(u(t))=\mu_{1+t}. \tag{1.34}$$

Namely, the family $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ *is an evolution system of measures for the stochastic modulated KdV equation* (1.29) *with the white noise initial data* u_0^ω *in* (1.28).

When $\rho>\frac{1}{2}$, convergence of the Galerkin approximation (see (9.15)) holds in the current stochastic setting; see Lemma 9.4.

The well-posedness issue of the stochastic KdV with an additive forcing:

$$\partial_t u+\partial_x^3 u+u\partial_x u=\phi\xi \tag{1.35}$$

has been studied on both the real line and on the torus; see [49, 50, 122, 129]. In particular, in [129], Quastel, Sosoe, and the fifth author proved an analogue of Theorem 1.23 (ii) for (1.35) with $\phi=\mathrm{Id}$ by implementing a variant of Bourgain's invariant measure argument [7] in the context of an evolution system of measures. The local well-posedness claim in Theorem 1.23 (i) follows from a slight modification of the proof of Theorem 1.3 (i). On the other hand, Theorem 1.23 (ii) follows from a straightforward adaptation of the argument developed in [129] to the current problem. See also [129, Corollary 1.3 (ii)] for a global existence result of (1.35) with initial data of the form $w_0+u_0^\omega$ for $w_0\in L^2(\mathbb{T})$ based on the ideas in [128], which also applies to the stochastic modulated KdV (1.29) with $\phi=\mathrm{Id}$.

The Duhamel formulation of (1.29) is given by

$$\begin{aligned}u(t)=U^w(t)u_0+U^w(t)\int_0^t(U^w(t'))^{-1}\mathcal{N}(u(t'))dt'\\ +U^w(t)\int_0^t(U^w(t'))^{-1}\phi\xi(dt'),\end{aligned}$$

where $U^w(t)=e^{-w(t)\partial_x^3}$, $\mathcal{N}(u)=\partial_x u^2$, and the last term on the right-hand side represents the effect of the stochastic forcing $\phi\xi$ in (1.29). As in the deterministic case, we study the following Duhamel formulation satisfied by the modulated interaction representation $\mathbf{u}(t)=U^w(t)^{-1}u(t)$:

$$\mathbf{u}(t)=u_0+\int_0^t U^w(t')^{-1}\mathcal{N}(U^w(t')\mathbf{u}(t'))dt'+\Psi(t), \tag{1.36}$$

where Ψ is a solution to the following stochastic equation:

$$\partial_t\Psi=U^w(t)^{-1}\phi\xi,\qquad \Psi(0)=0,$$

formally given by the following stochastic integral:

$$\Psi(t)=\int_0^t(U^w(t'))^{-1}\phi\xi(dt'). \tag{1.37}$$

See (9.1) for a precise definition of Ψ. Note that, due to the roughness-in-time of the noise, the stochastic term Ψ has temporal regularity $\alpha < \frac{1}{2}$ (see Lemma 9.1) and thus we expect that a solution $\mathbf{u}$ to (1.36) has temporal regularity $\alpha < \frac{1}{2}$. We point out that our nonlinear Young integral theory (Proposition 3.7) and a general local well-posedness result for the YDE (1.12) are adapted to the case with temporal regularity $\alpha < \frac{1}{2}$ (as long as $\alpha+\gamma > 1$; see the proof of Proposition 3.8). Hence, once we establish regularity properties of the stochastic term Ψ (Lemma 9.1), Theorem 1.23 (i) follows from a minor modification of the proof of Proposition 3.8. See Section 9 for details.

Next, we briefly discuss a proof of the almost sure global well-posedness result in Theorem 1.23 (ii), based on the argument in [129]. We first provide a heuristic argument for the claim (1.34) by viewing the stochastic modulated KdV (1.29) as a superposition of the deterministic modulated KdV (1.2) and the stochastic flow:

$$\partial_t u = \xi \tag{1.38}$$

(at the level of infinitesimal generators). While Theorem 1.20 states that the white noise (with any variance) is invariant under (1.2), it is easy to see that the stochastic flow (1.38) with a white noise initial data (with any variance) increases the variance by the length of the time interval under consideration. Then, at a formal level, the claim (1.34) follows from these observations together with the Lie-Trotter product formula [139, Section VIII.8]:

$$e^{t(A+B)} = \lim_{n\to\infty} \big[e^{\frac{t}{n}A} e^{\frac{t}{n}B}\big]^n \tag{1.39}$$

(which holds, for example, for finite-dimensional matrices A, B). Obviously, the Lie-Trotter product formula (1.39) is not directly applicable to our problem, and we justify the heuristic argument above by the Galerkin approximation (Lemma 9.4). See Section 9 for further details.

As in Subsection 1.5, by a straightforward modification of the proof of Theorem 1.23, we obtain the corresponding results for the following stochastic modulated BO equation on the circle:

$$\partial_t u - \mathcal{H}\partial_x^2 u \cdot \partial_t w = \partial_x u^2 + \phi\xi. \tag{1.40}$$

Theorem 1.24. (i) *Given $\rho \geq 1$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Suppose that $s \in \mathbb{R}$ satisfies* (1.21) *and that $\phi \in \mathrm{HS}(L^2_0(\mathbb{T}); H^s_0(\mathbb{T}))$. Then, the stochastic modulated BO equation* (1.40) *is locally well-posed in $H^s(\mathbb{T})$. In particular, when $\rho > 1$, the stochastic modulated BO equation* (1.40) *with an additive space-time white noise (namely, $\phi = \mathrm{Id}$) is locally well-posed in $H^s(\mathbb{T})$.*

(ii) *Given $\rho > 1$ and $\frac{1}{2} < \gamma < 1$, let w be (ρ,γ)-irregular on $\mathbb{R}_+$. Then, the stochastic modulated BO equation* (1.40) *on $\mathbb{T}$ with an additive space-time white noise (namely, $\phi = \mathrm{Id}$) is globally well-posed with the (spatial) white noise initial data u_0^ω in* (1.28)*, independent of the space-time white noise forcing ξ. Moreover, the family $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ is an evolution system of measures for the stochastic modulated BO equation* (1.40) *with the white noise initial data u_0^ω in* (1.28).

The same results also hold for the stochastic modulated ILW equation on the circle:

$$\partial_t u - \mathcal{G}_\delta\partial_x^2 u \cdot \partial_t w = \partial_x u^2 + \phi\xi \tag{1.41}$$

and the stochastic modulated scaled ILW equation on the circle:

$$\partial_t v - \widetilde{\mathcal{G}_\delta}\partial_x^2 v \cdot \partial_t w = \partial_x v^2 + \phi\xi. \tag{1.42}$$

In particular, deep-water and shallow-water convergence analogous to those in Theorems 1.14 and 1.21 also holds for (1.41) and (1.42). We omit details.

1.7. **Organization.** In Section 2, we introduce basic notations and function spaces. We also introduce various classes of drivers such as $\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$ in (2.9) and (2.11) which play a crucial role in the remaining part of this paper. In Section 3, after providing simple criteria for checking regularity of drivers (Lemma 3.1), we go over the basic theory for nonlinear Young integrals (Lemma 3.4 and Proposition 3.7). We then present a general local well-posedness result for the YDE (1.12) (Proposition 3.8). In Section 4, by establishing regularity properties of the associated drivers, we prove local well-posedness and related properties for the modulated KdV-type equations. In Section 5, we establish local well-posedness and related properties for the modulated BO and ILW equations. In Section 6, we prove the conservation of the L^2-norm for the modulated dispersive equations considered in this paper along with their L^2-global well-posedness in the regime where they are locally well-posed in $L^2(\mathcal{M})$. In Section 7, we adapt the I-method to the current modulated setting. By combining a crucial commutator estimate (Proposition 7.6) with the sewing lemma (Lemma 3.3), we prove global well-posedness of the modulated KdV (1.2) in negative Sobolev spaces (Theorems 1.3 (ii) and 1.7 (ii)). In Section 8, we provide a brief discussion for the proof of Theorem 1.20 on invariance of the white noise under the modulated KdV on the circle. Finally, in Section 9, we study the stochastic modulated KdV (1.29) on the circle and prove Theorem 1.23.

2. Notations and preliminary tools

2.1. **Basic notations.** Let $A \lesssim B$ denote an estimate of the form $A \le CB$ for some constant $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$, while $A \ll B$ denotes $A \le cB$ for some small constant $c > 0$. We may write $\lesssim_\alpha$ and $\sim_\alpha$ to emphasize the dependence on an external parameter α. We use $C > 0$ to denote various constants, which may vary line by line.

In expressing the dependence of a function u on the time variable, we often use the shorthand notation $u_t = u(t)$, which is standard in probability theory and stochastic analysis.

Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. We use $\widehat{f}$ and $\mathcal{F}(f) = \mathcal{F}_{\mathcal{M}}(f)$ to denote the Fourier transform of a function f on $\mathcal{M}$, defined by

$$\widehat{f}(n) = \int_{\mathcal{M}} f(x) e^{-inx} dx_{\mathcal{M}}.$$

Here, when $\mathcal{M} = \mathbb{R}$, $dx_{\mathbb{R}} = (2\pi)^{-1}dx$ denotes the scaled Lebesgue measure on $\mathbb{R}$, while, when $\mathcal{M} = \mathbb{T}$, $dx_{\mathbb{T}}$ denotes the normalized Lebesgue measure $dx_{\mathbb{T}} = (2\pi)^{-1}dx$. With this

convention, we have

$$f(x) = \int_{\mathcal{M}^*} \widehat{f}(\xi) e^{ix\xi} d\xi_{\mathcal{M}^*},$$
$$\|f\|^2_{L^2(\mathcal{M})} = \int_{\mathcal{M}} |f(x)|^2 dx_{\mathcal{M}} = \int_{\mathcal{M}^*} |\widehat{f}(\xi)|^2 d\xi_{\mathcal{M}^*},$$
$$\mathcal{F}_{\mathcal{M}}(fg)(\xi) = \int\limits_{\xi=\xi_1+\xi_2} \widehat{f}(\xi_1)\widehat{g}(\xi_2)(d\xi_1)_{\mathcal{M}^*},$$

where $\mathcal{M}^*$ is the Pontryagin dual of $\mathcal{M}$ (namely, $\mathcal{M}^* = \mathbb{R}$ when $\mathcal{M} = \mathbb{R}$, while $\mathcal{M}^* = \mathbb{Z}$ when $\mathcal{M} = \mathbb{T}$) and $d\xi_{\mathbb{R}}$ denotes the standard Lebesgue measure on $\mathbb{R}$, while $d\xi_{\mathbb{Z}}$ denotes the counting measure on $\mathbb{Z}$. On the circle, we impose the mean-zero assumption for the equations we consider in this paper and thus we set $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Given $N \in \mathbb{N}$, we denote by $\mathbf{P}_N$ the Dirichlet projector onto the frequencies $\{|\xi| \leq N\}$ defined by

$$\widehat{\mathbf{P}_N f}(\xi) = \mathbf{1}_{\{|\xi|\leq N\}} \widehat{f}(\xi). \tag{2.1}$$

We set $\mathbf{P}_N^{\perp} = \mathrm{Id} - \mathbf{P}_N$.

2.2. **Function spaces.** We use $\mathcal{S}(\mathbb{R})$ to denote the space of Schwartz functions on $\mathbb{R}$ and set $\mathcal{S}(\mathbb{T}) = C^\infty(\mathbb{T})$.

Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. Given $s \in \mathbb{R}$, we define the non-homogeneous and homogeneous Sobolev spaces $H^s(\mathcal{M})$ and $\dot{H}^s(\mathcal{M})$ via the norms:

$$\|f\|^2_{H^s(\mathcal{M})} = \int_{\mathcal{M}^*} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi,$$
$$\|f\|^2_{\dot{H}^s(\mathcal{M})} = \int_{\mathcal{M}^*} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi.$$

Given $k \in \mathbb{N}$, let $V, V_1, \dots, V_k$ be separable Hilbert spaces. We use

$$\mathcal{L}_k\Big(\bigotimes_{j=1}^k V_j; V\Big)$$

to denote the Banach space of bounded k-linear operators on $\bigotimes_{j=1}^k V_j$ with values in V. When $V_j = V$ for $j = 1, \dots, k$, we simply set $\mathcal{L}_k(V) = \mathcal{L}_k(V^{\otimes k}; V)$.

Let V be a Banach space and $T > 0$. For $n \in \mathbb{N}$, we denote

$$\Delta_{n,T} = \big\{(t_1, \dots, t_n) \in [0,T]^n : \ t_i > t_j \text{ for } i < j\big\}.$$

We denote by $C_{n,T}V$ the space of continuous functions from $\Delta_{n,T}$ to V. When $n = 1$, we may write $C_T V$ for simplicity, and equip this space with the supremum norm:

$$\|f\|_{C_T V} = \|f\|_{L^\infty_T V} = \sup_{0\leq t\leq T} \|f(t)\|_V.$$

We define the coboundary operator $\delta : C_{n,T}V \to C_{n+1,T}V$ as follows; given $f \in C_{n,T}V$ and $(t_1, \dots, t_{n+1}) \in \Delta_{n+1,T}$, we set

$$(\delta f)_{t_1,\dots,t_{n+1}} = \sum_{k=1}^{n+1} (-1)^k f_{t_1,\dots,t_{k-1},t_{k+1},\dots,t_{n+1}}.$$

For example, for $f \in C_T V$ and $g \in C_{2,T} V$, we have

$$\begin{aligned} (\delta f)_{t,r} &= f_t - f_r, \\ (\delta g)_{t_1,t_2,t_3} &= g_{t_1,t_3} - g_{t_1,t_2} - g_{t_2,t_3} \end{aligned} \tag{2.2}$$

for $(t,r) \in \Delta_{2,T}$ and $(t_1,t_2,t_3) \in \Delta_{3,T}$. As noted in [81], the sequence

$$0 \longrightarrow V \longrightarrow C_{1,T} V \xrightarrow{\delta} C_{2,T} V \xrightarrow{\delta} C_{3,T} V \xrightarrow{\delta} \cdots$$

is exact. In particular, we have $\delta \circ \delta = 0$ and if $f \in C_{n,T} V$ with $\delta f = 0$, then there exists a $g \in C_{n-1,T} V$ such that $f = \delta g$; see, for example, [81, Lemma 2.1].

Given $0 < \gamma < 1$, we denote by $C_T^\gamma V = C^\gamma([0,T]; V)$ the space of γ-Hölder continuous functions taking values in V, endowed with the seminorm:

$$\|f\|_{C_T^\gamma V} = \sup_{(t,r) \in \Delta_{2,T}} \frac{\|(\delta f)_{t,r}\|_V}{|t-r|^\gamma}.$$

We also define $\mathcal{C}_T^\gamma V = \mathcal{C}^\gamma([0,T]; V)$ via the norm:

$$\|f\|_{\mathcal{C}_T^\gamma V} = \|f\|_{L_T^\infty V} + \|f\|_{C_T^\gamma V}. \tag{2.3}$$

We also introduce the spaces $C_{n,T}^\gamma V$, $n = 2, 3$, equipped with the following Hölder-type norms; for $g \in C_{2,T} V$ and $h \in C_{3,T} V$, we set

$$\begin{aligned} \|g\|_{C_{2,T}^\gamma V} &= \sup_{(t,r) \in \Delta_{2,T}} \frac{\|g_{t,r}\|_V}{|t-r|^\gamma}, \\ \|h\|_{C_{3,T}^\gamma V} &= \inf \left\{ \sum_{n=1}^N \sup_{(t_1,t_2,t_3) \in \Delta_{3,T}} \frac{\|h^{(n)}_{t_1,t_2,t_3}\|_V}{|t_1 - t_2|^{\alpha_n} |t_2 - t_3|^{\gamma - \alpha_n}} \right\}, \end{aligned} \tag{2.4}$$

where the infimum is taken over all finite representations $h = \sum_{n=1}^N h^{(n)}$, $h^{(n)} \in C_{3,T} V$, of h and over all choices of $\alpha_n \in (0, \gamma)$.

Let V and W be Banach spaces. Given $k \in \mathbb{N}$, we use $\mathrm{Lip}_k(V; W)$ to denote the space of locally Lipschitz maps $f : V \to W$ with polynomial growth of order k such that

$$\|f\|_{\mathrm{Lip}_k(V;W)} = \sup_{x,y \in V} \frac{\|f(x) - f(y)\|_W}{\|x - y\|_V \big(1 + \|x\|_V + \|y\|_V\big)^{k-1}} < \infty. \tag{2.5}$$

When $V = W$, we simply set $\mathrm{Lip}_k(V) = \mathrm{Lip}_k(V; V)$. Given an integer $k \geq 2$, we say that $f \in \mathrm{Lip}_k^2(V; W)$ if

(i) $f \in \mathrm{Lip}_k(V; W)$,

(ii) f is Fréchet differentiable with $Df \in \mathrm{Lip}_{k-1}(V; \mathcal{L}_1(V; W))$.

From (2.4) and (2.5), we have

$$\|f\|_{C_{2,T}^\gamma \mathrm{Lip}_k(V;W)} = \sup_{(t,r) \in \Delta_{2,T}} \frac{1}{|t-r|^\gamma} \sup_{x,y \in V} \frac{\|f_{t,r}(x) - f_{t,r}(y)\|_W}{\|x - y\|_V \big(1 + \|x\|_V + \|y\|_V\big)^{k-1}}. \tag{2.6}$$

In addition, suppose that $V_0 \hookrightarrow V$ is a Banach subspace of V. Given an integer $k \geq 2$, we use $\mathrm{Lip}_k(V, V_0; V_0) \subset \mathrm{Lip}_k(V)$ to denote the space of maps $f \in \mathrm{Lip}_k(V)$ such that $f(V_0) \subset V_0$ and

$$\|f\|_{\mathrm{Lip}_k(V,V_0;V_0)} = \sup_{x,y \in V_0} \frac{\|f(x) - f(y)\|_{V_0}}{G_{V,V_0}(x,y)} < \infty, \tag{2.7}$$

where $G_{V,V_0}(x,y)$ is given by

$$\begin{aligned} G_{V,V_0}(x,y) &= \|x-y\|_{V_0}\big(1+\|x\|_V+\|y\|_V\big)^{k-1} \\ &\quad + \|x-y\|_V\big(1+\|x\|_V+\|y\|_V\big)^{k-2}\big(1+\|x\|_{V_0}+\|y\|_{V_0}\big). \end{aligned}$$

Then, from (2.4) and (2.7), we have

$$\|f\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)} = \sup_{(t,r)\in\Delta_{2,T}} \frac{1}{|t-r|^\gamma} \sup_{x,y\in V_0} \frac{\|f_{t,r}(x)-f_{t,r}(y)\|_{V_0}}{G_{V,V_0}(x,y)}. \tag{2.8}$$

Given $s\in\mathbb{R}$, $0<\gamma<1$, $k\in\mathbb{N}$, and $T>0$, we define the space $\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$ of drivers as follows; for $k\geq 2$, we set

$$\begin{aligned} &\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M}) \\ &\quad = \big\{X\in C^\gamma_{2,T}\mathrm{Lip}^2_k(H^s(\mathcal{M})) : X(0)=DX[0]=0 \text{ and } \delta X=0\big\}, \end{aligned} \tag{2.9}$$

endowed with the norm

$$\|X\|_{\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})} = \|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(H^s(\mathcal{M}))} + \|DX\|_{C^\gamma_{2,T}\mathrm{Lip}_{k-1}(H^s(\mathcal{M});\mathcal{L}_1(H^s(\mathcal{M})))}. \tag{2.10}$$

When $k=1$, we set

$$\mathcal{X}^{s,\gamma}_1([0,T]\times\mathcal{M}) = \big\{X\in C^\gamma_{2,T}\mathcal{L}(H^s(\mathcal{M})) : \delta X=0\big\}. \tag{2.11}$$

In (2.9), $X(0)=0$ (and $DX[0]=0$) means $X_{t,r}(0)=0$ (and $DX_{t,r}[0]=0$, respectively) for any $(t,r)\in\Delta_{2,T}$, where $DX_{t,r}[0]$ denotes the Fréchet derivative of $X_{t,r}$ at $u=0\in H^s(\mathcal{M})$. Similarly, we define $\dot{\mathcal{X}}^{s,\gamma}_k([0,T]\times\mathcal{M})$ by setting

$$\begin{aligned} &\dot{\mathcal{X}}^{s,\gamma}_k([0,T]\times\mathcal{M}) \\ &\quad = \big\{X\in C^\gamma_{2,T}\mathrm{Lip}^2_k(\dot{H}^s(\mathcal{M})) : X(0)=DX[0]=0 \text{ and } \delta X=0\big\}, \end{aligned} \tag{2.12}$$

endowed with the norm

$$\|X\|_{\dot{\mathcal{X}}^{s,\gamma}_k([0,T]\times\mathcal{M})} = \|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(\dot{H}^s(\mathcal{M}))} + \|DX\|_{C^\gamma_{2,T}\mathrm{Lip}_{k-1}(\dot{H}^s(\mathcal{M});\mathcal{L}_1(\dot{H}^s(\mathcal{M})))}$$

for $k\geq 2$. For $k=1$, we set $\dot{\mathcal{X}}^{s,\gamma}_1([0,T]\times\mathcal{M}) = C^\gamma_{2,T}\mathrm{Lip}_1(\dot{H}^s(\mathcal{M}))$. We may simply write $\mathcal{X}^{s,\gamma}_k([0,T])$ or $\mathcal{X}^{s,\gamma}_k(T)$, where there is no confusion about an underlying spatial domain. We write $X\in\mathcal{X}^{s,\gamma}_k(\mathbb{R}_+)=\mathcal{X}^{s,\gamma}_k(\mathbb{R}_+\times\mathcal{M})$ if $X\in\mathcal{X}^{s,\gamma}_k(T)$ for any $T>0$ (but $\sup_{T>0}\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}$ may be infinite). A similar comment applies to the homogenous space $\dot{\mathcal{X}}^{s,\gamma}_k([0,T]\times\mathcal{M})$.

More generally, given $s,s_0\in\mathbb{R}$, $0<\gamma<1$, $k\in\mathbb{N}$, and $T>0$ with $s_0>s$, we define the space $\mathcal{X}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M})$ of drivers (for establishing nonlinear smoothing) by setting

$$\mathcal{X}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M}) = C^\gamma_{2,T}\mathrm{Lip}_k(H^s(\mathcal{M});H^{s_0}(\mathcal{M})). \tag{2.13}$$

Similarly, we set

$$\dot{\mathcal{X}}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M}) = C^\gamma_{2,T}\mathrm{Lip}_k(\dot{H}^s(\mathcal{M});\dot{H}^{s_0}(\mathcal{M})).$$

In proving persistence of regularity, we need the following class of drivers. Given $s,s_0\in\mathbb{R}$, $0<\gamma<1$, an integer $k\geq 2$, and $T>0$ with $s_0>s$, we define $\mathcal{Y}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M})$ by

$$\mathcal{Y}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M}) = C^\gamma_{2,T}\mathrm{Lip}_k(H^s(\mathcal{M}),H^{s_0}(\mathcal{M});H^{s_0}(\mathcal{M})). \tag{2.14}$$

Similarly, we set

$$\dot{\mathcal{Y}}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M}) = C^\gamma_{2,T}\mathrm{Lip}_k(\dot{H}^s(\mathcal{M}),\dot{H}^{s_0}(\mathcal{M});\dot{H}^{s_0}(\mathcal{M})). \tag{2.15}$$

We often use short-hand notations such as $C^\gamma_T H^s_x = C^\gamma\big([0,T]; H^s(\mathcal{M})\big)$, when there is no ambiguity.

2.3. **On continuous and discrete convolutions.** We recall the following basic lemma on continuous and discrete convolutions.

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha \geq \beta \geq 0 \qquad \text{and} \qquad \alpha + \beta > 1. \tag{2.16}$$

Then, we have

$$\int_{\xi=\xi_1+\xi_2} \frac{d\xi_1}{\langle \xi_1\rangle^\alpha \langle \xi_2\rangle^\beta} \lesssim \frac{1}{\langle \xi\rangle^{\beta-\lambda}},$$
$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1\rangle^\alpha \langle n_2\rangle^\beta} \lesssim \frac{1}{\langle n\rangle^{\beta-\lambda}}$$

for any $\xi \in \mathbb{R}$ and $n \in \mathbb{Z}$, where $\lambda = \max(1-\alpha, 0)$ when $\alpha \neq 1$ and $\lambda = \varepsilon$ when $\alpha = 1$ for any $\varepsilon > 0$. In particular, if (2.16) holds, then we have

$$\sup_{\xi\in\mathbb{R}} \int_{\xi=\xi_1+\xi_2} \frac{d\xi_1}{\langle \xi_1\rangle^\alpha \langle \xi_2\rangle^\beta} + \sup_{n\in\mathbb{Z}} \sum_{n=n_1+n_2} \frac{1}{\langle n_1\rangle^\alpha \langle n_2\rangle^\beta} < \infty.$$

Lemma 2.1 follows from elementary computations. See, for example, [72, Lemma 4.2] and [118, Lemma 4.1].

3. Young solution theory

Our main goal in this paper is to study well-posedness of modulated dispersive equations in terms of the Duhamel formulation (1.8) satisfied by the modulated interaction representation $\mathbf{u}(t) = U^w(t)^{-1}u(t)$ defined in (1.7). Here, the main task is to make sense of the integral on the right-hand side of (1.8). In this section, we go over the construction of such an integral as a nonlinear Young integral under appropriate assumptions.

Let p be a function on $\mathbb{R}$, smooth on $\mathbb{R}\setminus\{0\}$, such that

$$p(0) = 0 \qquad \text{and} \qquad \overline{p(i\xi)} = -p(i\xi), \quad \xi \in \mathbb{R}\setminus\{0\}. \tag{3.1}$$

Namely, $p(\partial_x)$ is formally anti self-adjoint on $L^2(\mathcal{M})$ (which may be unbounded). Then, our goal is to give a proper meaning to the following integral:

$$\mathcal{I}^X(\mathbf{u})(t) = \int_0^t U^w(t')^{-1}\mathcal{N}(U^w(t')\mathbf{u}(t'))dt', \tag{3.2}$$

where $U^w(t) = e^{w(t)p(\partial_x)}$. For this purpose, we introduce a driver X, a priori defined on $\mathcal{S}(\mathcal{M})$, associated with the integral $\mathcal{I}^X(\mathbf{u})$ by setting

$$X_{t,r}(f) = \int_r^t U^w(t')^{-1}\mathcal{N}(U^w(t')f)dt', \quad t > r \geq 0. \tag{3.3}$$

In the previous work [34], the first and second authors introduced an approach to study modulated dispersive equations, where they made sense of the integral in (3.2) as a nonlinear Young integral. While our approach is equivalent to that presented in [34], we construct the integral in (3.2) as a nonlinear Young integral, using the sewing lemma (Lemma 3.3). Moreover, we have made our presentation in a pedestrian manner, accessible to non-experts.

In Subsection 3.1, we provide simple criteria to verify regularity of a given driver. In Subsection 3.2, we go over the construction of a nonlinear Young integral via the sewing lemma in an abstract setting. In Subsection 3.3, we present a basic local well-posedness result along with related properties for the YDE (1.12) in our functional framework.

3.1. **Regularity of drivers.** As we see in the next subsection (see Proposition 3.7), in our functional framework, we can construct the integral $\mathcal{I}^X(\mathbf{u})$ in (3.2) with the driver X in (3.3) as a nonlinear Young integral, provided that the driver X belongs to the class $\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$ defined in (2.9). In this subsection, by restricting our attention to the k-linear setting, we provide simple criteria for checking regularity of drivers.

In this paper, we consider modulated dispersive equation with a nonlinearity of the form:[16]

$$\mathcal{N}(u) = \partial_x u^k \tag{3.4}$$

for an integer $k \geq 2$. With a slight abuse of notation, by setting

$$\mathcal{N}(u_1, \dots, u_k) = \partial_x \Big(\prod_{j=1}^k u_j \Big), \tag{3.5}$$

we can view $\mathcal{N}$ as a k-linear operator, a priori bounded on $\mathcal{S}(\mathcal{M})$. With this notation, we can view the driver X defined in (3.3) as a k-linear operator given by

$$X_{t,r}(f_1, \dots, f_k) = \int_r^t U^w(t')^{-1} \mathcal{N}(U^w(t')f_1, \dots, U^w(t')f_k) dt', \quad t > r \geq 0, \tag{3.6}$$

for functions $f_1, \dots, f_k$ on $\mathcal{M}$. Note that we have

$$X_{t,r}(f) = X_{t,r}(f, \dots, f) \quad \text{and} \quad DX_{t,r}[f](g) = kX_{t,r}(f, \dots, f, g) \tag{3.7}$$

for functions f and g on $\mathcal{M}$, where $DX_{t,r}$ denotes the Fréchet derivative of $X_{t,r}$.

In the current k-linear setting, the following lemma provides simple criteria, allowing us to verify that a driver X in (3.3) belongs to the class $\mathcal{X}_k^{s,\gamma}([0,T] \times \mathcal{M})$ and other related classes of drivers. Since its proof easily follows from (3.7), the k-linearity of X in (3.6) (see also (3.5)), (2.9), (2.10), (2.13), and (2.14) with (2.6) and (2.8), we omit details.

Lemma 3.1. *Let X be a driver in (3.3), where $U^w(t) = e^{p(\partial_x)w(t)}$ with p as in (3.1) and $\mathcal{N}$ is as in (3.4) and (3.5). Then, given $s, s_0 \in \mathbb{R}$, $\frac{1}{2} < \gamma < 1$, an integer $k \geq 2$, and $T \geq 1$ such that $s_0 > s$, the following statements hold*:

(i) *If we have*

$$\|X_{t,r}\|_{\mathcal{L}_k(H^s(\mathcal{M}))} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma$$

for any $0 \leq r < t \leq T$, then X belongs to $\mathcal{X}_k^{s,\gamma}([0,T] \times \mathcal{M})$ defined in (2.9) with the bound

$$\|X\|_{\mathcal{X}_k^{s,\gamma}(T)} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}.$$

[16]For the renormalized modulated mKdV (1.14), the nonlinearity is not $\partial_x u^3$ but can still be viewed as a trilinear operator. See (4.16) and (4.17) below.

(ii) (persistence of regularity). *Suppose that X can be written as $X = \sum_{j=1}^k X^j$ such that for each $j = 1, \dots, k$, we have*

$$\begin{aligned} X^j &\in \mathcal{L}^{s,s_0}_{k,j} \\ &:= \mathcal{L}_k\Big(\big(\bigotimes_{i=1}^{j-1} H^s(\mathcal{M})\big) \otimes H^{s_0}(\mathcal{M}) \otimes \big(\bigotimes_{i=j+1}^{k} H^s(\mathcal{M})\big); H^{s_0}(\mathcal{M})\Big) \end{aligned} \tag{3.8}$$

with the bound

$$\|X^j_{t,r}\|_{\mathcal{L}^{s,s_0}_{k,j}} \lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma.$$

Then, X belongs to $\mathcal{Y}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M})$ defined in (2.14) with the bound

$$\|X\|_{\mathcal{Y}^{s,s_0,\gamma}_k(T)} \lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}.$$

(iii) (nonlinear smoothing). *If we have*

$$\|X_{t,r}\|_{\mathcal{L}_k(H^s(\mathcal{M});H^{s_0}(\mathcal{M}))} \lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma$$

for any $0 \le r < t \le T$, then X belongs to $\mathcal{X}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M})$ defined in (2.13) with the bound

$$\|X\|_{\mathcal{X}^{s,s_0,\gamma}_k(T)} \lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}.$$

(iv) (convergence). *Given $N \in \mathbb{N}$, define the truncated driver X^N by*

$$X^N_{t,r}(f) = \int_r^t U^w(t')^{-1} \mathbf{P}_N \mathcal{N}(\mathbf{P}_N U^w(t') f) dt'.$$

If we have

$$\|X^N_{t,r} - X_{t,r}\|_{\mathcal{L}_k(H^s(\mathcal{M}))} \lesssim o(1)|t-r|^\gamma,$$

as $N \to \infty$, uniformly in $0 \le r < t \le T$, then, X^N converges to X in $\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$.

Remark 3.2. (i) While we stated Part (iv) of Lemma 3.1 for a discrete parameter $N \in \mathbb{N}$, it also holds for a continuous parameter. In particular, we will use Lemma 3.1 (iv) not only for establishing convergence of the Galerkin approximations but also for establishing deep-water and shallow-water convergence of the (scaled) modulated ILW (1.16) (and (1.19)) claimed in Theorem 1.14. See Subsections 5.3 and 5.4.

(ii) By replacing the non-homogeneous Sobolev spaces with the corresponding homogeneous ones, Lemma 3.1 yields criteria to check if a driver belongs to the homogeneous version of the classes of drivers such as $X \in \dot{\mathcal{X}}^{s,\gamma}_k([0,T]\times\mathcal{M})$ and $\dot{\mathcal{Y}}^{s,s_0,\gamma}_k([0,T]\times\mathcal{M})$ defined in (2.12) and in (2.15), respectively.

3.2. **Sewing lemma and Young integrals.** In this subsection, we go over the construction of the integral $\mathcal{I}^X(\mathbf{u})$ in (3.2) as a nonlinear Young integral, using the sewing lemma. The following version of the sewing lemma is taken from [81] with slight modifications; see [81, Proposition 2.3, Corollary 2.4, Corollary 2.5]. See also [63, Lemma 4.2].

We set $C^{1+}_{n,T}V = \bigcup_{\gamma>1} C^\gamma_{n,T}V$.

Lemma 3.3 (sewing lemma)**.** *Let V be a Banach space and fix $T > 0$. Then, there exists a unique linear map (called the sewing map) $\Lambda : C^{1+}_{3,T}V \cap \operatorname{Ker}\delta|_{C_{3,T}V} \to C^{1+}_{2,T}V$ such that*

(i) *We have* $\delta\Lambda h = h$ *for each* $h \in C^{1+}_{3,T}V \cap \operatorname{Ker}\delta|_{C_{3,T}V}$.

(ii) *For each* $\zeta > 1$, *the sewing map* Λ *is continuous from* $C^{\zeta}_{3,T}V \cap \operatorname{Ker}\delta|_{C_{3,T}V}$ *to* $C^{\zeta}_{2,T}V$ *such that*

$$\|\Lambda h\|_{C^{\zeta}_{2,T}V} \le \frac{1}{2^{\zeta}-2}\|h\|_{C^{\zeta}_{3,T}V} \tag{3.9}$$

for any $h \in C^{\zeta}_{3,T}V$.

(iii) *Given any* $g \in C_{2,T}V$ *with* $\delta g \in C^{\zeta}_{3,T}V$, *there exists unique* $f \in C([0,T];V)$ (*modulo an additive constant*) *such that* $\delta f = (\mathrm{Id} - \Lambda\delta)g$. *In addition, we have*

$$(\delta f)_{t,r} = \lim_{|\Pi([r,t])|\to 0}\sum_{j=0}^{n-1} g_{t_j,t_{j+1}} \tag{3.10}$$

for any $(t,r) \in \Delta_{2,T}$, *where the limit is over any partition* $\Pi([r,t])$ *of* $[r,t]$:

$$\Pi([r,t]) = \{r = t_n < \cdots < t_1 < t_0 = t\}$$

whose mesh size $|\Pi([r,t])| = \sup_j |t_j - t_{j+1}|$ *tends to* 0.

As a consequence of the sewing lemma, we have the following lemma on the construction of nonlinear Young integrals. See also [90, 65] (which appeared after the first version of the current paper) for the construction of nonlinear Young integrals in a more general setting.

Lemma 3.4 (nonlinear Young integral)**.** *Let* V *be a Banach space. Given* $0 < \gamma < 1$, $T > 0$, *and* $k \in \mathbb{N}$, *let* $X \in C^{\gamma}_{2,T}\mathrm{Lip}_k(V)$, *where* $\mathrm{Lip}_k(V)$ *is as in* (2.5), *such that*

$$X_{t,r}(0) = 0 \tag{3.11}$$

for any $(t,r) \in \Delta_{2,T}$, *and*

$$(\delta X)_{t_1,t_2,t_3} = 0 \tag{3.12}$$

for any $(t_1,t_2,t_3) \in \Delta_{3,T}$. *Given* $f \in C^{\alpha}([0,T];V)$ *for some* $0 < \alpha < 1$ *such that* $f(0) \in V$, *define* Θ *on* $\Delta_{2,T}$ *by setting*

$$\Theta_{t,r} = X_{t,r}(f(r)), \quad (t,r) \in \Delta_{2,T}. \tag{3.13}$$

Suppose that $\zeta := \alpha + \gamma > 1$, *Then, the following statements hold*:

(i) *We have* $\Theta \in C^{\gamma}_{2,T}V$ *and* $\delta\Theta \in C^{\zeta}_{3,T}V$.

(ii) *There exists a unique function* $\mathcal{I} \in C^{\gamma}([0,T];V)$ *such that* $\mathcal{I}(0) = 0$ *and*

$$\delta\mathcal{I} = (\mathrm{Id} - \Lambda\delta)\Theta, \tag{3.14}$$

where Λ *is the sewing map in Lemma 3.3. Moreover, we have*

$$\|\delta\mathcal{I} - \Theta\|_{C^{\zeta}_{2,T}V} \le C_1(f)\|X\|_{C^{\gamma}_{2,T}\mathrm{Lip}_k(V)}, \tag{3.15}$$

$$\|\mathcal{I}\|_{C^{\gamma}_{T}V} \le C_2(f)\|X\|_{C^{\gamma}_{2,T}\mathrm{Lip}_k(V)}, \tag{3.16}$$

where $C_1(f)$ *and* $C_2(f)$ *are given by*

$$\begin{aligned} C_1(f) &= \frac{1}{2^{\zeta}-2}\big(1 + 2\|f\|_{L^{\infty}_{T}V}\big)^{k-1}\|f\|_{C^{\alpha}_{T}V}, \\ C_2(f) &= (1 \vee T)^{\alpha}C_1(f) + \big(1 + \|f\|_{L^{\infty}_{T}V}\big)^{k-1}\|f\|_{L^{\infty}_{T}V}. \end{aligned} \tag{3.17}$$

Here, $a \vee b := \max(a, b)$. Finally, for $(t, r) \in \Delta_{2,T}$, we have

$$(\delta\mathcal{I})_{t,r} = \lim_{|\Pi([r,t])|\to 0} \sum_{j=0}^{n-1} \Theta_{t_j,t_{j+1}}, \tag{3.18}$$

where the limit is in the sense of Lemma 3.3 (iii). In particular, we have

$$\mathcal{I}(t) = \lim_{|\Pi([0,t])|\to 0} \sum_{j=0}^{n-1} \Theta_{t_j,t_{j+1}}.$$

In the following, we will refer to $\mathcal{I}$ as the nonlinear Young integral *of f with the driver X and denote it by $\mathcal{I}^X(f)$.*

(iii) (persistence of regularity). *Let $k \geq 2$ and $V_0 \hookrightarrow V$ be a Banach subspace of V. In addition, suppose that $X \in C^\gamma_{2,T}\mathrm{Lip}_k(V, V_0; V_0)$, where $\mathrm{Lip}_k(V, V_0; V_0)$ is as in (2.7) and that $f \in C^\alpha([0, T]; V_0)$ such that $f(0) \in V_0$. Then, we have*

$$\Theta \in C^\gamma_{2,T}V_0 \qquad \text{and} \qquad \delta\Theta \in C^\zeta_{3,T}V_0. \tag{3.19}$$

Furthermore, we have $\mathcal{I} \in C^\gamma([0, T]; V_0)$, satisfying

$$\|\delta\mathcal{I} - \Theta\|_{C^\zeta_{2,T}V_0} \leq C_3(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)}, \tag{3.20}$$

$$\|\mathcal{I}\|_{C^\gamma_T V_0} \leq C_4(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)}, \tag{3.21}$$

where $C_3(f)$ and $C_4(f)$ are given by

$$\begin{aligned} C_3(f) = \frac{1}{2^\zeta - 2}\Big(&\big(1 + 2\|f\|_{L^\infty_T V}\big)^{k-1}\|f\|_{C^\alpha_T V_0} \\ &+ \big(1 + 2\|f\|_{L^\infty_T V}\big)^{k-2}\|f\|_{C^\alpha_T V}\big(1 + 2\|f\|_{L^\infty_T V_0}\big)\Big), \\ C_4(f) = (1 \vee T)^\alpha C_3 + 2&\big(1 + \|f\|_{L^\infty_T V}\big)^{k-1}\big(1 + \|f\|_{L^\infty_T V_0}\big). \end{aligned} \tag{3.22}$$

(iv) (convergence). *Suppose in addition that a sequence $\{X^N\}_{N\in\mathbb{N}} \subset C^\gamma_{2,T}\mathrm{Lip}_k(V)$ with $X^N(0) = 0$ and $\delta X^N = 0$ converges to X in $C^\gamma_{2,T}\mathrm{Lip}_k(V)$ as $N \to \infty$. Then, given any $f \in \mathcal{C}^\alpha([0, T]; V)$, the nonlinear Young integral $\mathcal{I}^{X^N}(f)$ of f with the driver X^N converges to the nonlinear Young integral $\mathcal{I}^X(f)$ of f with the driver X in $\mathcal{C}^\gamma([0, T]; V)$ as $N \to \infty$. Moreover, given any $r > 0$, the rate of convergence is uniform in $f \in B_r$, where B_r denotes the ball in $\mathcal{C}^\alpha([0, T]; V)$ of radius $r > 0$ centered at the origin.*

Remark 3.5. The nonlinear Young integral $\mathcal{I}^X(f)$ of f with a driver X is sometimes denoted by

$$\int_0^t X_{dt'}(f(t')).$$

See, for example, [34].

Remark 3.6. Let X be as in Lemma 3.4. Suppose in addition that $X \in C^\gamma_{2,T}\mathrm{Lip}_k(V; V_0)$, where $V_0 \hookrightarrow V$ is a Banach subspace of V. Then, given $f \in \mathcal{C}^\alpha([0, T]; V)$ for some $0 < \alpha < 1$ such that $\zeta = \alpha + \gamma > 1$, it follows from a straightforward modification of the proof of Lemma 3.4 that $\mathcal{I}^X(f) \in C^\gamma([0, T]; V_0)$ with a bound

$$\|\mathcal{I}^X(f)\|_{C^\gamma_T V_0} \leq C_2(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V;V_0)},$$

where $C_2(f)$ is as in (3.17).

Proof of Lemma 3.4. (i) By the definition (3.13) of Θ, (3.11), and (2.6), we have

$$\begin{aligned}\|\Theta_{t,r}\|_V &= \|X_{t,r}(f_r) - X_{t,r}(0)\|_V \\ &\le \|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t-r|^\gamma\big(1+\|f_r\|_V\big)^{k-1}\|f_r\|_V \\ &\le A_1(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t-r|^\gamma,\end{aligned} \tag{3.23}$$

for any $(t,r)\in\Delta_{2,T}$, where $A_1(f)$ is given by

$$A_1(f) = \big(1+\|f\|_{L^\infty_T V}\big)^{k-1}\|f\|_{L^\infty_T V} < \infty. \tag{3.24}$$

This proves $\Theta\in C^\gamma_{2,T}V$, since $X\in C^\gamma_{2,T}\mathrm{Lip}_k(V)$.

From (3.13) and (3.12), we have

$$\begin{aligned}(\delta\Theta)_{t_1,t_2,t_3} &= X_{t_1,t_3}(f_{t_3}) - X_{t_1,t_2}(f_{t_2}) - X_{t_2,t_3}(f_{t_3}) \\ &= X_{t_1,t_2}(f_{t_3}) - X_{t_1,t_2}(f_{t_2}),\end{aligned} \tag{3.25}$$

for any $(t_1,t_2,t_3)\in\Delta_{3,T}$. Then, from (2.6) and the α-Hölder continuity of f, we have

$$\begin{aligned}\|(\delta\Theta)_{t_1,t_2,t_3}\|_V &\le \|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t_1-t_2|^\gamma\big(1+\|f_{t_2}\|_V+\|f_{t_3}\|_V\big)^{k-1} \\ &\quad\times\|f_{t_2}-f_{t_3}\|_V \\ &\le A_2(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t_1-t_2|^\gamma|t_2-t_3|^\alpha,\end{aligned} \tag{3.26}$$

where $A_2(f)$ is given by

$$A_2(f) = \big(1+2\|f\|_{L^\infty_T V}\big)^{k-1}\|f\|_{C^\alpha_T V}. \tag{3.27}$$

Consequently, from (2.4) and $\zeta=\alpha+\gamma$, we have

$$\|\delta\Theta\|_{C^\zeta_{3,T}V} \le A_2\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)} < \infty, \tag{3.28}$$

yielding $\delta\Theta\in C^\zeta_{3,T}V$.

(ii) In view of (i) with $\zeta=\alpha+\gamma>1$, we apply the sewing lemma (Lemma 3.3) and conclude that there exists $\mathcal{I}\in C([0,T];V)$ such that $\delta\mathcal{I} = (\mathrm{Id}-\Lambda\delta)\Theta$. Moreover, from (3.14) and (3.9), we have

$$\|\delta\mathcal{I}-\Theta\|_{C^\zeta_{2,T}V} = \|\Lambda(\delta\Theta)\|_{C^\zeta_{2,T}V} \le \frac{1}{2^\zeta-2}\|\delta\Theta\|_{C^\zeta_{3,T}V}. \tag{3.29}$$

Hence, by setting

$$C_1(f) = \frac{1}{2^\zeta-2}A_2(f), \tag{3.30}$$

the bound (3.15) follows from (3.29) and (3.28).

From (3.15), we have

$$\|\mathcal{I}_t-\mathcal{I}_r-\Theta_{t,r}\|_V \le C_1(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t-r|^\zeta, \tag{3.31}$$

for any $(t,r)\in\Delta_{2,T}$. Then, from (3.31), (3.23), and the triangle inequality, we conclude that $\mathcal{I}\in C^\gamma_T V$, satisfying the bound (3.16). Lastly, we note that the identity (3.18) follows from (3.10).

(iii) Proceeding as in (3.23) with (2.8), we have

$$\begin{aligned}\|\Theta_{t,r}\|_{V_0} &= \|X_{t,r}(f_r) - X_{t,r}(0)\|_{V_0}\\ &\le A_3(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)}|t-r|^\gamma\end{aligned} \tag{3.32}$$

for any $(t,r)\in\Delta_{2,T}$, where $A_3 = A_3(f)$ is given by

$$A_3(f) = 2\big(1+\|f\|_{L^\infty_T V}\big)^{k-1}\big(1+\|f\|_{L^\infty_T V_0}\big) < \infty.$$

This proves $\Theta\in C^\gamma_{2,T}V_0$, since $X\in C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)$. Proceeding as in (3.26) with (3.25) and (2.8), we have

$$\begin{aligned}\|(\delta\Theta)_{t_1,t_2,t_3}\|_{V_0} &\le \|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)}|t_1-t_2|^\gamma\\ &\quad\times\Big\{\big(1+\|f_{t_2}\|_V+\|f_{t_3}\|_V\big)^{k-1}\|f_{t_2}-f_{t_3}\|_{V_0}\\ &\qquad+\big(1+\|f_{t_2}\|_V+\|f_{t_3}\|_V\big)^{k-2}\\ &\qquad\quad\times\big(1+\|f_{t_2}\|_{V_0}+\|f_{t_3}\|_{V_0}\big)\|f_{t_2}-f_{t_3}\|_V\Big\}\\ &\le A_4(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)}|t_1-t_2|^\gamma|t_2-t_3|^\alpha,\end{aligned}$$

where $A_4(f)$ is given by

$$\begin{aligned}A_4(f) &= \big(1+2\|f\|_{L^\infty_T V}\big)^{k-1}\|f\|_{C^\alpha_T V_0}\\ &\quad+\big(1+2\|f\|_{L^\infty_T V}\big)^{k-2}\|f\|_{C^\alpha_T V}\big(1+2\|f\|_{L^\infty_T V_0}\big).\end{aligned}$$

Consequently, from (2.4) and $\zeta=\alpha+\gamma$, we have

$$\|\delta\Theta\|_{C^\zeta_{3,T}V_0} \le A_4(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)} < \infty, \tag{3.33}$$

yielding $\delta\Theta\in C^\zeta_{3,T}V_0$. This proves (3.19). Then, from the sewing lemma (Lemma 3.3), we obtain $\mathcal{I}\in C([0,T];V_0)$. Proceeding as in (3.29), the bound (3.20) with $C_3(f) = \frac{1}{2^\zeta-2}A_4(f)$ follows from (3.9) and (3.33). From (3.20), we have

$$\|\mathcal{I}_t - \mathcal{I}_r - \Theta_{t,r}\|_{V_0} \le C_3(f)\|X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V,V_0;V_0)}|t-r|^\zeta, \tag{3.34}$$

for any $(t,r)\in\Delta_{2,T}$. Then, the bound (3.21) follows from (3.34), (3.32), and the triangle inequality.

(iv) Let $\mathcal{I}^N(f) = \mathcal{I}^{X^N}(f)$, $N\in\mathbb{N}$, and $\mathcal{I}(f) = \mathcal{I}^X(f)$ be the nonlinear Young integrals of f associated with the drivers X^N and X, respectively, constructed in Part (i). Then, from (3.14) and the linearity of the sewing map Λ, we have

$$\delta\mathcal{I}^N(f) - \delta\mathcal{I}(f) = \Theta^N - \Theta - \Lambda(\delta\Theta^N - \delta\Theta), \tag{3.35}$$

where Θ is as in (3.13) and Θ^N is given by

$$\Theta^N_{t,r} = X^N_{t,r}(f_r), \quad (t,r)\in\Delta_{2,T}. \tag{3.36}$$

Proceeding as in (3.23), we have

$$\|\Theta^N_{t,r} - \Theta_{t,r}\|_V \le A_1(f)\|X^N - X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t-r|^\gamma, \tag{3.37}$$

where $A_1(f)$ is as in (3.24). On the other hand, from (3.9) in Lemma 3.3, we have

$$\begin{aligned}\|(\Lambda(\delta\Theta^N - \delta\Theta))_{t,r}\|_V &\le |t-r|^\zeta \|\Lambda(\delta\Theta^N - \delta\Theta)\|_{C^\zeta_{2,T}V}\\ &\le \frac{1}{2^\zeta - 2}|t-r|^\zeta \|\delta\Theta^N - \delta\Theta\|_{C^\zeta_{3,T}V}.\end{aligned} \tag{3.38}$$

Proceeding as in (3.26) with (3.36) and (3.25) with (3.30), we have

$$\begin{aligned}&\|\delta\Theta^N_{t_1,t_2,t_3} - \delta\Theta_{t_1,t_2,t_3}\|_V\\ &\quad = \big\|(X^N_{t_1,t_2} - X_{t_1,t_2})(f_{t_3}) - (X^N_{t_1,t_2} - X_{t_1,t_2})(f_{t_2})\big\|_V\\ &\quad \le A_2(f)\|X^N - X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t_1-t_2|^\gamma |t_2-t_3|^\alpha,\end{aligned} \tag{3.39}$$

where $A_2(f)$ is as in (3.27). Thus, from (3.38) and (3.39), we obtain

$$\|(\Lambda(\delta\Theta^N - \delta\Theta))_{t,r}\|_V \le C_1(f)\|X^N - X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}|t-r|^\zeta. \tag{3.40}$$

Hence, from (3.35), (3.37), and (3.40), we have

$$\begin{aligned}&\|\delta(\mathcal{I}^N(f))_{t,r} - \delta(\mathcal{I}(f))_{t,r}\|_V\\ &\quad \le \Big(A_1(f)|t-r|^\gamma + C_1(f)|t-r|^\zeta\Big)\|X^N - X\|_{C^\gamma_{2,T}\mathrm{Lip}_k(V)}.\end{aligned} \tag{3.41}$$

from which we conclude that $\mathcal{I}^N(f)$ converges to $\mathcal{I}(f)$ in $\mathcal{C}^\gamma([0,T];V)$.

Given $r>0$, it follows from (3.24) and (3.27) that $A_1(f)$ and $A_2(f)$ are uniformly bounded for $f \in B_r \subset \mathcal{C}^\alpha([0,T];V)$. Then, together with (3.41), we conclude that the rate of convergence is uniform in $f \in B_r$. □

In the following proposition, we summarize the properties of the nonlinear Young integral $\mathcal{I}^X(\mathbf{u})$ for a driver $X \in \mathcal{X}^{s,\gamma}_k$ and $\mathbf{u} \in \mathcal{C}^\alpha([0,T];H^s(\mathcal{M}))$.

Proposition 3.7. *Given $s\in\mathbb{R}$, $0<\gamma<1$, $k\in\mathbb{N}$, and $T>0$, let $X \in \mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$, where $\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$ is defined in (2.9) and (2.11). Let $0<\alpha<1$ such that $\zeta = \alpha+\gamma>1$.*
(i) *Let $\mathbf{u}\in\mathcal{C}^\alpha([0,T];H^s(\mathcal{M}))$. Then, the nonlinear Young integral $\mathcal{I}^X(\mathbf{u})$ with the driver X exists as the unique function $\mathcal{I}^X(\mathbf{u}) \in C^\gamma([0,T];H^s(\mathcal{M}))$ with $\mathcal{I}^X(\mathbf{u})(0)=0$ such that*

$$\delta\mathcal{I}^X(\mathbf{u}) = (\mathrm{Id} - \Lambda\delta)\Theta, \tag{3.42}$$

where Θ is given by

$$\Theta_{t,r} = X_{t,r}(\mathbf{u}(r)), \quad (t,r)\in\Delta_{2,T}.$$

Moreover, we have

$$\|\delta\mathcal{I}^X(\mathbf{u}) - \Theta\|_{C^\zeta_{2,T}H^s_x} \le C_1(\mathbf{u})\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}, \tag{3.43}$$

$$\|\mathcal{I}^X(\mathbf{u})\|_{C^\gamma_T H^s_x} \le C_2(\mathbf{u})\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}, \tag{3.44}$$

where $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ are as in (3.17) with $V = H^s(\mathcal{M})$, and

$$(\delta\mathcal{I}^X(\mathbf{u}))_{t,r} = \lim_{|\Pi([r,t])|\to 0}\sum_{j=0}^{n-1}\Theta_{t_j,t_{j+1}}$$

for any $(t,r)\in\Delta_{2,T}$.
(ii) *Suppose that a sequence $\{X^N\}_{N\in\mathbb{N}} \subset \mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$ converges to X in $\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$ as $N\to\infty$. Then, given any $\mathbf{u}\in\mathcal{C}^\alpha([0,T];H^s(\mathcal{M}))$, the nonlinear Young integral $\mathcal{I}^{X^N}(\mathbf{u})$*

of $\mathbf{u}$ *with the driver* X^N *converges to the nonlinear Young integral* $\mathcal{I}^X(\mathbf{u})$ *of* $\mathbf{u}$ *with the driver* X *in* $\mathcal{C}^\gamma([0,T];H^s(\mathcal{M}))$ *as* $N\to\infty$. *Moreover, given any* $r>0$, *the rate of convergence is uniform in* $\mathbf{u}\in B_r$ *where* B_r *denotes the ball in* $\mathcal{C}^\alpha([0,T];H^s(\mathcal{M}))$ *of radius* $r>0$ *centered at the origin.*

Proof. By noting from (2.9) and (2.10) (see also (2.11)) that $\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M}) \subset C^\gamma_{2,T}\mathrm{Lip}_k(H^s(\mathcal{M}))$, the claims follow from Lemma 3.4 with $V=H^s(\mathcal{M})$. □

3.3. **Young differential equation.** Our goal in this paper is to study the following YDE:

$$\mathbf{u}(t) = u_0 + \mathcal{I}^X(\mathbf{u})(t), \tag{3.45}$$

where $\mathcal{I}^X(\mathbf{u})$ is the nonlinear Young integral of $\mathbf{u}$ with a driver X of the form (3.3). In the following proposition, we establish local well-posedness and related properties of the YDE (3.45) in the abstract setting with a driver $X\in\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$. We refer interested readers to [65] for a recent review on nonlinear YDEs. In Sections 4 and 5, by verifying the hypotheses in Lemma 3.1, we apply Proposition 3.8 to prove local well-posedness and other related properties of modulated dispersive equations.

Proposition 3.8. (i) (local well-posedness). *Given* $s\in\mathbb{R}$, $\frac12<\gamma<1$, $k\in\mathbb{N}$, *and* $T\geq 1$, *let* $X\in\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$, *where* $\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$ *is defined in* (2.9) *and* (2.11). *Then, the Young differential equation* (3.45) *with the driver* X *is locally well-posed in* $H^s(\mathcal{M})$. *More precisely, given* $u_0\in H^s(\mathcal{M})$, *there exist* $C_0>0$ *and* $\theta>0$, *independent of* u_0 *and* X, *and a unique solution* $\mathbf{u}\in\mathcal{C}^\gamma([0,\tau];H^s(\mathcal{M}))$ *to* (3.45) *with* $\mathbf{u}|_{t=0}=u_0$, *where the local existence time* $\tau=\tau\big(\|u_0\|_{H^s(\mathcal{M})},\|X\|_{\mathcal{X}_k^{s,\gamma}(T)}\big)\in(0,1]$ *satisfies*

$$\tau\geq C_0\Big(\big(1+\|X\|_{\mathcal{X}_k^{s,\gamma}(T)}\big)\big(1+\|u_0\|_{H^s(\mathcal{M})}\big)^{k-1}\Big)^{-\theta}. \tag{3.46}$$

Moreover, given $0<\alpha<\gamma$ *with* $\alpha+\gamma>1$, *there exists* $C_\alpha>0$ *such that*

$$\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}\leq C_\alpha\|u_0\|_{H^s}, \tag{3.47}$$

while we have

$$\|\mathbf{u}\|_{C^\gamma_\tau H^s_x}\lesssim\|X\|_{\mathcal{X}_k^{s,\gamma}(T)}(1+\|u_0\|_{H^s})^k. \tag{3.48}$$

(ii) (persistence of regularity). *In addition, suppose that* $u_0\in H^{s_0}(\mathcal{M})$ *for some* $s_0>s$ *and that* $X\in\mathcal{Y}_k^{s,s_0,\gamma}([0,T]\times\mathcal{M})$, *where* $\mathcal{Y}_k^{s,s_0,\gamma}([0,T]\times\mathcal{M})$ *is as in* (2.14). *Then, by possibly making* $\tau>0$ *smaller by a multiplicative constant* (*still satisfying* (3.46); *in particular* $\tau>0$ *depends on the* H^s*-norm of the initial data* u_0 *but not on its* H^{s_0}*-norm*), *we have* $\mathbf{u}\in\mathcal{C}^\gamma([0,\tau];H^{s_0}(\mathcal{M}))$.

(iii) (nonlinear smoothing). *In addition to the hypotheses in Part (i), suppose that* $X\in\mathcal{X}_k^{s,s_0,\gamma}([0,T]\times\mathcal{M})$, *where* $\mathcal{X}_k^{s,s_0,\gamma}([0,T]\times\mathcal{M})$ *is as in* (2.13). *Then, we have* $\mathcal{I}^X(\mathbf{u})\in\mathcal{C}^\gamma([0,\tau];H^{s_0}(\mathcal{M}))$, *where* $\mathbf{u}$ *is the solution to* (3.45) *constructed in Part (i).*

(iv) (convergence). *In addition to the hypotheses in Part (i), suppose that a sequence* $\{X^N\}_{N\in\mathbb{N}}\subset\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$ *converges to* X *in* $\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$ *as* $N\to\infty$. *Then, by possibly making* $\tau>0$ *smaller by a multiplicative constant* (*still satisfying* (3.46)), *the solution* $\mathbf{u}^N$ *to the following Young differential equation*:

$$\mathbf{u}^N(t)=u_0+\mathcal{I}^{X^N}(\mathbf{u}^N)(t) \tag{3.49}$$

converges to the solution $\mathbf{u}$ *to* (3.45) *constructed in Part (i) in* $\mathcal{C}^\gamma([0,\tau];H^s(\mathcal{M}))$ *as* $N\to\infty$. *Moreover, given any* $r>0$, *the rate of convergence of* $\mathbf{u}^N$ *to* $\mathbf{u}$ *is uniform in* $u_0\in B_r$ *where* B_r *denotes the ball in* $H^s(\mathcal{M})$ *of radius* $r>0$ *centered at the origin.*

In Proposition 3.8, we assumed $T\geq 1$ for simplicity. With the same proof, we have the same results even when $0<T<1$, in which case, the local existence time $\tau>0$ needs to satisfy an additional constraint $\tau\leq T$.

Remark 3.9. (i) Suppose that $X\in\mathcal{X}^{s,\gamma}_k(\mathbb{R}_+\times\mathcal{M})$. Then, from (3.46), we obtain the following blowup alternative; let $T_*\in(0,\infty]$ be the maximal time of existence of the solution $\mathbf{u}$ to the YDE (3.45). Then, we have either

$$T_*=\infty \qquad \text{or} \qquad \lim_{t\nearrow T_*}\|\mathbf{u}(t)\|_{H^s}=\infty. \tag{3.50}$$

(ii) In Proposition 3.8, we presented a local well-posedness result for the YDE (3.45) in the non-homogeneous Sobolev space $H^s(\mathcal{M})$. Given $X\in\dot{\mathcal{X}}^{s,\gamma}_k([0,T]\times\mathcal{M})$ defined in (2.12) and $u_0\in\dot{H}^s(\mathcal{M})$, the same proof yields the corresponding local well-posedness result in the homogeneous Sobolev spaces.

(iii) Given $\rho>0$, $\frac{1}{2}<\gamma<1$, and $T>0$, let w and w_n, $n\in\mathbb{N}$, be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Let $X=X^w$ and $X^n=X^{w_n}$ be the associated drivers defined in (3.3) with the nonlinearity of the form (3.4). Suppose that convergence of w_n to w in $\mathcal{W}^{\rho,\gamma}_T$ implies convergence of X^n to X in $\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$. Then, from (the proof of) Proposition 3.8, we see that the solution map:

$$(u_0,w)\in H^s(\mathcal{M})\times\mathcal{W}^{\rho,\gamma}_T\longmapsto\mathbf{u}\in\mathcal{C}^\gamma([0,\tau];H^s(\mathcal{M}))$$

is continuous. In our application, convergence of w_n to w in $\mathcal{W}^{\rho,\gamma}_T$ indeed implies convergence of the corresponding driver X^n to X in $\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$. See Remark 4.2.

Proof of Proposition 3.8. (i) Let $0<\alpha<\gamma$ such that $\alpha+\gamma>1$. Given $0<\tau\leq 1$ (to be chosen later), define the map $\Gamma=\Gamma_{X,u_0}$ on $\mathcal{C}^\alpha([0,\tau];H^s(\mathcal{M}))$ by setting

$$\Gamma(\mathbf{u})(t)=u_0+\mathcal{I}^X(\mathbf{u})(t), \tag{3.51}$$

where $\mathcal{I}^X(\mathbf{u})$ is the nonlinear Young integral of $\mathbf{u}$ with a driver $X\in\mathcal{X}^{s,\gamma}_k([0,T]\times\mathcal{M})$, constructed in Proposition 3.7. We show that Γ is a contraction on the ball $B_R\subset\mathcal{C}^\alpha([0,\tau];H^s(\mathcal{M}))$ of radius $R>0$ (to be chosen later) centered at the origin.

Recalling that $\alpha+\gamma>1$, it follows from (3.44) in Proposition 3.7 with (3.17) that

$$\|\mathcal{I}^X(\mathbf{u})(t)-\mathcal{I}^X(\mathbf{u})(r)\|_{H^s}\lesssim\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}|t-r|^\gamma\big(1+\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}\big)^{k-1}\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}$$

for any $0\leq r<t\leq\tau$. Thus, we obtain

$$\|\mathcal{I}^X(\mathbf{u})\|_{C^\alpha_\tau H^s_x}\lesssim\tau^{\gamma-\alpha}\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}\big(1+\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}\big)^{k-1}\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}. \tag{3.52}$$

Hence, from (3.51) and (3.52) with (2.3) and $\mathcal{I}^X(\mathbf{u})(0)=0$, we have

$$\|\Gamma(\mathbf{u})\|_{\mathcal{C}^\alpha_\tau H^s_x}\leq\|u_0\|_{H^s}+C\tau^{\gamma-\alpha}\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}\big(1+\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}\big)^{k-1}\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x} \tag{3.53}$$

for any $0 < \tau \leq 1$. Then, by setting[17]

$$R = 2\|u_0\|_{H^s} \tag{3.54}$$

and choosing $\tau = \tau\big(R, \|X\|_{\mathcal{X}_k^{s,\gamma}(T)}\big) = \tau\big(\|u_0\|_{H^s}, \|X\|_{\mathcal{X}_k^{s,\gamma}(T)}\big) > 0$ sufficiently small, we see that Γ maps the ball $B_R \subset \mathcal{C}^\alpha([0,\tau]; H^s(\mathcal{M}))$ of radius $R > 0$ centered at the origin into itself.

Next, we show that Γ is a contraction. Given $\mathbf{u}^1, \mathbf{u}^2 \in B_R$, let

$$\Theta^1_{t,r} = X_{t,r}(\mathbf{u}^1_r) \qquad \text{and} \qquad \Theta^2_{t,r} = X_{t,r}(\mathbf{u}^2_r). \tag{3.55}$$

Then, from (3.51), we have

$$\|\Gamma(\mathbf{u}^1) - \Gamma(\mathbf{u}^2)\|_{\mathcal{C}^\alpha_\tau H^s_x} = \|\mathcal{I}^X(\mathbf{u}^1) - \mathcal{I}^X(\mathbf{u}^2)\|_{\mathcal{C}^\alpha_\tau H^s_x} \tag{3.56}$$

for any $0 < \tau \leq 1$. From (3.42) and the linearity of Λ, we have

$$\begin{aligned} &\|\delta(\mathcal{I}^X(\mathbf{u}^1))_{t,r} - \delta(\mathcal{I}^X(\mathbf{u}^2))_{t,r}\|_{H^s} \\ &\quad \leq \|\Theta^1_{t,r} - \Theta^2_{t,r}\|_{H^s} + \|(\Lambda(\delta\Theta^1 - \delta\Theta^2))_{t,r}\|_{H^s} \end{aligned} \tag{3.57}$$

for any $0 \leq r < t \leq \tau$. From (3.55) and (2.6), we have

$$\begin{aligned} \|\Theta^1_{t,r} - \Theta^2_{t,r}\|_{H^s} &\leq \|X\|_{\mathcal{X}_k^{s,\gamma}(T)} |t-r|^\gamma \\ &\quad \times \big(1 + \|\mathbf{u}^1_r\|_{H^s} + \|\mathbf{u}^2_r\|_{H^s}\big)^{k-1} \|\mathbf{u}^1_r - \mathbf{u}^2_r\|_{H^s} \\ &\lesssim \|X\|_{\mathcal{X}_k^{s,\gamma}(T)} |t-r|^\gamma (1+R)^{k-1} \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathcal{C}^\alpha_\tau H^s_x} \end{aligned} \tag{3.58}$$

for any $0 \leq r < t \leq \tau$ and $\mathbf{u}^1, \mathbf{u}^2 \in B_R$.

Now, let us estimate the second term on the right-hand side of (3.57). Let $\zeta = \alpha + \gamma > 1$. Then, from (3.9) in Lemma 3.3, we have

$$\begin{aligned} \|(\Lambda(\delta\Theta^1 - \delta\Theta^2))_{t,r}\|_{H^s} &\leq |t-r|^\zeta \|\Lambda(\delta\Theta^1 - \delta\Theta^2)\|_{C^\zeta_{2,\tau} H^s_x} \\ &\lesssim |t-r|^\zeta \|\delta\Theta^1 - \delta\Theta^2\|_{C^\zeta_{3,\tau} H^s_x}. \end{aligned} \tag{3.59}$$

In the following, we first consider the case $k \geq 2$. From (3.55), (3.25), and the fundamental theorem of calculus, we have

$$\begin{aligned} &\|\delta\Theta^1_{t_1,t_2,t_3} - \delta\Theta^2_{t_1,t_2,t_3}\|_{H^s} \\ &\quad = \big\|X_{t_1,t_2}(\mathbf{u}^1_{t_3}) - X_{t_1,t_2}(\mathbf{u}^2_{t_3}) - \big(X_{t_1,t_2}(\mathbf{u}^1_{t_2}) - X_{t_1,t_2}(\mathbf{u}^2_{t_2})\big)\big\|_{H^s} \\ &\quad = \bigg\| \int_0^1 DX_{t_1,t_2}[\lambda \mathbf{u}^1_{t_3} + (1-\lambda)\mathbf{u}^2_{t_3}](\mathbf{u}^1_{t_3} - \mathbf{u}^2_{t_3}) d\lambda \\ &\qquad - \int_0^1 DX_{t_1,t_2}[\lambda \mathbf{u}^1_{t_2} + (1-\lambda)\mathbf{u}^2_{t_2}](\mathbf{u}^1_{t_2} - \mathbf{u}^2_{t_2}) d\lambda \bigg\|_{H^s} \end{aligned} \tag{3.60}$$

[17] In the following, we assume that $u_0 \neq 0$. If $u_0 = 0$, we can simply set $R = 1$ and carry out the contraction argument to show that $\mathbf{u} \equiv 0$ is a unique solution to (3.45).

for any $(t_1, t_2, t_3) \in \Delta_{3,\tau}$, where DX_{t_1,t_2} denotes the Fréchet derivative of X_{t_1,t_2}. Then, from (2.10) with (2.5), we bound the right-hand side of (3.60) by

$$\begin{aligned}
&\bigg\| \int_0^1 DX_{t_1,t_2}[\lambda \mathbf{u}^1_{t_3} + (1-\lambda)\mathbf{u}^2_{t_3}]\big((\delta\mathbf{u}^1)_{t_2,t_3} - (\delta\mathbf{u}^2)_{t_2,t_3}\big) d\lambda \bigg\|_{H^s} \\
&\quad + \bigg\| \int_0^1 \Big(DX_{t_1,t_2}[\lambda \mathbf{u}^1_{t_3} + (1-\lambda)\mathbf{u}^2_{t_3}] \\
&\qquad\qquad - DX_{t_1,t_2}[\lambda \mathbf{u}^1_{t_2} + (1-\lambda)\mathbf{u}^2_{t_2}]\Big)(\mathbf{u}^1_{t_2} - \mathbf{u}^2_{t_2}) d\lambda \bigg\|_{H^s} \\
&\lesssim \|X\|_{\mathcal{X}^{s,\gamma}_k(T)} |t_1 - t_2|^\gamma (1+R)^{k-1} \|(\delta\mathbf{u}^1)_{t_2,t_3} - (\delta\mathbf{u}^2)_{t_2,t_3}\|_{H^s} \\
&\quad + \|X\|_{\mathcal{X}^{s,\gamma}_k(T)} |t_1 - t_2|^\gamma (1+R)^{k-2} \big(\|\delta\mathbf{u}^1_{t_2,t_3}\|_{H^s} + \|\delta\mathbf{u}^2_{t_2,t_3}\|_{H^s}\big) \|\mathbf{u}^1_{t_2} - \mathbf{u}^2_{t_2}\|_{H^s} \\
&\lesssim \|X\|_{\mathcal{X}^{s,\gamma}_k(T)} |t_1 - t_2|^\gamma |t_2 - t_3|^\alpha (1+R)^{k-1} \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathcal{C}^\alpha_\tau H^s_x}
\end{aligned} \tag{3.61}$$

for $\mathbf{u}^1, \mathbf{u}^2 \in B_R \subset \mathcal{C}^\alpha([0,\tau]; H^s(\mathcal{M}))$. Thus, from (3.60) and (3.61) with $\zeta = \alpha + \gamma > 1$, we obtain

$$\|\delta(\Theta^1 - \Theta^2)\|_{C^\zeta_{3,\tau} H^s_x} \lesssim \|X\|_{\mathcal{X}^{s,\gamma}_k(T)} (1+R)^{k-1} \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathcal{C}^\alpha_\tau H^s_x}. \tag{3.62}$$

When $k = 1$, the conclusion (3.62) follows from the second line of (3.60) and the linearity of X_{t_1,t_2}.

Hence, putting (3.56), (3.57), (3.58), (3.59), and (3.62) together, we obtain

$$\begin{aligned}
\|\Gamma(\mathbf{u}^1) - \Gamma(\mathbf{u}^2)\|_{\mathcal{C}^\alpha_\tau H^s_x} &= \|\mathcal{I}^X(\mathbf{u}^1) - \mathcal{I}^X(\mathbf{u}^2)\|_{\mathcal{C}^\alpha_\tau H^s_x} \\
&\lesssim \tau^{\gamma-\alpha} \|X\|_{\mathcal{X}^{s,\gamma}_k(T)} (1+R)^{k-1} \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathcal{C}^\alpha_\tau H^s_x}.
\end{aligned} \tag{3.63}$$

Then, by choosing $\tau = \tau\big(R, \|X\|_{\mathcal{X}^{s,\gamma}_k(T)}\big) = \tau\big(\|u_0\|_{H^s}, \|X\|_{\mathcal{X}^{s,\gamma}_k(T)}\big) > 0$ sufficiently small, we conclude that Γ is a contraction on B_R. The bound (3.46) follows from (3.53) and (3.63).

By the Banach fixed point theorem, there exists a unique fixed point $\mathbf{u} \in B_R$ such that

$$\mathbf{u}(t) = \Gamma(\mathbf{u})(t) = u_0 + \mathcal{I}^X(\mathbf{u})(t). \tag{3.64}$$

Namely, $\mathbf{u}$ is a solution to the YDE (3.45). Here, the uniqueness a priori holds only in the ball B_R but, by a standard continuity argument, we can extend uniqueness to the entire space $\mathcal{C}^\alpha([0,\tau]; H^s(\mathcal{M}))$ (by possibly making τ smaller by a multiplicative constant). Moreover, from Proposition 3.7, we know that the right-hand side of (3.64) belongs to $\mathcal{C}^\gamma([0,\tau]; H^s(\mathcal{M}))$, from which we conclude that $\mathbf{u} \in \mathcal{C}^\gamma([0,\tau]; H^s(\mathcal{M}))$.

Finally the bound (3.47) follows from the fact that $\mathbf{u} \in B_R$ with (3.54), while the bound (3.48) follows from (3.64), (3.44), and (3.17) with the fact that $\mathbf{u} \in B_R$ and (3.54).

(ii) In Part (i), we constructed a solution to (3.45) on the time interval $[0,\tau]$ with the bound

$$\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x} \le R.$$

Proceeding as in (3.52) and (3.53) with (3.64), (3.21), (3.22), and (2.14), we have

$$\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^{s_0}_x} \le \|u_0\|_{H^{s_0}} + C\tau^{\gamma-\alpha} \|X\|_{\mathcal{Y}^{s,s_0,\gamma}_k(T)} (1+R)^{k-1} \|\mathbf{u}\|_{C^\alpha_\tau H^{s_0}_x}.$$

Then, by possibly making $\tau > 0$ smaller (by a multiplicative constant, and hence (3.46) still holds), we conclude that $\mathbf{u} \in \mathcal{C}^\alpha([0,\tau]; H^{s_0}(\mathcal{M}))$. Then, from Lemma 3.4 (iii), we obtain $\mathcal{I}^X(\mathbf{u}) \in \mathcal{C}^\gamma([0,\tau]; H^{s_0}(\mathcal{M}))$ and thus $\mathbf{u} \in \mathcal{C}^\gamma([0,\tau]; H^{s_0}(\mathcal{M}))$ in view of (3.64).

(iii) Let $\mathbf{u} \in \mathcal{C}^\gamma([0,\tau]; H^s(\mathcal{M}))$ be the solution constructed in Part (i). Then, since $2\gamma > 1$, it follows from Remark 3.6 that $\mathcal{I}^X(\mathbf{u}) \in \mathcal{C}^\gamma([0,\tau]; H^{s_0}(\mathcal{M}))$.

(iv) For simplicity, we assume that $X \neq 0$. We first note that the convergence of X^N to X in $\mathcal{X}_k^{s,\gamma}([0,T]\times\mathcal{M})$ implies that there exists $N_0 \in \mathbb{N}$ such that

$$\|X^N\|_{\mathcal{X}_k^{s,\gamma}(T)} \leq 2\|X\|_{\mathcal{X}_k^{s,\gamma}(T)} \tag{3.65}$$

for any $N \geq N_0$. Thus, by making τ smaller by a multiplicative constant, Part (i) guarantees existence of a unique solution $\mathbf{u}^N \in \mathcal{C}^\gamma([0,\tau]; H^s(\mathcal{M}))$ to (3.49) for each $N \geq N_0$. Moreover, we have

$$\|\mathbf{u}^N\|_{\mathcal{C}_\tau^\alpha H_x^s}, \|\mathbf{u}\|_{\mathcal{C}_\tau^\alpha H_x^s} \leq R \tag{3.66}$$

for any $N \geq N_0$, where R is as in Part (i).

From (3.45) and (3.49), we have

$$\begin{aligned} \mathbf{u}(t) - \mathbf{u}^N(t) &= \mathcal{I}^X(\mathbf{u})(t) - \mathcal{I}^{X^N}(\mathbf{u}^N)(t) \\ &= \Big(\mathcal{I}^X(\mathbf{u})(t) - \mathcal{I}^{X^N}(\mathbf{u})(t)\Big) + \Big(\mathcal{I}^{X^N}(\mathbf{u})(t) - \mathcal{I}^{X^N}(\mathbf{u}^N)(t)\Big). \end{aligned} \tag{3.67}$$

From Proposition 3.7 (ii), we have

$$\lim_{N\to\infty} \|\mathcal{I}^X(\mathbf{u}) - \mathcal{I}^{X^N}(\mathbf{u})\|_{\mathcal{C}_\tau^\gamma H_x^s} = 0. \tag{3.68}$$

By applying (3.57), (3.58), (3.59), and (3.62) with (3.65) and (3.66), we obtain

$$\|\mathcal{I}^{X^N}(\mathbf{u}) - \mathcal{I}^{X^N}(\mathbf{u}^N)\|_{\mathcal{C}_\tau^\alpha H_x^s} \lesssim \tau^{\gamma-\alpha}\|X\|_{\mathcal{X}_k^{s,\gamma}(T)}(1+R)^{k-1}\|\mathbf{u} - \mathbf{u}^N\|_{\mathcal{C}_\tau^\alpha H_x^s} \tag{3.69}$$

for any $N \geq N_0$. Hence, by possibly taking $\tau > 0$ smaller (by a multiplicative constant), it follows from (3.67), (3.68), and (3.69) that

$$\|\mathbf{u} - \mathbf{u}^N\|_{\mathcal{C}_\tau^\alpha H_x^s} \lesssim \|\mathcal{I}^X(\mathbf{u}) - \mathcal{I}^{X^N}(\mathbf{u})\|_{\mathcal{C}_\tau^\gamma H_x^s} \longrightarrow 0, \tag{3.70}$$

as $N \to \infty$. This proves convergence of $\mathbf{u}^N$ to $\mathbf{u}$ in $\mathcal{C}^\alpha([0,\tau]; H^s(\mathcal{M}))$.

We now upgrade this convergence to that in $\mathcal{C}^\gamma([0,\tau]; H^s(\mathcal{M}))$. By applying (3.57), (3.58), (3.59), and (3.62) with (3.65) and (3.66) once again, we have

$$\|\mathcal{I}^{X^N}(\mathbf{u}) - \mathcal{I}^{X^N}(\mathbf{u}^N)\|_{\mathcal{C}_\tau^\gamma H_x^s} \lesssim \|X\|_{\mathcal{X}_k^{s,\gamma}(T)}(1+R)^{k-1}\|\mathbf{u} - \mathbf{u}^N\|_{\mathcal{C}_\tau^\alpha H_x^s}. \tag{3.71}$$

Then, the claimed convergence of $\mathbf{u}^N$ to $\mathbf{u}$ in $\mathcal{C}^\gamma([0,\tau]; H^s(\mathcal{M}))$ follows from (3.67), (3.68), and (3.71) with (3.70). Moreover, given $r > 0$, the uniform convergence rate for $\|u_0\|_{H^s} \leq r$ follows from the fact that the rate of the convergence (3.68) is uniform in $\|u_0\|_{H^s} \leq r$ (Proposition 3.7 (ii)) and the bound (3.66) which is uniform with respect to the initial data u_0 with $\|u_0\|_{H^s} \leq r$.

This completes the proof of Proposition 3.8. □

4. Local well-posedness of the modulated KdV and mKdV

In this section, we prove local well-posedness and related properties (persistence of regularity, nonlinear smoothing, and convergence of the Galerkin approximation) of the modulated KdV-type equations:

- Theorem 1.3 (i) and Theorem 1.16 for the modulated KdV (1.2) on the circle,
- Theorems 1.10 and 1.17 for the modulated mKdV (1.13) on the circle,

- Theorem 1.7 (i) for the modulated KdV (1.2) on the real line.

These results follow from Proposition 3.8, once we verify the conditions in Lemma 3.1 for each of the equations.

4.1. **Modulated KdV equation on the circle.** We consider the modulated KdV (1.2) on $\mathbb{T}$. More precisely, as we discussed in Section 1, we consider the following equation satisfied by the modulated interaction representation $\mathbf{u}$ of u defined in (1.7):

$$\partial_t \mathbf{u} = U^w(t)^{-1}\partial_x\big((U^w(t)\mathbf{u})^2\big), \tag{4.1}$$

where $U^w(t) = e^{-w(t)\partial_x^3}$ denotes the modulated linear propagator for (1.2). By writing (4.1) (with initial data u_0) in the integral form, we have

$$\mathbf{u}(t) = u_0 + \int_0^t U^w(t')^{-1}\partial_x\big((U^w(t')\mathbf{u}(t'))^2\big)dt'. \tag{4.2}$$

Then, the bilinear driver $X = X^{\mathrm{KdV}}$ associated with the modulated KdV is given by

$$X_{t,r}(f_1, f_2) = X^{\mathrm{KdV}}_{t,r}(f_1, f_2) = \int_r^t U^w(t')^{-1}\partial_x\big((U^w(t')f_1)(U^w(t')f_2)\big)dt' \tag{4.3}$$

for functions f_1 and f_2 on $\mathbb{T}$. By taking the Fourier transform, we have

$$\mathcal{F}\big(X_{t,r}(f_1, f_2)\big)(n) = in \sum_{\substack{n_1,n_2\in\mathbb{Z}^*\\ n=n_1+n_2}} \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2), \tag{4.4}$$

where $\Phi^w_{t,r}$ is as in (1.6) and $\Xi_{\mathrm{KdV}}(\bar{n})$ denotes the *resonance function*[18] for KdV given by

$$\Xi_{\mathrm{KdV}}(\bar{n}) = \Xi_{\mathrm{KdV}}(n, n_1, n_2) = -n^3 + n_1^3 + n_2^3. \tag{4.5}$$

Recall that, under $n = n_1 + n_2$, we have

$$\Xi_{\mathrm{KdV}}(\bar{n}) = -3nn_1n_2. \tag{4.6}$$

The following proposition establishes basic mapping properties of the bilinear driver X defined in (4.3).

Proposition 4.1. *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1.*

(i) *Suppose that $\rho \geq \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy* (1.9). *Then, the driver X defined in* (4.3) *belongs to $\mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{T})$ defined in* (2.9).

(ii) (persistence of regularity). *Suppose that $\rho \geq \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy* (1.9). *Then, for any $s_0 > s$, the driver X belongs to $\mathcal{Y}_2^{s,s_0,\gamma}([0,T]\times\mathbb{T})$ defined in* (2.14).

(iii) (nonlinear smoothing). *In addition, suppose that $s_0 > s$ satisfies* (1.23). *Then, the driver X belongs to $\mathcal{X}_2^{s,s_0,\gamma}([0,T]\times\mathbb{T})$ defined in* (2.13).

[18]Here, we follow the terminology in [150]. We point out that $\Xi_{\mathrm{KdV}}(\bar{n})$ is also called the modulation function (see [105]) but we do not use this latter terminology to avoid confusion with a modulation function $w(t)$.

(iv) (Galerkin approximation). *Suppose that* $\rho \geq \frac{1}{2}$ *and* $s \in \mathbb{R}$ *satisfy* (1.24). *Then, the truncated driver* $X^N = X^{\mathrm{KdV},N}$, *defined by*

$$X^N_{t,r}(f_1, f_2) = \int_r^t U^w(t')^{-1} \mathbf{P}_N \partial_x \big((\mathbf{P}_N U^w(t') f_1)(\mathbf{P}_N U^w(t') f_2) \big) dt', \tag{4.7}$$

converges to X *in* $\mathcal{X}_2^{s,\gamma}([0,T] \times \mathbb{T})$ *as* $N \to \infty$.

Once we prove Proposition 4.1, local well-posedness of the modulated KdV (1.2) on $\mathbb{T}$ and other related properties (Theorem 1.3 (i) and Theorem 1.16) follow from Proposition 3.8 with Proposition 4.1 applied to (4.2). We omit details.

Proof of Proposition 4.1. In the following, we verify the hypotheses in Lemma 3.1 with $k = 2$. Then, the claimed results follow from Lemma 3.1.

(i) From (4.4) and Hölder's inequality, we have

$$\begin{aligned}
\|X_{t,r}(f_1, f_2)\|_{H^s}^2 &\leq \sum_{n \in \mathbb{Z}^*} \langle n \rangle^{2s+2} \bigg| \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ n = n_1 + n_2}} \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n})) \widehat{f_1}(n_1) \widehat{f_2}(n_2) \bigg|^2 \\
&\leq \sum_{n \in \mathbb{Z}^*} \langle n \rangle^{2s+2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ n = n_1 + n_2}} \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))|^2}{\langle n_2 \rangle^{2s}} |\widehat{f_1}(n_1)|^2 \|f_2\|_{H^s(\mathbb{T})}^2 \\
&\leq \bigg(\sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n, n_2 \in \mathbb{Z}^* \\ n_1 = n - n_2}} \langle n \rangle^{2s+2} \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))|^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s}} \bigg) \|f_1\|_{H^s}^2 \|f_2\|_{H^s}^2
\end{aligned} \tag{4.8}$$

for any $0 \leq r < t \leq T$. Then, from (4.8), (1.5), and (4.6), we have

$$\begin{aligned}
\|X_{t,r}\|_{\mathcal{L}_2(H^s)} &\lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t - r|^\gamma \\
&\quad \times \bigg(\sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n, n_2 \in \mathbb{Z}^* \\ n_1 = n - n_2}} \langle n \rangle^{2-4\rho} \frac{\langle n \rangle^{2s+2\rho}}{\langle n_1 \rangle^{2s+2\rho} \langle n_2 \rangle^{2s+2\rho}} \bigg)^{\frac{1}{2}}
\end{aligned} \tag{4.9}$$

for any $0 \leq r < t \leq T$. Without loss of generality, assume that $|n_1| \gtrsim |n_2|$ in (4.9). Then, we have $|n_1| \gtrsim |n|$ under $n = n_1 + n_2$. Thus, we have

$$\sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n, n_2 \in \mathbb{Z}^* \\ n_1 = n - n_2}} \langle n \rangle^{2-4\rho} \frac{\langle n \rangle^{2s+2\rho}}{\langle n_1 \rangle^{2s+2\rho} \langle n_2 \rangle^{2s+2\rho}} \lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{n \in \mathbb{Z}^*} \frac{1}{\langle n \rangle^{4\rho-2} \langle n - n_1 \rangle^{2s+2\rho}}, \tag{4.10}$$

provided that $s + \rho \geq 0$. In view of Lemma 2.1, we see that the right-hand side of (4.10) is finite if

$$4\rho - 2 \geq 0, \quad 2s + 2\rho \geq 0, \quad \text{and} \quad 4\rho - 2 + 2s + 2\rho > 1,$$

Namely,

$$\rho \geq \frac{1}{2}, \quad s \geq -\rho, \quad \text{and} \quad s > \frac{3}{2} - 3\rho, \tag{4.11}$$

which is satisfied under (1.9). Hence, from Lemma 3.1 (i), we conclude that $X \in \mathcal{X}_2^{s,\gamma}([0,T] \times \mathbb{T})$ under (1.9).

(ii) Write $X = X^1 + X^2$, where X^1 is the contribution from $|n_1| \geq |n_2|$ in (4.4) and $X^2 = X - X^1$. We only consider X^1 since a similar bound holds for X^2 by symmetry. Proceeding as in (4.8), (4.9), and (4.10) with $|n_1| \gtrsim |n|$ and (3.8), we have

$$\begin{aligned}\|X^1_{t,r}\|_{\mathcal{L}^{s,s_0}_{2,1}} &\lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma \\ &\quad \times \bigg(\sup_{n_1\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{1}{\langle n\rangle^{4\rho-2}\langle n-n_1\rangle^{2s+2\rho}}\bigg)^{\frac{1}{2}} \\ &\lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma\end{aligned}$$

for any $0 \leq r < t \leq T$, provided that (1.9) holds. Hence, from Lemma 3.1 (ii), we conclude that, if (1.9) holds, then $X \in \mathcal{Y}^{s,s_0,\gamma}_2([0,T]\times\mathbb{T})$ for any $s_0 > s$.

(iii) Without loss of generality, let us assume $|n_1| \gtrsim |n|$ in (4.4). Then, proceeding as in Part (i), we have

$$\begin{aligned}\|X_{t,r}\|_{\mathcal{L}_2(H^s;H^{s_0})} &\lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma \\ &\quad \times \bigg(\sup_{n_1\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{1}{\langle n\rangle^{4\rho-2s_0+2s-2}\langle n-n_1\rangle^{2s+2\rho}}\bigg)^{\frac{1}{2}}\end{aligned} \tag{4.12}$$

for any $0 \leq r < t \leq T$. In view of Lemma 2.1, we see that the right-hand side of (4.12) is finite if

$$s_0 \leq s + 2\rho - 1, \quad s \geq -\rho, \quad \text{and} \quad s_0 < 2s + 3\rho - \frac{3}{2},$$

which is satisfied under (1.23). Hence, from Lemma 3.1 (iii), we conclude that $X \in \mathcal{X}^{s,s_0,\gamma}_2([0,T]\times\mathbb{T})$ under (1.23).

(iv) From (4.4) with (4.7), we have

$$\begin{aligned}&\mathcal{F}\big(X_{t,r}(f_1,f_2)\big)(n) - \mathcal{F}\big(X^N_{t,r}(f_1,f_2)\big)(n) \\ &\quad = in\sum_{\substack{n_1,n_2\in\mathbb{Z}^*\\ n=n_1+n_2}} \mathbf{1}_{\{\max(|n|,|n_1|,|n_2|)>N\}}\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2).\end{aligned}$$

Note that

$$\mathbf{1}_{\{\max(|n|,|n_1|,|n_2|)>N\}} \lesssim N^{-\varepsilon}\max(|n|,|n_1|,|n_2|)^\varepsilon. \tag{4.13}$$

Assume that $|n_1| \gtrsim |n_2|$ as in Part (i) (which implies $|n_1| \sim \max(|n|,|n_1|,|n_2|)$), Then, by proceeding as in (4.9) and (4.10)

$$\begin{aligned}&\|X_{t,r} - X^N_{t,r}\|_{\mathcal{L}_2(H^s)} \\ &\lesssim N^{-\varepsilon}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma\bigg(\sup_{n_1\in\mathbb{Z}^*}\sum_{\substack{n,n_2\in\mathbb{Z}^*\\ n_1=n-n_2}}\langle n\rangle^{2-4\rho+2\varepsilon}\frac{\langle n\rangle^{2s+2\rho-2\varepsilon}}{\langle n_1\rangle^{2s+2\rho-2\varepsilon}\langle n_2\rangle^{2s+2\rho}}\bigg)^{\frac{1}{2}} \\ &\lesssim N^{-\varepsilon}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma\bigg(\sup_{n_1\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{1}{\langle n\rangle^{4\rho-2-2\varepsilon}\langle n-n_1\rangle^{2s+2\rho}}\bigg)^{\frac{1}{2}}.\end{aligned} \tag{4.14}$$

for any $0 \le r < t \le T$, provided that $s + \rho \ge \varepsilon$. Moreover, in view of Lemma 2.1, we see that the last factor on the right-hand side of (4.14) is finite if

$$\rho \ge \frac{1}{2} + \frac{1}{2}\varepsilon, \quad s \ge -\rho, \quad \text{and} \quad s > \frac{3}{2} - 3\rho,$$

which is satisfied under (1.24). Hence, from Lemma 3.1 (iv), we conclude that X^N converges to X in $\mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{T})$ under (1.24) by choosing $\varepsilon > 0$ sufficiently small.

This concludes the proof of Proposition 4.1. □

Remark 4.2. Given $\rho \ge \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w and w_n, $n \in \mathbb{N}$, be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Let $X = X^w$ and $X^n = X^{w_n}$ be the bilinear drivers for the modulated KdV on the circle defined in (4.3) associated with the modulations w and w_n, respectively. Let $s \in \mathbb{R}$ satisfy (1.9) such that Proposition 4.1 (i) guarantees $X, X^n \in \mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{T})$. Now, suppose that w_n converges to w in $\mathcal{W}_T^{\rho,\gamma}$ as $n \to \infty$. Then, a straightforward modification of (4.8) and (4.9) with the finiteness of the sum on the right-hand side of (4.9) under (1.9) yields

$$\|X^n_{t,r} - X_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim \|\Phi^{w_n} - \Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma = o(1)|t-r|^\gamma \tag{4.15}$$

as $n \to \infty$. Hence, we conclude from Lemma 3.1 (iv) that X^n converges to X in $\mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{T})$ with the Lipschitz bound (4.15). Then, from Remark 3.9 (iii) with (3.41) and(4.15), we see that the solution map:

$$(u_0, w) \in H^s(\mathbb{T}) \times \mathcal{W}_T^{\rho,\gamma} \longmapsto \mathbf{u} \in \mathcal{C}^\gamma([0,\tau]; H^s(\mathbb{T}))$$

is in fact locally Lipschitz continuous. A similar comment applies to all the modulated dispersion equations we consider in this paper.

4.2. **Modulated mKdV equation on the circle.** In this subsection, we consider the modulated (renormalized) mKdV (1.14) on $\mathbb{T}$. The renormalized nonlinearity in (1.14) is given by

$$\mathcal{N}(f) = 3\big(f^2 - \|f\|_{L^2}^2\big)\partial_x f. \tag{4.16}$$

By taking the Fourier transform and symmetrization, we have

$$\widehat{\mathcal{N}(f)}(n) = in \sum_{A_n} \widehat{f}(n_1)\widehat{f}(n_2)\widehat{f}(n_3) - 3in|\widehat{f}(n)|^2\widehat{f}(n), \tag{4.17}$$

where

$$A_n = \big\{(n_1,n_2,n_3) \in (\mathbb{Z}^*)^3 : n = n_1 + n_2 + n_3,\ n \ne n_1, n_2, n_3\big\}. \tag{4.18}$$

See [6, Lemma 8.16] and [148, (1.9)]. With a slight abuse of notation, define the trilinear version of $\mathcal{N}(f)$ by setting

$$\begin{aligned}\mathcal{F}\big(\mathcal{N}(f_1,f_2,f_3)\big)(n) = in \sum_{A_n} \widehat{f}_1(n_1)\widehat{f}_2(n_2)\widehat{f}_3(n_3)\\ - 3in\widehat{f}_1(n)\overline{\widehat{f}_2(n)}\widehat{f}_3(n).\end{aligned} \tag{4.19}$$

Then, the trilinear driver associated with the modulated mKdV (1.14) is given by

$$X_{t,r}(f_1,f_2,f_3) = \int_r^t U^w(t')^{-1}\mathcal{N}(U^w(t')f_1, U^w(t')f_2, U^w(t')f_3)dt', \tag{4.20}$$

where $U^w(t) = e^{-w(t)\partial_x^3}$ and $\mathcal{N}$ is as in (4.19). By taking the Fourier transform, we have

$$\begin{aligned}\mathcal{F}\big(X_{t,r}(f_1,f_2,f_3)\big)(n) &= in\sum_{A_n}\Phi^w_{t,r}(\Xi_{\mathrm{mKdV}})\widehat{f_1}(n_1)\widehat{f_2}(n_2)\widehat{f_3}(n_3)\\ &\quad - 3i(t-r)n\widehat{f_1}(n)\overline{\widehat{f_2}(n)}\widehat{f_3}(n)\\ &=: \mathcal{F}\big(X^{(1)}_{t,r}(f_1,f_2,f_3)\big)(n) + \mathcal{F}\big(X^{(2)}_{t,r}(f_1,f_2,f_3)\big)(n),\end{aligned} \tag{4.21}$$

where $\Phi^w_{t,r}(\cdot)$ is as in (1.6) and $X^{(1)}$ corresponds to the non-resonant part of X (namely $\Xi_{\mathrm{mKdV}}(\bar{n}) \neq 0$), while $X^{(2)}$ corresponds to the resonant part of X (namely $\Xi_{\mathrm{mKdV}}(\bar{n}) = 0$). Here, $\Xi_{\mathrm{mKdV}}(\bar{n})$ denotes the resonance function for mKdV given by

$$\Xi_{\mathrm{mKdV}}(\bar{n}) = \Xi_{\mathrm{mKdV}}(n,n_1,n_2,n_3) = -n^3 + n_1^3 + n_2^3 + n_3^3.$$

Recall that, under $n = n_1 + n_2 + n_3$, we have

$$\Xi_{\mathrm{mKdV}}(\bar{n}) = -3(n-n_1)(n-n_2)(n-n_3). \tag{4.22}$$

As in the case of the modulated KdV studied in the previous subsection, once we prove the following proposition on basic mapping properties of the trilinear driver X defined in (4.20), local well-posedness of the modulated mKdV and other related properties (Theorem 1.10 and Theorem 1.17) follow from Proposition 3.8.

Proposition 4.3. *Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1.*

(i) *Suppose that $s \geq \frac{1}{2}$. Then, the driver X defined in* (4.20) *belongs to $\mathcal{X}_3^{s,\gamma}([0,T]\times\mathbb{T})$ defined in* (2.9).

(ii) (persistence of regularity). *Suppose that $s \geq \frac{1}{2}$. Then, for any $s_0 > s$, the driver X belongs to $\mathcal{Y}_3^{s,s_0,\gamma}([0,T]\times\mathbb{T})$ defined in* (2.14).

(iii) (nonlinear smoothing). *In addition, suppose that $s_0 > s \geq \frac{1}{2}$ satisfies* (1.25). *Then, the driver X belongs to $\mathcal{X}_3^{s,s_0,\gamma}([0,T]\times\mathbb{T})$ defined in* (2.13).

(iv) (Galerkin approximation). *Suppose that $\rho > \frac{1}{2}$ and $s > \frac{1}{2}$. Then, the truncated driver X^N, defined by*

$$X^N_{t,r}(f_1,f_2,f_3) = \int_r^t U^w(t')^{-1}\mathbf{P}_N\mathcal{N}(\mathbf{P}_N U^w(t')f_1, \mathbf{P}_N U^w(t')f_2, \mathbf{P}_N U^w(t')f_3)dt', \tag{4.23}$$

converges to X in $\mathcal{X}_3^{s,\gamma}([0,T]\times\mathbb{T})$ as $N\to\infty$, where $\mathcal{N}$ is as in (4.19).

Proof. Let $\rho, s \geq \frac{1}{2}$. As in the proof of Proposition 4.1, we verify the hypotheses in Lemma 3.1 with $k = 3$.

(i) We first consider the resonant part $X^{(2)}$. From (4.21), we have

$$\|X^{(2)}_{t,r}\|_{\mathcal{L}_3(H^s)} \lesssim |t-r|$$

for any $0 \leq r < t \leq T$, provided that $s \geq \frac{1}{2}$, just as in the usual (unmodulated) mKdV case.

Next, we consider the non-resonant part $X^{(1)}$. From (4.21) and Hölder's inequality, we have

$$\begin{aligned}\|X^{(1)}_{t,r}(f_1,f_2,f_3)\|^2_{H^s} &\le \sum_{n\in\mathbb{Z}^*}\langle n\rangle^{2s+2}\bigg|\sum_{A_n}\Phi^w_{t,r}(\Xi_{\mathrm{mKdV}}(\bar n))\prod_{j=1}^3\widehat{f_j}(n_j)\bigg|^2\\ &\lesssim \bigg(\sup_{n\in\mathbb{Z}^*}\sum_{A_n}\frac{\langle n\rangle^{2s+2}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}}|\Phi^w_{t,r}(\Xi_{\mathrm{mKdV}}(\bar n))|^2\bigg)\prod_{j=1}^3\|f_j\|^2_{H^s}.\end{aligned}\tag{4.24}$$

By symmetry, assume that $|n_1|\gtrsim\max(|n_1|,|n_2|,|n_3|)$, which implies $|n_1|\gtrsim|n|$. Then, from (1.5) and (4.22) with $s\ge0$ and (4.18), we have

$$\begin{aligned}\|X^{(1)}_{t,r}\|_{\mathcal{L}_3(H^s)} &\le \bigg(\sup_{n\in\mathbb{Z}^*}\sum_{A_n}\frac{\langle n\rangle^{2}}{\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}}|\Phi^w_{t,r}(\Xi_{\mathrm{mKdV}}(\bar n))|^2\bigg)^{\frac12}\\ &\le A_{\mathrm{mKdV}}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma\end{aligned}\tag{4.25}$$

for any $0\le r<t\le T$, where A_{mKdV} is given by

$$A^2_{\mathrm{mKdV}}=\sup_{n\in\mathbb{Z}^*}\sum_{A_n}\frac{\langle n\rangle^2}{\langle n_2+n_3\rangle^{2\rho}\prod_{j=2}^3\langle n_j\rangle^{2s}\langle n-n_j\rangle^{2\rho}}.$$

Fix small $\varepsilon>0$. From Hölder's inequality, Young's inequality (with $\rho\ge\frac12$), and Lemma 2.1, we have

$$\begin{aligned}A^2_{\mathrm{mKdV}} &\le \sup_{n\in\mathbb{Z}^*}\langle n\rangle^2\bigg\|\frac{1}{\langle n_2\rangle^{2s}\langle n-n_2\rangle^{2\rho}}\bigg\|^2_{\ell^{\frac{2(1+\varepsilon)}{1+2\varepsilon}}_{n_2}}\bigg\|\frac{1}{\langle n_2\rangle^{2\rho}}\bigg\|_{\ell^{1+\varepsilon}_{n_2}}\\ &\lesssim \sup_{n\in\mathbb{Z}^*}\langle n\rangle^2\bigg\|\frac{1}{\langle n_2\rangle^{2s}\langle n-n_2\rangle^{2\rho}}\bigg\|^2_{\ell^{\frac{2(1+\varepsilon)}{1+2\varepsilon}}_{n_2}}\\ &\lesssim \sup_{n\in\mathbb{Z}^*}\langle n\rangle^{2-4\min(s,\rho)}<\infty,\end{aligned}\tag{4.26}$$

provided that $\min(s,\rho)\ge\frac12$.

(ii) As for the resonant part $X^{(2)}$, it suffices to note that

$$\frac{\langle n\rangle^{2s_0+2}}{\langle n\rangle^{2s_0+4s}}\lesssim1,$$

uniformly in $n\in\mathbb{Z}^*$, provided that $s\ge\frac12$. As for the non-resonant part $X^{(1)}$, without loss of generality, assume that $|n_1|\gtrsim\max(|n_1|,|n_2|,|n_3|)$, which implies $|n_1|\gtrsim|n|$. Then, proceeding as in (4.24) and (4.25) with $s\ge0$ (see also (3.8)), we have

$$\begin{aligned}\|X^{(1)}_{t,r}\|_{\mathcal{L}^{s,s_0}_{3,1}} &\le \bigg(\sup_{n\in\mathbb{Z}^*}\sum_{\substack{A_n\\ |n_1|\gtrsim|n|}}\frac{\langle n\rangle^{2s_0+2}}{\langle n_1\rangle^{2s_0}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}}|\Phi^w_{t,r}(\Xi_{\mathrm{mKdV}}(\bar n))|^2\bigg)^{\frac12}\\ &\le A_{\mathrm{mKdV}}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma\\ &\lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma\end{aligned}$$

for any $0\le r<t\le T$, provided that $\min(\rho,s)\ge\frac12$.

(iii) As for the resonant part $X^{(2)}$, it suffices to note that

$$\frac{\langle n\rangle^{2s_0+2}}{\langle n\rangle^{6s}} \lesssim 1,$$

uniformly in $n \in \mathbb{Z}^*$, provided that $s_0 \le 3s-1$. As for the non-resonant part $X^{(1)}$, without loss of generality, assume that $|n_1| \gtrsim \max(|n_1|, |n_2|, |n_3|)$, which implies $|n_1| \gtrsim |n|$. Then, proceeding as in (4.24) and (4.25) with $s_0 > s \ge 0$, we have

$$\|X^{(1)}_{t,r}\|_{\mathcal{L}_3(H^s;H^{s_0})} \le A^{(1)}_{\mathrm{mKdV}} \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma$$

for any $0 \le r < t \le T$, where $A^{(1)}_{\mathrm{mKdV}}$ is given by

$$(A^{(1)}_{\mathrm{mKdV}})^2 = \sup_{n\in\mathbb{Z}^*} \sum_{A_n} \frac{\langle n\rangle^{2s_0-2s+2}}{\langle n_2+n_3\rangle^{2\rho} \prod_{j=2}^3 \langle n_j\rangle^{2s} \langle n-n_j\rangle^{2\rho}}.$$

Here, by a crude estimate, we have dropped the factor $\langle n-n_1\rangle^{2\rho}$ (see Remark 1.18 below). By proceeding as in (4.26) with Lemma 2.1, we have

$$(A^{(1)}_{\mathrm{mKdV}})^2 \lesssim \sup_{n\in\mathbb{Z}^*} \langle n\rangle^{2s_0-2s+2-4\min(\rho,s)} < \infty,$$

provided that (1.25) holds.

(iv) From (4.21) with (4.23), we have

$$\begin{aligned}
&\mathcal{F}\big(X_{t,r}(f_1,f_2,f_3)\big)(n) - \mathcal{F}\big(X^N_{t,r}(f_1,f_2,f_3)\big)(n)\\
&\quad = in \sum_{\widetilde{A}_n} \Phi^w_{t,r}(\Xi_{\mathrm{mKdV}}) \widehat{f_1}(n_1)\widehat{f_2}(n_2)\widehat{f_3}(n_3)\\
&\qquad - \mathbf{1}_{\{|n|>N\}} \cdot 3i(t-r) n \widehat{f_1}(n) \overline{\widehat{f_2}(n)} \widehat{f_3}(n).
\end{aligned}$$

Here, $\widetilde{A}_n$ is given by

$$\widetilde{A}_n = \big\{(n_1,n_2,n_3) \in A_n : \max(|n|,|n_1|,|n_2|,|n_3|) > N\big\},$$

where A_n is as in (4.18). Let $X^{N,(1)}$ and $X^{N,(2)}$ denote the non-resonant and resonant parts of X^N, respectively. Then, we have

$$\|X^{(2)}_{t,r} - X^{N,(2)}_{t,r}\|_{\mathcal{L}_3(H^s)} \lesssim N^{-\varepsilon}|t-r|$$

for any $0 \le r < t \le T$, provided that $s \ge \frac{1}{2} + \frac{1}{2}\varepsilon$.

Let us consider the non-resonant part. From (4.22), we have

$$\max(|n|,|n_1|,|n_2|,|n_3|) \lesssim \langle \Xi_{\mathrm{mKdV}}(\bar{n})\rangle$$

on A_n defined in (4.18). Consequently, we have

$$\mathbf{1}_{\widetilde{A}_n} \lesssim N^{-\varepsilon_0} \langle \Xi_{\mathrm{mKdV}}(\bar{n})\rangle^{\varepsilon_0} \tag{4.27}$$

for $\varepsilon_0 > 0$. By repeating the computations in (4.25) and (4.26) with $\min(s, \rho-\varepsilon_0) \ge \frac{1}{2}$, we obtain

$$\|X^{(1)}_{t,r} - X^{N,(1)}_{t,r}\|_{\mathcal{L}_3(H^s)} \lesssim N^{-\varepsilon_0} \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma$$

for any $0 \le r < t \le T$. □

4.3. **Modulated KdV equation on the real line.** We conclude this section by considering the modulated KdV (1.2) on the real line. The bilinear driver associated with the modulated KdV is given by X in (4.3), where we view all the operators and functions as those defined on $\mathbb{R}$. By taking the Fourier transform, we have

$$\mathcal{F}_{\mathbb{R}}\big(X_{t,r}(f_1,f_2)\big)(\xi) = i\xi \int_{\xi=\xi_1+\xi_2} \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)d\xi_1, \tag{4.28}$$

where $\Phi^w_{t,r}$ is as in (1.6) and

$$\Xi_{\mathrm{KdV}}(\bar{\xi}) = \Xi_{\mathrm{KdV}}(\xi,\xi_1,\xi_2) = -\xi^3+\xi_1^3+\xi_2^3.$$

As in (4.6), we have

$$\Xi_{\mathrm{KdV}}(\bar{\xi}) = -3\xi\xi_1\xi_2 \tag{4.29}$$

under $\xi=\xi_1+\xi_2$. Then, as in the previous subsections, local well-posedness of the modulated KdV (1.2) on $\mathbb{R}$ (Theorem 1.7 (i)) follows from Proposition 3.8 once we prove the following proposition.

Proposition 4.4. *Given $\rho>\frac{1}{2}$, $\frac{1}{2}<\gamma<1$, and $T>0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1.*

(i) *Suppose that $\rho>\frac{1}{2}$ and $s\in\mathbb{R}$ satisfy* (1.11). *Then, the driver X defined in* (4.3) *belongs to $\mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{R})$ defined in* (2.9).

(ii) (persistence of regularity). *Suppose that $\rho>\frac{1}{2}$ and $s\in\mathbb{R}$ satisfy* (1.11). *Then, for any $s_0>s$, the driver X belongs to $\mathcal{Y}_2^{s,s_0,\gamma}([0,T]\times\mathbb{R})$ defined in* (2.14).

The regularity restriction $s>-\frac{3}{2}$ in (1.11) (for $\rho\geq\frac{3}{2}$) is (essentially) sharp. See Remark 4.5 below.

Proof of Proposition 4.4. While we proceed as in the proof of Proposition 4.1 for the modulated KdV on $\mathbb{T}$, we need to pay particular attention to low frequencies in the current real line case.

(i) Proceeding as in (4.8), we have

$$\|X_{t,r}\|^2_{\mathcal{L}_2(H^s(\mathbb{R}))} \leq \sup_{\xi_1\in\mathbb{R}}\int_{\mathbb{R}}\langle\xi\rangle^{2s}\xi^2\frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}}d\xi. \tag{4.30}$$

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}\|X_{t,r}(f_1,f_2)\|^2_{H^s(\mathbb{R})} &\leq \int_{\mathbb{R}}\langle\xi\rangle^{2s}\xi^2\bigg(\int_{\xi=\xi_1+\xi_2}\frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}}d\xi_1\bigg)\\ &\quad\times\bigg(\int_{\xi=\xi_1+\xi_2}\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}|\widehat{f_1}(\xi_1)|^2|\widehat{f_2}(\xi_2)|^2d\xi_1\bigg)d\xi\\ &\leq\bigg(\sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}}\langle\xi\rangle^{2s}\xi^2\frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}}d\xi_1\bigg)\|f_1\|^2_{H^s(\mathbb{R})}\|f_2\|^2_{H^s(\mathbb{R})}.\end{aligned} \tag{4.31}$$

Thus, we have

$$\|X_{t,r}\|^2_{\mathcal{L}_2(H^s(\mathbb{R}))} \le \sup_{\xi\in\mathbb{R}} \int_{\mathbb{R}} \langle\xi\rangle^{2s}\xi^2 \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}} d\xi_1. \tag{4.32}$$

Then, by dividing the frequency space $\mathbb{R}^2_{\xi,\xi_1}$ into the following four regions:

$$\begin{aligned} D_1 &= \{|\xi|<1, |\xi_1|<2\},\\ D_2 &= \{|\xi|<1, |\xi_1|\ge 2\},\\ D_3 &= \Big\{|\xi|\ge 1, |\xi_1|\ge \frac{1}{2}, |\xi_2|\ge \frac{1}{2}\Big\},\\ D_4 &= \Big\{|\xi|\ge 1, |\xi_1|\ge \frac{1}{2}, |\xi_2|< \frac{1}{2}\Big\} \cup \Big\{|\xi|\ge 1, |\xi_1|< \frac{1}{2}, |\xi_2|\ge \frac{1}{2}\Big\} \end{aligned} \tag{4.33}$$

and applying (4.30) and (4.32) (with symmetry $\xi_1 \leftrightarrow \xi_2$ on D_4), we obtain

$$\begin{aligned} \|X_{t,r}\|^2_{\mathcal{L}_2(H^s(\mathbb{R}))} \le{}& \sup_{|\xi|<1} \int_{|\xi_1|<2} \langle\xi\rangle^{2s}\xi^2 \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}} d\xi_1\\ &+ \sup_{|\xi_1|\ge 2} \int_{|\xi|<1} \langle\xi\rangle^{2s}\xi^2 \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}} d\xi\\ &+ \sup_{|\xi_1|\ge \frac{1}{2}} \int_{\substack{|\xi|\ge 1\\ |\xi-\xi_1|\ge\frac{1}{2}}} \langle\xi\rangle^{2s}\xi^2 \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}} d\xi\\ &+ 2\sup_{|\xi|\ge 1} \int_{|\xi-\xi_1|<\frac{1}{2}} \langle\xi\rangle^{2s}\xi^2 \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}} d\xi_1\\ =:{}& A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{4.34}$$

From (1.5) with (4.29), we have

$$A_1 \lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \tag{4.35}$$

for any $0\le r<t\le T$ without any restriction on $\rho>0$ and $s\in\mathbb{R}$.

Next, we consider A_2. When $s\ge 0$, it follows from the triangle inequality and (1.5) that

$$A_2 \lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \tag{4.36}$$

for any $0\le r<t\le T$ without any restriction on $\rho>0$ (and $s\ge 0$). Now, suppose that $s<0$. Note from (1.5) that

$$|\Phi^w_{t,r}(a)| \le \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} \frac{|t-r|^\gamma}{\langle a\rangle^\rho} \le \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} \frac{|t-r|^\gamma}{\langle a\rangle^{\rho'}} \tag{4.37}$$

for any $a\in\mathbb{R}$ and $0<\rho'\le\rho$. Under $|\xi|<1$ and $|\xi_1|\ge 2$, we have $|\xi-\xi_1|>1$. In particular, we have $|\xi_1|\sim\langle\xi_1\rangle$ and $|\xi-\xi_1|\sim\langle\xi-\xi_1\rangle$. Moreover, we have $\langle\xi\xi_1(\xi-\xi_1)\rangle \ge |\xi\xi_1(\xi-\xi_1)|$.

Then, with (4.37), we have

$$\begin{aligned} A_2 &\lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \int_{|\xi|<1} \xi^2 \sup_{\substack{|\xi_1|\geq 2\\ |\xi-\xi_1|>1}} \frac{1}{\langle \xi\xi_1(\xi-\xi_1)\rangle^{2\rho'}\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}}d\xi \\ &\lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \int_{|\xi|<1} |\xi|^{2-2\rho'} \sup_{\substack{|\xi_1|\geq 2\\ |\xi-\xi_1|>1}} \frac{1}{|\xi_1(\xi-\xi_1)|^{2s+2\rho'}}d\xi \\ &\lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \end{aligned} \tag{4.38}$$

for any $0\leq r<t\leq T$, provided that $2-2\rho'>-1$ and $s+\rho'\geq 0$, which can be satisfied by choosing suitable $0<\rho'\leq\rho$ if

$$\rho>0, \quad s>-\frac{3}{2}, \quad \text{and} \quad s\geq -\rho. \tag{4.39}$$

As for A_3, there is no low frequency issue. Thus, by proceeding as in (4.9) and (4.10), we have

$$\begin{aligned} A_3 &\lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \sup_{|\xi_1|\geq\frac{1}{2}} \int_{\substack{|\xi|\geq 1\\ |\xi-\xi_1|\geq\frac{1}{2}}} \frac{\langle\xi\rangle^{2s+2-2\rho}}{\langle\xi_1\rangle^{2s+2\rho}\langle\xi-\xi_1\rangle^{2s+2\rho}}d\xi \\ &\lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \end{aligned} \tag{4.40}$$

for any $0\leq r<t\leq T$, provided that (4.11) holds.

As for A_4, we first note that $|\xi_1|\sim|\xi|\geq 1$ and $\langle\xi-\xi_1\rangle\sim 1$ under $|\xi|\geq 1$ and $|\xi-\xi_1|<\frac{1}{2}$. Then, from (1.5), (4.29), and a change of variables $\zeta_2=\xi^2(\xi-\xi_1)$, we have

$$\begin{aligned} A_4 &\lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \sup_{|\xi|\geq 1}\xi^2 \int_{|\xi-\xi_1|<\frac{1}{2}} \frac{1}{\langle\xi^2(\xi-\xi_1)\rangle^{2\rho}}d\xi_1 \\ &= \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \sup_{|\xi|\geq 1} \int_{|\zeta_2|<\frac{\xi^2}{2}} \frac{1}{\langle\zeta_2\rangle^{2\rho}}d\zeta_2 \lesssim \|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma} \end{aligned} \tag{4.41}$$

for any $0\leq r<t\leq T$, provided that $\rho>\frac{1}{2}$ (and any $s\in\mathbb{R}$).

Therefore, putting (4.34), (4.35), (4.36), (4.38), (4.40), and (4.41) together, we conclude from Lemma 3.1 (i) that $X\in\mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{R})$ under (1.11).

(ii) Write $X=X^1+X^2$, where X^1 is the contribution from $|\xi_1|\geq|\xi_2|$ in (4.28) and $X^2=X-X^1$. We only consider X^1 since X^2 can be handled in an analogous manner. From (4.33) with (3.8), we have

$$\|X^1_{t,r}\|^2_{\mathcal{L}^{s,s_0}_{2,1}}\leq \widetilde{A}_1+\widetilde{A}_2+\widetilde{A}_3+\widetilde{A}_4,$$

where $\widetilde{A}_j$, $j=1,\dots,4$, is obtained from A_j in (4.34) by replacing $\langle\xi\rangle^{2s}$ and $\langle\xi_1\rangle^{2s}$ with $\langle\xi\rangle^{2s_0}$ and $\langle\xi_1\rangle^{2s_0}$, respectively.

Suppose that (1.11) holds and that $s_0>s$. From (4.35), a slight modification of (4.40) (as in the proof of Proposition 4.1 (ii)), and (4.41), we have

$$\widetilde{A}_1+\widetilde{A}_3+\widetilde{A}_4\lesssim\|\Phi^w\|^2_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^{2\gamma}$$

for any $0 \le r < t \le T$. When $s \ge 0$, we also have

$$\widetilde{A}_2 \lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma}$$

for any $0 \le r < t \le T$. When $s < 0$, a slight modification of (4.38) with $|\xi_1| \sim |\xi - \xi_1|$ yields

$$\begin{aligned}
\widetilde{A}_2 &\lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \int_{|\xi|<1} \xi^2 \sup_{\substack{|\xi_1|\ge 2\\ |\xi-\xi_1|>1}} \frac{1}{\langle \xi\xi_1(\xi-\xi_1)\rangle^{2\rho'} \langle \xi_1\rangle^{2s_0} \langle \xi-\xi_1\rangle^{2s}} d\xi \\
&\lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \int_{|\xi|<1} |\xi|^{2-2\rho'} \sup_{|\xi_1|\ge 2} \frac{1}{|\xi_1|^{2s_0+2s+4\rho'}} d\xi \\
&\lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma}
\end{aligned}$$

for any $0 \le r < t \le T$, provided that $2-2\rho' > -1$ and $s_0 + s + 2\rho' \ge 0$, which can be satisfied by choosing suitable $0 < \rho' \le \rho$ under (1.11) and $s_0 > s$. Hence, from Lemma 3.1 (ii), we conclude that, if (1.11) holds, then $X \in \mathcal{Y}_2^{s,s_0,\gamma}([0,T]\times\mathbb{R})$ for any $s_0 > s$. □

Remark 4.5. The regularity restriction $s > -\frac{3}{2}$ appears in the analysis of A_2 in (4.38); see (4.39). In this remark, we show that the regularity restriction $s > -\frac{3}{2}$ is essentially necessary (modulo the endpoint $s = -\frac{3}{2}$), regardless of the value of ρ.

Given $N \gg 1$, define a function f on $\mathbb{R}$ by setting

$$\widehat{f}(\xi) = N^{-s-\frac{1}{2}} \cdot \mathbf{1}_{|\xi|\sim N} \tag{4.42}$$

such that

$$\|f\|_{H^s} \sim 1. \tag{4.43}$$

We now estimate the H^s-norm of $X_{t,r}(f,f)$ from below by considering the contribution coming from $|\xi| \le c_0 N^{-2}$ for some small constant $c_0 > 0$. Namely,

$$|\Xi_{\mathrm{KdV}}(\bar{\xi})| = 3|\xi\xi_1\xi_2| \lesssim c_0 \ll 1. \tag{4.44}$$

In this case, it follows from (1.6) and the continuity of the modulation function w that

$$\operatorname{Re} \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi})) \gtrsim_w |t-r|, \tag{4.45}$$

provided that t is sufficiently close to (given) r (and by choosing $c_0 > 0$ sufficiently small, depending on the value of $w(r)$). Then, from (4.28) with (4.42) and (4.45), we have

$$\begin{aligned}
\|X_{t,r}(f,f)\|^2_{H^s(\mathbb{R})} &\gtrsim_{w,t,r} N^{-4s-2} \int_{|\xi|\le c_0 N^{-2}} |\xi|^2 \bigg(\int \mathbf{1}_{|\xi_1|\sim N} \cdot \mathbf{1}_{|\xi-\xi_1|\sim N} d\xi_1 \bigg)^2 d\xi \\
&\gtrsim N^{-4s} \int_{|\xi|\le c_0 N^{-2}} |\xi|^2 d\xi \sim N^{-4s-6}.
\end{aligned} \tag{4.46}$$

Hence, from (4.43) and (4.46) and taking $N \to \infty$, we conclude that if $X_{t,r} \in \mathcal{L}_2(H^s(\mathbb{R}))$, we must have $s \ge -\frac{3}{2}$, showing the essential sharpness of the condition $s > -\frac{3}{2}$ for $\rho \ge \frac{3}{2}$ in (1.11) (modulo the endpoint $s = -\frac{3}{2}$). We point out that by following [9, Section 6], the argument above (with $r = 0$) yields the failure of C^2-smoothness of the solution map for the modulated KdV (1.2) on $H^s(\mathbb{R})$ for $s < -\frac{3}{2}$.

In the argument presented above, we considered the case where the resonance function $\Xi_{\mathrm{KdV}}(\bar{\xi})$ is small; see (4.44). In this case, we can not make use of the irregularity of the

modulation function w, thus reducing the problem to the (unmodulated) KdV case. This is the reason why we encountered the scaling-critical regularity $s = -\frac{3}{2}$ for the (unmodulated) KdV.

The argument presented above exploits the high-to-low energy transfer, leading to the divergence of the Picard second iterate for $s < -\frac{3}{2}$. Such an argument also appears in showing norm inflation for dispersive equations in negative regularity; see, for example, [124, 32]. By considering a (suitably weighted) lacunary series of initial data of the form (4.42), it may be possible to construct a counterexample for the endpoint case $s = -\frac{3}{2}$. See [32, Section 4] for such an argument. For simplicity of the presentation, however, we do not pursue this issue here.

Remark 4.6. In order to prove analogues of Proposition 4.1 (iii) and (iv) as a modification of the proof of Proposition 4.4, we would need to (at least) bound

$$A_4' = \sup_{|\xi|\geq 1} \int_{|\xi-\xi_1|<\frac{1}{2}} \langle \xi \rangle^{2(s+\varepsilon)} \xi^2 \frac{|\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} d\xi_1$$

for some $\varepsilon > 0$. By proceeding as in (4.41), we have

$$\begin{aligned} A_4' &\lesssim \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{|\xi|\geq 1} |\xi|^{2+2\varepsilon} \int_{|\xi-\xi_1|<\frac{1}{2}} \frac{1}{\langle \xi^2(\xi-\xi_1)\rangle^{2\rho}} d\xi_1 \\ &= \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{|\xi|\geq 1} |\xi|^{2\varepsilon} \int_{|\xi_2|<\frac{\xi^2}{2}} \frac{1}{\langle \xi_2 \rangle^{2\rho}} d\xi_2 = \infty. \end{aligned}$$

This shows that we can not establish nonlinear smoothing or convergence of the Galerkin approximation as a straightforward modification of the proof of Proposition 4.4.

5. Modulated BO and ILW equations on the circle

In this section, we consider the modulated BO and (scaled) ILW equations on the circle. By establishing regularity properties of the associated drivers, we first prove their local well-posedness and related properties (Theorems 1.12 and 1.19) in Subsections 5.1 and 5.2, where the claims for the modulated (scaled) ILW follow from a perturbation of the BO case. This perturbative viewpoint allows us to easily prove deep-water convergence (Theorem 1.14 (i)) in Subsection 5.3. On the other hand, careful analysis on the resonance function $\widetilde{\Xi}_\delta(\bar{n})$ for the scaled ILW defined in (5.40) is required to establish shallow-water convergence (Theorem 1.14 (ii)); see Subsection 5.4.

5.1. Modulated BO equation.

The bilinear driver associated with the modulated BO (1.15) on $\mathbb{T}$ is given by

$$X^{\mathrm{BO}}_{t,r}(f_1, f_2) = \int_r^t U^w(t')^{-1} \partial_x \big((U^w(t') f_1)(U^w(t') f_2) \big) dt', \tag{5.1}$$

where

$$U^w(t) = e^{w(t)\mathcal{H}\partial_x^2} \tag{5.2}$$

denotes the modulated linear propagator for (1.15). By taking the Fourier transform, we have

$$\mathcal{F}\big(X^{\mathrm{BO}}_{t,r}(f_1,f_2)\big)(n) = in \sum_{\substack{n_1,n_2\in\mathbb{Z}^*\\ n=n_1+n_2}} \Phi^w_{t,r}(\Xi_{\mathrm{BO}}(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2) \tag{5.3}$$

where $\Phi^w_{t,r}$ is as in (1.6) and $\Xi_{\mathrm{BO}}(\bar{n})$ denotes the resonance function for BO given by

$$\Xi_{\mathrm{BO}}(\bar{n}) = \Xi_{\mathrm{BO}}(n,n_1,n_2) = -|n|n + |n_1|n_1 + |n_2|n_2. \tag{5.4}$$

As in the previous section, Theorem 1.12 (i) and Theorem 1.19 (i) and (ii) follow from Proposition 3.8, once we prove the following proposition.

Proposition 5.1. *Given $\rho \geq 1$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1.*

(i) *Suppose that $\rho \geq 1$ and $s \in \mathbb{R}$ satisfy (1.21). Then, the driver X^{BO} defined in (5.1) belongs to $\mathcal{X}^{s,\gamma}_2([0,T]\times\mathbb{T})$ defined in (2.9).*

(ii) (persistence of regularity). *Suppose that $\rho \geq 1$ and $s \in \mathbb{R}$ satisfy (1.21). Then, for any $s_0 > s$, the driver X^{BO} belongs to $\mathcal{Y}^{s,s_0,\gamma}_2([0,T]\times\mathbb{T})$ defined in (2.14).*

(iii) (nonlinear smoothing). *In addition, suppose that $s_0 > s$ satisfies (1.26). Then, the driver X^{BO} belongs to $\mathcal{X}^{s,s_0,\gamma}_2([0,T]\times\mathbb{T})$ defined in (2.13).*

(iv) (Galerkin approximation). *Define the truncated driver $X^{\mathrm{BO},N}$ as in (4.7), where we replace $U^w(t)$ by the modulated linear propagator for BO defined in (5.2). Suppose that $\rho > 1$ and $s > -\frac{1}{2}\rho$. Then, $X^{\mathrm{BO},N}$ converges to X^{BO} in $\mathcal{X}^{s,\gamma}_2([0,T]\times\mathbb{T})$ as $N\to\infty$.*

Proof. As in the proof of Proposition 4.1, we verify the hypotheses in Lemma 3.1 with $k = 2$.

(i) Proceeding as in (4.8) (see also (4.9)), we have

$$\|X^{\mathrm{BO}}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim A_{\mathrm{BO}}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma$$

for any $0 \leq r < t \leq T$, where A_{BO} is given by

$$A_{\mathrm{BO}} = \bigg(\sup_{n_1\in\mathbb{Z}^*}\sum_{\substack{n,n_2\in\mathbb{Z}^*\\ n_1=n-n_2}} \langle n\rangle^{2s+2}\frac{1}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}|\Xi_{\mathrm{BO}}(\bar{n})|^{2\rho}}\bigg)^{\frac{1}{2}}.$$

From (5.4), we have, under $n = n_1 + n_2$,

(a) If $n_1n_2 > 0$ (i.e. n, n_1, and n_2 all have the same sign), then $|\Xi_{\mathrm{BO}}(\bar{n})| = 2|n_1n_2|$.

(b) If $n, n_1 > 0 > n_2$, then $|\Xi_{\mathrm{BO}}(\bar{n})| = 2|nn_2|$.

(c) If $n, n_2 > 0 > n_1$, then $|\Xi_{\mathrm{BO}}(\bar{n})| = 2|nn_1|$.

(d) If $n_1 > 0 > n, n_2$, then $|\Xi_{\mathrm{BO}}(\bar{n})| = 2|nn_1|$.

(e) If $n_2 > 0 > n, n_1$, then $|\Xi_{\mathrm{BO}}(\bar{n})| = 2|nn_2|$.

By symmetry, we only consider Cases (a) and (b) in the following.

• **Case (i.a):** $n_1 n_2 > 0$.

Without loss of generality, assume that $|n_1| \gtrsim |n_2|$. Then, we have $|n_1| \gtrsim |n|$ under $n = n_1 + n_2$. In view of Lemma 2.1, we have

$$\begin{aligned} A_{\mathrm{BO}}^2 &\lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n \in \mathbb{Z}^* \\ n_1 = n - n_2}} |n|^{2-2\rho} \frac{|n|^{2s+2\rho}}{|n_1 n_2|^{2s+2\rho}} \\ &\lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{n \in \mathbb{Z}^*} \frac{1}{|n|^{2\rho-2} |n - n_1|^{2s+2\rho}} < \infty, \end{aligned} \tag{5.5}$$

provided that

$$\rho \geq 1, \quad s \geq -\rho, \quad \text{and} \quad s > \frac{3}{2} - 2\rho. \tag{5.6}$$

• **Case (i.b):** $n, n_1 > 0 > n_2$.

In this case, we have $|n_1| > |n_2|$ and thus $|n_1| \gtrsim |n|$. When $s \geq 0$, it follows from Lemma 2.1 that

$$A_{\mathrm{BO}}^2 \lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n \in \mathbb{Z}^* \\ n_1 = n - n_2}} \frac{1}{|n|^{2\rho-2} |n_2|^{2s+2\rho}} < \infty, \tag{5.7}$$

provided that (5.6) holds.

Next, we consider the case $s < 0$. We separately consider the cases $|n_1| \sim |n_2|$ and $|n_1| \sim |n|$ ($\gg |n_2|$). When $|n_1| \sim |n_2|$, it follows from Lemma 2.1 that

$$A_{\mathrm{BO}}^2 \lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n \in \mathbb{Z}^* \\ n_1 = n - n_2}} \frac{1}{|n|^{2\rho-2s-2} |n_2|^{4s+2\rho}} < \infty, \tag{5.8}$$

provided that

$$\rho \geq s + 1, \quad s \geq -\frac{1}{2}\rho, \quad \text{and} \quad s > \frac{3}{2} - 2\rho. \tag{5.9}$$

When $|n_1| \sim |n|$, (5.5) holds under (5.6).

A simple computation shows that (5.6) and (5.9) (with $s < 0$ for the latter) are satisfied under (1.21). Hence, from Lemma 3.1 (i), we conclude that $X^{\mathrm{BO}} \in \mathcal{X}_2^{s,\gamma}([0,T] \times \mathbb{T})$ under (1.21).

(ii) Write $X^{\mathrm{BO}} = X^{\mathrm{BO},1} + X^{\mathrm{BO},2}$, where $X^{\mathrm{BO},1}$ is the contribution from $|n_1| \geq |n_2|$ in (5.3) and $X^{\mathrm{BO},2} = X^{\mathrm{BO}} - X^{\mathrm{BO},1}$. We only consider $X^{\mathrm{BO},1}$ since a similar bound holds for $X^{\mathrm{BO},2}$ by symmetry. Since $|n_1| \geq |n_2|$, we only need to consider the cases (a), (b), and (e) in Part (i). Moreover, by symmetry, it suffices to consider the cases (a) and (b).

As before, we have

$$\|X_{t,r}^{\mathrm{BO},1}\|_{\mathcal{L}_{2,1}^{s,s_0}} \lesssim A_{\mathrm{BO}}^{(1)} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t - r|^{\gamma},$$

where $A_{\mathrm{BO}}^{(1)}$ is given by

$$A_{\mathrm{BO}}^{(1)} = \bigg(\sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n, n_2 \in \mathbb{Z}^* \\ n_1 = n - n_2 \\ |n_1| \geq |n_2|}} \langle n \rangle^{2s_0+2} \frac{1}{\langle n_1 \rangle^{2s_0} \langle n_2 \rangle^{2s} |\Xi_{\mathrm{BO}}(\bar{n})|^{2\rho}} \bigg)^{\frac{1}{2}}. \tag{5.10}$$

• **Case (ii.a):** $n_1 n_2 > 0$.

In this case, we can proceed as in Case (i.a) and obtain $A^{(1)}_{\mathrm{BO}} < \infty$ under (1.21).

• **Case (ii.b):** $n, n_1 > 0 > n_2$.

When $s_0 \geq 0$ or $|n_1| \sim |n|$, we can proceed as in Case (i.b) and obtain $A^{(1)}_{\mathrm{BO}} < \infty$ under (5.6). Thus, it suffices to consider the case $s_0 < 0$ and $|n_1| \sim |n_2|$. From (5.10) and Lemma 2.1, we have

$$(A^{(1)}_{\mathrm{BO}})^2 \lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n \in \mathbb{Z}^* \\ n_1 = n - n_2}} \frac{1}{|n|^{2\rho - 2s_0 - 2} |n_2|^{2s_0 + 2s + 2\rho}} < \infty,$$

provided that

$$\rho \geq s_0 + 1, \quad s_0 + s \geq -\rho, \quad \text{and} \quad s > \frac{3}{2} - 2\rho$$

which is satisfied under (1.21). Hence, from Lemma 3.1 (ii), we conclude that, if (1.21) holds, then $X^{\mathrm{BO}} \in \mathcal{Y}_2^{s, s_0, \gamma}([0,T] \times \mathbb{T})$ for any $s_0 > s$.

(iii) As in Part (i), by symmetry, we only consider Cases (a) and (b). As before, we have

$$\|X^{\mathrm{BO}}_{t,r}\|_{\mathcal{L}_2(H^s; H^{s_0})} \lesssim A^{(2)}_{\mathrm{BO}} \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} |t - r|^{\gamma},$$

where $A^{(2)}_{\mathrm{BO}}$ is given by

$$A^{(2)}_{\mathrm{BO}} = \bigg(\sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n, n_2 \in \mathbb{Z}^* \\ n_1 = n - n_2}} \langle n \rangle^{2s_0 + 2} \frac{1}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} |\Xi_{\mathrm{BO}}(\bar{n})|^{2\rho}} \bigg)^{\frac{1}{2}}. \tag{5.11}$$

• **Case (iii.a):** $n_1 n_2 > 0$.

Without loss of generality, assume that $|n_1| \geq |n_2|$. Then, we have $|n_1| \gtrsim |n|$ under $n = n_1 + n_2$. In view of Lemma 2.1, we have

$$\begin{aligned}(A^{(2)}_{\mathrm{BO}})^2 &\lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n \in \mathbb{Z}^* \\ n_1 = n - n_2}} |n|^{2s_0 - 2s + 2 - 2\rho} \frac{|n|^{2s + 2\rho}}{|n_1 n_2|^{2s + 2\rho}} \\ &\lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{n \in \mathbb{Z}^*} \frac{1}{|n|^{2\rho - 2s_0 + 2s - 2} |n - n_1|^{2s + 2\rho}} < \infty,\end{aligned}$$

provided that

$$s_0 \leq s + \rho - 1, \quad s \geq -\rho, \quad \text{and} \quad s_0 < 2s + 2\rho - \frac{3}{2}. \tag{5.12}$$

• **Case (iii.b):** $n, n_1 > 0 > n_2$.

When $s_0 \geq 0$ or $|n_1| \sim |n|$, we can proceed as in Case (iii.a) and obtain $A^{(2)}_{\mathrm{BO}} < \infty$ under (5.6). Thus, it suffices to consider the case $s_0 < 0$ and $|n_1| \sim |n_2|$. From (5.11) and Lemma 2.1, we have

$$(A^{(2)}_{\mathrm{BO}})^2 \lesssim \sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n \in \mathbb{Z}^* \\ n_1 = n - n_2}} \frac{1}{|n|^{2\rho - 2s_0 - 2} |n_2|^{4s + 2\rho}} < \infty,$$

provided that

$$s_0 \leq \rho - 1, \quad s \geq -\frac{1}{2}\rho, \quad \text{and} \quad s_0 < 2s + 2\rho - \frac{3}{2}. \tag{5.13}$$

By separately considering the cases $s \geq 0$ and $s < 0$, we obtain (1.26) from (5.12) and (5.13). Hence, from Lemma 3.1 (iii), we conclude that $X \in \mathcal{X}_2^{s,s_0,\gamma}([0,T] \times \mathbb{T})$ under (1.26).

(iv) By proceeding as in the proof of Proposition 4.1 (iv) with (4.13), we see that the claim follows from a slight modification of Part (i), provided that

$$\rho > 1, \quad s > -\rho, \quad \text{and} \quad s > \frac{3}{2} - 2\rho.$$

and

$$\rho \geq s + 1, \quad s > -\frac{1}{2}\rho, \quad \text{and} \quad s > \frac{3}{2} - 2\rho.$$

where the latter set of the conditions is for $s < 0$. Note that these conditions are satisfied if $\rho > 1$ and $s > -\frac{1}{2}\rho$.

This concludes the proof of Proposition 5.1. □

5.2. **Modulated ILW equation.** In this subsection, we consider the well-posedness and related issues for the modulated ILW (1.16) and the modulated scaled ILW (1.19) for fixed $0 < \delta < \infty$. Since these two equations are related by the scaling transform (1.18), we only consider the modulated ILW (1.16).

Fix $0 < \delta < \infty$. The bilinear driver associated with the modulated ILW (1.16) is given by

$$X^\delta_{t,r}(f_1, f_2) = \int_r^t U^w(t')^{-1} \partial_x \big((U^w(t') f_1)(U^w(t') f_2) \big) dt', \tag{5.14}$$

where

$$U^w(t) = e^{w(t)\mathcal{G}_\delta \partial_x^2} \tag{5.15}$$

denotes the modulated linear propagator for (1.16). By taking the Fourier transform, we have

$$\mathcal{F}\big(X^\delta_{t,r}(f_1, f_2)\big)(n) = in \sum_{\substack{n_1, n_2 \in \mathbb{Z}^* \\ n = n_1 + n_2}} \Phi^w_{t,r}(\Xi_\delta(\bar{n})) \widehat{f_1}(n_1) \widehat{f_2}(n_2) \tag{5.16}$$

where $\Phi^w_{t,r}$ is as in (1.6) and $\Xi_\delta(\bar{n})$ denotes the resonance function for ILW given by

$$\Xi_\delta(\bar{n}) = \Xi_\delta(n, n_1, n_2) = -p_\delta(n) + p_\delta(n_1) + p_\delta(n_2). \tag{5.17}$$

Here, $p_\delta(n)$ denotes the multiplier for $-i\mathcal{G}_\delta \partial_x^2$ given by

$$p_\delta(n) = n^2 \coth(\delta n) - \frac{n}{\delta}. \tag{5.18}$$

See (1.17). Then, under $n = n_1 + n_2$, we have

$$\Xi_\delta(\bar{n}) = -n^2 \coth(\delta n) + n_1^2 \coth(\delta n_1) + n_2^2 \coth(\delta n_2). \tag{5.19}$$

By applying [27, (2.7)] and the fact that $x^{\alpha+1} \leq C_\alpha(e^{2x} - 1)$ for any $x \geq 0$ and $\alpha \geq 0$, we have

$$n^2 \coth(\delta n) - |n| n = \operatorname{sgn}(n) \cdot \frac{2n^2}{e^{2|\delta n|} - 1} = O(\delta^{-2}) \tag{5.20}$$

for any $n \in \mathbb{Z}^*$. Hence, from (5.4), (5.19) and (5.20), we have

$$\Xi_\delta(\bar{n}) = \Xi_{\mathrm{BO}}(\bar{n}) + O(\delta^{-2}), \tag{5.21}$$

uniformly in $\bar{n} = (n, n_1, n_2) \in (\mathbb{Z}^*)^3$. Then, a slight modification of the proof of Proposition 5.1 for the modulated BO yields the following proposition on the driver X^δ for the modulated ILW, from which Theorem 1.12 (ii) and Theorem 1.19 (iii) follow.

Proposition 5.2. *Given $\rho \geq 1$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1. Fix $0 < \delta < \infty$.*

(i) *Suppose that $\rho \geq 1$ and $s \in \mathbb{R}$ satisfy* (1.21). *Then, the driver X^δ defined in* (5.14) *belongs to $\mathcal{X}_2^{s,\gamma}([0,T] \times \mathbb{T})$ defined in* (2.9).

(ii) (persistence of regularity). *Suppose that $\rho \geq 1$ and $s \in \mathbb{R}$ satisfy* (1.21). *Then, for any $s_0 > s$, the driver X^δ belongs to $\mathcal{Y}_2^{s,s_0,\gamma}([0,T] \times \mathbb{T})$ defined in* (2.14).

(iii) (nonlinear smoothing). *In addition, suppose that $s_0 > s$ satisfies* (1.26). *Then, the driver X^δ belongs to $\mathcal{X}_2^{s,s_0,\gamma}([0,T] \times \mathbb{T})$ defined in* (2.13).

(iv) (Galerkin approximation). *Define the truncated driver $X^{\delta,N}$ as in* (4.7), *where we replace $U^w(t)$ by the modulated linear propagator for ILW defined in* (5.15). *Suppose that $\rho > 1$ and $s > -\frac{1}{2}\rho$. Then, the truncated driver $X^{\delta,N}$ converges to X^δ in $\mathcal{X}_2^{s,\gamma}([0,T] \times \mathbb{T})$ as $N \to \infty$.*

Proof. We only show how the claim in (i) follows as a modification of the proof of Proposition 5.1 (i). The claims in (ii), (iii), and (iv) follow from similar modifications.

Proceeding as in (4.8) (see also (4.9)), we have

$$\|X^\delta_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim A_\delta \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t - r|^\gamma \tag{5.22}$$

for any $0 \leq r < t \leq T$, where A_δ is given by

$$A_\delta = \bigg(\sup_{n_1 \in \mathbb{Z}^*} \sum_{\substack{n, n_2 \in \mathbb{Z}^* \\ n_1 = n - n_2}} \langle n \rangle^{2s+2} \frac{1}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle \Xi_\delta(\bar{n}) \rangle^{2\rho}} \bigg)^{\frac{1}{2}}. \tag{5.23}$$

From (a)-(e) in the proof of Proposition 5.1 (i) we have

$$|\Xi_{\mathrm{BO}}(\bar{n})| \gtrsim \max(|n|, |n_1|, |n_2|) \tag{5.24}$$

for any $\bar{n} = (n, n_1, n_2) \in (\mathbb{Z}^*)^3$ with $n = n_1 + n_2$. In particular, from (5.21) and (5.24), we obtain

$$|\Xi_\delta(\bar{n})| \geq |\Xi_{\mathrm{BO}}(\bar{n})| - O(\delta^{-2}) \sim |\Xi_{\mathrm{BO}}(\bar{n})| \tag{5.25}$$

when $\max(|n|, |n_1|, |n_2|) \gg \delta^{-2}$. Now, write

$$A_\delta = A_\delta^1 + A_\delta^2, \tag{5.26}$$

where A_δ^1 is the restriction of A_δ to the case $\max(|n|, |n_1|, |n_2|) \gg \delta^{-2}$ and $A_\delta^2 = A_\delta - A_\delta^1$. Then, from (5.23), (5.25), and the proof of Proposition 5.1 (i), we have

$$A_\delta^1 < \infty, \tag{5.27}$$

provided that (1.21) holds. On the other hand, by a crude estimate, we have

$$A_\delta^2 \leq \bigg(\sup_{|n_1|\lesssim \delta^{-2}} \sum_{|n|\lesssim \delta^{-2}} \frac{\langle n\rangle^{2s+2}}{\langle n_1\rangle^{2s}\langle n-n_1\rangle^{2s}} \bigg)^{\frac{1}{2}} \lesssim 1 + \delta^{-6|s|-3} < \infty. \tag{5.28}$$

Hence, from Lemma 3.1 (i) with (5.22), (5.26), (5.27), and (5.28), we conclude that $X \in \mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{T})$ under (1.21). □

Remark 5.3. It is clear from the proof that the claims in Proposition 5.2 (such as the size of the $\mathcal{X}_2^{s,\gamma}(T)$-norm, convergence speed, etc.) are uniform in $\delta \gg 1$. However, there is no uniform control for $0 < \delta \ll 1$, which presents an additional difficulty in establishing shallow-water convergence (as compared to deep-water convergence); see Subsection 5.4.

5.3. **Deep-water convergence of the modulated ILW.** In this subsection, we establish deep-water convergence ($\delta \to \infty$) of the modulated ILW (1.16) to the modulated BO (1.15) (Theorem 1.14 (i)).

Given $u_0 \in H^s(\mathbb{T})$, let u^δ, $0 < \delta < \infty$, and u denote the local-in-time solutions to the modulated ILW (1.16) and the modulated BO (1.15) with $u^\delta|_{t=0} = u|_{t=0} = u_0$, respectively. Let $\mathbf{u}^\delta(t) = e^{-w(t)\mathcal{G}_\delta\partial_x^2}u^\delta(t)$ and $\mathbf{u}(t) = e^{-w(t)\mathcal{H}\partial_x^2}u(t)$ be the modulated interaction representations of u^δ and u (under the modulated ILW and the modulated BO, respectively). Then, by the unitarity of $e^{-w(t)\mathcal{G}_\delta\partial_x^2}$ on $H^s(\mathbb{T})$, we have

$$\|u^\delta - u\|_{C_\tau H_x^s} \leq \|\mathbf{u}^\delta - \mathbf{u}\|_{C_\tau H_x^s} + \|(e^{-w(t)\mathcal{G}_\delta\partial_x^2} - e^{-w(t)\mathcal{H}\partial_x^2})u\|_{C_\tau H_x^s} \tag{5.29}$$

From the mean value theorem and [107, Lemma 2.1], we have, for each $n \in \mathbb{Z}^*$,

$$\begin{aligned}\big|\mathcal{F}\big((e^{-w(t)\mathcal{G}_\delta\partial_x^2} - e^{-w(t)\mathcal{H}\partial_x^2})f\big)(n)\big| &\leq |w(t)|\big|n^2\widehat{\mathcal{G}_\delta}(n) - (-i|n|n)\big||\widehat{f}(n)| \\ &\longrightarrow 0,\end{aligned}$$

as $\delta \to \infty$. Then, from the dominated convergence theorem, we see that the second term on the right-hand side of (5.29) tends to 0 as $\delta \to \infty$.

Hence, Theorem 1.14 (i) follows once we show

$$\lim_{\delta\to\infty} \|\mathbf{u}^\delta - \mathbf{u}\|_{C_\tau H_x^s} = 0. \tag{5.30}$$

In view of Lemma 3.1 (iv), Remark 3.2 (i), and Proposition 3.8 (iv), it then suffices to prove

$$\|X_{t,r}^\delta - X_{t,r}^{\mathrm{BO}}\|_{\mathcal{L}_2(H^s)} \lesssim o(1)|t-r|^\gamma, \tag{5.31}$$

as $\delta \to \infty$, uniformly in $0 \leq r < t \leq T$, where X^δ and X^{BO} are as in (5.14) and (5.1), respectively.

Proceeding as in (4.8) with (5.16) and (5.3) we have

$$\|X_{t,r}^\delta - X_{t,r}^{\mathrm{BO}}\|_{\mathcal{L}_2(H^s)} \leq \bigg(\sup_{n_1\in\mathbb{Z}^*} \sum_{\substack{n,n_2\in\mathbb{Z}^* \\ n_1 = n-n_2}} \langle n\rangle^{2s+2} \frac{|\Phi_{t,r}^w(\Xi_\delta(\bar{n})) - \Phi_{t,r}^w(\Xi_{\mathrm{BO}}(\bar{n}))|^2}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \bigg)^{\frac{1}{2}}. \tag{5.32}$$

From (1.5) and (5.21), we have

$$\begin{aligned}|\Phi_{t,r}^w(\Xi_\delta(\bar{n})) - \Phi_{t,r}^w(\Xi_{\mathrm{BO}}(\bar{n}))| &\leq |\Phi_{t,r}^w(\Xi_\delta(\bar{n}))| + |\Phi_{t,r}^w(\Xi_{\mathrm{BO}}(\bar{n}))| \\ &\lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^\gamma \frac{1}{\langle \Xi_{\mathrm{BO}}(\bar{n})\rangle^\rho}\end{aligned} \tag{5.33}$$

for any $\delta \gg 1$ and $0 \leq r < t \leq T$. On the other hand, from (1.6), (5.21), and the mean value theorem, we have

$$\begin{aligned} |\Phi^w_{t,r}(\Xi_\delta(\bar{n})) - \Phi^w_{t,r}(\Xi_{\mathrm{BO}}(\bar{n}))| &= \bigg| \int_r^t e^{i\Xi_{\mathrm{BO}}(\bar{n})w(t')} \big(e^{iO(\delta^{-2})w(t')} - 1\big) dt' \bigg| \\ &\lesssim \delta^{-2}|t-r|. \end{aligned} \tag{5.34}$$

By interpolating (5.33) and (5.34), we have

$$|\Phi^w_{t,r}(\Xi_\delta(\bar{n})) - \Phi^w_{t,r}(\Xi_{\mathrm{BO}}(\bar{n}))| \lesssim_T \delta^{-2\varepsilon} \|\Phi^w\|^{1-\varepsilon}_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma \frac{1}{\langle \Xi_{\mathrm{BO}}(\bar{n})\rangle^{\rho(1-\varepsilon)}} \tag{5.35}$$

for any $0 \leq \varepsilon \leq 1$. Hence, by a slight modification of (5.5), (5.7), and (5.8) (where we replace ρ by $\rho(1-\varepsilon)$ for small $\varepsilon > 0$) with (5.32) and (5.35), we obtain

$$\|X^\delta_{t,r} - X^{\mathrm{BO}}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim_T \delta^{-2\varepsilon} \|\Phi^w\|^{1-\varepsilon}_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma = o(1)|t-r|^\gamma,$$

as $\delta \to \infty$, uniformly in $0 \leq r < t \leq T$, provided that

$$\rho > 1 \quad \text{and} \quad s > -\frac{1}{2}\rho.$$

This proves (5.31) (and (5.30)) and thus concludes the proof of Theorem 1.14 (i).

5.4. **Shallow-water convergence of the modulated scaled ILW.** In this subsection, we establish shallow-water convergence ($\delta \to 0$) of the modulated scaled ILW (1.19) to the modulated KdV (1.2) (Theorem 1.14 (ii)). In the previous subsection, we proved the deep-water convergence by viewing the modulated ILW as a perturbation of the modulated BO. In the shallow-water regime, we can not employ this simple perturbative viewpoint and thus, we need to carry out careful analysis on the resonance function $\widetilde{\Xi}_\delta(\bar{n})$ for the scaled ILW (see (5.40)). In fact, in the low frequency regime, we view the modulated scaled ILW as a perturbation of the modulated KdV, while we view it as a perturbation of the modulated BO in the high frequency regime. When there is a mixture of high and low frequencies, we proceed with a direct computation, exhibiting an extra smoothing property. In the remaining part of this section, we assume that $0 < \delta \ll 1$.

Let $\widetilde{\mathcal{G}}_\delta$ be as in (1.20) (see also (1.17)). Then, the Fourier multiplier $\widetilde{p}_\delta(n)$ of $-i\widetilde{\mathcal{G}}_\delta\partial_x^2$ is given by

$$\widetilde{p}_\delta(n) = \frac{3}{\delta} p_\delta(n) = n^3\big(1 - h(n,\delta)\big), \tag{5.36}$$

where $p_\delta(n)$ is as in (5.18) and $h(n,\delta)$ is given by[19]

$$h(n,\delta) = \sum_{k=1}^\infty \frac{6\delta^2 n^2}{\pi^2 k^2(\pi^2 k^2 + \delta^2 n^2)} \in (0,1). \tag{5.37}$$

See [1, Lemma 8.2.1], [107, (the proof of) Lemma 2.3], and [26, Lemma 2.3]. Fix $N \in \mathbb{N}$. Then, from (5.37), we have

$$h(n,\delta) \lesssim \delta^2 N^2 \tag{5.38}$$

for any $|n| \leq N$.

[19] Recall that $\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$.

Let $\widetilde{X}^\delta$ be the bilinear driver associated with the modulated scaled ILW (1.19), given by

$$\widetilde{X}^\delta_{t,r}(f_1,f_2)=\int_r^t U^w(t')^{-1}\partial_x\big((U^w(t')f_1)(U^w(t')f_2)\big)dt', \tag{5.39}$$

where $U^w(t)=e^{w(t)\widetilde{\mathcal{G}}_\delta\partial_x^2}$. By taking the Fourier transform, we have

$$\mathcal{F}\big(\widetilde{X}^\delta_{t,r}(f_1,f_2)\big)(n)=in\sum_{\substack{n_1,n_2\in\mathbb{Z}^*\\ n=n_1+n_2}}\Phi^w_{t,r}(\widetilde{\Xi}_\delta(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2)$$

where $\Phi^w_{t,r}$ is as in (1.6) and $\widetilde{\Xi}_\delta(\bar{n})$ denotes the resonance function for the scaled ILW given by

$$\begin{aligned}\widetilde{\Xi}_\delta(\bar{n})=\widetilde{\Xi}_\delta(n,n_1,n_2)&=-\widetilde{p}_\delta(n)+\widetilde{p}_\delta(n_1)+\widetilde{p}_\delta(n_2)\\ &=-n^3\big(1-h(n,\delta)\big)+n_1^3\big(1-h(n_1,\delta)\big)+n_2^3\big(1-h(n_2,\delta)\big).\end{aligned} \tag{5.40}$$

Given $u_0\in H^s(\mathbb{T})$, let v^δ, $0<\delta\ll 1$, and u denote the local-in-time solutions to the modulated scaled ILW (1.19) and the modulated KdV (1.2) with $v^\delta|_{t=0}=u|_{t=0}=u_0$, respectively. Let $\mathbf{v}^\delta(t)=e^{-w(t)\widetilde{\mathcal{G}}_\delta\partial_x^2}v^\delta(t)$ and $\mathbf{u}(t)=e^{w(t)\partial_x^3}u(t)$ be the modulated interaction representations of v^δ and u (under the modulated scaled ILW and the modulated KdV, respectively). Then, by the unitarity of $e^{-w(t)\widetilde{\mathcal{G}}_\delta\partial_x^2}$ on $H^s(\mathbb{T})$, we have

$$\|v^\delta-u\|_{C_\tau H^s_x}\le\|\mathbf{v}^\delta-\mathbf{u}\|_{C_\tau H^s_x}+\|(e^{-w(t)\widetilde{\mathcal{G}}_\delta\partial_x^2}-e^{w(t)\partial_x^3})u\|_{C_\tau H^s_x} \tag{5.41}$$

From the mean value theorem and [107, Lemma 2.3], we have, for each $n\in\mathbb{Z}^*$,

$$\begin{aligned}\big|\mathcal{F}\big((e^{-w(t)\widetilde{\mathcal{G}}_\delta\partial_x^2}-e^{w(t)\partial_x^3})f\big)(n)\big|&\le|w(t)|\big|n^2\widehat{\widetilde{\mathcal{G}}_\delta}(n)-(-in^3)\big||\widehat{f}(n)|\\ &\longrightarrow 0,\end{aligned}$$

as $\delta\to 0$. Then, from the dominated convergence theorem, we see that the second term on the right-hand side of (5.41) tends to 0 as $\delta\to 0$. Hence, Theorem 1.14 (ii) follows once we show

$$\lim_{\delta\to 0}\|\mathbf{v}^\delta-\mathbf{u}\|_{C_\tau H^s_x}=0. \tag{5.42}$$

As in the case of the deep-water convergence, in view of Lemma 3.1 (iv), Remark 3.2, and Proposition 3.8 (iv), (5.42) follows once we prove

$$\|\widetilde{X}^\delta_{t,r}-X^{\mathrm{KdV}}_{t,r}\|_{\mathcal{L}_2(H^s)}\lesssim o(1)|t-r|^\gamma, \tag{5.43}$$

as $\delta\to 0$, uniformly in $0\le r\le t\le T$, where X^{KdV} is the driver for the modulated KdV defined in (4.3).

From (4.5) and (5.40), we have

$$\Delta\widetilde{\Xi}_\delta(\bar{n}):=\widetilde{\Xi}_\delta(\bar{n})-\Xi_{\mathrm{KdV}}(\bar{n})=n^3h(n,\delta)-n_1^3h(n_1,\delta)-n_2^3h(n_2,\delta). \tag{5.44}$$

Then, from (5.38), and (5.44), we have

$$|\Delta\widetilde{\Xi}_\delta(\bar{n})|\lesssim\delta^2N^5, \tag{5.45}$$

uniformly in $|n|,|n_1|,|n_2|\le N$. Thus, we have

$$\langle\widetilde{\Xi}_\delta(\bar{n})\rangle\sim\langle\Xi_{\mathrm{KdV}}(\bar{n})\rangle, \tag{5.46}$$

uniformly in $|n|, |n_1|, |n_2| \le N$ and $\delta \ll N^{-\frac{5}{2}}$. Then, from (1.5) and (5.46), we have

$$\begin{aligned} |\Phi^w_{t,r}(\widetilde{\Xi}_\delta(\bar{n})) - \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))| &\le |\Phi^w_{t,r}(\widetilde{\Xi}_\delta(\bar{n}))| + |\Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))| \\ &\lesssim \|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma \frac{1}{\langle \Xi_{\mathrm{KdV}}(\bar{n})\rangle^\rho} \end{aligned} \tag{5.47}$$

uniformly in $|n|, |n_1|, |n_2| \le N$, $\delta \ll N^{-\frac{5}{2}}$, and $0 \le r < t \le T$. Moreover, from (1.6), (5.45), and the mean value theorem, we have

$$\begin{aligned} |\Phi^w_{t,r}(\widetilde{\Xi}_\delta(\bar{n})) - \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))| &= \Big| \int_r^t e^{i\Xi_{\mathrm{KdV}}(\bar{n})w(t')} \big(e^{i\Delta\widetilde{\Xi}_\delta(\bar{n})w(t')} - 1\big) dt' \Big| \\ &\lesssim \delta^2 N^5 |t-r|, \end{aligned} \tag{5.48}$$

uniformly in $|n|, |n_1|, |n_2| \le N$. By interpolating (5.47) and (5.48), we have

$$\begin{aligned} &|\Phi^w_{t,r}(\widetilde{\Xi}_\delta(\bar{n})) - \Phi^w_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))| \cdot \mathbf{1}_{\{|n|,|n_1|,|n_2|\le N\}} \\ &\lesssim_T \delta^{2\varepsilon} N^{5\varepsilon} \|\Phi^w\|^{1-\varepsilon}_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma \frac{\mathbf{1}_{\{|n|,|n_1|,|n_2|\le N\}}}{\langle \Xi_{\mathrm{KdV}}(\bar{n})\rangle^{\rho(1-\varepsilon)}}. \end{aligned} \tag{5.49}$$

for any $0 \le \varepsilon \le 1$.

Given $N \in \mathbb{N}$, let $X^{\mathrm{KdV},N}_{t,r}$ be the frequency-truncated driver defined in (4.7). Similarly, let $\widetilde{X}^{\delta,N}_{t,r}$ be the frequency-truncated version of $\widetilde{X}^\delta$ in (5.39). Then, proceeding as in the proof of Proposition 4.1 (i) (where we replace ρ by $\rho(1-\varepsilon)$ for small $\varepsilon > 0$) with (5.49),

$$\|\widetilde{X}^{\delta,N}_{t,r} - X^{\mathrm{KdV},N}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim_T \delta^{2\varepsilon} N^{5\varepsilon} \|\Phi^w\|^{1-\varepsilon}_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^\gamma = o(1)|t-r|^\gamma, \tag{5.50}$$

as $\delta \to 0$, uniformly in $0 \le r < t \le T$, provided that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy (1.24). From the proof of Proposition 4.1 (iv), we have

$$\|X^{\mathrm{KdV},N}_{t,r} - X^{\mathrm{KdV}}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim o(1)|t-r|^\gamma, \tag{5.51}$$

as $N \to \infty$, uniformly in $0 \le r < t \le T$. Hence, by the triangle inequality with (5.50) and (5.51), the bound (5.43) follows once we prove the following uniform bound:

$$\sup_{0<\delta\ll 1} \|\widetilde{X}^\delta_{t,r} - \widetilde{X}^{\delta,N}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim o(1)|t-r|^\gamma, \tag{5.52}$$

as $N \to \infty$.

Let us first take a close look at the resonance function $\widetilde{\Xi}_\delta(\bar{n})$ defined in (5.40). A direct computation with (5.36) and (5.37) yields

$$\widetilde{\Xi}_\delta(\bar{n}) = -6nn_1n_2 \sum_{k=1}^\infty \frac{\pi^2 k^2 \big(3\pi^2 k^2 + \delta^2(n_1^2 + n_1n_2 + n_2^2)\big)}{\prod_{j=0}^2 (\pi^2 k^2 + \delta^2 n_j^2)} \tag{5.53}$$

under $n = n_1 + n_2$, where $n_0 := n$. Write $|n|$, $|n_1|$, and $|n_2|$ in the non-increasing order and denote them by $n_{\max}$, n_{med}, and $n_{\min}$, respectively.

• **Case 1:** $n_{\max} \lesssim \delta^{-1}$.

Denote by $\widetilde{X}^{\delta,(1)}$ and $\widetilde{X}^{\delta,N,(1)}$ the contributions to $\widetilde{X}^\delta$ and $\widetilde{X}^{\delta,N}$, respectively, from the frequency region $n_{\max} \lesssim \delta^{-1}$. In this case, from (5.53), we have

$$|\widetilde{\Xi}_\delta(\bar{n})| \sim |nn_1n_2| \sum_{k=1}^\infty \frac{1}{\pi^2 k^2} \sim |nn_1n_2| \sim |\Xi_{\mathrm{KdV}}(\bar{n})|, \tag{5.54}$$

uniformly in $0 < \delta \ll 1$. Hence, by repeating the proof of Proposition 4.1 (iv) with (5.54), we have

$$\sup_{0<\delta\ll 1} \|\widetilde{X}^{\delta,(1)}_{t,r} - \widetilde{X}^{\delta,N,(1)}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim o(1)|t-r|^\gamma, \tag{5.55}$$

as $N \to \infty$, provided that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy (1.24).

• **Case 2:** $n_{\min} \gg \delta^{-1}$.

Denote by $\widetilde{X}^{\delta,(2)}$ and $\widetilde{X}^{\delta,N,(2)}$ the contributions to $\widetilde{X}^\delta$ and $\widetilde{X}^{\delta,N}$, respectively, from the frequency region $n_{\min} \gg \delta^{-1}$. From (a)-(e) in the proof of Proposition 5.1 (i) we have

$$|\Xi_{\mathrm{BO}}(\bar{n})| \gtrsim n_{\max} n_{\min} \gg \delta^{-2}. \tag{5.56}$$

Then, from (5.21) and (5.56), we obtain

$$|\Xi_\delta(\bar{n}) - \Xi_{\mathrm{BO}}(\bar{n})| = O(\delta^{-2}) \ll |\Xi_{\mathrm{BO}}(\bar{n})|.$$

from which we conclude that

$$|\Xi_\delta(\bar{n})| \sim |\Xi_{\mathrm{BO}}(\bar{n})| \tag{5.57}$$

uniformly in $0 < \delta \ll 1$. Thus, from (5.57) with (5.40), (5.36), and (5.17), we obtain

$$|\widetilde{\Xi}_\delta(\bar{n})| \sim \delta^{-1}|\Xi_{\mathrm{BO}}(\bar{n})| \gg |\Xi_{\mathrm{BO}}(\bar{n})|, \tag{5.58}$$

uniformly in $0 < \delta \ll 1$. Hence, by repeating the proof of Proposition 5.1 (iv) with (5.58), we have

$$\sup_{0<\delta\ll 1} \|\widetilde{X}^{\delta,(2)}_{t,r} - \widetilde{X}^{\delta,N,(2)}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim o(1)|t-r|^\gamma, \tag{5.59}$$

as $N \to \infty$, provided that $\rho > 1$ and $s > -\frac{1}{2}\rho$.

• **Case 3:** $n_{\max} \gg \delta^{-1} \gtrsim n_{\min}$.

Denote by $\widetilde{X}^{\delta,(3)}$ and $\widetilde{X}^{\delta,N,(3)}$ the contributions to $\widetilde{X}^\delta$ and $\widetilde{X}^{\delta,N}$, respectively, in this case. Assume that $|n_1| \le |n_2|$. Since $n_{\mathrm{med}} \sim n_{\max}$ under $n = n_1 + n_2$, it follows from (5.53) with $n_1 n_2 \le \frac{1}{2}n_1^2 + \frac{1}{2}n_2^2$ that

$$|\widetilde{\Xi}_\delta(\bar{n})| \sim |n n_1 n_2| \sum_{k=1}^\infty \frac{\pi^2 k^2}{\prod_{j=0}^1 (\pi^2 k^2 + \delta^2 n_j^2)}$$

Without loss of generality, assume that $|n| = |n_0| \le |n_1|$. Then, we have $|n_1| \sim n_{\max} \gg \delta^{-1} \gtrsim n_{\min} = |n|$. In particular, we have $\pi^2 k^2 + \delta^2 n^2 \sim \pi^2 k^2$. Then, by estimating from below by an integral, we have

$$\begin{aligned} |\widetilde{\Xi}_\delta(\bar{n})| &\sim \delta^{-1}|n n_2| \sum_{k=1}^\infty \frac{1}{\pi^2(\frac{k}{\delta n_1})^2 + 1} \frac{1}{\delta |n_1|} \\ &\gtrsim n_{\min}^2 n_{\max} \int_0^\infty \frac{1}{\pi^2 x^2 + 1} dx \\ &\sim n_{\min}^2 n_{\max}, \end{aligned} \tag{5.60}$$

uniformly in $0 < \delta \ll 1$. Hence, by repeating the proof of Proposition 5.1 (iv) with (5.60), we have

$$\sup_{0<\delta\ll 1} \|\widetilde{X}^{\delta,(3)}_{t,r} - \widetilde{X}^{\delta,N,(3)}_{t,r}\|_{\mathcal{L}_2(H^s)} \lesssim o(1)|t-r|^\gamma, \tag{5.61}$$

as $N \to \infty$, provided that $\rho > 1$ and $s > -\frac{1}{2}\rho$.

Putting (5.55), (5.59), and (5.61) together, we obtain (5.52), provided that $\rho > 1$ and $s > -\frac{1}{2}\rho$. Therefore, under the same condition, we obtain (5.43) from the triangle inequality, (5.50), (5.51), and (5.52), where we first take $\delta \to 0$ and then $N \to \infty$. This concludes the proof of Theorem 1.14 (ii).

6. Global well-posedness in L^2

In this section, our main goal is to establish the following proposition on the conservation of the L^2-norm of a solution to the modulated dispersive equations we considered in Sections 4 and 5.

Proposition 6.1. *Given $\rho > 0$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1. Given $s \geq 0$, suppose that the driver X defined in* (4.3) *belongs to $\mathcal{X}_2^{s,\gamma}([0,T] \times \mathbb{T})$ and that u is a solution to the modulated KdV equation* (1.2) *on $\mathbb{T}$, whose modulated interaction representation $\mathbf{u}$ defined in* (1.7) *belongs to $\mathcal{C}^\gamma([0,T]; H^s(\mathbb{T}))$, satisfying*

$$\mathbf{u} = u_0 + \mathcal{I}^X(\mathbf{u}). \tag{6.1}$$

Here, $\mathcal{I}^X(\mathbf{u})$ denotes the nonlinear Young integral of $\mathbf{u}$ with the driver X. Then, we have

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2} \tag{6.2}$$

for any $0 \leq t \leq T$.

With appropriate minor modifications on the driver space $X \in \mathcal{X}_k^{s,\gamma}([0,T] \times \mathcal{M})$, the L^2-conservation (6.2) *also holds for the modulated KdV equation on $\mathbb{R}$, the modulated mKdV, BO, ILW, and scaled ILW equations on $\mathbb{T}$.*

Once we prove Proposition 6.1, global well-posedness in $H^s(\mathcal{M})$, $s \geq 0$, holds for the following cases:

(i) the modulated KdV (1.2) on $\mathcal{M} = \mathbb{T}$ for $\rho > \frac{1}{2}$,

(ii) the modulated KdV (1.2) on $\mathcal{M} = \mathbb{R}$ for $\rho > \frac{1}{2}$,

(iii) the modulated BO (1.15), the modulated ILW (1.16), and the modulated scaled ILW (1.19) on $\mathcal{M} = \mathbb{T}$ for $\rho \geq 1$ (Theorem 1.12 (ii)).

These global well-posedness results follow from Proposition 6.1 together with the persistence-of-regularity results (Proposition 3.8 (ii) with Propositions 4.1 (ii), 4.4 (ii), 5.1 (ii), and 5.2 (ii)) which state that the local existence time of an H^s-solution depends only on the L^2-norm of initial data (besides the $\mathcal{X}_k^{s,\gamma}(T)$-norm of the associated driver X). Since the argument is standard, we omit details.

Proof of Proposition 6.1. We only consider the periodic modulated KdV case. The claim for the other equations follows from a similar computation.

Let X be as in (4.3). For simplicity, we set $X_{t,r}(f) = X_{t,r}(f,f)$. Then, from the unitarity of $U^w(t)$ on $L^2(\mathbb{T})$, we have

$$\begin{aligned}\langle f, X_{t,r}(f)\rangle_{L^2} &= \int_r^t \int_{\mathbb{T}} f \cdot U^w(t')^{-1}\partial_x\big((U^w(t')f)^2\big)dxdt' \\ &= \int_r^t \int_{\mathbb{T}} U^w(t')f \cdot \partial_x\big((U^w(t')f)^2\big)dxdt' \\ &= \frac{2}{3}\int_r^t \int_{\mathbb{T}} \partial_x\big((U^w(t')f)^3\big)dxdt' = 0\end{aligned} \tag{6.3}$$

for any $0 \le r < t \le T$ and $f \in L^2(\mathbb{T})$. Consequently, we obtain

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^2}^2 &= \|\mathbf{u}(r)\|_{L^2}^2 + 2\big\langle \mathbf{u}(r), X_{t,r}(\mathbf{u}(r))\big\rangle_{L^2} + \|\mathbf{u}(t)-\mathbf{u}(r)\|_{L^2}^2 + R_{t,r} \\ &= \|\mathbf{u}(r)\|_{L^2}^2 + \|\mathbf{u}(t)-\mathbf{u}(r)\|_{L^2}^2 + R_{t,r},\end{aligned} \tag{6.4}$$

where

$$R_{t,r} = 2\big\langle \mathbf{u}(r), \mathbf{u}(t)-\mathbf{u}(r)-X_{t,r}(\mathbf{u}(r))\big\rangle_{L^2}.$$

By assumption, we have $\mathbf{u} \in \mathcal{C}^\gamma([0,T];L^2(\mathbb{T}))$ with $\gamma > \frac{1}{2}$. Then, from (6.1) and (3.43) in Proposition 3.7 (see also (3.17)), we have

$$\begin{aligned}|R_{t,r}| &\le 2\|\mathbf{u}(r)\|_{L^2}\|\mathbf{u}(t)-\mathbf{u}(r)-X_{t,r}(\mathbf{u}(r))\|_{L^2} \\ &= 2\|\mathbf{u}(r)\|_{L^2}\|(\delta\mathcal{I}^X(\mathbf{u}))_{t,r} - X_{t,r}(\mathbf{u}(r))\|_{L^2} \\ &\le C\big(\|\mathbf{u}\|_{\mathcal{C}_T^\gamma L_x^2}\big)\|X\|_{\mathcal{X}_k^{s,\gamma}(T)}|t-r|^{2\gamma}.\end{aligned} \tag{6.5}$$

In addition, we have

$$\|\mathbf{u}(t)-\mathbf{u}(r)\|_{L^2}^2 \le \|\mathbf{u}\|_{C_T^\gamma L_x^2}^2|t-r|^{2\gamma} \tag{6.6}$$

for any $0 \le r < t \le T$. Hence, from (6.4), (6.5), and (6.6), we obtain

$$\big|\|\mathbf{u}(t)\|_{L^2}^2 - \|\mathbf{u}(r)\|_{L^2}^2\big| \lesssim C\big(\|\mathbf{u}\|_{\mathcal{C}_T^\gamma L_x^2}, \|X\|_{\mathcal{X}_k^{s,\gamma}(T)}\big)|t-r|^{2\gamma}. \tag{6.7}$$

By recalling $2\gamma > 1$, the L^2-conservation (6.2) then follows from (6.7). □

7. Global well-posedness of the modulated KdV in negative Sobolev spaces

In this section, we prove global well-posedness of the periodic modulated KdV (1.2) in negative Sobolev spaces (Theorem 1.3 (ii)). Our strategy is to adapt the so-called I-method (= the method of almost conservation laws), introduced by Colliander, Keel, Staffilani, Takaoka, and Tao [38], to the current modulated setting.

Fix $s < 0$ in the remaining part of this section. Given $N \ge 1$, we define a smooth, even, non-increasing (in $|\xi|$) function $m_{s,N} \in C^\infty(\mathbb{R};[0,1])$ by setting

$$m_{s,N}(\xi) = \begin{cases} 1, & \text{for } |\xi| \le N, \\ \frac{|\xi|^s}{N^s}, & \text{for } |\xi| \ge 2N. \end{cases} \tag{7.1}$$

Then, we define the so-called I-operator $I = I_{s,N}$ to be the Fourier multiplier operator with the multiplier $m_{s,N}$:

$$\widehat{I_{s,N}f}(\xi) = m_{s,N}(\xi)\widehat{f}(\xi). \tag{7.2}$$

Note that $I_{s,N}$ acts as the ***i***dentity operator on low frequencies $\{|\xi| \le N\}$, while it acts as a (fractional) ***i***ntegration operator (recall $s < 0$) on high frequencies $\{|\xi| \ge 2N\}$ with the following bound:

$$\|f\|_{H^s(\mathbb{T})} \le \|If\|_{L^2(\mathbb{T})} \lesssim N^{-s}\|f\|_{H^s(\mathbb{T})}, \tag{7.3}$$

and hence the name ***I***-method.

We proved local well-posedness of the modulated KdV (1.2) on $\mathbb{T}$ by applying the abstract subcritical local well-posedness result (Proposition 3.8). Thus, it follows from the blowup alternative (3.50) in Remark 3.9 (i) that global well-posedness of the periodic modulated KdV follows once we show that the H^s-norm of a solution $u(t)$ remains finite for each finite time $t > 0$. In view of (7.3), it then suffices to control the so-called modified energy $\|Iu(t)\|_{L^2}^2$.[20]

In Subsection 7.1, we go over basic scaling properties of the periodic modulated KdV. In Subsection 7.2, we then study local well-posedness of the so-called (scaled) modulated I-KdV equation (7.20). After establishing a crucial commutator estimate (Proposition 7.6) in Subsection 7.3, we establish global well-posedness by combining the commutator estimate (Proposition 7.6) with the sewing lemma (Lemma 3.3) in Subsection 7.4. It is worthwhile to note that we need to make use of the sewing lemma even in the globalization argument via the I-method which is a new feature in the current modulated setting. We also discuss the real line case in Subsection 7.5.

7.1. **Scaling properties of the modulated KdV.** In this subsection, we discuss the scaling properties of the modulated KdV (1.2) on $\mathbb{T}$ which will be used to make the problem at hand into a small data problem; see (7.10), (7.48), and (7.50). We first recall the scaling invariance of the (unmodulated) KdV (1.3); a function u is a solution to KdV (1.3) on $[0,T] \times \mathbb{T}$ if and only if the scaled function

$$u^\lambda(t,x) = \lambda^{-2}u(\lambda^{-3}t, \lambda^{-1}x) \tag{7.4}$$

is a solution to KdV on $[0, \lambda^3 T] \times \mathbb{T}_\lambda$, where $\mathbb{T}_\lambda = \mathbb{R}/(2\pi\lambda\mathbb{Z})$ denotes the dilated circle. We note that, unlike the real line setting, the spatial domain is dilated under the scaling (7.4).

In the current modulated setting, we formally see that u is a solution to the modulated KdV (1.2) on $[0,T] \times \mathbb{T}$ with initial data u_0 if and only if u^λ in (7.4) is a solution to the following scaled modulated KdV on $[0, \lambda^3 T] \times \mathbb{T}_\lambda$:

$$\partial_t u^\lambda + \partial_x^3 u^\lambda \cdot \partial_t w^\lambda = \partial_x (u^\lambda)^2 \tag{7.5}$$

with the scaled initial data

$$u_0^\lambda(x) = \lambda^{-2}u_0(\lambda^{-1}x), \tag{7.6}$$

where the scaled modulation w^λ is given by

$$w^\lambda(t) = \lambda^3 w(\lambda^{-3}t) \tag{7.7}$$

such that $\partial_t w^\lambda(t) = \partial_t w(\lambda^{-3}t)$. Namely, the scaling (7.7) on the modulation function is $\dot{W}^{1,\infty}$-invariant. See Lemma 7.3 for equivalence of the unscaled and scaled modulated KdV equations. In the following, we study the scaled modulated KdV (7.5) on the dilated circle $\mathbb{T}_\lambda$ with $\lambda \ge 1$.

[20] As we see below, given a target time $T > 0$, we in fact apply a scaling first and study the scaled modified energy in (7.27).

Before proceeding further, we introduce some notations. Set $\mathbb{Z}_\lambda = \mathbb{Z}/\lambda$ and $\mathbb{Z}_\lambda^* = \mathbb{Z}^*/\lambda$. We define the Fourier transform of a function f on $\mathbb{T}_\lambda$ by

$$\widehat{f}(n) = \int_{\mathbb{T}_\lambda} f(x) e^{-inx} \frac{dx}{2\pi}$$

for $n \in \mathbb{Z}_\lambda$. Then, we have

$$\begin{aligned} f(x) &= \frac{1}{\lambda} \sum_{n\in\mathbb{Z}_\lambda} \widehat{f}(n) e^{inx}, \\ \int_{\mathbb{T}_\lambda} |f(x)|^2 \frac{dx}{2\pi} &= \frac{1}{\lambda} \sum_{n\in\mathbb{Z}_\lambda} |\widehat{f}(n)|^2, \\ \widehat{fg}(n) &= \frac{1}{\lambda} \sum_{\substack{n_1,n_2\in\mathbb{Z}_\lambda \\ n=n_1+n_2}} \widehat{f}(n_1)\widehat{g}(n_2). \end{aligned} \tag{7.8}$$

Given $s \in \mathbb{R}$, we define the non-homogeneous and homogeneous Sobolev spaces $H^s(\mathbb{T}_\lambda)$ and $\dot{H}^s(\mathbb{T}_\lambda)$ via the (semi-)norms:

$$\begin{aligned} \|f\|_{H^s(\mathbb{T}_\lambda)}^2 &= \frac{1}{\lambda} \sum_{n\in\mathbb{Z}_\lambda} \langle n\rangle^{2s} |\widehat{f}(n)|^2, \\ \|f\|_{\dot{H}^s(\mathbb{T}_\lambda)}^2 &= \frac{1}{\lambda} \sum_{n\in\mathbb{Z}_\lambda} |n|^{2s} |\widehat{f}(n)|^2. \end{aligned} \tag{7.9}$$

When $\lambda = 1$, the $H^s(\mathbb{T})$-norm and $\dot{H}^s(\mathbb{T})$-norm are equivalent in the current mean-zero setting. When $\lambda \gg 1$, we still have $\|f\|_{H^s(\mathbb{T}_\lambda)} \sim_\lambda \|f\|_{\dot{H}^s(\mathbb{T}_\lambda)}$ for a mean-zero function f on $\mathbb{T}_\lambda$ but the implicit constant diverges as $\lambda \to \infty$. In the following, we will work with the homogeneous Sobolev spaces, since they enjoy a better scaling property; noting from (7.6) that $\mathcal{F}_{\mathbb{T}_\lambda}(u_0^\lambda)(n) = \lambda^{-1}\mathcal{F}_{\mathbb{T}}(u_0)(\lambda n)$, $n \in \mathbb{Z}_\lambda$, we have

$$\|u_0^\lambda\|_{\dot{H}^s(\mathbb{T}_\lambda)} = \lambda^{-\frac{3}{2}-s} \|u_0\|_{\dot{H}^s(\mathbb{T})} \tag{7.10}$$

for any $s \in \mathbb{R}$, while, for the non-homogeneous Sobolev space, we have

$$\|u_0^\lambda\|_{H^s(\mathbb{T}_\lambda)} \le \lambda^{-\frac{3}{2}-s} \|u_0\|_{H^s(\mathbb{T})}$$

for $s \le 0$; see also Remark 7.2. In particular, when $s > -\frac{3}{2}$ (namely, in the scaling subcritical regime), we can make the $\dot{H}^s(\mathbb{T}_\lambda)$-norm of the scaled initial data u_0^λ small by choosing $\lambda \gg_{\|u_0\|_{H^s(\mathbb{T})}} 1$.

Now, we are ready to study the scaled modulated KdV (7.5) on $\mathbb{T}_\lambda$ with the scaled initial data u_0^λ in (7.6). The Duhamel formulation for the modulated interaction representation

$$\mathbf{u}^\lambda(t) = U^{w^\lambda}(t)^{-1} u^\lambda(t) \tag{7.11}$$

is given by

$$\mathbf{u}^\lambda(t) = u_0^\lambda + \int_0^t U^{w^\lambda}(t')^{-1} \partial_x \big((U^{w^\lambda}(t') \mathbf{u}^\lambda(t'))^2 \big) dt' \tag{7.12}$$

with the associated bilinear driver X^λ given by

$$X_{t,r}^\lambda(f_1, f_2) = \int_r^t U^{w^\lambda}(t')^{-1} \partial_x \big((U^{w^\lambda}(t') f_1)(U^{w^\lambda}(t') f_2) \big) dt' \tag{7.13}$$

for functions f_1 and f_2 on $\mathbb{T}_\lambda$. Then, local well-posedness of the scaled modulated KdV (7.5) in $\dot{H}^s(\mathbb{T}_\lambda)$ follows from Proposition 3.8 and the following lemma.

Lemma 7.1. *Let $\lambda \geq 1$. Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1, and let X^λ be as in (7.13). Suppose that $\rho \geq \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy (1.9). Then, the driver X^λ in (7.13) belongs to $\dot{\mathcal{X}}_2^{s,\gamma}([0, \lambda^3 T] \times \mathbb{T}_\lambda)$ defined in (2.12), satisfying the bound:*

$$\|X^\lambda_{t,r}\|_{\mathcal{L}_2(\dot{H}^s(\mathbb{T}_\lambda))} \lesssim \lambda^{\frac{3}{2}+s-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma \tag{7.14}$$

for any $0 \leq r < t \leq \lambda^3 T$.

Proof. Proceeding as in (4.8) with (7.8) and (7.9), we have

$$\begin{aligned} \|X^\lambda_{t,r}(f_1, f_2)\|^2_{\dot{H}^s(\mathbb{T}_\lambda)} &= \lambda^{-3} \sum_{n \in \mathbb{Z}^*_\lambda} |n|^{2s+2} \bigg| \sum_{\substack{n_1, n_2 \in \mathbb{Z}^*_\lambda \\ n = n_1 + n_2}} \Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n})) \widehat{f_1}(n_1) \widehat{f_2}(n_2) \bigg|^2 \\ &\leq \lambda^{-1} \bigg(\sup_{n_1 \in \mathbb{Z}^*_\lambda} \sum_{\substack{n, n_2 \in \mathbb{Z}^*_\lambda \\ n_1 = n - n_2}} |n|^{2s+2} \frac{|\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))|^2}{|n_1|^{2s} |n_2|^{2s}} \bigg) \|f_1\|^2_{\dot{H}^s(\mathbb{T}_\lambda)} \|f_2\|^2_{\dot{H}^s(\mathbb{T}_\lambda)}, \end{aligned} \tag{7.15}$$

where $\Phi^{w^\lambda}_{t,r}$ and $\Xi_{\mathrm{KdV}}(\bar{n})$ are as in (1.6) and (4.5), respectively. From (1.6), (7.7), (1.5), and (4.6), we have

$$\begin{aligned} |\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))| &= \bigg| \int_r^t e^{i \Xi_{\mathrm{KdV}}(\bar{n}) w^\lambda(t')} dt' \bigg| = \lambda^3 \bigg| \int_{\lambda^{-3} r}^{\lambda^{-3} t} e^{i \lambda^3 \Xi_{\mathrm{KdV}}(\bar{n}) w(t')} dt' \bigg| \\ &\lesssim \lambda^{3-3\rho-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma |n n_1 n_2|^{-\rho} \end{aligned} \tag{7.16}$$

for any $n, n_1, n_2 \in \mathbb{Z}^*_\lambda$, satisfying $n = n_1 + n_2$.

Given $n, n_1, n_2 \in \mathbb{Z}^*_\lambda$, let $m = \lambda n$ and $m_j = \lambda n_j$, $j = 1, 2$. Then, by proceeding as in the proof of Proposition 4.1 (i) with (7.15) and (7.16), we have

$$\begin{aligned} \|X^\lambda_{t,r}\|_{\mathcal{L}_2(\dot{H}^s(\mathbb{T}_\lambda))} &\lesssim \lambda^{\frac{5}{2}-3\rho-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma \\ &\quad \times \bigg(\sup_{n_1 \in \mathbb{Z}^*_\lambda} \sum_{\substack{n, n_2 \in \mathbb{Z}^*_\lambda \\ n_1 = n - n_2}} |n|^{2-4\rho} \frac{|n|^{2s+2\rho}}{|n_1|^{2s+2\rho} |n_2|^{2s+2\rho}} \bigg)^{\frac{1}{2}} \\ &\lesssim \lambda^{\frac{3}{2}+s-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma \\ &\quad \times \bigg(\sup_{m_1 \in \mathbb{Z}^*} \sum_{\substack{m, m_2 \in \mathbb{Z}^* \\ m_1 = m - m_2}} |m|^{2-4\rho} \frac{|m|^{2s+2\rho}}{|m_1|^{2s+2\rho} |m_2|^{2s+2\rho}} \bigg)^{\frac{1}{2}} \\ &\lesssim \lambda^{\frac{3}{2}+s-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma \end{aligned}$$

for any $0 \leq r < t \leq \lambda^3 T$, provided that (1.9) holds. Hence, from Lemma 3.1 (i) and Remark 3.2 (ii), we conclude that $X^\lambda \in \dot{\mathcal{X}}_2^{s,\gamma}([0, \lambda^3 T] \times \mathbb{T}_\lambda)$ under (1.9). □

Remark 7.2. By repeating the proof for the non-homogeneous Sobolev space $H^s(\mathbb{T}_\lambda)$ with

$$\lambda^{-1} \langle m \rangle \lesssim \langle n \rangle \lesssim \langle m \rangle,$$

we obtain

$$\|X^\lambda_{t,r}\|_{\mathcal{L}_2(H^s(\mathbb{T}_\lambda))} \lesssim \lambda^{\frac{3}{2}-s-3\gamma}\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^\gamma,$$

for $\lambda \geq 1$, which gives a worse power of λ as compared to (7.14), yielding a worse regularity threshold for global well-posedness on the unscaled modulated KdV (1.2).

The next lemma shows equivalence of the unscaled and scaled problems. See Definition 1.2 for the definition of the class $\dot{\mathscr{D}}^s_w(I\times\mathcal{M})$.

Lemma 7.3. *Let $\lambda \geq 1$. Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Fix $s \in \mathbb{R}$ satisfying (1.9) and $u_0 \in H^s(\mathbb{T})$. Then, $u \in \dot{\mathscr{D}}^s_w([0,\tau]\times\mathbb{T})$ is a solution to the modulated KdV (1.2) on $[0,\tau]\times\mathbb{T}$ with $u|_{t=0} = u_0$ for some $0 < \tau \leq T$ if and only if the scaled function u^λ defined in (7.4) belongs to $\dot{\mathscr{D}}^s_{w^\lambda}([0,\lambda^3\tau]\times\mathbb{T}_\lambda)$ and is a solution to the scaled modulated KdV (7.5) on $[0,\lambda^3\tau]\times\mathbb{T}_\lambda$ with the scaled initial data $u^\lambda|_{t=0} = u^\lambda_0$ defined in (7.6).*

Proof. We only show the forward direction. Suppose that $u \in \dot{\mathscr{D}}^s_w([0,\tau]\times\mathbb{T})$ is a solution to the modulated KdV (1.2) on $[0,\tau]\times\mathbb{T}$ with $u|_{t=0} = u_0$. From Definition 1.2, the modulated interaction representation $\mathbf{u}(t) = U^w(t)^{-1}u(t)$ belongs to $\mathcal{C}^\gamma([0,\tau];\dot{H}^s(\mathbb{T}))$. Then, a direct computation with (7.4) shows that

$$\mathbf{u}^\lambda(t) = U^{w^\lambda}(t)^{-1}u^\lambda(t) \in \mathcal{C}^\gamma([0,\lambda^3\tau];\dot{H}^s(\mathbb{T}_\lambda)).$$

Namely, $u^\lambda \in \dot{\mathscr{D}}^s_{w^\lambda}([0,\lambda^3\tau]\times\mathbb{T}_\lambda)$.

Now, we need to check that u^λ is indeed a solution to the scaled modulated KdV (7.5) on the time interval $[0,\lambda^3\tau]$ in the sense that its modulated interaction representation $\mathbf{u}^\lambda$ satisfies (7.12), where we interpret the integral as a nonlinear Young integral.

Let f_1 and f_2 be functions on $\mathbb{T}_\lambda$. Then, from (7.13) with (7.8), (1.6), (7.7), a change of variables, (4.5), and (4.4), we have

$$\begin{aligned}
\mathcal{F}_{\mathbb{T}_\lambda}\big(X^\lambda_{t,r}(f_1,f_2)\big)(n) &= \frac{in}{\lambda}\sum_{\substack{n_1,n_2\in\mathbb{Z}^*_\lambda\\ n=n_1+n_2}} \Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2)\\
&= in\lambda^2\sum_{\substack{m_1,m_2\in\mathbb{Z}^*\\ \lambda n=m_1+m_2}} \Phi^{w}_{\lambda^{-3}t,\lambda^{-3}r}(\Xi_{\mathrm{KdV}}(\lambda n,m_1,m_2))\widehat{f_1}(\lambda^{-1}m_1)\widehat{f_2}(\lambda^{-1}m_2) \qquad (7.17)\\
&= \lambda^{-1}\mathcal{F}_{\mathbb{T}}\big(X_{\lambda^{-3}t,\lambda^{-3}r}(f_1^{\lambda^{-1}},f_2^{\lambda^{-1}})\big)(\lambda n)
\end{aligned}$$

for any $n \in \mathbb{Z}^*_\lambda$ and $0 \leq r < t \leq \lambda^3\tau$, where $m_j = \lambda n_j$ and $f_j^{\lambda^{-1}}(x) = \lambda^2 f_j(\lambda x)$ is a function on $\mathbb{T}$, $j = 1,2$. Thus, from (7.17) with (7.4), we have

$$X^\lambda_{t,r}(\mathbf{u}^\lambda(r),\mathbf{u}^\lambda(r))(x) = \lambda^{-2}X_{\lambda^{-3}t,\lambda^{-3}r}(\mathbf{u}(\lambda^{-3}r),\mathbf{u}(\lambda^{-3}r))(\lambda^{-1}x) \qquad (7.18)$$

for $x \in \mathbb{T}_\lambda$. Since $\mathbf{u}$ is a solution to (7.12) on the time interval $[0,\tau]$ with $\lambda = 1$, where we interpret the integral as a nonlinear Young integral, it follows from (3.43) in Proposition 3.7 (adapted to the homogeneous Sobolev space) that

$$\|\mathbf{u}(\lambda^{-3}t) - \mathbf{u}(\lambda^{-3}r) - X_{\lambda^{-3}t,\lambda^{-3}r}(\mathbf{u}(\lambda^{-3}r),\mathbf{u}(\lambda^{-3}r))\|_{\dot{H}^s(\mathbb{T})} \lesssim |t-r|^{2\gamma} \qquad (7.19)$$

for any $0 \leq \lambda^{-3}r < \lambda^{-3}t \leq \tau$. Then, from (7.19) with (7.4), (7.18), and (7.10), we have

$$\|\mathbf{u}^\lambda(t) - \mathbf{u}^\lambda(r) - X^\lambda_{t,r}(\mathbf{u}^\lambda(r),\mathbf{u}^\lambda(r))\|_{\dot{H}^s(\mathbb{T}_\lambda)} \lesssim \lambda^{-\frac{3}{2}-s}|t-r|^{2\gamma}$$

for any $0 \le r < t \le \lambda^3\tau$. Let $\Theta_{t,r} = X^\lambda_{t,r}(\mathbf{u}^\lambda(r), \mathbf{u}^\lambda(r))$. Noting that $2\gamma > 1$, we then obtain

$$\mathbf{u}^\lambda(t) - u_0^\lambda = \lim_{|\Pi([0,t])|\to 0} \sum_{j=0}^{n-1} \Theta_{t_j,t_{j+1}} = \mathcal{I}^{X^\lambda}(\mathbf{u}^\lambda)(t)$$

for any $0 < t \le \lambda^3\tau$, where the limit is in the sense of Lemma 3.3 (iii) and the second equality follows from the uniqueness of the nonlinear Young integral with the driver X^λ. This proves that $\mathbf{u}^\lambda$ is a solution to (7.12) on $[0, \lambda^3\tau]$. □

7.2. **Modulated I-KdV equation.** In this subsection, we go over local well-posedness of the following scaled modulated I-KdV on the dilated circle $\mathbb{T}_\lambda$:

$$\partial_t I u^\lambda + \partial_x^3 I u^\lambda \cdot \partial_t w^\lambda = \partial_x I\big((u^\lambda)^2\big). \tag{7.20}$$

By writing $\partial_x I\big((u^\lambda)^2\big) = \partial_x I\big((I^{-1}Iu^\lambda)(I^{-1}Iu^\lambda)\big)$, we define the associated bilinear driver Y^λ by setting

$$Y^\lambda(f_1, f_2) = IX^\lambda(I^{-1}f_1, I^{-1}f_2) \tag{7.21}$$

for functions f_1 and f_2 on $\mathbb{T}_\lambda$, where X^λ is as in (7.13).

Lemma 7.4. *Let $\lambda \ge 1$. Given $\rho \ge \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1, and let Y^λ be as in (7.21). Then, for any $s < 0$ satisfying (1.9), the driver Y^λ in (7.21) belongs to $\mathcal{X}_2^{0,\gamma}([0,\lambda^3 T]\times\mathbb{T}_\lambda)$ defined in (2.9), satisfying the bound:*

$$\|Y^\lambda_{t,r}\|_{\mathcal{L}_2(L^2(\mathbb{T}_\lambda))} \lesssim \lambda^{\frac{3}{2}-3\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^\gamma \tag{7.22}$$

for any $0 \le r < t \le \lambda^3 T$.

Proof. In view of Lemma 3.1 (i), it suffices to prove (7.22). Moreover, by the interpolation lemma (see [39, Lemma 12.1]), it suffices to prove (7.22) for $N = 1$.

Proceeding as in (7.15) with (7.21), we have

$$\begin{aligned}
&\|Y^\lambda_{t,r}(f_1,f_2)\|^2_{L^2(\mathbb{T}_\lambda)}\\
&= \lambda^{-3}\sum_{n\in\mathbb{Z}^*_\lambda}|n|^2\bigg|\sum_{\substack{n_1,n_2\in\mathbb{Z}^*_\lambda\\ n=n_1+n_2}} M(\bar n)\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar n))\widehat{f_1}(n_1)\widehat{f_2}(n_2)\bigg|^2\\
&\le \lambda^{-1}\bigg(\sup_{n_1\in\mathbb{Z}^*_\lambda}\sum_{\substack{n,n_2\in\mathbb{Z}^*_\lambda\\ n_1=n-n_2}}|n|^2|M(\bar n)|^2|\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar n))|^2\bigg)\|f_1\|^2_{L^2(\mathbb{T}_\lambda)}\|f_2\|^2_{L^2(\mathbb{T}_\lambda)},
\end{aligned} \tag{7.23}$$

where $M(\bar n) = M(n, n_1, n_2)$ is defined by

$$M(\bar n) = M(n,n_1,n_2) = \frac{m_{s,1}(n)}{m_{s,1}(n_1)m_{s,1}(n_2)}. \tag{7.24}$$

Then, by comparing (7.23) with (7.15), the bound (7.22) follows from a slight modification of the proof of Lemma 7.1 with

$$M(\bar n) \lesssim \lambda^{-s}\frac{|n|^s}{|n_1|^s|n_2|^s}$$

for any $n, n_1, n_2 \in \mathbb{Z}^*_\lambda$ with $n = n_1 + n_2$, which follows from (7.24), (7.1), and case by case analysis (recall that $s < 0$). □

By noting that we have $\lambda^{\frac{3}{2}-3\gamma} \leq 1$ for any $\lambda \geq 1$ in (7.22), it follows from Lemma 7.4 and Lemma 3.1 (i) (more precisely, see (2.9) and (2.10)) that

$$\|Y^\lambda\|_{\mathcal{X}_2^{0,\gamma}([0,\lambda^3 T]\times\mathbb{T}_\lambda)} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}, \tag{7.25}$$

where the implicit constant is independent of $\lambda \geq 1$. Then, as a corollary to Lemma 7.4 (with (7.25)) and Proposition 3.8, we obtain the following local well-posedness result for the scaled modulated I-KdV (7.20).

Corollary 7.5. *Let $\lambda \geq 1$. Given $\rho \geq \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Fix $s < 0$ satisfying (1.9). Then, given any $v_0 \in L^2(\mathbb{T}_\lambda)$ and $t_0 \in [0,\lambda^3 T]$, there exist $C_0 > 0$, $\theta > 0$, both independent of v_0 and t_0, and a unique solution Iu^λ to the scaled modulated I-KdV equation (7.20) with $Iu^\lambda(t_0) = v_0$ on the time interval $[t_0, t_0+\tau] \cap [0,\lambda^3 T]$, belonging to the class:*

$$\mathscr{D}^0_{w^\lambda}\big(([t_0,t_0+\tau]\cap[0,\lambda^3 T])\times\mathbb{T}_\lambda\big),$$

where the local existence time $\tau = \tau\big(\|v_0\|_{L^2(\mathbb{T}_\lambda)}\big) > 0$ satisfies

$$\tau \geq C_0 \|Y^\lambda\|^{-\theta}_{\mathcal{X}_2^{0,\gamma}([0,\lambda^3 T]\times\mathbb{T}_\lambda)}\big(1+\|v_0\|_{L^2(\mathbb{T}_\lambda)}\big)^{-\theta}. \tag{7.26}$$

Moreover, given any $0 < \alpha < \gamma$ with $\alpha + \gamma > 1$, there exists $C_\alpha > 0$ such that

$$\|I\mathbf{u}^\lambda\|_{\mathcal{C}^\alpha([t_0,t_0+\tau];L^2(\mathbb{T}_\lambda))} \leq C_\alpha \|v_0\|_{L^2(\mathbb{T}_\lambda)},$$

where $\mathbf{u}^\lambda$ denotes the modulated interaction representation of u^λ defined in (7.11).

7.3. **Commutator estimate.** In view of the blowup alternative (3.50) with (7.3) and (7.10), our main task is to control the growth of the (scaled) modified energy:

$$\|Iu^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)} = \|I\mathbf{u}^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)}, \tag{7.27}$$

where $\mathbf{u}^\lambda$ denotes the modulated interaction representation of u^λ defined in (7.11). Recall from (6.3) that

$$\big\langle I\mathbf{u}^\lambda(r), X^\lambda_{t,r}(I\mathbf{u}^\lambda(r), I\mathbf{u}^\lambda(r))\big\rangle_{L^2(\mathbb{T}_\lambda)} = 0. \tag{7.28}$$

Using (7.28), we then have

$$\begin{aligned} &\|I\mathbf{u}^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)} - \|I\mathbf{u}^\lambda(r)\|^2_{L^2(\mathbb{T}_\lambda)} \\ &\quad = 2\big\langle I\mathbf{u}^\lambda(r), \mathrm{com}_{t,r}(\mathbf{u}^\lambda(r),\mathbf{u}^\lambda(r))\big\rangle_{L^2(\mathbb{T}_\lambda)} + R^\lambda_{t,r} \end{aligned} \tag{7.29}$$

for any $t > r \geq 0$, where the commutator $\mathrm{com}_{t,r}$ and the remainder $R^\lambda_{t,r}$ are given by

$$\begin{aligned} \mathrm{com}_{t,r}(f_1,f_2) &= IX^\lambda_{t,r}(f_1,f_2) - X^\lambda_{t,r}(If_1,If_2), \\ R^\lambda_{t,r} &= \|I\mathbf{u}^\lambda(t) - I\mathbf{u}^\lambda(r)\|^2_{L^2(\mathbb{T}_\lambda)} \\ &\quad + 2\big\langle I\mathbf{u}^\lambda(r), I\mathbf{u}^\lambda(t) - I\mathbf{u}^\lambda(r) - IX^\lambda_{t,r}(\mathbf{u}^\lambda(r),\mathbf{u}^\lambda(r))\big\rangle_{L^2(\mathbb{T}_\lambda)}. \end{aligned} \tag{7.30}$$

In the next proposition, we establish a key commutator estimate, controlling the first term on the right-hand side of (7.29). In particular, note that the bound (7.31) on the commutator is controlled by a negative power of N (see (7.32)), yielding almost conservation of the modified energy $\|I\mathbf{u}^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)}$, which is the reason that this approach is called the method of almost conservation laws.

Proposition 7.6. *Let $\lambda \geq 1$. Given $\rho > \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1, and let X^λ be as in (7.13). Fix $s < 0$ satisfying (1.9). Then, we have*

$$\begin{aligned} \| \operatorname{com}_{t,r}(f_1, f_2)\|_{L^2(\mathbb{T}_\lambda)} &= \|IX^\lambda_{t,r}(f_1, f_2) - X^\lambda_{t,r}(If_1, If_2)\|_{L^2(\mathbb{T}_\lambda)} \\ &\lesssim K(\lambda, N)\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t - r|^\gamma \|If_1\|_{L^2(\mathbb{T}_\lambda)}\|If_2\|_{L^2(\mathbb{T}_\lambda)} \end{aligned} \tag{7.31}$$

for any $0 \leq r < t \leq \lambda^3 T$, $\lambda \geq 1$, and $N \in \mathbb{N}$, where $K(\lambda, N)$ is given by

$$K(\lambda, N) = \begin{cases} \lambda^{3-3\rho-3\gamma} N^{\frac{3}{2}-3\rho}, & \text{if } \frac{1}{2} < \rho < \frac{3}{2}, \\ \lambda^{-\frac{3}{2}-3\gamma} N^{-3}\sqrt{\log\langle N\rangle + \log\langle\lambda\rangle}, & \text{if } \rho = \frac{3}{2}, \\ \lambda^{\frac{3}{2}-2\rho-3\gamma} N^{-2\rho}, & \text{if } \rho > \frac{3}{2}. \end{cases} \tag{7.32}$$

Proof. From (7.13) with (7.2) and (7.8), we have

$$\begin{aligned} &\|IX^\lambda_{t,r}(f_1, f_2) - X^\lambda_{t,r}(If_1, If_2)\|^2_{L^2(\mathbb{T}_\lambda)} \\ &= \lambda^{-3} \sum_{n \in \mathbb{Z}^*_\lambda} |n|^2 \bigg| \sum_{\substack{n_1, n_2 \in \mathbb{Z}^*_\lambda \\ n = n_1 + n_2}} M_N(\bar{n}) \Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2)\bigg|^2, \end{aligned} \tag{7.33}$$

where $\Phi^{w^\lambda}_{t,r}$ and $\Xi_{\mathrm{KdV}}(\bar{n})$ are as in (1.6) and (4.5), respectively, and

$$M_N(\bar{n}) = M_N(n, n_1, n_2) = m_{s,N}(n) - m_{s,N}(n_1)m_{s,N}(n_2). \tag{7.34}$$

By symmetry, assume $|n_1| \geq |n_2|$ in the following. Then, given $n \in \mathbb{Z}^*_\lambda$, we split

$$\Sigma_n = \big\{(n_1, n_2) \in (\mathbb{Z}^*_\lambda)^2 : n = n_1 + n_2,\ |n_1| \geq |n_2|\big\}$$

into four regions:

$$\Sigma_n = \bigcup_{j=1,2,3,4} D_j(n),$$

where $D_j(n)$, $j = 1, \dots, 4$, is given by

$$\begin{aligned} D_1(n) &= \Big\{(n_1, n_2) \in \Sigma_n : |n_1| < \frac{N}{2}, |n_2| < \frac{N}{2}\Big\}, \\ D_2(n) &= \Big\{(n_1, n_2) \in \Sigma_n : |n_1| \geq \frac{N}{2}, |n_2| < \frac{N}{2}, |n| < \frac{N}{4}\Big\}, \\ D_3(n) &= \Big\{(n_1, n_2) \in \Sigma_n : |n_1| \geq \frac{N}{2}, |n_2| < \frac{N}{2}, |n| \geq \frac{N}{4}\Big\}, \\ D_4(n) &= \Big\{(n_1, n_2) \in \Sigma_n : |n_1| \geq \frac{N}{2}, |n_2| \geq \frac{N}{2}\Big\}. \end{aligned} \tag{7.35}$$

On $D_1(n)$ and $D_2(n)$, we have $M_N(\bar{n}) = 0$, since $m_{s,N}(n) = m_{s,N}(n_1) = m_{s,N}(n_2) = 1$.

Proceeding as in (4.8) and (4.31) with (7.9), we have

$$\text{LHS of (7.33)} \lesssim \sum_{j=3}^4 A^{(j)}_{\lambda,N}\|If_1\|^2_{L^2(\mathbb{T}_\lambda)}\|If_2\|^2_{L^2(\mathbb{T}_\lambda)}$$

where $A^{(j)}_{\lambda,N}$, $j = 3, 4$, is given by

$$A^{(3)}_{\lambda,N} = \lambda^{-1} \sup_{\substack{n\in\mathbb{Z}^*_\lambda \\ |n|\geq \frac{N}{4}}} |n|^2 \sum_{D_3(n)} \frac{|\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))|^2 |M_N(\bar{n})|^2}{|m_{s,N}(n_1)|^2 |m_{s,N}(n_2)|^2}, \tag{7.36}$$

$$A^{(4)}_{\lambda,N} = \lambda^{-1} \sup_{n_1\in\mathbb{Z}^*_\lambda} \sum_{n\in\mathbb{Z}^*_\lambda} \mathbf{1}_{D_4(n)} \cdot |n|^2 \frac{|\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{n}))|^2 |M_N(\bar{n})|^2}{|m_{s,N}(n_1)|^2 |m_{s,N}(n_2)|^2}. \tag{7.37}$$

• **Case 1:** on D_3.

In this case, we have $m_{s,N}(n_2) = 1$ and $|n| \sim |n_1| \gtrsim N \gtrsim |n_2|$. Then, from (7.1) and the mean value theorem, we have

$$|M_N(\bar{n})| = |m_{s,N}(n) - m_{s,N}(n_1)| \lesssim \frac{|n|^{s-1}|n_2|}{N^s}. \tag{7.38}$$

Then, from (7.36), (7.16), (7.38), and a Riemann sum approximation, we have

$$\begin{aligned} A^{(3)}_{\lambda,N} &\lesssim \lambda^{5-6\rho-6\gamma} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{|n|\geq \frac{N}{4}} |n|^{2s-2\rho} \sum_{D_3(n)} \frac{1}{|n_1|^{2s+2\rho}|n_2|^{2\rho-2}} \\ &\lesssim \lambda^{6-6\rho-6\gamma} N^{-4\rho} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sum_{\substack{m_2\in\mathbb{Z}^* \\ |m_2|\lesssim \lambda N}} \Big|\frac{m_2}{\lambda}\Big|^{2-2\rho} \frac{1}{\lambda} \\ &\lesssim \lambda^{6-6\rho-6\gamma} N^{-4\rho} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \int_{\lambda^{-1}}^{2N} x^{2-2\rho} dx \\ &\lesssim K(\lambda, N)^2 \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \end{aligned} \tag{7.39}$$

for any $0 \leq r < t \leq \lambda^3 T$, where $K(\lambda, N)$ is as in (7.32). Here, the last step follows from separately estimating the cases when $\rho < \frac{3}{2}$, $\rho = \frac{3}{2}$, and $\rho > \frac{3}{2}$. This proves (7.31) in this case.

• **Case 2:** on D_4.

From (7.34), we have

$$|M_N(\bar{n})| \leq |m_{s,N}(n)| + |m_{s,N}(n_1) m_{s,N}(n_2)|. \tag{7.40}$$

Then, from (7.37) with (7.16) and (7.40), we have

$$\begin{aligned} A^{(4)}_{\lambda,N} &\lesssim \lambda^{5-6\rho-6\gamma} N^{4s} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \\ &\qquad \times \sup_{n_1\in\mathbb{Z}^*_\lambda} \sum_{n\in\mathbb{Z}^*_\lambda} \mathbf{1}_{D_4(n)} \cdot |n|^{2-2\rho} \frac{|m_{s,N}(n)|^2}{|n_1|^{2s+2\rho}|n_2|^{2s+2\rho}} \\ &\quad + \lambda^{5-6\rho-6\gamma} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \\ &\qquad \times \sup_{n_1\in\mathbb{Z}^*_\lambda} \sum_{n\in\mathbb{Z}^*_\lambda} \mathbf{1}_{D_4(n)} \cdot |n|^{2-2\rho} |n_1|^{-2\rho} |n_2|^{-2\rho} \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{7.41}$$

Let us first estimate I.

◦ **Subcase 2.a:** $|n| \ll N$.

In this case, we have $m_{s,N}(n) = 1$. Then, proceeding as in (7.39) with $|n_1| \sim |n_2| \gtrsim N$, we have

$$\begin{aligned} \mathrm{I} &\lesssim \lambda^{5-6\rho-6\gamma} N^{-4\rho} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sum_{\substack{n\in\mathbb{Z}^*_\lambda \\ |n|\ll N}} |n|^{2-2\rho} \\ &\lesssim K(\lambda, N)^2 \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \end{aligned} \tag{7.42}$$

for any $0 \le r < t \le \lambda^3 T$, thus yielding (7.31).

◦ **Subcase 2.b:** $|n| \gtrsim N$.

In this case, we have $|m_N(n)| \lesssim N^{-s}|n|^s$. Without loss of generality, assume that $|n| \lesssim |n_1|$. We note that, for $\rho \ge \frac{3}{2}$, we have

$$\lambda^{3-3\rho-3\gamma} N^{\frac{3}{2}-3\rho} \le \lambda^{\frac{3}{2}-2\rho-3\gamma} N^{-2\rho}. \tag{7.43}$$

Then, from (7.41), a Riemann sum approximation, Lemma 2.1, and (7.43), we have

$$\begin{aligned} \mathrm{I} &\lesssim \lambda^{5-6\rho-6\gamma} N^{2s} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{\substack{n_1\in\mathbb{Z}^*_\lambda \\ |n_1|\gtrsim N}} \sum_{\substack{n\in\mathbb{Z}^*_\lambda \\ |n|,|n-n_1|\gtrsim N}} \frac{1}{\langle n\rangle^{4\rho-2}\langle n-n_1\rangle^{2s+2\rho}} \\ &\lesssim \lambda^{6-6\rho-6\gamma} N^{2s} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \\ &\quad \times \sup_{\substack{n_1\in\mathbb{Z}^*_\lambda \\ |n_1|\gtrsim N}} \sum_{\substack{m\in\mathbb{Z}^* \\ |\frac{m}{\lambda}|,|\frac{m}{\lambda}-n_1|\gtrsim N}} \frac{1}{\langle \frac{m}{\lambda}\rangle^{4\rho-2}\langle \frac{m}{\lambda}-n_1\rangle^{2s+2\rho}} \frac{1}{\lambda} \\ &\lesssim \lambda^{6-6\rho-6\gamma} N^{2s} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{\substack{n_1\in\mathbb{Z}^*_\lambda \\ |n_1|\gtrsim N}} \int_{|\xi|,|\xi-n_1|\gtrsim N} \frac{d\xi}{\langle \xi\rangle^{4\rho-2}\langle \xi-n_1\rangle^{2s+2\rho}} \\ &\lesssim \lambda^{6-6\rho-6\gamma} N^{3-6\rho} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \\ &\lesssim K(\lambda, N)^2 \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma}, \end{aligned} \tag{7.44}$$

provided that (4.11) holds. First, suppose that $s > -\rho$. In applying Lemma 2.1 in the fourth step of (7.44), we used

$$\begin{aligned} &\int_{|\xi|,|\xi-n_1|\gtrsim N} \frac{d\xi}{\langle \xi\rangle^{4\rho-2}\langle \xi-n_1\rangle^{2s+2\rho}} \\ &\qquad \lesssim N^{2-2s-6\rho+\alpha+\beta} \int_{|\xi|,|\xi-n_1|\gtrsim N} \frac{d\xi}{\langle \xi\rangle^{\alpha}\langle \xi-n_1\rangle^{\beta}} \\ &\qquad \lesssim N^{3-2s-6\rho}, \end{aligned} \tag{7.45}$$

uniformly in $n_1 \in \mathbb{Z}^*_\lambda$, by taking some $0 \le \alpha, \beta < 1$ such that $\alpha \le 4\rho - 2$ and $\beta \le 2s + 2\rho$ satisfying $\alpha+\beta > 1$. This choice is possible since $\rho > \frac{1}{2}$ and (4.11) give $4\rho-2 > 0$, $2s+2\rho > 0$, and $(4\rho - 2) + (2s + 2\rho) > 1$.

When $s = -\rho$, the condition (1.9) implies $\rho > \frac{3}{4}$. In this case, we simply use

$$\int_{|\xi|,|\xi-n_1|\gtrsim N} \frac{d\xi}{\langle\xi\rangle^{4\rho-2}\langle\xi-n_1\rangle^{0}} \leq \int_{|\xi|\gtrsim N} \frac{d\xi}{\langle\xi\rangle^{4\rho-2}} \lesssim N^{3-4\rho} = N^{3-2s-6\rho}, \tag{7.46}$$

uniformly in $n_1 \in \mathbb{Z}^*_\lambda$.

Next, we estimate II.

◦ **Subcase 2.c:** $|n| \ll N$.

By substituting $|n_1|, |n_2| \gtrsim N$ in (7.41), we see that (7.42) also holds for II in this case.

◦ **Subcase 2.d:** $|n| \gtrsim N$.

Without loss of generality, assume that $N \lesssim |n| \lesssim |n_1|$. From (7.41), a Riemann sum approximation, Lemma 2.1, and (7.43), we have

$$\begin{aligned}
\text{II} &\lesssim \lambda^{5-6\rho-6\gamma}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \sup_{\substack{n_1\in\mathbb{Z}^*_\lambda\\ |n_1|\gtrsim N}} \sum_{\substack{n\in\mathbb{Z}^*_\lambda\\ |n|,|n-n_1|\gtrsim N}} \frac{1}{\langle n\rangle^{4\rho-2}\langle n-n_1\rangle^{2\rho}} \\
&\lesssim \lambda^{6-6\rho-6\gamma}N^{3-6\rho}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \\
&\lesssim K(\lambda,N)^2\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma},
\end{aligned}$$

provided that $\rho > \frac{1}{2}$, where, at the second inequality, we proceeded as in (7.45). □

7.4. **Global well-posedness on the circle.** We are now ready to put together all the ingredients from the previous subsections and prove global well-posedness of the periodic modulated KdV (1.2) in negative Sobolev spaces as claimed in Theorem 1.3 (ii).

Fix $u_0 \in H^s(\mathbb{T})$ for some $s < 0$ (to be determined later). Let u be a solution to the modulated KdV (1.2) on $\mathbb{T}$ with $u|_{t=0} = u_0$ whose existence for short times is guaranteed by Theorem 1.3 (i). In view of the blowup alternative (3.50), our goal is to show that

$$\sup_{0\leq t\leq T}\|u(t)\|^2_{H^s(\mathbb{T})} \leq C(T) < \infty \tag{7.47}$$

for each finite $T > 0$. In view of (7.3) (which also holds true on $\mathbb{T}_\lambda$), (7.4), (7.10), and (7.27), the bound (7.47) follows once we prove the following bound on the modified energy:

$$\sup_{0\leq t\leq \lambda^3 T}\|I\mathbf{u}^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)} \leq C(T,\lambda) < \infty$$

for some $\lambda \geq 1$.

From (7.3) and (7.10), we have

$$\|Iu^\lambda_0\|_{L^2(\mathbb{T}_\lambda)} \lesssim \lambda^{-s-\frac{3}{2}}N^{-s}\|u_0\|_{H^s(\mathbb{T})}. \tag{7.48}$$

Then, given small $\varepsilon_0 > 0$, we can choose

$$\lambda \sim N^{-\frac{2s}{3+2s}}, \tag{7.49}$$

such that

$$\|Iu^\lambda_0\|^2_{L^2(\mathbb{T}_\lambda)} \leq \varepsilon_0, \tag{7.50}$$

provided that $s > -\frac{3}{2}$. Here, the implicit constant in (7.49) depends only on $\|u_0\|_{H^s(\mathbb{T})}$.

We now study the increment of the modified energy. Let us first make preliminary analysis on the remainder term $R^\lambda_{t,r}$ defined in (7.30). By rewriting (7.29), we have

$$\begin{aligned} R^\lambda_{t,r} &= \|I\mathbf{u}^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)} - \|I\mathbf{u}^\lambda(r)\|^2_{L^2(\mathbb{T}_\lambda)} \\ &\quad - 2\big\langle I\mathbf{u}^\lambda(r), \mathrm{com}_{t,r}(\mathbf{u}^\lambda(r), \mathbf{u}^\lambda(r))\big\rangle_{L^2(\mathbb{T}_\lambda)}, \end{aligned} \tag{7.51}$$

where $\mathrm{com}_{t,r}$ is as in (7.30). Then, from (2.2) and (7.51) with

$$\mathrm{com}_{t_1,t_3}(\mathbf{u}^\lambda(t_3), \mathbf{u}^\lambda(t_3)) - \mathrm{com}_{t_2,t_3}(\mathbf{u}^\lambda(t_3), \mathbf{u}^\lambda(t_3)) = \mathrm{com}_{t_1,t_2}(\mathbf{u}^\lambda(t_3), \mathbf{u}^\lambda(t_3)),$$

which follows from (7.30), we have

$$\begin{aligned} (\delta R^\lambda)_{t_1,t_2,t_3} &= R^\lambda_{t_1,t_3} - R^\lambda_{t_1,t_2} - R^\lambda_{t_2,t_3} \\ &= 2\big\langle I\mathbf{u}^\lambda(t_2) - I\mathbf{u}^\lambda(t_3), \mathrm{com}_{t_1,t_2}(\mathbf{u}^\lambda(t_2), \mathbf{u}^\lambda(t_2))\big\rangle_{L^2(\mathbb{T}_\lambda)} \\ &\quad + 2\big\langle I\mathbf{u}^\lambda(t_3), \mathrm{com}_{t_1,t_2}(\mathbf{u}^\lambda(t_2) - \mathbf{u}^\lambda(t_3), \mathbf{u}^\lambda(t_2))\big\rangle_{L^2(\mathbb{T}_\lambda)} \\ &\quad + 2\big\langle I\mathbf{u}^\lambda(t_3), \mathrm{com}_{t_1,t_2}(\mathbf{u}^\lambda(t_3), \mathbf{u}^\lambda(t_2) - \mathbf{u}^\lambda(t_3))\big\rangle_{L^2(\mathbb{T}_\lambda)} \end{aligned} \tag{7.52}$$

for any $t_1 > t_2 > t_3 \geq 0$.

Fix an interval $J \subset [0, \lambda^3 T]$ and $0 < \alpha < \gamma$ such that $\alpha + \gamma > 1$. Then, it follows from (7.52), Cauchy-Schwarz's inequality, and Proposition 7.6 that

$$|(\delta R^\lambda)_{t_1,t_2,t_3}| \lesssim K(\lambda, N)\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t_1 - t_2|^\gamma |t_2 - t_3|^\alpha \|I\mathbf{u}^\lambda\|^3_{\mathcal{C}^\alpha(J;L^2(\mathbb{T}_\lambda))}$$

for any $t_1 \geq t_2 \geq t_3 \geq 0$ belonging to the interval J, where $K(\lambda, N)$ is as in (7.32). Since $\alpha + \gamma > 1$, it follows from the sewing lemma (Lemma 3.3) that

$$|R^\lambda_{t_1,t_2}| \lesssim K(\lambda, N)\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T}|t_1 - t_2|^{\alpha+\gamma} \|I\mathbf{u}^\lambda\|^3_{\mathcal{C}^\alpha(J;L^2(\mathbb{T}_\lambda))} \tag{7.53}$$

for any $t_1 > t_2 \geq 0$ belonging to the interval J.

We now put everything together. Let $\tau \sim 1$ be the local existence time for the scaled modulated I-KdV (7.20) given by (7.26) in Corollary 7.5, corresponding to initial data v_0 with $\|v_0\|^2_{L^2(\mathbb{T}_\lambda)} = 2\varepsilon_0$. Namely, if we have

$$\|I\mathbf{u}^\lambda(t_0)\|^2_{L^2(\mathbb{T}_\lambda)} \leq 2\varepsilon_0 \tag{7.54}$$

for some $0 \leq t_0 \leq \lambda^3 T$, then Corollary 7.5 guarantees that the solution Iu^λ to the scaled modulated I-KdV (7.20) exists on the time interval $[t_0, t_0 + \tau] \cap [0, \lambda^3 T]$, satisfying the bound:

$$\|I\mathbf{u}^\lambda\|^2_{\mathcal{C}^\alpha([t_0,t_0+\tau];L^2(\mathbb{T}_\lambda))} \leq C_\alpha \varepsilon_0. \tag{7.55}$$

for some $C_\alpha > 0$. In particular, in view of (7.50) (which states that (7.54) is satisfied with $t_0 = 0$), we see that the solution Iu^λ exists on $[0, \tau]$.

Suppose now that by choosing $N \gg 1$ (and thus $\lambda = \lambda(N) \gg 1$ in view of (7.49)), we can make $K(\lambda, N)$ in (7.32) as small as we need; see (7.60) and (7.61). Then, from (7.29), (7.50), Proposition 7.6, (7.53), and (7.55) with $t_0 = 0$, we have

$$\begin{aligned} \sup_{0\leq t\leq \tau} \|I\mathbf{u}^\lambda(t)\|^2_{L^2(\mathbb{T}_\lambda)} &\leq \varepsilon_0 + C_2 K(\lambda, N)\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} \\ &\leq 2\varepsilon_0. \end{aligned} \tag{7.56}$$

In particular, the solution Iu^λ exists on $[\tau, 2\tau]$, satisfying (7.54) and (7.55) with $t_0 = \tau$. Then, from (7.29), (7.56) Proposition 7.6, (7.53), and (7.55) with $t_0 = \tau$, we have

$$\begin{aligned}\sup_{\tau\le t\le 2\tau} \|I\mathbf{u}^\lambda(t)\|_{L^2(\mathbb{T}_\lambda)}^2 &\le \varepsilon_0 + 2C_2K(\lambda,N)\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}\\ &\le 2\varepsilon_0,\end{aligned} \tag{7.57}$$

provided that we can make $K(\lambda, N)$ small by choosing $N \gg 1$ (and thus $\lambda = \lambda(N) \gg 1$). If (7.57) holds, then the solution Iu^λ exists on $[2\tau, 3\tau]$, satisfying (7.54) and (7.55) with $t_0 = 2\tau$, which allows us to iterate this process.

After j steps, this iterative process shows that the solution Iu^λ exists on $[0, (j+1)\tau]$, satisfying (7.54) and (7.55) with $t_0 = j\tau$, and moreover, we have

$$\sup_{0\le t\le j\tau} \|I\mathbf{u}^\lambda(t)\|_{L^2(\mathbb{T}_\lambda)}^2 \le \varepsilon_0 + jC_2K(\lambda,N)\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}. \tag{7.58}$$

Then, proceeding as in (7.57) with (7.58), we obtain

$$\begin{aligned}\sup_{0\le t\le (j+1)\tau} \|I\mathbf{u}^\lambda(t)\|_{L^2(\mathbb{T}_\lambda)}^2 &\le \varepsilon_0 + (j+1)C_2K(\lambda,N)\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}\\ &\le 2\varepsilon_0,\end{aligned} \tag{7.59}$$

provided that we can make $K(\lambda, N)$ small by choosing $N \gg 1$ (and thus $\lambda = \lambda(N) \gg 1$ for $-\frac{3}{2} < s < 0$).

Our goal is to construct the solution Iu^λ on the time interval $[0, \lambda^3 T]$, for which we need to iterate this process $j \sim \lambda^3 T$ times, since $\tau \sim 1$. From (7.59), we see that it is possible to reach the target time $\lambda^3 T$ (such that $\lambda^3 T$ is (at most) the doubling time of the modified energy), provided that

$$\lambda^3 T \lesssim K(\lambda,N)^{-1} = \begin{cases} \lambda^{-3+3\rho+3\gamma}N^{-\frac{3}{2}+3\rho}, & \text{if } \frac{1}{2} < \rho < \frac{3}{2},\\ \lambda^{\frac{3}{2}+3\gamma}N^3(\log\langle N\rangle + \log\langle\lambda\rangle)^{-\frac{1}{2}}, & \text{if } \rho = \frac{3}{2},\\ \lambda^{-\frac{3}{2}+2\rho+3\gamma}N^{2\rho}, & \text{if } \rho > \frac{3}{2}. \end{cases} \tag{7.60}$$

In view of (7.49), we can guarantee (7.60) by choosing $N \gg 1$, provided that

$$s > \begin{cases} \max\left(\frac{3-6\rho}{6-4\gamma}, -\frac{3}{2}\right) & \text{if } \frac{1}{2} < \rho < \frac{3}{2},\\ -\frac{3}{2}, & \text{if } \rho \ge \frac{3}{2}, \end{cases} \tag{7.61}$$

where we used the fact that $\frac{-2\rho}{3-2\gamma} < -\frac{3}{2}$ for $\rho \ge \frac{3}{2}$ and $\frac{1}{2} < \gamma < 1$. Hence, from (1.9) and (7.61) with $s > -\frac{3}{2}$, we conclude global well-posedness of the modulated KdV in $H^s(\mathbb{T})$, provided that $\rho > \frac{1}{2}$ and $s \in \mathbb{R}$ satisfy (1.10). This concludes the proof of Theorem 1.3 (ii).

7.5. **Real line case.** We conclude this section by discussing the real line case. In this case, we need to work with the non-homogeneous Sobolev spaces whose scaling property is worse (for example, compare (7.16) and (7.63)) and thus we need to proceed with care. In particular, we overcome the issue of the worse scaling properties by changes of variables (which are not available in the periodic case). While we used the same notations from Subsections 7.1-7.4, all the objects (equations, functions, and operators) are to be understood as those defined on the real line.

Let $s < 0$. Then, we have

$$\|u_0^\lambda\|_{H^s(\mathbb{R})} \le \lambda^{-\frac{3}{2}-s}\|u_0\|_{H^s(\mathbb{R})}, \tag{7.62}$$

where u_0^λ is as in (7.6). As in the periodic case, our goal is to study the scaled modulated I-KdV (7.20) on the real line. Given small $\varepsilon_0 > 0$, it follows from (7.3) (which also holds true on $\mathbb{R}$) and (7.62) that

$$\|Iu_0^\lambda\|_{L^2(\mathbb{R})} \lesssim \lambda^{-s-\frac{3}{2}} N^{-s} \|u_0\|_{H^s(\mathbb{R})} \leq \varepsilon_0,$$

where the second step follows from choosing $\lambda \sim N^{-\frac{2s}{3+2s}}$ as in (7.49), provided that $s > -\frac{3}{2}$. This allows us to reduce the study to the small data regime as in the periodic case.

Given $\lambda \geq 1$, let X^λ be the driver defined in (7.13), where we view all the operators and functions interpreted as those defined on $\mathbb{R}$. Then, we have the following analogue of Lemma 7.1 on the regularity of the driver X^λ whose proof follows from a slight modification of the proof of Proposition 4.4 (i).

Lemma 7.7. *Let $\lambda \geq 1$. Given $\rho > \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1, and let X^λ be as in (7.13). Suppose that $\rho > \frac{1}{2}$ and $s < 0$ satisfy (1.11). Then, the driver X^λ belongs to $\mathcal{X}_2^{s,\gamma}([0, \lambda^3 T] \times \mathbb{R})$ defined in (2.9), satisfying the bound*:

$$\|X_{t,r}^\lambda\|_{\mathcal{L}_2(H^s(\mathbb{R}))} \lesssim \lambda^{\frac{3}{2}-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma$$

for any $0 \leq r < t \leq \lambda^3 T$.

Once we prove Lemma 7.7, an analogue of Lemma 7.3 on equivalence of the unscaled and scaled problems in the current real line case follows from the same proof.

Proof of Lemma 7.7. Proceeding as in (7.16), we have

$$\begin{aligned} |\Phi_{t,r}^{w^\lambda}(\Xi_{\mathrm{KdV}}(\bar{\xi}))| &= \left| \int_r^t e^{i\Xi_{\mathrm{KdV}}(\bar{\xi}) w^\lambda(t')} dt' \right| = \lambda^3 \left| \int_{\lambda^{-3} r}^{\lambda^{-3} t} e^{i\lambda^3 \Xi_{\mathrm{KdV}}(\bar{\xi}) w(t')} dt' \right| \\ &\lesssim \lambda^{3-3\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma \langle \lambda^3 \xi \xi_1 \xi_2 \rangle^{-\rho} \end{aligned} \tag{7.63}$$

for any $\xi, \xi_1, \xi_2 \in \mathbb{R}$ satisfying $\xi = \xi_1 + \xi_2$.

Proceeding as in (4.30), (4.32), and (4.34), we have

$$\begin{aligned} \|X_{t,r}^\lambda\|_{\mathcal{L}_2(H^s(\mathbb{R}))}^2 \leq{}& \sup_{|\xi|<1} \int_{|\xi_1|<2} \langle \xi \rangle^{2s} \xi^2 \frac{|\Phi_{t,r}^{w^\lambda}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} d\xi_1 \\ &+ \sup_{|\xi_1| \geq 2} \int_{|\xi|<1} \langle \xi \rangle^{2s} \xi^2 \frac{|\Phi_{t,r}^{w^\lambda}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} d\xi \\ &+ \sup_{|\xi_1| \geq \frac{1}{2}} \int_{\substack{|\xi| \geq 1 \\ |\xi - \xi_1| \geq \frac{1}{2}}} \langle \xi \rangle^{2s} \xi^2 \frac{|\Phi_{t,r}^{w^\lambda}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} d\xi \\ &+ 2 \sup_{|\xi| \geq 1} \int_{|\xi - \xi_1| < \frac{1}{2}} \langle \xi \rangle^{2s} \xi^2 \frac{|\Phi_{t,r}^{w^\lambda}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} d\xi_1 \\ =:{}& A_1^\lambda + A_2^\lambda + A_3^\lambda + A_4^\lambda. \end{aligned} \tag{7.64}$$

We first estimate A_1^λ. In this case, we have $|\xi|, |\xi_1|, |\xi - \xi_1| \lesssim 1$. Without loss of generality, assume that $|\xi| \lesssim |\xi - \xi_1|$. Then, from (7.63) and a change of variable $\zeta_1 = \lambda^3 \xi^2 \xi_1$, we have

$$\begin{aligned} A_1^\lambda &\lesssim \lambda^{6-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \sup_{|\xi|<1} \xi^2 \int_{|\xi_1|<2} \frac{1}{\langle \lambda^3 \xi^2 \xi_1 \rangle^{2\rho}} d\xi_1 \\ &\le \lambda^{3-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \sup_{|\xi|<1} \int_{\mathbb{R}} \frac{1}{\langle \zeta_1 \rangle^{2\rho}} d\zeta_1 \\ &\lesssim \lambda^{3-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \end{aligned} \tag{7.65}$$

for any $0 \le r < t \le \lambda^3 T$, provided that $\rho > \frac{1}{2}$.

From (7.63), a change of variable $\zeta = \lambda \xi$, and relabelling $\zeta_1 = \lambda \xi_1$, we have

$$\begin{aligned} A_2^\lambda &\lesssim \lambda^{6-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \\ &\quad \times \sup_{|\xi_1|\ge 2} \int_{|\xi|<1} \xi^2 \frac{1}{|\xi_1(\xi-\xi_1)|^{2s} \langle \lambda^3 \xi \xi_1 (\xi - \xi_1) \rangle^{2\rho}} d\xi \\ &= \lambda^{3+4s-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \\ &\quad \times \sup_{|\zeta_1|\ge 2\lambda} \int_{|\zeta|<\lambda} \zeta^2 \frac{1}{|\zeta_1(\zeta-\zeta_1)|^{2s} \langle \zeta \zeta_1 (\zeta - \zeta_1) \rangle^{2\rho}} d\zeta \end{aligned} \tag{7.66}$$

for any $0 \le r < t \le \lambda^3 T$. Denote by $A_2^{\lambda,(1)}$ the contribution to A_2^λ from $|\zeta| \lesssim 1$. Then, proceeding as in (4.38), we have

$$\begin{aligned} A_2^{\lambda,(1)} &\lesssim \lambda^{3+4s-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \\ &\lesssim \lambda^{3-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \end{aligned} \tag{7.67}$$

for any $0 \le r < t \le \lambda^3 T$, provided that

$$\rho > 0, \quad -\frac{3}{2} < s < 0, \quad \text{and} \quad s \ge -\rho.$$

Let $A_2^{\lambda,(2)} = A_2^\lambda - A_2^{\lambda,(1)}$. Using $|\zeta_1| \sim |\zeta - \zeta_1| \gtrsim \lambda$, we have

$$\begin{aligned} A_2^{\lambda,(2)} &\lesssim \lambda^{3+4s-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \sup_{|\zeta_1|\ge 2\lambda} \int_{1 \ll |\zeta| < \lambda} |\zeta|^{2-2\rho} \frac{1}{|\zeta_1(\zeta-\zeta_1)|^{2s+2\rho}} d\zeta \\ &\lesssim \lambda^{3-4\rho-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \sup_{|\zeta_1|\ge 2\lambda} \int_{1 \ll |\zeta| < \lambda} |\zeta|^{2-2\rho} d\zeta \\ &\lesssim \lambda^{3-6\gamma} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2 |t-r|^{2\gamma} \end{aligned} \tag{7.68}$$

for any $0 \le r < t \le \lambda^3 T$, provided that $s + \rho \ge 0$ and $\rho \ge \frac{1}{2}$.

As for A_3^λ, there is no low frequency issue. Thus, by proceeding as in (4.9) and (4.10) with (7.63), we have

$$\begin{aligned} A_3^\lambda &\lesssim \lambda^{6-6\rho-6\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2|t-r|^{2\gamma}\sup_{|\xi_1|\geq\frac{1}{2}}\int_{\substack{|\xi|\geq 1\\ |\xi-\xi_1|\geq\frac{1}{2}}}\frac{|\xi|^{2s+2-2\rho}}{|\xi_1(\xi-\xi_1)|^{2s+2\rho}}d\xi \\ &\lesssim \lambda^{3-6\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2|t-r|^{2\gamma} \end{aligned} \tag{7.69}$$

for any $0\leq r<t\leq\lambda^3 T$, provided that (4.11) holds.

As for A_4^λ, we first note that $|\xi_1|\sim|\xi|\geq 1$ and $\langle\xi-\xi_1\rangle\sim 1$ under $|\xi|\geq 1$ and $|\xi-\xi_1|<\frac{1}{2}$. Then, from (7.63), (4.29), and a change of variables $\zeta_2=\lambda^3\xi^2(\xi-\xi_1)$, we have

$$\begin{aligned} A_4^\lambda &\lesssim \lambda^{6-6\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2|t-r|^{2\gamma}\sup_{|\xi|\geq 1}\xi^2\int_{|\xi-\xi_1|<\frac{1}{2}}\frac{1}{\langle\lambda^3\xi^2(\xi-\xi_1)\rangle^{2\rho}}d\xi_1 \\ &= \lambda^{3-6\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2|t-r|^{2\gamma}\sup_{|\xi|\geq 1}\int_{|\zeta_2|<\frac{\lambda^3\xi^2}{2}}\frac{1}{\langle\zeta_2\rangle^{2\rho}}d\zeta_2 \\ &\lesssim \lambda^{3-6\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}^2|t-r|^{2\gamma} \end{aligned} \tag{7.70}$$

for any $0\leq r<t\leq\lambda^3 T$, provided that $\rho>\frac{1}{2}$ (and any $s\in\mathbb{R}$).

Therefore, putting (7.64), (7.65), (7.66), (7.67), (7.68), (7.69), and (7.70) together, we conclude from Lemma 3.1 (i) that $X^\lambda\in\mathcal{X}_2^{s,\gamma}([0,\lambda^3T]\times\mathbb{R})$ under $s<0$ and (1.11). □

We now consider the scaled modulated I-KdV (7.20) on $\mathbb{R}$. Let Y^λ be the associated driver defined in (7.21), where we view all the operators and functions interpreted as those defined on $\mathbb{R}$. Then, as a corollary to Lemma 7.7, we obtain the following regularity property of the driver Y^λ and local well-posedness of the scaled modulated I-KdV (7.20) on the real line. See Lemma 7.4 and Corollary 7.5 for the corresponding statements in the periodic setting studied earlier.

Corollary 7.8. *Let $\lambda\geq 1$. Given $\rho>\frac{1}{2}$, $\frac{1}{2}<\gamma<1$, and $T>0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1, and let X^λ be as in (7.13). Suppose that $\rho>\frac{1}{2}$ and $s<0$ satisfy (1.11). Then, the driver Y^λ in (7.21) belongs to $\mathcal{X}_2^{0,\gamma}([0,\lambda^3T]\times\mathbb{R})$ defined in (2.9), satisfying the bound*:

$$\|Y_{t,r}^\lambda\|_{\mathcal{L}_2(L^2(\mathbb{R}))}\lesssim\lambda^{\frac{3}{2}-3\gamma}\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}|t-r|^\gamma \tag{7.71}$$

for any $0\leq r<t\leq\lambda^3T$.

Moreover, given any $v_0\in L^2(\mathbb{R})$ and $t_0\in[0,\lambda^3T]$, there exist $C_0>0$, $\theta>0$, both independent of v_0 and t_0, and a unique solution Iu to the scaled modulated I-KdV equation (7.20) on the real line with $Iu^\lambda(t_0)=v_0$ on the time interval $[t_0,t_0+\tau]\cap[0,\lambda^3T]$, belonging to the class:

$$\mathscr{D}_{w^\lambda}^0\big(([t_0,t_0+\tau]\cap[0,\lambda^3T])\times\mathbb{R}\big),$$

where the local existence time $\tau=\tau\big(\|v_0\|_{L^2(\mathbb{R})}\big)>0$ satisfies

$$\tau\geq C_0\|Y^\lambda\|_{\mathcal{X}_2^{0,\gamma}([0,\lambda^3T]\times\mathbb{R})}^{-\theta}\big(1+\|v_0\|_{L^2(\mathbb{R})}\big)^{-\theta}.$$

Moreover, given any $0 < \alpha < \gamma$ with $\alpha + \gamma > 1$, there exists $C_\alpha > 0$ such that

$$\|I\mathbf{u}^\lambda\|_{\mathcal{C}^\alpha([t_0,t_0+\tau];L^2(\mathbb{R}))} \le C_\alpha \|v_0\|_{L^2(\mathbb{R})}.$$

Proof. The bound (7.71) follows from Lemma 7.7, the interpolation lemma (see [39, Lemma 12.1]), and the bound

$$M(\bar{\xi}) \lesssim \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s}$$

for any $\xi, \xi_1, \xi_2 \in \mathbb{R}$ with $\xi = \xi_1 + \xi_2$, where $M(\bar{\xi}) = M(\xi, \xi_1, \xi_2)$ is as in (7.24). Then, from Lemma 3.1 (i), we conclude that $Y^\lambda \in \mathcal{X}_2^{0,\gamma}([0, \lambda^3 T] \times \mathbb{R})$.

By noting that $\lambda^{\frac{3}{2}-3\gamma} \le 1$ for any $\lambda \ge 1$, the local well-posedness claim follows from the regularity of the driver Y^λ and Proposition 3.8. We omit details. □

Next, we present a key commutator estimate. While we proceed as in the proof of Proposition 7.6, we provide details since it involves changes of variables, unique to the current real line setting.

Lemma 7.9. *Let $\lambda \ge 1$. Given $\rho > \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ, γ)-irregular on $[0, T]$ in the sense of Definition 1.1, and let X^λ be as in (7.13). Fix $s < 0$ satisfying (1.11). Then, we have*

$$\begin{aligned} \|\operatorname{com}_{t,r}(f_1, f_2)\|_{L^2(\mathbb{R})} &= \|IX^\lambda_{t,r}(f_1, f_2) - X^\lambda_{t,r}(If_1, If_2)\|_{L^2(\mathbb{R})} \\ &\lesssim \widetilde{K}(\lambda, N) \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-r|^\gamma \|If_1\|_{L^2(\mathbb{R})} \|If_2\|_{L^2(\mathbb{R})} \end{aligned} \tag{7.72}$$

for any $0 \le r < t \le \lambda^3 T$, $\lambda \ge 1$, and $N \in \mathbb{N}$, where $\widetilde{K}(\lambda, N)$ is given by

$$\widetilde{K}(\lambda, N) = \begin{cases} \lambda^{3-3\rho-3\gamma} N^{\frac{3}{2}-3\rho}, & \text{if } \frac{1}{2} < \rho < \frac{3}{2}, \\ \lambda^{-\frac{3}{2}-3\gamma} N^{-3} \sqrt{\log\langle N \rangle + \log\langle \lambda \rangle}, & \text{if } \rho = \frac{3}{2}, \\ \lambda^{-\frac{3}{2}-3\gamma} N^{-3}, & \text{if } \rho > \frac{3}{2}. \end{cases} \tag{7.73}$$

Proof. Proceeding as in the proof of Proposition 7.6, we have

$$\text{LHS of (7.72)} \le \bigg(\sum_{j=3}^4 B^{(j)}_{\lambda,N} \|If_1\|^2_{L^2(\mathbb{R})} \|If_2\|^2_{L^2(\mathbb{R})} \bigg)^{\frac{1}{2}},$$

where $B^{(j)}_{\lambda,N}$, $j = 3, 4$, is given by

$$B^{(3)}_{\lambda,N} = \sup_{|\xi| \ge \frac{N}{4}} |\xi|^2 \int_{D_3(\xi)} \frac{|\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2 |M_N(\bar{\xi})|^2}{|m_{s,N}(\xi_1)|^2 |m_{s,N}(\xi_2)|^2} d\xi_1, \tag{7.74}$$

$$B^{(4)}_{\lambda,N} = \sup_{\xi_1 \in \mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{D_4(\xi)} \cdot |\xi|^2 \frac{|\Phi^{w^\lambda}_{t,r}(\Xi_{\mathrm{KdV}}(\bar{\xi}))|^2 |M_N(\bar{\xi})|^2}{|m_{s,N}(\xi_1)|^2 |m_{s,N}(\xi_2)|^2} d\xi. \tag{7.75}$$

Here, $M_N(\bar{\xi}) = M_N(\xi, \xi_1, \xi_2)$ and $D_j(\xi)$, $j = 3, 4$, are as in (7.34) and (7.35), respectively, with obvious modifications to the current real line setting.

• **Case 1:** on D_3.

In this case, we have $|\xi| \sim |\xi_1| \gtrsim N \gtrsim |\xi_2|$. Then, from (7.74), (7.63), (7.38), and a change

of variables $\zeta_2 = \lambda^3\xi^2\xi_2$, we have

$$\begin{aligned} B^{(3)}_{\lambda,N} &\lesssim \lambda^{6-6\gamma}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \sup_{|\xi|\geq\frac{N}{4}} \int_{D_3(\xi)} \frac{|\xi_2|^2}{\langle\lambda^3\xi^2\xi_2\rangle^{2\rho}}\, d\xi_2 \\ &\lesssim \lambda^{-3-6\gamma}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \sup_{|\xi|\geq\frac{N}{4}} \xi^{-6} \int_{|\zeta_2|\lesssim\lambda^3\xi^2N} \frac{|\zeta_2|^2}{\langle\zeta_2\rangle^{2\rho}}\, d\zeta_2 \\ &\lesssim \widetilde{K}(\lambda,N)^2\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \end{aligned} \tag{7.76}$$

for any $0 \leq r < t \leq \lambda^3 T$, where $\widetilde{K}(\lambda, N)$ is as in (7.73). This proves (7.72) in this case.

• **Case 2:** on D_4.

From (7.75) with (7.63) and (7.40), we have

$$\begin{aligned} B^{(4)}_{\lambda,N} &\lesssim \lambda^{6-6\gamma}N^{4s}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \\ &\qquad \times \sup_{\xi_1\in\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{D_4(\xi)} \cdot \frac{|\xi|^2|m_{s,N}(\xi)|^2}{|\xi_1|^{2s}|\xi_2|^{2s}\langle\lambda^3\xi\xi_1\xi_2\rangle^{2\rho}} d\xi \\ &\quad + \lambda^{6-6\gamma}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \sup_{\xi_1\in\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{D_4(\xi)} \frac{|\xi|^2}{\langle\lambda^3\xi\xi_1\xi_2\rangle^{2\rho}} d\xi \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{7.77}$$

Let us first estimate I.

◦ **Subcase 2.a:** $|\xi| \ll N$.

In this case, we have $m_{s,N}(\xi) = 1$ and $|\xi_1| \sim |\xi_2| \gtrsim N$. Then, by a change of variables $\zeta = \lambda^3\xi_1^2\xi$, we have

$$\begin{aligned} \mathrm{I} &\lesssim \lambda^{-3-6\gamma}N^{4s}\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \sup_{|\xi_1|\gtrsim N} |\xi_1|^{-6-4s} \int_{|\zeta|\lesssim\lambda^3\xi_1^2N} \frac{\zeta^2}{\langle\zeta\rangle^{2\rho}} d\zeta \\ &\lesssim \widetilde{K}(\lambda,N)^2\|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T}|t-r|^{2\gamma} \end{aligned}$$

for any $0 \leq r < t \leq \lambda^3 T$, thus yielding (7.72). Here, in estimating the integral, we used the fact that (i) $s + \rho \geq 0$ when $\frac{1}{2} < \rho < \frac{3}{2}$ and (ii) $s > -\frac{3}{2}$ when $\rho \geq \frac{3}{2}$.

◦ **Subcase 2.b:** $|\xi| \gtrsim N$.

In this case, we have $|m_N(\xi)| \lesssim N^{-s}|\xi|^s$. Without loss of generality, assume that $|\xi| \lesssim |\xi_1|$. We note that, for $\rho \geq \frac{3}{2}$, we have

$$\lambda^{3-3\rho-3\gamma}N^{\frac{3}{2}-3\rho} \leq \lambda^{-\frac{3}{2}-3\gamma}N^{-3}. \tag{7.78}$$

Then, proceeding as in (7.44) with (7.77), (7.45), (7.46), and (7.78) with (7.73), we have

$$\begin{aligned}
\mathrm{I} &\lesssim \lambda^{6-6\rho-6\gamma} N^{2s} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{\xi_1\in\mathbb{R}} \int_{|\xi|\gtrsim N} \mathbf{1}_{D_4(\xi)} \cdot \frac{|\xi|^{2+2s-2\rho}}{|\xi_1|^{2s+2\rho}|\xi_2|^{2s+2\rho}} d\xi \\
&\lesssim \lambda^{6-6\rho-6\gamma} N^{2s} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{\xi_1\in\mathbb{R}} \int_{|\xi|\gtrsim N} \mathbf{1}_{D_4(\xi)} \cdot \frac{1}{\langle\xi\rangle^{4\rho-2}\langle\xi_2\rangle^{2s+2\rho}} d\xi \qquad (7.79)\\
&\lesssim \lambda^{6-6\rho-6\gamma} N^{3-6\rho} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \\
&\lesssim \widetilde{K}(\lambda,N)^2 \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma},
\end{aligned}$$

provided that (4.11) holds.

Next, we estimate II.

∘ **Subcase 2.c:** $|\xi| \ll N$.

In this case, we have $|\xi_1| \sim |\xi_2| \gtrsim N$. Then, proceeding as in (7.76) with a change of variables $\zeta = \lambda^3\xi_1^2\xi$, we have

$$\begin{aligned}
\mathrm{II} &\lesssim \lambda^{-3-6\gamma} \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma} \sup_{|\xi_1|\gtrsim N} |\xi_1|^{-6} \int_{|\zeta|\lesssim \lambda^3\xi_1^2 N} \frac{\zeta^2}{\langle\zeta\rangle^{2\rho}} d\zeta \\
&\lesssim \widetilde{K}(\lambda,N)^2 \|\Phi^w\|^2_{\mathcal{W}^{\rho,\gamma}_T} |t-r|^{2\gamma}
\end{aligned}$$

for any $0 \le r < t \le \lambda^3 T$, thus yielding (7.72).

∘ **Subcase 2.d:** $|\xi| \gtrsim N$.

Without loss of generality, assume that $N \lesssim |\xi| \lesssim |\xi_1|$. Then, by noting that $\langle\lambda^3\xi\xi_1\xi_2\rangle \sim \lambda^3|\xi\xi_1\xi_2| \gtrsim \lambda^3\xi^2|\xi_2|$, we can repeat the computation in (7.79) and obtain (7.72). □

We conclude this section by briefly discussing a proof of Theorem 1.7 (ii). In view of (7.62), Corollary 7.8, and Lemma 7.9 (compare (7.73) with (7.32)), there is no difference from the periodic case, when $\frac{1}{2} < \rho \le \frac{3}{2}$, yielding global well-posedness of the modulated KdV (1.2) on the real line for the same range of s. When $\rho > \frac{3}{2}$, by substituting $\rho = \frac{3}{2}$ in the last case of (7.60), we obtain the condition $s > -\frac{3}{3-2\gamma}$ (which is smaller than $-\frac{3}{2}$ for $\frac{1}{2} < \gamma < 1$). In view of the regularity restriction $s > -\frac{3}{2}$ from the local well-posedness (see (1.11)), we conclude that global well-posedness holds for $s > -\frac{3}{2}$. This concludes the proof of Theorem 1.7 (ii).

8. Invariance of the white noise under the modulated KdV

In this section, we briefly discuss invariance of the white noise under the modulated KdV, BO, ILW, and scaled ILW equations on the circle (Theorems 1.20 and 1.21). For simplicity, we restrict our attention to the case of the modulated KdV (1.2) on $\mathbb{T}$. As mentioned in Subsection 1.5, Theorem 1.20 follows once we prove the following proposition.

Proposition 8.1. *Let $N \in \mathbb{N}$. The white noise μ defined in* (1.27) *is invariant under the truncated modulated KdV dynamics* (1.22).

Given $N \in \mathbb{N}$, set

$$u_N = \mathbf{P}_N u^N \qquad \text{and} \qquad u_N^\perp = \mathbf{P}_N^\perp u^N,$$

where $\mathbf{P}_N^\perp = \mathrm{Id} - \mathbf{P}_N$. Then, the truncated modulated KdV (1.22) decouples into the low frequency nonlinear but finite-dimensional dynamics, satisfied by u_N:

$$\begin{cases} \partial_t u_N + \partial_x^3 u_N \cdot \partial_t w = \partial_x \mathbf{P}_N(u_N^2) \\ u_N(0) = \mathbf{P}_N u^N(0) \end{cases} \tag{8.1}$$

and the high frequency linear dynamics, satisfied by $u_N^\perp$:

$$\begin{cases} \partial_t u_N^\perp + \partial_x^3 u_N^\perp \cdot \partial_t w = 0 \\ u_N^\perp(0) = \mathbf{P}_N^\perp u^N(0). \end{cases} \tag{8.2}$$

Let $\rho > \frac{2}{3}$. Then, by a slight modification of the proof of Theorem 1.3 applied to the low frequency part (8.1) along with the conservation of the (truncated) L^2-norm, we see that the truncated modulated KdV (1.22) is globally well-posed in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$, since the high frequency dynamics is linear (which is globally well-posed in $\mathbf{P}_N^\perp H^s(\mathbb{T})$ for any $s \in \mathbb{R}$).

Let us write the white noise μ as

$$\mu = \mu^{N,\text{low}} \otimes \mu^{N,\text{high}}, \tag{8.3}$$

where $\mu^{N,\text{low}} = (\mathbf{P}_N)_* \mu$ and $\mu^{N,\text{high}} = (\mathbf{P}_N^\perp)_* \mu$ are the pushforward image measures under the maps $\mathbf{P}_N$ and $\mathbf{P}_N^\perp$, respectively. From the rotational invariance of complex-valued Gaussian random variables, we see that $\mu^{N,\text{high}}$ is invariant under the high frequency linear dynamics (8.2) whose action on each Fourier coefficient is given by $\mathcal{F}(u_N^\perp)(0,n) = g_n \mapsto \mathcal{F}(u_N^\perp)(t,n) = e^{in^3 w(t)} g_n$.

By Plancherel's theorem with $\widehat{u}(-n) = \overline{\widehat{u}(n)}$, we can write the truncated white noise $\mu^{N,\text{low}} = (\mathbf{P}_N)_* \mu$ as

$$\begin{aligned} d\mu^{N,\text{low}} &= Z_N^{-1} \exp\bigg(-\frac{1}{2} \int_{\mathbb{T}} (\mathbf{P}_N u)^2 dx_{\mathbb{T}} \bigg) d\mathbf{P}_N u \\ &= Z_N^{-1} \exp\bigg(-\sum_{n=1}^N |\widehat{u}(n)|^2 \bigg) \prod_{n=1}^N d\widehat{u}(n), \end{aligned}$$

where $d\widehat{u}(n)$, $n = 1, \dots, N$, is the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. Arguing as in Proposition 6.1, we see that the truncated L^2-norm $\|\mathbf{P}_N u\|_{L^2}$ (= the $\mathbb{C}^N$-Euclidean distance on the Fourier side) is conserved under the flow of (8.1). Hence, invariance of the truncated white noise $\mu^{N,\text{low}}$ follows once we prove invariance of the Lebesgue measure $\prod_{n=1}^N d\widehat{u}(n)$.

Very formally, we can write the (time-dependent) Hamiltonian for (8.1) as

$$H_N(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x \mathbf{P}_N u)^2 \partial_t w(t) dx_{\mathbb{T}} + \frac{1}{3} \int_{\mathbb{T}} (\mathbf{P}_N u)^3 dx_{\mathbb{T}}.$$

Unfortunately, the expression above does not make sense since $\partial_t w(t)$ is not well defined for fixed t. Similarly, by viewing (8.1) on the Fourier side, one may try to verify invariance of the Lebesgue measure $\prod_{n=1}^N d\widehat{u}(n)$ by directly computing the divergence of the vector field; see, for example, [119, Lemma 4.1]. However, the divergence involves $\partial_t w(t)$ and hence this approach fails as well. (If we pretend that $\partial_t w(t)$ makes sense for fixed t, the divergence is indeed 0 since the terms with $\partial_t w(t)$ cancel out.)

A correct approach is to study the equations satisfied by the modulated interaction representation. Fix $N \in \mathbb{N}$. By applying $U^w(t)^{-1}$ to the truncated modulated KdV (1.22), we obtain the following truncated version of (4.1):

$$\partial_t \mathbf{u}^N = \mathbf{P}_N U^w(t)^{-1} \partial_x \big((\mathbf{P}_N U^w(t) \mathbf{u}^N)^2 \big). \tag{8.4}$$

Since the action of $U^w(t)^{-1}$ on each Fourier coefficient is the multiplication by $e^{-in^3 w(t)}$, we see that the white noise is invariant under $U^w(t)^{-1}$, namely

$$\mu = (U^w(t)^{-1})_* \mu.$$

See also Section 4 in [132]. Thus, Proposition 8.1 follows once we prove invariance of the white noise μ under (8.4).

As before, we decouple the truncated modulated KdV dynamics (8.4) into the finite-dimensional nonlinear dynamics for the low frequency part $\mathbf{u}_N = \mathbf{P}_N \mathbf{u}^N$:

$$\begin{cases} \partial_t \mathbf{u}_N = \mathbf{P}_N U^w(t)^{-1} \partial_x \big((U^w(t) \mathbf{u}_N)^2 \big) \\ \mathbf{u}_N|_{t=0} = \mathbf{P}_N u_0^\omega, \end{cases} \tag{8.5}$$

and the constant dynamics for the high frequency part $\mathbf{u}_N^\perp = \mathbf{P}_N^\perp \mathbf{u}^N$:

$$\begin{cases} \partial_t \mathbf{u}_N^\perp = 0 \\ \mathbf{u}_N^\perp|_{t=0} = \mathbf{P}_N^\perp u_0^\omega. \end{cases} \tag{8.6}$$

Thanks to (8.3), invariance of the white noise $\mu = \mu^{N,\text{low}} \otimes \mu^{N,\text{high}}$ under (8.4) follows once we prove invariance of $\mu^{N,\text{low}}$ (and $\mu^{N,\text{high}}$) under (8.5) (and (8.6), respectively). Obviously, $\mu^{N,\text{high}}$ is invariant under the constant dynamics (8.6). In view of the L^2-conservation for (8.5) (which follows from the L^2-conservation for (8.1) and the L^2-unitarity of $U^w(t)^{-1}$), invariance of the truncated white noise $\mu^{N,\text{low}}$ (and hence Proposition 8.1 and Theorem 1.20) follows once we prove invariance of the Lebesgue measure $\prod_{n=1}^N d\widehat{\mathbf{u}}_N(n)$ under (8.5).

Lemma 8.2. *The Lebesgue measure $\prod_{n=1}^N d\widehat{\mathbf{u}}_N(n)$ on $\mathbb{C}^N$ is invariant under the truncated dynamics* (8.5) *viewed on the Fourier side.*

Proof. Let $p_n(t) = \operatorname{Re} \widehat{\mathbf{u}}_N(t,n)$ and $q_n(t) = \operatorname{Im} \widehat{\mathbf{u}}_N(t,n)$ for $1 \le |n| \le N$. Since $\mathbf{u}_N$ is real-valued, we have

$$p_{-n} = p_n \qquad \text{and} \qquad q_{-n} = -q_n. \tag{8.7}$$

Then, (8.5) can be rewritten as

$$\partial_t p_n = P_n \qquad \text{and} \qquad \partial_t q_n = Q_n \tag{8.8}$$

for $n = 1, \dots N$, where P_n and Q_n are defined by

$$\begin{aligned} P_n = \frac{in}{2}\Bigg[& \sum_{\substack{1\le |n_1|,|n_2|\le N\\ n=n_1+n_2}} e^{i\Xi_{\mathrm{KdV}}(\bar{n})w(t)}(p_{n_1}+iq_{n_1})(p_{n_2}+iq_{n_2}) \\ & - \sum_{\substack{1\le |n_1|,|n_2|\le N\\ n=n_1+n_2}} e^{-i\Xi_{\mathrm{KdV}}(\bar{n})w(t)}(p_{n_1}-iq_{n_1})(p_{n_2}-iq_{n_2})\Bigg], \\ Q_n = \frac{n}{2}\Bigg[& \sum_{\substack{1\le |n_1|,|n_2|\le N\\ n=n_1+n_2}} e^{i\Xi_{\mathrm{KdV}}(\bar{n})w(t)}(p_{n_1}+iq_{n_1})(p_{n_2}+iq_{n_2}) \\ & + \sum_{\substack{1\le |n_1|,|n_2|\le N\\ n=n_1+n_2}} e^{-i\Xi_{\mathrm{KdV}}(\bar{n})w(t)}(p_{n_1}-iq_{n_1})(p_{n_2}-iq_{n_2})\Bigg], \end{aligned} \tag{8.9}$$

where $\Xi_{\mathrm{KdV}}(\bar{n})$ is as in (4.6). From (8.9) with (8.7) (with $p_0 = q_0 = 0$), we have

$$\begin{aligned} \partial_{p_n} P_n &= in\Big(e^{i\Xi_{\mathrm{KdV}}(n,-n,2n)w(t)}(p_{2n}+iq_{2n}) \\ &\qquad - e^{-i\Xi_{\mathrm{KdV}}(n,-n,2n)w(t)}(p_{2n}-iq_{2n})\Big), \\ \partial_{q_n} Q_n &= n\Big(e^{i\Xi_{\mathrm{KdV}}(n,-n,2n)w(t)}(-i)(p_{2n}+iq_{2n}) \\ &\qquad + e^{-i\Xi_{\mathrm{KdV}}(n,-n,2n)w(t)}i(p_{2n}-iq_{2n})\Big). \end{aligned} \tag{8.10}$$

Let $A(\bar{p},\bar{q}) = A(p_1,\dots,p_N,q_1,\dots,q_N)$ be the vector field of the dynamics (8.8) defined by

$$A(\bar{p},\bar{q}) = (P_1,\dots,P_N,Q_1,\dots,Q_N).$$

Then, from (8.10), we have

$$\operatorname{div}_{\bar{p},\bar{q}} A(\bar{p},\bar{q}) = \sum_{n=1}^{N}(\partial_{p_n}P_n + \partial_{q_n}Q_n) = 0.$$

Therefore from Liouville's theorem, we conclude that the Lebesgue measure $\prod_{n=1}^N d\widehat{\mathbf{u}}_N(n) = \prod_{n=1}^N dp_n dq_n$ is invariant under the truncated dynamics (8.5). □

9. Stochastic modulated KdV

9.1. **Local well-posedness.** In this subsection, we prove local well-posedness of the stochastic modulated KdV (1.29) (Theorem 1.23 (i)).

Let us first give a precise meaning of the stochastic term Ψ in (1.37). Let W denote a cylindrical Wiener process on $L^2_0(\mathbb{T})$:

$$W(t) = \sum_{n\in\mathbb{Z}^*} B_n(t)e_n,$$

where $\{B_n\}_{n\in\mathbb{Z}^*}$ is defined by $B_n(t) = \langle \xi, \mathbf{1}_{[0,t]}\cdot e_n\rangle_{t,x}$. Here, $\langle\cdot,\cdot\rangle_{t,x}$ denotes the duality pairing on $\mathbb{R}_+\times\mathbb{T}$. As a result, we see that $\{B_n\}_{n\in\mathbb{Z}^*}$ is a family of mutually independent

complex-valued Brownian motions conditioned that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^*$. Note that we have, for any $n \in \mathbb{Z}^*$,

$$\mathrm{Var}(B_n(t)) = \mathbb{E}\big[\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{t,x} \overline{\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{t,x}}\big] = \|\mathbf{1}_{[0,t]} \cdot e_n\|_{L^2_{t,x}}^2 = t.$$

Then, we can write the stochastic term Ψ in (1.37) as

$$\Psi(t) = \int_0^t U^w(t')^{-1}\phi dW(t') = \sum_{m,n\in\mathbb{Z}^*} \phi_{mn} I_{t,0}(m,n) e_m, \tag{9.1}$$

where $I_{t_1,t_2}(m,n)$, $m,n \in \mathbb{Z}^*$, is the Wiener integral given by

$$I_{t_1,t_2}(m,n) = \int_{t_2}^{t_1} e^{-im^3 w(t')} dB_n(t'). \tag{9.2}$$

Note that $I_{t_1,t_2}(-m,-n) = \overline{I_{t_1,t_2}(m,n)}$, $m,n \in \mathbb{Z}^*$.

Lemma 9.1. *Given $\rho > 0$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Given $s \in \mathbb{R}$, let $\phi \in \mathrm{HS}(L^2_0(\mathbb{T}); H^s_0(\mathbb{T}))$. Then, given any $0 < \alpha < \frac{1}{2}$, Ψ defined in* (9.1) *almost surely belongs to $\mathcal{C}^\alpha([0,T]; H^s(\mathbb{T}))$.*

Proof. By writing

$$\phi(e_n) = \sum_{m\in\mathbb{Z}^*} \phi_{mn} e_m, \quad n \in \mathbb{Z}^*, \tag{9.3}$$

the Hilbert-Schmidt norm of ϕ is given by

$$\|\phi\|_{\mathrm{HS}(L^2_0;H^s_0)} = \bigg(\sum_{n\in\mathbb{Z}^*} \|\phi(e_n)\|_{H^s_0}^2\bigg)^{\frac{1}{2}} = \bigg(\sum_{n,m\in\mathbb{Z}^*} \langle m\rangle^{2s} |\phi_{mn}|^2\bigg)^{\frac{1}{2}}. \tag{9.4}$$

Then, from the equivalence of the $L^p(\Omega)$-norms of a Gaussian random variable, (9.1), (9.3), (9.2), (9.4), and the Wiener isometry, we have

$$\begin{aligned}
\big\| \|\Psi(t_1) - \Psi(t_2)\|_{H^s} \big\|_{L^p(\Omega)} &\sim_p \big\| \|\Psi(t_1) - \Psi(t_2)\|_{H^s} \big\|_{L^2(\Omega)} \\
&= \bigg(\sum_{m\in\mathbb{Z}^*} \langle m\rangle^{2s} \mathbb{E}\Big[\Big| \sum_{n\in\mathbb{Z}^*} I_{t_1,t_2}(m,n)\phi_{mn} \Big|^2 \Big] \bigg)^{\frac{1}{2}} \\
&= \bigg(\sum_{n,m\in\mathbb{Z}^*} \langle m\rangle^{2s} |\phi_{mn}|^2 \mathbb{E}\big[|I_{t_1,t_2}(m,n)|^2 \big] \bigg)^{\frac{1}{2}} \\
&\sim \|\phi\|_{\mathrm{HS}(L^2_0;H^s_0)} |t_1 - t_2|^{\frac{1}{2}}
\end{aligned} \tag{9.5}$$

for any $t_1 \geq t_2 \geq 0$ and any finite $p \geq 1$. Then, given $0 < \alpha < \frac{1}{2}$, by applying the Kolmogorov continuity criterion ([3, Theorem 2.1]) to (9.5) with $p > \frac{2}{1-2\alpha}$, we conclude that $\Psi \in \mathcal{C}^\alpha([0,T]; H^s(\mathbb{T}))$, almost surely. □

We now present a proof of Theorem 1.23 (i).

Proof of Theorem 1.23 (i). Fix $\alpha = \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$ such that $\alpha + \gamma > 1$. Define the map $\Gamma^\Psi = \Gamma^\Psi_{X,u_0}$ on $\mathcal{C}^\alpha([0,\tau]; H^s(\mathbb{T}))$ by setting

$$\begin{aligned}
\Gamma^\Psi(\mathbf{u})(t) &= u_0 + \mathcal{I}^X(\mathbf{u})(t) + \Psi(t) \\
&= \Gamma(\mathbf{u}) + \Psi(t),
\end{aligned}$$

where $\Gamma(\mathbf{u})$ is as in (3.51). Then, by setting an almost surely finite radius $R = 2\big(\|u_0\|_{H^s} + \|\Psi\|_{\mathcal{C}^\alpha_T H^s_x}\big)$ and choosing an almost surely positive local existence time $\tau = \tau\big(R, \|X\|_{\mathcal{X}^{s,\gamma}_k(T)}\big) = \tau\big(\|u_0\|_{H^s}, \|\Psi\|_{\mathcal{C}^\alpha_T H^s_x}, \|X\|_{\mathcal{X}^{s,\gamma}_k(T)}\big)$ sufficiently small, it follows from (3.53) and (3.63) that

$$\begin{aligned}\|\Gamma^\Psi(\mathbf{u})\|_{\mathcal{C}^\alpha_\tau H^s_x} &\le \|u_0\|_{H^s} + \|\Psi\|_{\mathcal{C}^\alpha_T H^s_x} + C\tau^{\gamma-\alpha}\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}(1+R)^{k-1}\|\mathbf{u}\|_{\mathcal{C}^\alpha_\tau H^s_x}\\ &\le R\end{aligned}$$

and

$$\begin{aligned}\|\Gamma^\Psi(\mathbf{u}^1) - \Gamma^\Psi(\mathbf{u}^2)\|_{\mathcal{C}^\alpha_\tau H^s_x} &\le C\tau^{\gamma-\alpha}\|X\|_{\mathcal{X}^{s,\gamma}_k(T)}(1+R)^{k-1}\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathcal{C}^\alpha_\tau H^s_x}\\ &\le \frac{1}{2}\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathcal{C}^\alpha_\tau H^s_x}\end{aligned}$$

for any $\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2 \in B_R \subset \mathcal{C}^\alpha([0,\tau]; H^s(\mathcal{M}))$, from which we conclude that Γ^Ψ is a contraction on B_R. This proves local well-posedness of the stochastic modulated KdV (1.29) on $\mathbb{T}$. □

9.2. **Global dynamics.** In this subsection, we briefly go over a proof of Theorem 1.23 (ii). Since the argument is identical to that in [129], we only sketch the main ideas.

Given $N \in \mathbb{N}$, consider the following truncated stochastic modulated KdV:

$$\begin{cases} \partial_t u^N + \partial_x^3 u^N \cdot \partial_t w = \partial_x \mathbf{P}_N\big((\mathbf{P}_N u^N)^2\big) + \xi\\ u^N|_{t=0} = u_0^\omega, \end{cases} \tag{9.6}$$

where u_0^ω is the white noise given in (1.28). In fact, as in Section 8, it is more convenient to work at the level of the modulated interaction representation $\mathbf{u}(t) = U^w(t)^{-1}u(t)$. Then, (9.6) becomes

$$\begin{cases} \partial_t \mathbf{u}^N = \mathbf{P}_N U^w(t)^{-1}\partial_x\big((\mathbf{P}_N U^w(t)\mathbf{u}^N)^2\big) + U^w(t)^{-1}\xi\\ \mathbf{u}^N|_{t=0} = u_0^\omega. \end{cases} \tag{9.7}$$

As in Section 8, the truncated dynamics (9.7) decouples into the finite-dimensional nonlinear dynamics for the low frequency part $\mathbf{u}_N = \mathbf{P}_N\mathbf{u}^N$:

$$\begin{cases} \partial_t \mathbf{u}_N = \mathbf{P}_N U^w(t)^{-1}\partial_x\big((U^w(t)\mathbf{u}_N)^2\big) + \mathbf{P}_N U^w(t)^{-1}\xi\\ \mathbf{u}_N|_{t=0} = \mathbf{P}_N u_0^\omega, \end{cases} \tag{9.8}$$

and the linear dynamics for the high frequency part $\mathbf{u}_N^\perp = \mathbf{P}_N^\perp\mathbf{u}^N$:

$$\begin{cases} \partial_t \mathbf{u}_N^\perp = \mathbf{P}_N^\perp U^w(t)^{-1}\xi\\ \mathbf{u}_N^\perp|_{t=0} = \mathbf{P}_N^\perp u_0^\omega. \end{cases} \tag{9.9}$$

By a minor modification of [129, Lemma 2.1], we see that both (9.8) and (9.9) are globally well-posed, which in particular implies that (9.7) is globally well-posed. For $t_1 \ge t_2 \ge 0$, we denote by $\Phi^{N,\mathrm{low}}_{t_1,t_2}$ and $\Phi^{N,\mathrm{high}}_{t_1,t_2}$ the solution maps for (9.8) and (9.9) sending data φ at time t_2 to the solutions $\Phi^{N,\mathrm{low}}_{t_1,t_2}\varphi$ and $\Phi^{N,\mathrm{high}}_{t_1,t_2}\varphi$ at time t_1. We let $P^{N,\mathrm{low}}_{t_1,t_2}$ and $P^{N,\mathrm{high}}_{t_1,t_2}$ denote the transition semigroups for (9.8) and (9.9), respectively, defined as in (1.31), where the expectation is taken over the noise restricted to the time interval $[t_2, t_1]$. We also use $\Phi^N_{t_1,t_2}$ and $P^N_{t_1,t_2}$ to denote the solution map and the transition semigroup for the truncated stochastic modulated KdV (9.7).

Given $\sigma \geq 0$, let μ_σ be the white noise measure (with variance σ) as in (1.30). Then, as in (8.3), we can write μ_σ as

$$\mu_\sigma = \mu_\sigma^{N,\text{low}} \otimes \mu_\sigma^{N,\text{high}},$$

where $\mu_\sigma^{N,\text{low}} = (\mathbf{P}_N)_*\mu_\sigma$ and $\mu_\sigma^{N,\text{high}} = (\mathbf{P}_N^\perp)_*\mu_\sigma$ are the pushforward image measures of μ_σ under $\mathbf{P}_N$ and $\mathbf{P}_N^\perp$, respectively. The high frequency dynamics (9.9) is linear and it is easy to verify that

$$(P_{t_1,t_2}^{N,\text{high}})^* \mu_{1+t_2}^{N,\text{high}} = \mu_{1+t_1}^{N,\text{high}}. \tag{9.10}$$

See (1.33) (and (1.32)) for the definition of the adjoint $(P_{t_1,t_2}^{N,\text{high}})^*$. By writing it on the Fourier side, we see that the low frequency dynamics (9.8) is nothing but a finite-dimensional system of SDEs, which can be viewed as the superposition of the finite-dimensional modulated KdV dynamics:

$$\partial_t \mathbf{u}_N = \mathbf{P}_N U^w(t)^{-1} \partial_x \big((U^w(t)\mathbf{u}_N)^2 \big) \tag{9.11}$$

and the linear stochastic dynamics:

$$\partial_t \mathbf{u}_N = \mathbf{P}_N U^w(t)^{-1} \xi. \tag{9.12}$$

As we saw in Section 8, the truncated white noise $\mu_\sigma^{N,\text{low}}$ (with any variance σ) is invariant under the former dynamics (9.11), while the latter dynamics (9.12) increases the variance of the white noise initial data by the length of the time interval under consideration. Then, in view of the Lie-Trotter product formula (1.39), we see that

$$(P_{t_1,t_2}^{N,\text{low}})^* \mu_{1+t_2}^{N,\text{low}} = \mu_{1+t_1}^{N,\text{low}}. \tag{9.13}$$

Putting (9.10) and (9.13) together with the invariance of the white noise μ_σ (with any variance σ) under $U^w(t)$, we then obtain the following proposition.

Proposition 9.2. *Let $N \in \mathbb{N}$. Then, for any $t_1 \geq t_2 \geq 0$, we have*

$$(P_{t_1,t_2}^N)^* \mu_{1+t_2} = \mu_{1+t_1},$$

where P_{t_1,t_2}^N is the transition semigroup for the truncated dynamics (9.7)*. Namely, $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ is an evolution system of measures for the truncated dynamics* (9.7) *and also for the truncated stochastic modulated KdV equation* (9.6)*.*

In [129], the authors proved an analogous claim for the truncated stochastic (unmodulated) KdV by studying the Kolmogorov forward equation (= the Fokker-Planck equation) for the distribution of the Fourier coefficients of a solution $u_N(t)$ to the low frequency dynamics and showing that the truncated white noise is the unique solution to the Kolmogorov forward equation; see [129, Section 2]. We point out that for such an argument, we need to make use of the computations in the proof of Lemma 8.2. Namely, in the current modulated setting, we need to work at the level of the modulated interaction representation.

Once we have Proposition 9.2 at hand, a variant of Bourgain's invariant measure argument [7] yields the following probabilistic uniform (in N) growth bound on the solutions to the truncated dynamics (9.7).

Proposition 9.3. *Given $\rho > \frac{2}{3}$ and $\frac{1}{2} < \gamma < 1$, let w be (ρ,γ)-irregular on $\mathbb{R}_+$. Let $s < -\frac{1}{2}$ satisfy* (1.24)*. Given $N \in \mathbb{N}$, let $\mathbf{u}^N$ be the global solution to the truncated dynamics* (9.7) *with the mean-zero white noise initial data u_0^ω in* (1.28)*. Then, given any $T \gg 1$ and $0 < \varepsilon \ll 1$, there exists a set $\Omega_{T,\varepsilon}(N)$ such that $\mathbb{P}(\Omega_{T,\varepsilon}(N)^c) < \varepsilon$ and*

$$\sup_{t\in[0,T]} \|\mathbf{u}^N(t)\|_{H^s} \le C\big(\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}}\big)\sqrt{\log\frac{1}{\varepsilon}}\sqrt{T\log T} \tag{9.14}$$

on $\Omega_{T,\varepsilon}(N)$, where $C : [0,\infty) \to (0,\infty)$ is a non-decreasing function, independent of $N \in \mathbb{N}$, $T \gg 1$, and $\varepsilon \ll 1$.

Proposition 9.3 follows from a straightforward modification of the proof of [129, Proposition 4.1] and thus we omit details. The probabilistic uniform (in N) growth bound (9.14) together with the following lemma on convergence of the Galerkin approximation yields the almost sure global well-posedness claimed in Theorem 1.23 (ii). Finally, (1.34) follows from Proposition 9.2 and the following lemma (with $\phi = \mathrm{Id}$).

Lemma 9.4 (Galerkin approximation)**.** *Given $\rho > \frac{1}{2}$, $\frac{1}{2} < \gamma < 1$, and $T > 0$, let w be (ρ,γ)-irregular on $[0,T]$ in the sense of Definition 1.1. Suppose that $s \in \mathbb{R}$ satisfies* (1.24) *and fix $u_0 \in H^s(\mathbb{T})$ and $\phi \in \mathrm{HS}(L^2_0(\mathbb{T}); H^s_0(\mathbb{T}))$. Then, as $N \to \infty$, the solution $\mathbf{u}^N$ to the following truncated dynamics*:

$$\begin{cases} \partial_t \mathbf{u}^N = \mathbf{P}_N U^w(t)^{-1}\partial_x\big((\mathbf{P}_N U^w(t)\mathbf{u}^N)^2\big) + U^w(t)^{-1}\phi\xi \\ \mathbf{u}^N|_{t=0} = u_0 \end{cases} \tag{9.15}$$

converges to the modulated interaction $\mathbf{u}(t) = U^w(t)^{-1}u(t)$ of the solution u to the stochastic modulated KdV equation (1.29) *with $u|_{t=0} = u_0$ in $\mathcal{C}^\gamma([0,\tau]; H^s(\mathbb{T}))$, where $\tau > 0$ denotes the almost surely positive local existence time from Theorem 1.23 (i)* (*but possibly made smaller by a multiplicative constant*)*.*

It is worthwhile to note that while we have $\mathbf{u}, \mathbf{u}^N \in \mathcal{C}^\alpha([0,\tau]; H^s(\mathbb{T}))$ only for $\alpha < \frac{1}{2}$, the difference $\mathbf{u} - \mathbf{u}^N$ belongs to a better space $\mathcal{C}^\gamma([0,\tau]; H^s(\mathbb{T}))$ (with $\gamma > \frac{1}{2}$).

Proof of Lemma 9.4. Let X and X^N be as in (4.3) and (4.7), respectively. Then, from Proposition 4.1 (iv), we see that X^N converges to X in $\mathcal{X}_2^{s,\gamma}([0,T]\times\mathbb{T})$ as $N \to \infty$.

Note that $\mathbf{u}$ and $\mathbf{u}^N$ satisfy the following equations:

$$\mathbf{u}(t) = u_0 + \mathcal{I}^X(\mathbf{u})(t) + \Psi,$$
$$\mathbf{u}^N(t) = u_0 + \mathcal{I}^{X^N}(\mathbf{u}^N)(t) + \Psi,$$

where Ψ is as in (1.37) and (9.1). In particular, $\mathbf{u} - \mathbf{u}^N$ satisfies (3.67). Then, by repeating the proof of Proposition 3.8 (iv), we obtain the desired convergence. □

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