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Ultraviolet Renormalization of the Nelson Hamiltonian through Functional Integration

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Abstract

Starting from the N-particle Nelson Hamiltonian defined by imposing an ultraviolet cutoff, we perform ultraviolet renormalization by showing that in the zero cutoff limit a self-adjoint operator exists after a logarithmically divergent term is subtracted from the original Hamiltonian. We obtain this term as the diagonal part of a pair interaction appearing in the density of a Gibbs measure derived from the Feynman-Kac representation of the Hamiltonian. Also, we show existence of a weak coupling limit of the renormalized Hamiltonian and derive an effective Yukawa interaction potential between the particles.

Keywords: N-particle Nelson model, ultraviolet cutoff, energy renormalization, Yukawa potential, Feynman-Kac representation, stochastic integrals, Gibbs measure

1 Introduction

In this paper we consider the N-particle Nelson model, which describes the interaction of N electrically charged spinless particles with a scalar boson field. In Fock space representation the model can be given by the Hamiltonian

$$H = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm I} \tag{1.1}$$

acting on

$$\mathscr{H} = L^2(\mathbb{R}^{3N}) \otimes \mathscr{F},$$

where $L^2(\mathbb{R}^{3N})$ is the particle space and $\mathscr{F}$ denotes the symmetric Fock space over $L^2(\mathbb{R}^3)$ describing the bosons. Recall that $\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)}$, where $\mathscr{F}^{(n)} = \otimes_{\rm sym}^n L^2(\mathbb{R}^3)$ is the n-boson subspace and $\mathscr{F}^{(0)} = \mathbb{C}$ (where the subscript indicates symmetrized tensor product), for which the infinite direct sum norm $\|F\|_{\mathscr{F}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathscr{F}^{(n)}}^2$ is finite. We denote the Fock vacuum by $\mathbb{1}_{\mathscr{F}} = 1 \oplus 0 \oplus 0 \oplus \ldots \in \mathscr{F}$, and write simply $\mathbb{1}$ when no confusion arises.

The components of the N-particle Nelson Hamiltonian are as follows. The N-particle Schrödinger operator

$$H_{\rm p} = -\frac{1}{2} \sum_{j=1}^{N} \Delta_j + V$$

is the Hamiltonian of the free particles with an external potential $V : \mathbb{R}^{3N} \to \mathbb{R}$, which acts as a multiplication operator, and where $\Delta_j = \Delta_{x_j}$ denotes the 3-dimensional Laplacian.

Let $a^*(f)$ and $a(f)$, $f \in L^2(\mathbb{R}^3)$, be the boson creation and annihilation operators, respectively, satisfying the canonical commutation relations $[a(f), a^*(g)] = (\bar{f}, g)$ and $[a(f), a(g)] = 0 = [a^*(f), a^*(g)]$. We formally denote $a^\sharp(f) = \int_{\mathbb{R}^3} a^\sharp(k) \widehat{f}(k) dk$, where $a^\sharp$ stands for a or a^*. Denote by $\omega(k)$ the dispersion relation, which we will choose in the main part of the paper to be $\omega(k) = |k|$, describing massless bosons. The free field Hamiltonian $H_{\rm f}$ of $\mathscr{F}$ is then defined by the second quantization of ω, i.e.,

$$H_{\rm f} \prod_{j=1}^{n} a^*(f_j)\mathbb{1} = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n)\mathbb{1} \quad \text{and} \quad H_{\rm f}\mathbb{1} = \mathbb{1},$$

formally expressed as

$$H_{\rm f} = \int_{\mathbb{R}^3} \omega(k) a^*(k) a(k) dk.$$

The interaction Hamiltonian is formally defined by

$$H_{\rm I}(x) = g \sum_{j=1}^{N} \int_{\mathbb{R}^3} \frac{1}{\sqrt{2\omega(k)}} \left(\widehat{\varphi}(k) {\rm e}^{ik \cdot x_j} a(k) + \widehat{\varphi}(-k) {\rm e}^{-ik \cdot x_j} a^*(k) \right) dk \tag{1.2}$$

for every $x \in \mathbb{R}^{3N}$. We make the identification $\mathscr{H} \cong L^2(\mathbb{R}^{3N}; \mathscr{F})$, i.e., $F \in \mathscr{H}$ will be regarded as a function $\mathbb{R}^{3N} \ni x \mapsto F(x) \in \mathscr{F}$ such that $\int_{\mathbb{R}^{3N}} \|F(x)\|_{\mathscr{F}}^2 dx < \infty$. Under this identification the interaction Hamiltonian becomes $(H_{\rm I}F)(x) = H_{\rm I}(x)F(x)$ on $\mathscr{H}$. The function φ (featured in its Fourier transform $\widehat{\varphi}$) is a function describing a charge distribution so that the total charge is $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. In the Hamiltonian it has a

regularising role, making the operator well defined, and physically it has the meaning of an ultraviolet (UV) cutoff. The prefactor g is a coupling constant between the particles and the field.

Under the assumptions

$$\widehat{\varphi}/\omega^{1/2}, \widehat{\varphi}/\omega \in L^2(\mathbb{R}^3), \quad \overline{\widehat{\varphi}(k)} = \widehat{\varphi}(-k) \tag{1.3}$$

the interaction $H_{\rm I}$ is well defined, symmetric and infinitesimally $\mathbb{1}\otimes H_{\rm f}$-bounded. Thus by the Kato-Rellich theorem H is a self-adjoint operator on the domain $D(H_{\rm p}\otimes\mathbb{1})\cap D(\mathbb{1}\otimes H_{\rm f})$. If, moreover, an infrared (IR) cutoff is imposed by the condition

$$\widehat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3), \tag{1.4}$$

then H has a unique ground state [Spo98, BFS98, Ger00, Ara01, Sas05], i.e., an eigenfunction $\Psi \in \mathscr{H}$ at the bottom of its spectrum. As it was shown in [LMS02, DG04, Hir06], condition (1.4) is also necessary for a ground state to exist in this space.

In this paper we are interested in an appropriate definition of H in the point charge limit, i.e., when $\varphi(x) \to (2\pi)^{3/2}\delta(x)$ or, equivalently, $\widehat{\varphi}(k) \to 1$. This is a physically interesting but mathematically singular case, when condition (1.3) fails to hold. In order to analyse this limit, we regularise the Hamiltonian by choosing the UV cutoff function $\widehat{\varphi}_\varepsilon(k) = {\rm e}^{-\varepsilon|k|^2/2}$. With this choice in (1.2) we define the Hamiltonian H_ε, and by regarding $\varepsilon > 0$ as a UV cutoff parameter we will analyse the limit of $H_\varepsilon - E_\varepsilon$ as $\varepsilon \downarrow 0$, where E_ε is an energy renormalization term, which will be determined below.

The main results of this paper are as follows.

(1) By using functional integration we derive the energy renormalization term E_ε from the diagonal part of a pair interaction, and show the existence of the renormalized Hamiltonian $H_{\rm ren} = \lim_{\varepsilon\downarrow 0}(H_\varepsilon - E_\varepsilon)$ in the sense of strong convergence of the related semigroups.

(2) We derive the pair interaction potential in the Boltzmann-Gibbs density of the path measure associated with $H_{\rm ren}$.

(3) We show existence of the weak coupling limit of ${\rm e}^{-tH_{\rm ren}}$ and compare it with that of the Nelson model with UV cutoff.

Here are some comments to these points.

(1) The first part of this problem was already investigated in [Nel64a] by using Gross transform. This is a unitary transform implemented by $e^{i\pi_\varepsilon(x)}$ with the generator

$$\pi_\varepsilon(x) = i\sum_{j=1}^N \int_{|k|>\Lambda} \frac{1}{\sqrt{2\omega(k)}}\frac{1}{\omega(k)+|k|^2/2}(-a(k)\widehat{\varphi}_\varepsilon(k)e^{ikx_j} + a^*(k)\widehat{\varphi}_\varepsilon(-k)e^{-ikx_j})dk. \tag{1.5}$$

Here $\Lambda > 0$ is introduced for π_ε to be well defined. Nelson has shown that the Gross transformed operator ${\rm e}^{i\pi_\varepsilon(x)}(H_\varepsilon - E_\varepsilon){\rm e}^{-i\pi_\varepsilon(x)}$ converges to a self-adjoint operator in norm resolvent sense as $\varepsilon \downarrow 0$, and ${\rm e}^{i\pi_\varepsilon(x)} \to {\rm e}^{i\pi_0(x)}$ in strong sense.

In contrast to this approach, in the following we present a UV renormalization by using path measure methods. Nelson did consider a path integral approach [Nel64b], however, this remained incomplete since the approach based on Gross transform may have appeared simpler and satisfactory for the purposes of his investigation. Taking the Gross transformation of $H_\varepsilon - E_\varepsilon$, a cancellation of diverging terms occurs and the limit $\varepsilon \downarrow 0$ can be analysed to define a UV renormalised Hamiltonian. However, in this paper we do not take Gross transform and derive E_ε from the diagonal part of the pair potential associated with a Gibbs measure instead.

Following our previous work on the one-particle Nelson model [LMS02, BHLMS02] we find it worthwhile to analyse this problem by using functional integration not just for the extra insights it gives (applicable also to other cases, e.g., UV renormalization of the Nelson model with variable coefficients [GHPS12]), but also because this approach allows to prove existence of a ground state of the UV renormalized Hamiltonian. As far as we are aware, the existence of a ground state of $H_{\rm ren}$ was shown for sufficiently weak couplings only [HHS05]. This problem is another illustration of the fact that direct analytic and path integral methods complement each other, and both have specific advantages.

A key point in this paper is to show that

$$\lim_{\varepsilon\downarrow 0}(f\otimes \mathbb{1}, \mathrm{e}^{-T(H_\varepsilon - E_\varepsilon)} g\otimes \mathbb{1}) \tag{1.6}$$

can be expressed in terms of a path measure representation (Lemma 2.16 below), and

$$H_\varepsilon - E_\varepsilon > C \tag{1.7}$$

holds with a constant C, uniformly in $\varepsilon > 0$ (Corollary 2.21). Although these were established by operator analysis methods in [Nel64a] by using the Gross transform, we derive them directly by using Feynman-Kac type formulae for $H_\varepsilon - E_\varepsilon$, so our strategy is completely different from Nelson's.

(2) By constructing a functional integral representation of the Nelson Hamiltonian with UV cutoff and integrating out the field part, an expression is obtained in terms of an expectation with respect to Wiener measure on Brownian paths under V and a pair interaction potential W (see Chapter 6 in [LHB11]). Using this as a density, the path measure can be seen as a Gibbs measure on paths. The pair interaction has the form

$$\int_{-T}^{T} ds \int_{-T}^{T} dt W(B_t - B_s, t-s), \tag{1.8}$$

where W depends on the UV cutoff [BHLMS02]. On the other hand, a similar construction for the Pauli-Fierz model in non-relativistic quantum electrodynamics yields a Gibbs measure with pair interaction formally given by [Spo87, BH09]

$$\int_{-T}^{T} dB_s^\mu \int_{-T}^{T} dB_t^\nu W_{\mu\nu}(B_t - B_s, t-s). \tag{1.9}$$

It is remarkable that the double Riemann integral in (1.8) is replaced by a double stochastic integral. In this paper we will consider the finite volume Gibbs measure associated with

the Nelson model without UV cutoff and obtain that the exponent in the Boltzmann-Gibbs density has the form (Corollary 2.23)

$$\int_{-T}^{T} ds \int_{s}^{T} dB_t W(B_t - B_s, t-s) + \int_{-T}^{T} ds W(B_T - B_s, T-s). \tag{1.10}$$

It is interesting to see that the Gibbs measure without a UV cutoff has the intermediate form of (1.8) and (1.9).

(3) Finally we consider the weak coupling limit, which is a scaling limit such that the creation operators a^* and the annihilation operators a are scaled to κa and κa^*, respectively, with a scaling parameter $\kappa > 0$. When $\varepsilon > 0$, the scaled Nelson Hamiltonian $H_\varepsilon(\kappa)$ converges in the limit $\kappa \to \infty$ to a Schrödinger operator with an effective potential, and furthermore it converges to a Schrödinger operator with Yukawa potential when in a subsequent limit $\varepsilon \downarrow 0$. We are interested in obtaining the weak coupling limit of $H_{\rm ren}(\kappa)$ and comparing it with that of the UV-regularized Hamiltonian (Corollary 3.3).

We note that in this paper the space dimension is fixed to the physical $d = 3$, however, our arguments can be carried out for any other dimension in a similar way. We will see that the energy renormalization term behaves like $E_\varepsilon \sim -\int_1^\infty e^{-\varepsilon r^2} r^{d-4} dr$. When $d = 1$ or $d = 2$, there is no need for an energy renormalization, and when $d \geq 3$, energy renormalization becomes important to deal with the UV divergences in the $\varepsilon \downarrow 0$ limit. Also, it will be seen that our arguments do not need the assumption of a pinning external potential, in fact, we do not need any. However, we keep a V in our formulae with the understanding that the results are valid also for $V \equiv 0$.

The plan of this paper is as follows. In the main Section 2 we perform renormalization on the level of the density of the Gibbs measure, determine the pair interaction potential, and prove existence of a UV renormalized Hamilton operator. In Section 3 we study a weak coupling limit of the renormalized Hamiltonian and derive an effective interaction potential between the particles.

2 Energy renormalization by path measures

2.1 Functional integral representation of the regularized Hamiltonian

First we define the version of the Nelson model which will be the main object studied in this paper. We choose $\omega(k) = |k|$. Let

$$\mathbb{1}_\Lambda(k) = \begin{cases} 1, & \omega(k) < \Lambda \\ 0, & \omega(k) \geq \Lambda \end{cases}$$

and $\mathbb{1}_\Lambda^\perp(k) = \mathbb{1} - \mathbb{1}_\Lambda(k)$. We assume that $\Lambda > 0$, which is needed in (2.29), Lemma 2.9 and Corollary 2.21 below. For simplicity we will use the following standing assumption throughout below.

Assumption 2.1 *The external potential V is a bounded continuous function. In particular, it is of Kato-class, i.e., it satisfies*

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^3}\mathbb{E}^x\left[\int_0^t |V(B_s)|ds\right]=0. \tag{2.1}$$

Note that in what follows $V\equiv 0$ is a possible choice without changing the statements. We summarize some properties of Kato-class potentials in Appendix A for the reader's convenience; they will be used below on some objects which take the role of a potential.

Consider the cutoff function

$$\widehat{\varphi}_\varepsilon(k)=\mathrm{e}^{-\varepsilon|k|^2/2}\mathbb{1}_\Lambda^\perp(k),\quad \varepsilon\geq 0 \tag{2.2}$$

and define the regularized Hamiltonian

$$H_\varepsilon=H_\mathrm{p}\otimes\mathbb{1}+\mathbb{1}\otimes H_\mathrm{f}+H_\mathrm{I}^\varepsilon,\quad \varepsilon>0, \tag{2.3}$$

by

$$H_\mathrm{I}^\varepsilon(x)=g\sum_{j=1}^N\int_{\mathbb{R}^3}\frac{1}{\sqrt{2\omega(k)}}\left(\widehat{\varphi}_\varepsilon(k)\mathrm{e}^{ik\cdot x_j}a(k)+\widehat{\varphi}_\varepsilon(-k)\mathrm{e}^{-ik\cdot x_j}a^*(k)\right)dk. \tag{2.4}$$

Here $\varepsilon>0$ is regarded as the UV cutoff parameter. The main purpose of this paper is to consider the limit $\varepsilon\downarrow 0$ of H_ε. We show that this limit can be sensibly defined by an energy renormalization. Define

$$E_\varepsilon=-\frac{g^2}{2}N\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\varepsilon|k|^2}}{\omega(k)}\beta(k)\mathbb{1}_\Lambda^\perp(k)dk, \tag{2.5}$$

where

$$\beta(k)=\frac{1}{\omega(k)+|k|^2/2}. \tag{2.6}$$

Notice that $E_\varepsilon\to-\infty$ as $\varepsilon\downarrow 0$.

Our main theorem states that $H_\varepsilon-E_\varepsilon$ converges in the $\varepsilon\downarrow 0$ limit to a non-trivial self-adjoint operator H_ren in a specific sense, which we call the UV renormalized Nelson Hamiltonian.

Theorem 2.2 *There exists a self-adjoint operator H_ren such that*

$$\mathrm{s}\text{-}\lim_{\varepsilon\downarrow 0}\mathrm{e}^{-t(H_\varepsilon-E_\varepsilon)}=\mathrm{e}^{-tH_\mathrm{ren}},\quad t\geq 0. \tag{2.7}$$

We carry out the proof by functional integration and will obtain E_ε as the diagonal term of a pair interaction potential on the paths of a Brownian motion.

Let $(B_t)_{t\in\mathbb{R}}=(B_t^1,...,B_t^N)_{t\in\mathbb{R}}$, where $(B_t^j)_{t\in\mathbb{R}}$, $j=1,...,N$, are N independent, two-sided, $\mathbb{R}^3$-valued standard Brownian motions on the space of continuous paths indexed by the whole real line, endowed with Wiener measure. We denote by $\mathbb{E}^x[\cdots]$ expectation with respect to the Wiener measure starting from x at $t=0$. It is known that a Feynman-Kac formula holds for $(F,\mathrm{e}^{-2TH_\varepsilon}G)$, see [LHB11, Theorem 6.3], given in terms of a two-sided Brownian motion. In particular, for $F=f\otimes\mathbb{1}$ and $G=h\otimes\mathbb{1}$ we have the following representation.

Proposition 2.3 *Let $f, h \in L^2(\mathbb{R}^3)$. Then it follows that*

$$(f \otimes \mathbb{1}, \mathrm{e}^{-2TH_\varepsilon} h \otimes \mathbb{1}) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) \mathrm{e}^{-\int_{-T}^T V(B_s) ds} \mathrm{e}^{\frac{q^2}{2} S_\varepsilon} \right],$$

where

$$S_\varepsilon = \sum_{i,j=1}^N \int_{-T}^T ds \int_{-T}^T dt W_\varepsilon(B_t^i - B_s^j, t-s) \tag{2.8}$$

is the pair interaction given by the pair interaction potential

$$W_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{1}{2\omega(k)} \mathrm{e}^{-\varepsilon|k|^2} \mathrm{e}^{-ik\cdot x} \mathrm{e}^{-\omega(k)|t|} \mathbb{1}_\Lambda^\perp(k) dk. \tag{2.9}$$

2.2 Renormalized action

In this section we perform UV renormalization on the level of the density of the path measure. Consider the function

$$\varphi_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\varepsilon|k|^2} \mathrm{e}^{-ik\cdot x - \omega(k)|t|}}{2\omega(k)} \beta(k) \mathbb{1}_\Lambda^\perp(k) dk, \quad \varepsilon \geq 0, \tag{2.10}$$

where $\beta(k)$ is given by (2.6).

Proposition 2.4 *There exists a functional S_0^{ren} such that*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^T V(B_s) ds} \mathrm{e}^{\frac{q^2}{2}(S_\varepsilon - 4NT\varphi_\varepsilon(0,0))} \right] = \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^T V(B_s) ds} \mathrm{e}^{\frac{q^2}{2} S_0^{\mathrm{ren}}} \right]. \tag{2.11}$$

Notice that $W_\varepsilon(x,t)$ is smooth, and $W_\varepsilon(x,t) \to W_0(x,t)$ as $\varepsilon \downarrow 0$ for every $(x,t) \neq (0,0)$, where

$$W_0(x,t) = \int_{\mathbb{R}^3} \frac{1}{2\omega(k)} \mathrm{e}^{-ik\cdot x} \mathrm{e}^{-\omega(k)|t|} \mathbb{1}_\Lambda^\perp(k) dk. \tag{2.12}$$

It is seen, however, that $W_\varepsilon(0,0) \to \infty$ as $\varepsilon \downarrow 0$, i.e., $W_0(x,t)$ has a power-like singularity at $(0,0)$, thus (2.11) is non-trivial to obtain. We will prove the above proposition through a sequence of lemmas below.

From now on we fix $T > 0$. As $\varepsilon \downarrow 0$ only the diagonal part of the interaction has a singular term. Also, fix $0 < \tau \leq T$ and denote $[t]_T = -T \vee t \wedge T$. We decompose the regularized interaction into diagonal and off-diagonal parts as

$$S_\varepsilon = S_\varepsilon^{\mathrm{D}} + S_\varepsilon^{\mathrm{OD}}, \tag{2.13}$$

where

$$S_\varepsilon^{\mathrm{D}} = 2 \sum_{i,j=1}^N \int_{-T}^T ds \int_s^{[s+\tau]_T} dt\, W_\varepsilon(B_t^i - B_s^j, t-s) \tag{2.14}$$

and

$$S_\varepsilon^{\mathrm{OD}} = 2 \sum_{i,j=1}^N \int_{-T}^T ds \int_{[s+\tau]_T}^T dt W_\varepsilon(B_t^i - B_s^j, t-s). \tag{2.15}$$

$S_\varepsilon^{\mathrm{D}}$ denotes the integral of S_ε in the neighborhood of the diagonal $\{(t,t) \in \mathbb{R}^2 | |t| \leq T\}$, and $S_\varepsilon^{\mathrm{OD}}$ on its complement. Notice that for $\tau = T$ we have $S_\varepsilon^{\mathrm{OD}} = 0$. The next lemma is easy to prove and we omit the details.

Lemma 2.5 *It follows pathwise that* $\lim_{\varepsilon \downarrow 0} S_\varepsilon^{\mathrm{OD}} = S_0^{\mathrm{OD}}$, *where* S_0^{OD} *is* $S_\varepsilon^{\mathrm{OD}}$ *with* $\varepsilon = 0$.

A representation in terms of a stochastic integral will help us deal with the more difficult term $S_\varepsilon^{\mathrm{D}}$. In the following lemma first we derive some bounds on $\varphi_\varepsilon(x,t)$ and its gradient.

Lemma 2.6 *There exists a constant* $c > 0$ *such that the bounds*

$$|\nabla \varphi_\varepsilon(x,t)| \leq c|t|^{-1}, \quad t \neq 0$$
$$|\nabla \varphi_\varepsilon(x,t)| \leq c|x|^{-1}, \quad |x| \neq 0$$

hold uniformly in ε. *Furthermore, similar bounds hold for the function* $\varphi_0 - \varphi_\varepsilon$ *with a constant* $\mathrm{c}_\varepsilon > 0$ *such that* $\lim_{\varepsilon \downarrow 0} \mathrm{c}_\varepsilon = 0$, *i.e.,*

$$|\nabla \varphi_\varepsilon(x,t) - \nabla \varphi_0(x,t)| \leq \mathrm{c}_\varepsilon |t|^{-1}, \quad t \neq 0,$$
$$|\nabla \varphi_\varepsilon(x,t) - \nabla \varphi_0(x,t)| \leq \mathrm{c}_\varepsilon |x|^{-1}, \quad |x| \neq 0.$$

Proof. The first bound on the gradient follows directly by

$$|\nabla \varphi_\varepsilon(x,t)| \leq \int_{\mathbb{R}^3} \frac{1}{2(\omega(k) + |k|^2/2)} \mathrm{e}^{-\varepsilon|k|^2} \mathrm{e}^{-\omega(k)|t|} \mathbb{1}_\Lambda^\perp(k) dk \leq c \int_\Lambda^\infty \mathrm{e}^{-rt} dr.$$

Next consider the second. After integration over the angular variables we obtain

$$\varphi_\varepsilon(x,t) = 2\pi \int_\Lambda^\infty \frac{\mathrm{e}^{-\varepsilon r^2 - r|t|}}{r(2+r)} \frac{\sin(r|x|)}{|x|} dr. \tag{2.16}$$

Differentiation in (2.16) gives

$$\nabla \varphi_\varepsilon(x,t) = \frac{2\pi x}{|x|^2} \int_\Lambda^\infty \frac{\mathrm{e}^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{r(2|x|+r)} (r\cos r - \sin r)\, dr, \tag{2.17}$$

and estimating the right-hand side we have

$$|\nabla \varphi_\varepsilon(x,t)| \leq \int_0^1 \frac{Cr^3}{r^2} dr + \left| \int_1^\infty \frac{\mathrm{e}^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{(2|x|+r)} \cos r dr \right| + \int_1^\infty \frac{1}{r^2} dr$$

using that $|r\cos r - \sin r| \leq Cr^3$, for a constant $C > 0$ and all $r \in [0,1]$. Since the integral in the middle term above is bounded, the lemma follows. □

Lemma 2.7 *If* $\varepsilon > 0$, *then the expression*

$$S_\varepsilon^{\mathrm{D}} = 2 \sum_{i,j=1}^N \int_{-T}^T \varphi_\varepsilon(B_s^i - B_s^j, 0) ds - 2 \sum_{i,j=1}^N \int_{-T}^T \varphi_\varepsilon(B_{[s+\tau]_T}^i - B_s^j, [s+\tau]_T - s) ds$$
$$+ 2 \sum_{i,j=1}^N \int_{-T}^T ds \int_s^{[s+\tau]_T} \nabla \varphi_\varepsilon(B_t^i - B_s^j, t-s) \cdot dB_t^i \tag{2.18}$$

holds.

PROOF. Notice that $\varphi_\varepsilon(x,t)$ solves the equation

$$\left(\partial_t + \frac{1}{2}\Delta\right)\varphi_\varepsilon(x,t) = -W_\varepsilon(x,t), \qquad x \in \mathbb{R}^3, \quad t \geq 0. \tag{2.19}$$

Fix i and j. Then the Itô formula yields that

$$\begin{aligned}
&\varphi_\varepsilon(B^i_{[s+\tau]_T} - B^j_s, [s+\tau]_T - s) - \varphi_\varepsilon(B^i_s - B^j_s, 0) \\
&= \int_s^{[s+\tau]_T} \nabla\varphi_\varepsilon(B^i_t - B^j_s, t-s)\cdot dB^i_t + \int_s^{[s+\tau]_T}\left(\partial_t + \frac{1}{2}\Delta\right)\varphi_\varepsilon(B^i_t - B^j_s, t-s)dt.
\end{aligned} \tag{2.20}$$

Hence by (2.19)

$$\begin{aligned}
&\int_s^{[s+\tau]_T} W_\varepsilon(B^i_t - B^j_s, t-s)dt \\
&= \varphi_\varepsilon(B^i_s - B^j_s, 0) - \varphi_\varepsilon(B^i_{[s+\tau]_T} - B^j_s, [s+\tau]_T - s) + \int_s^{[s+\tau]_T}\nabla\varphi_\varepsilon(B^i_t - B^j_s, t-s)\cdot dB^i_t
\end{aligned} \tag{2.21}$$

follows. Inserting this expression into $S^{\mathrm{D}}_\varepsilon$ proves the claim. □

Lemma 2.7 suggests the definition

$$S^{\mathrm{ren}}_\varepsilon = S_\varepsilon - 4NT\varphi_\varepsilon(0,0), \quad \varepsilon > 0 \tag{2.22}$$

as a renormalized action. This can be expressed as

$$S^{\mathrm{ren}}_\varepsilon = S^{\mathrm{OD}}_\varepsilon + X_\varepsilon + Y_\varepsilon + Z_\varepsilon \tag{2.23}$$

where

$$X_\varepsilon = 2\sum_{i\neq j}^N \int_{-T}^T \varphi_\varepsilon(B^i_s - B^j_s, 0)ds, \tag{2.24}$$

$$Y_\varepsilon = 2\sum_{i,j=1}^N \int_{-T}^T ds \int_s^{[s+\tau]_T} \nabla\varphi_\varepsilon(B^i_t - B^j_s, t-s)\cdot dB^i_t, \tag{2.25}$$

$$Z_\varepsilon = -2\sum_{i,j=1}^N \int_{-T}^T \varphi_\varepsilon(B^i_{[s+\tau]_T} - B^j_s, [s+\tau]_T - s)ds. \tag{2.26}$$

Lemma 2.8 *There exist constants c_z and c_s such that $|Z_\varepsilon| \leq c_z T$ and $|S^{\mathrm{OD}}_\varepsilon| \leq c_s T^2$, uniformly in the paths and $\varepsilon \geq 0$.*

PROOF. We see that

$$|Z_\varepsilon| \leq 4\pi N^2\left(\int_{-T}^{T-\tau} ds\int_\Lambda^\infty \frac{\mathrm{e}^{-r\tau}}{1+r/2}dr + \int_{T-\tau}^T ds\int_\Lambda^\infty \frac{\mathrm{e}^{-r(T-s)}}{1+r/2}dr\right) \leq c_z T$$

with some $c_z > 0$. The bound $|S^{\mathrm{OD}}_\varepsilon| \leq c_s T^2$ is derived similarly. □

Lemma 2.9 *It follows that* $\sup_{x\in\mathbb{R}^3}\mathbb{E}^x[|X_\varepsilon|]<\infty$, *for all* $\varepsilon\geq 0$.

Proof. We notice that

$$X_\varepsilon=\sum_{i\neq j}^{N}\int_{-T}^{T}ds\frac{2\pi}{|B_s^i-B_s^j|}\int_\Lambda^\infty\frac{\sin\sqrt{r|B_s^i-B_s^j|}}{r+r^2/2}\mathrm{e}^{-\varepsilon r^2}dr,\quad \varepsilon\geq 0. \tag{2.27}$$

Hence

$$\mathbb{E}^x[|X_\varepsilon|]\leq\sum_{i\neq j}^{N}\int_{-T}^{T}ds\mathbb{E}^x\left[\frac{2\pi}{|B_s^i-B_s^j|}\right]\int_\Lambda^\infty\frac{1}{r+r^2/2}dr. \tag{2.28}$$

The assumption $\Lambda>0$ implies

$$\int_\Lambda^\infty\frac{1}{r+r^2/2}dr<\infty. \tag{2.29}$$

Since $\sum_{i\neq j}^N|x^i-x^j|^{-1}$ is a Kato-class potential on $L^2(\mathbb{R}^{3N})$ (see Appendix A), we have

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}^x\left[\sum_{i\neq j}^{N}\int_{-T}^{T}\frac{ds}{|B_s^i-B_s^j|}\right]<\infty.$$

Thus (2.28) is finite and the lemma follows. □

Lemma 2.10 *It follows that* $\sup_{x\in\mathbb{R}^3}\mathbb{E}^x[|Y_\varepsilon|]<\infty$ *for all* $\varepsilon\geq 0$, *and* $\lim_{\varepsilon\downarrow 0}\sup_{x\in\mathbb{R}^3}\mathbb{E}^x[|Y_\varepsilon-Y_0|]=0$.

Proof. By the Fubini theorem the stochastic and Lebesgue integrals can be interchanged, thus Y_ε has the representation

$$Y_\varepsilon=2\sum_{i,j=1}^{N}\int_{-T}^{T}\left(\int_{[t-\tau]_T}^{t}\nabla\varphi_\varepsilon(B_t^i-B_s^j,t-s)ds\right)\cdot dB_t^i. \tag{2.30}$$

This is a well-defined random variable for every $\varepsilon\geq 0$. The Itô isometry gives

$$\begin{aligned}
&\mathbb{E}^x\big[|Y_\varepsilon-Y_0|^2\big]\\
&=4\sum_{i=1}^{N}\int_{-T}^{T}\mathbb{E}^0\left[\left|\sum_{j=1}^{N}\int_{[t-\tau]_T}^{t}(\nabla\varphi_\varepsilon-\nabla\varphi_0)(B_t^i-B_s^j+x^i-x^j,t-s)ds\right|^2\right]dt\\
&\leq 4\mathrm{c}_\varepsilon\sqrt{N}\sum_{i,j=1}^{N}\int_{-T}^{T}\mathbb{E}^0\left[\left(\int_{[t-\tau]_T}^{t}|B_t^i-B_s^j+x^i-x^j|^{-\theta}|t-s|^{-(1-\theta)}ds\right)^2\right]dt,
\end{aligned}$$

where we used Lemma 2.6 and an interpolation to obtain the bound

$$|\nabla\varphi_\varepsilon(x,t)-\nabla\varphi_0(x,t)|\leq\mathrm{c}_\varepsilon|x|^{-\theta}|t|^{-(1-\theta)},\quad\theta\in[0,1]. \tag{2.31}$$

Here $\mathrm{c}_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. With some $\frac{1}{2} < \theta < 1$, the Schwarz inequality applied to the latter integral gives

$$\begin{aligned}
&\mathbb{E}^x[|Y_\varepsilon - Y_0|^2] \\
&\leq 4\mathrm{c}_\varepsilon \sqrt{N} \sum_{i,j=1}^N \int_{-T}^T \mathbb{E}^0 \left[\int_{[t-\tau]_T}^t |B_t^i - B_s^j + x^i - x^j||^{-2\theta} ds \right] \left(\int_{[t-\tau]_T}^t |t-s|^{-2(1-\theta)} ds \right) dt \\
&\leq 4\mathrm{c}_\varepsilon \tau^{2\theta-1} \sqrt{N} \sum_{i,j=1}^N \int_{-T}^T \left(\int_{[t-\tau]_T}^t \mathbb{E}^0[|B_t^i - B_s^j + x^i - x^j|^{-2\theta}] ds \right) dt. \qquad (2.32)
\end{aligned}$$

Write $x^i - x^j = X$. Then we can compute as

$$\begin{aligned}
\int_{-T}^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} |u - v + X|^{-2\theta} p_t(u) p_s(v) du dv &\leq \int_{-\infty}^\infty dt \int_{\mathbb{R}^3} |u|^{-2\theta} p_{t+s}(u - X) du \\
&= \int_{\mathbb{R}^3} \frac{1}{|u - X|} \frac{1}{|u|^{2\theta}} du.
\end{aligned}$$

Since $|x|^{-2\theta} \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for $p > 3/2$, the potential $|x|^{-2\theta}$ is Kato-class. Thus

$$\sup_{X \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|u - X|} \frac{1}{|u|^{2\theta}} du < \infty,$$

see also Appendix A. Hence $\sup_{x \in \mathbb{R}^3} \int_{-T}^T dt \int_{[t-\tau]_T}^t ds \mathbb{E}^x[|B_t^i - B_s^j|^{-2\theta}] < \infty$ follows. Therefore we obtain that

$$\sup_{x \in \mathbb{R}^3} \mathbb{E}^x[|Y_\varepsilon - Y_0|^2] \to 0 \qquad (2.33)$$

as $\varepsilon \downarrow 0$ by (2.32) and the fact that $\mathrm{c}_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Moreover, $\sup_{x \in \mathbb{R}^3} \mathbb{E}^x[|Y_\varepsilon|] < \infty$ follows from the inequality

$$\mathbb{E}^x[|Y_\varepsilon|^2] \leq 4c\tau^{2\theta-1} \sqrt{N} \sum_{i,j=1}^N \int_{-T}^T dt \int_{[t-\tau]_T}^t \mathbb{E}^x[|B_t^i - B_s^j + x^i - x^j|^{-2\theta}] ds.$$

□

For $x = (x^1, ..., x^N) \in \mathbb{R}^{3N}$ define

$$\begin{aligned}
S_\varepsilon^{\mathrm{OD},T}(x) &= 2 \sum_{i,j=1}^N \int_0^{2T} ds \int_{[s+\tau]_T}^T W_\varepsilon(B_t^i - B_s^j, t-s) dt, \\
X_\varepsilon^T(x) &= 2 \sum_{i,j=1}^N \int_0^{2T} \varphi_\varepsilon(B_s^i - B_s^j + x^i - x^j, 0) ds, \\
Y_\varepsilon^T(x) &= 2 \sum_{i,j=1}^N \int_0^{2T} ds \int_s^{[s+\tau]_T} \nabla \varphi_\varepsilon(B_t^i - B_s^j + x^i - x^j, t-s) \cdot dB_t^i, \\
Z_\varepsilon^T(x) &= -2 \sum_{i,j=1}^N \int_0^{2T} \varphi_\varepsilon(B_{[s+\tau]_T}^i - B_s^j + x^i - x^j, [s+\tau]_T - s) ds.
\end{aligned}$$

Lemma 2.11 *Let $\alpha \in \mathbb{R}$. Then there exists a constant $c_U(\alpha) > 0$, independent of $\varepsilon \geq 0$, such that*

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}^0[\mathrm{e}^{\alpha U_\varepsilon^T(x)}] < c_U(\alpha), \quad \varepsilon \geq 0, \qquad U = S^{\mathrm{OD}}, X, Y, Z. \tag{2.34}$$

Proof. Let $U = X$. By the inequality $|X_\varepsilon^T(x)| \leq C \sum_{i\neq j}^N \int_0^{2T} |B_s^i - B_s^j|^{-1} ds$ and the fact that $\sum_{i\neq j}^N |x^i - x^j|^{-1}$ is a Kato-class potential, we see that

$$\sup_{x\in\mathbb{R}^3} \left|\mathbb{E}^0[\mathrm{e}^{\alpha X_\varepsilon^T(x)}]\right| \leq \sup_{x\in\mathbb{R}^3} \mathbb{E}^x\left[\mathrm{e}^{|\alpha| \sum_{i,j=1}^N \int_0^{2T} |B_s^i - B_s^j|^{-1} ds}\right] < \infty. \tag{2.35}$$

Take $U = Y$, and let $\Phi_t = \Phi_t(x) = (\Phi_t^1, \dots, \Phi_t^N)$ and

$$\Phi_t^i = \Phi_t^i(x) = 2\sum_{j=1}^N \int_{[t-\tau]_T}^t \nabla\varphi_\varepsilon(B_t^i - B_s^j + x^i - x^j, t-s) ds.$$

Thus $Y_\varepsilon^T(x)$ has the representation

$$Y_\varepsilon^T(x) = \int_0^{2T} \Phi_t \cdot dB_t. \tag{2.36}$$

Using the Girsanov theorem, we obtain $\mathbb{E}^0\left[\mathrm{e}^{2\alpha \int_0^{2T} \Phi_t \cdot dB_t - \frac{1}{2}(2\alpha)^2 \int_0^{2T} |\Phi_t|^2 dt}\right] = 1$ and the bound

$$\begin{aligned}\left(\mathbb{E}^0\left[\mathrm{e}^{\alpha Y_\varepsilon^T(x)}\right]\right)^2 &\leq \mathbb{E}^0\left[\mathrm{e}^{2\alpha \int_0^{2T} \Phi_t \cdot dB_t - \frac{1}{2}(2\alpha)^2 \int_0^{2T} |\Phi_t|^2 dt}\right] \mathbb{E}^0\left[\mathrm{e}^{2\alpha^2 \int_2^{2T} |\Phi_t|^2 dt}\right] \\ &= \mathbb{E}^0\left[\mathrm{e}^{2\alpha^2 \int_0^{2T} |\Phi_t|^2 dt}\right].\end{aligned}$$

In a similar way to (2.31) we have

$$\int_0^{2T} |\Phi_t|^2 dt \leq 4c\tau^{2\theta-1}\sqrt{N} Q(x),$$

where c is the constant in Lemma 2.6, which is independent of ε,

$$Q(x) = \sum_{i,j=1}^N \int_0^{2T} ds \int_s^{[s+\tau]_T} |B_t^i - B_s^j + x^i - x^j|^{-2\theta} dt$$

and we recall that we have chosen $\frac{1}{2} < \theta < 1$. Thus we have

$$\left(\mathbb{E}^0\left[\mathrm{e}^{\alpha Y_\varepsilon^T(x)}\right]\right)^2 \leq \mathbb{E}^0\left[\mathrm{e}^{\gamma Q(x)}\right], \tag{2.37}$$

where $\gamma = 8c\sqrt{N}\alpha^2\tau^{2\theta-1}$. The Jensen inequality gives

$$\mathbb{E}^0\left[\mathrm{e}^{\gamma Q(x)}\right] \leq \int_0^{2T} \frac{ds}{2T} \mathbb{E}^0\left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_s^{[s+\tau]_T} |B_t^i - B_s^j + x^i - x^j|^{-2\theta} dt}\right]. \tag{2.38}$$

We split up the right-hand side of (2.38) by $\int_0^{2T} = \int_0^{2T-\tau} + \int_{2T-\tau}^{2T}$. Consider the first term. Since $[s+\tau]_T = s+\tau$, we have

$$\begin{aligned}&\int_0^{2T-\tau} \frac{ds}{2T} \mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_s^{[s+\tau]_T} |B_t^i - B_s^j + x^i - x^j|^{-2\theta} dt} \right] \\ &\quad = \int_0^{2T-\tau} \tfrac{ds}{2T} \mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_0^{\tau} |B_{s+t}^i - B_s^j + x^i - x^j|^{-2\theta} dt} \right]. \end{aligned} \tag{2.39}$$

Let $\mathscr{F}_s = \sigma(B_r^j; 0 \le r \le s,\, j = 1, ..., N)$ be the natural filtration of the $3N$-dimensional Brownian motion $(B_t)_{t\ge 0}$. Taking conditional expectation and using the Markov property, we see that

$$\begin{aligned}\mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_0^{\tau} |B_{s+t}^i - B_s^j + x^i - x^j|^{-2\theta} dt} \right] &= \mathbb{E}^0 \left[\mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_0^{\tau} |B_{s+t}^i - B_s^j + x^i - x^j|^{-2\theta} dt} | \mathscr{F}_s \right] \right] \\ &= \mathbb{E}^0 \left[\mathbb{E}^{B_s} \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_0^{\tau} |B_t^i - B_0^j + x^i - x^j|^{-2\theta} dt} \right] \right].\end{aligned}$$

We can further compute this as

$$\begin{aligned}&= (2\pi s)^{-3N/2} \int_{\mathbb{R}^{3N}} \mathrm{e}^{-|y|^2/2s} \mathbb{E}^y \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_0^{\tau} |B_t^i - B_0^j + x^i - x^j|^{-2\theta} dt} \right] dy \\ &= (2\pi s)^{-3N/2} \int_{\mathbb{R}^{3N}} \mathrm{e}^{-|y|^2/2s} \prod_{i=1}^N \mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{j=1}^N \int_0^{\tau} |B_t^i + (x+y)^i - (x+y)^j|^{-2\theta} dt} \right] dy.\end{aligned}$$

Since $|x|^{-2\theta}$ is a Kato-class potential, we see that

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}^0 [\mathrm{e}^{\beta \int_0^{\tau} |B_s^i + x|^{-2\theta} ds}] = \sup_{x\in\mathbb{R}^3} \mathbb{E}^x [\mathrm{e}^{\beta \int_0^{\tau} |B_s^i|^{-2\theta} ds}] < \infty,$$

for all $\beta \in \mathbb{R}$ (see Appendix A). From this it follows that

$$\sup_{x,y\in\mathbb{R}^3} \mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{j=1}^N \int_0^{\tau} |B_t^i + (x+y)^i - (x+y)^j|^{-2\theta} dt} \right] < \infty. \tag{2.40}$$

In particular,

$$\begin{aligned}&\int_0^{2T-\tau} \frac{1}{2T} \mathbb{E}^x \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_s^{[s+\tau]_T} |B_t^i - B_s^j + x^i - x^j|^{-2\theta} dt} \right] ds \\ &\le \frac{2T-\tau}{2T} \sup_{x,y\in\mathbb{R}^3} \prod_{i=1}^N \mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{j=1}^N \int_0^{\tau} |B_t^i + (x+y)^i - (x+y)^j|^{-2\theta} dt} \right] < \infty. \end{aligned} \tag{2.41}$$

Next consider the second term and note that $[s+\tau]_T = 2T$. In the same way as above, we have

$$\begin{aligned}&\int_{2T-\tau}^{2T} \frac{1}{2T} \mathbb{E}^x \left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^N \int_s^{[s+\tau]_T} |B_t^i - B_s^j + x^i - x^j|^{-2\theta} dt} \right] ds \\ &\le \frac{\tau}{2T} \sup_{x,y\in\mathbb{R}^3} \prod_{i=1}^N \mathbb{E}^0 \left[\mathrm{e}^{2T\gamma \sum_{j=1}^N \int_0^{\tau} |B_t^i + (x+y)^i - (x+y)^j|^{-2\theta} dt} \right] < \infty. \end{aligned} \tag{2.42}$$

Hence by (2.41) and (2.42) we obtain

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}^0[\mathrm{e}^{2\alpha Y_\varepsilon^T(x)}] \le \left(\sup_{x,y\in\mathbb{R}^3}\prod_{i=1}^N \mathbb{E}^0\left[\mathrm{e}^{2T\gamma\sum_{j=1}^N\int_0^\tau |B_t^i+(x+y)^i-(x+y)^j|^{-2\theta}dt}\right]\right)^{1/2} < \infty. \quad (2.43)$$

Finally, consider the cases $U = Z$ and $U = S^{\mathrm{OD}}$. From $|Z_\varepsilon^T(x)| \le c_z T$ and $|S_\varepsilon^{\mathrm{OD},T}(x)| \le cT^2$ we obtain $\sup_{x\in\mathbb{R}^3}\mathbb{E}[\mathrm{e}^{\alpha Z_\varepsilon^T(x)}] < \infty$ and $\sup_{x\in\mathbb{R}^3}\mathbb{E}[\mathrm{e}^{\alpha S_\varepsilon^{\mathrm{OD}T}(x)}] < \infty$. □

Lemma 2.12 *Let $\alpha \in \mathbb{R}$ and $f, h \in L^2(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} dx\mathbb{E}^x[f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\alpha S_\varepsilon^{\mathrm{ren}}}] < \infty$$

for all $\varepsilon \ge 0$.

Proof. Recall the decomposition $S_\varepsilon^{\mathrm{ren}} = S_\varepsilon^{\mathrm{OD}} + X_\varepsilon + Y_\varepsilon + Z_\varepsilon$. By the Schwarz inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} dx\mathbb{E}^x[|f(B_{-T})h(B_T)|\mathrm{e}^{\alpha S_\varepsilon^{\mathrm{ren}}}] \\
&\le \int_{\mathbb{R}^3} dx\mathbb{E}^0\left[|f(x)h(B_{2T}+x)|\mathrm{e}^{-\int_0^{2T}V(B_s+x)ds}\mathrm{e}^{\alpha(S_\varepsilon^{\mathrm{OD},T}(x)+X_\varepsilon^T(x)+Y_\varepsilon^T(x)+Z_\varepsilon^T(x))}\right] \\
&\le \|f\|\|h\|\sup_{x\in\mathbb{R}^3}\left(\mathbb{E}^0\left[\mathrm{e}^{-2\int_0^{2T}V(B_s+x)ds}\mathrm{e}^{2\alpha(S_\varepsilon^{\mathrm{OD},T}(x)+X_\varepsilon^T(x)+Y_\varepsilon^T(x)+Z_\varepsilon^T(x))}\right]\right)^{1/2}. \quad (2.44)
\end{aligned}$$

By Lemma 2.11 and the fact that V is Kato-class, we see that

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}^0\left[\mathrm{e}^{-2\int_0^{2T}V(B_s+x)ds}\mathrm{e}^{2\alpha(S_\varepsilon^{\mathrm{OD},T}(x)+X_\varepsilon^T(x)+Y_\varepsilon^T(x)+Z_\varepsilon^T(x))}\right] < \infty,$$

and the lemma follows. □

2.3 Renormalized Hamiltonian

In this section we show that $H_\varepsilon + g^2 N\varphi_\varepsilon(0,0)$ converges to a self-adjoint operator H_{ren} as $\varepsilon \downarrow 0$.

2.3.1 Convergence of the renormalized action

Lemma 2.13 *If $\alpha \in \mathbb{R}$, then*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}^0\left[|\mathrm{e}^{\alpha X_\varepsilon^T(x)} - \mathrm{e}^{\alpha X_0^T(x)}|\right] = 0, \quad x \in \mathbb{R}^{3N}. \quad (2.45)$$

Proof. We obtain that $|X_\varepsilon^T(x)| \le \int_0^{2T} V_C(B_s^1 + x^1, ..., B_s^N + x^N)ds$ and

$$\mathbb{E}^0\left[|\mathrm{e}^{\alpha X_\varepsilon^T(x)} - \mathrm{e}^{\alpha X_0^T(x)}|\right] \le 2\mathbb{E}^0\left[|\mathrm{e}^{\alpha\int_{-T}^T V_C(B_s^1+x^1,....,B_s^N+x^N)ds}\right] < \infty.$$

Since $X_\varepsilon^T(x) \to X_0^T(x)$ a.s. for every x, the Lebesgue dominated convergence theorem implies (2.45). □

Lemma 2.14 *If $\alpha \in \mathbb{R}$, then*

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^3} \mathbb{E}^0[|\mathrm{e}^{\alpha U_\varepsilon^T(x)} - \mathrm{e}^{\alpha U_0^T(x)}|] = 0, \quad U = S^{\mathrm{OD}}, Y, Z. \tag{2.46}$$

Proof. Let $U = Y$. It suffices to show that $\sup_{x \in \mathbb{R}^3} \mathbb{E}^0\Big[|\mathrm{e}^{\alpha(Y_\varepsilon^T(x) - Y_0^T(x))} - 1|\Big] \to 0$. We have

$$\mathbb{E}^0\left[\left(\mathrm{e}^{\alpha(Y_\varepsilon^T(x) - Y_0^T(x))} - 1\right)^2\right] = \mathbb{E}^0\left[\mathrm{e}^{2\alpha(Y_\varepsilon^T(x) - Y_0^T(x))}\right] + 1 - 2\mathbb{E}^0\left[\mathrm{e}^{\alpha(Y_\varepsilon^T(x) - Y_0^T(x))}\right]. \tag{2.47}$$

We will show below that $\lim_{\varepsilon \downarrow 0} \mathbb{E}^0[\mathrm{e}^{\alpha(Y_\varepsilon^T(x) - Y_0^T(x))}] = 1$. Define the random process $\delta\Phi_t = \delta\Phi_t(x) = (\delta\Phi_t^1, \dots, \delta\Phi_t^N)$ and

$$\delta\Phi_t^i = \delta\Phi_t^i(x) = 2\sum_{j=1}^N \int_{[t-\tau]_T}^t (\nabla\varphi_\varepsilon - \nabla\varphi_0)(B_t^i - B_s^j + x^i - x^j, t-s)ds$$

so that

$$Y_\varepsilon^T(x) - Y_0^T(x) = \int_0^{2T} \delta\Phi_t(x)dt.$$

By the Girsanov theorem

$$1 = \mathbb{E}^0[\mathrm{e}^{\alpha \int_0^{2T} \delta\Phi_t \cdot dB_t - \frac{\alpha^2}{2}\int_0^{2T} |\delta\Phi_t|^2 dt}] \tag{2.48}$$

for every $\alpha \in \mathbb{R}$, hence it follows that

$$\sup_{x \in \mathbb{R}^3} \left(\mathbb{E}^0\left[\mathrm{e}^{\alpha(Y_\varepsilon^T(x) - Y_0^T(x))}\right] - 1\right)^2 \le \sup_{x \in \mathbb{R}^3} \mathbb{E}^0\left[\mathrm{e}^{2\alpha \int_0^{2T} \delta\Phi_t \cdot dB_t}\right] \mathbb{E}^0\left[\left(1 - \mathrm{e}^{-\frac{\alpha^2}{2}\int_0^{2T} |\delta\Phi_t|^2 dt}\right)^2\right]. \tag{2.49}$$

We see that by (2.48) again

$$\sup_{x \in \mathbb{R}^3} \mathbb{E}^0\left[\mathrm{e}^{2\alpha \int_0^{2T} \delta\Phi_t \cdot dB_t}\right] \le \sup_{x \in \mathbb{R}^3} \left(\mathbb{E}^0\left[\mathrm{e}^{4\alpha^2 \int_0^{2T} |\delta\Phi_t|^2 dt}\right]\right)^{1/2} \tag{2.50}$$

and furthermore

$$\sup_{x \in \mathbb{R}^3} \mathbb{E}^0\left[\left(1 - \mathrm{e}^{-\frac{\alpha^2}{2}\int_0^{2T} |\delta\Phi_t|^2 dt}\right)^2\right] \le \sup_{x \in \mathbb{R}^3} \mathbb{E}^0\left[\left|\frac{\alpha^2}{2}\int_0^{2T} |\delta\Phi_t|^2 dt\right|^2\right] \to 0 \tag{2.51}$$

as $\varepsilon \downarrow 0$. Here (2.51) can be shown by Lemma 2.10. The right-hand side of (2.50) is uniformly bounded in ε, which can be proven in the same way as in the proof of Lemma 2.12. Hence (2.49) converges to zero as $\varepsilon \downarrow 0$ and (2.46) for $U = Y$ follows.

Let $U = Z$. It suffices to show that $\sup_{x \in \mathbb{R}^3} \mathbb{E}^x\big[|\mathrm{e}^{\alpha(Z_\varepsilon - Z_0)} - 1|\big] \to 0$. We have

$$\begin{aligned} &Z_\varepsilon(x) - Z_0(x) \\ &= 2\sum_{i,j=1}^N \int_{-T}^T ds \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-ik\cdot(B^i_{[s+\tau]_T - s} + x^i - B^j_{[s+\tau]_T - s} - x^j)} \mathrm{e}^{-([s+\tau]_T - s)\omega(k)}}{\omega(k)} \beta(k) \mathbb{1}_\Lambda^\perp(k)(1 - \mathrm{e}^{-\varepsilon|k|^2})dk. \end{aligned}$$

Let $\eta_\varepsilon(x) = \alpha(Z_\varepsilon(x) - Z_0(x))$. It can be directly seen that

$$|\eta_\varepsilon(x)|^n \le c^n \alpha^n T^n \varepsilon^n$$

for a constant c independent of x. Then we have $\mathbb{E}^0[\mathrm{e}^{\eta_\varepsilon(x)}] = 1 + \sum_{n\ge 1} \frac{1}{n!}\mathbb{E}^0[\eta_\varepsilon(x)^n]$ and

$$\sum_{n\ge 1} \frac{1}{n!}\mathbb{E}^0[|\eta_\varepsilon(x)|^n] \le \sum_{n\ge 1} \frac{1}{n!} c^n T^n \varepsilon^n \to 0$$

as $\varepsilon \downarrow 0$, uniformly in x. Thus (2.46) for $U = Z$ follows. For $U = S^{\mathrm{OD}}$ we obtain (2.46) in a similar way. □

Lemma 2.15 *Let $\alpha \in \mathbb{R}$ and $f, h \in L^2(\mathbb{R}^3)$. Then*

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\alpha S_\varepsilon^{\mathrm{ren}}}] \\ = \int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\alpha S_0^{\mathrm{ren}}}]. \end{aligned} \tag{2.52}$$

Proof. Write $A_\varepsilon = A_\varepsilon(x) = \alpha(S_\varepsilon^{\mathrm{OD},T}(x) + Y_\varepsilon^T(x) + Z_\varepsilon^T(x))$. Then by shift invariance of Brownian motion with respect to the measure $dxdP^x$ and by telescoping we have that

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} dx \mathbb{E}^x\left[f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\left(\mathrm{e}^{\alpha S_\varepsilon^{\mathrm{ren}}} - \mathrm{e}^{\alpha S_0^{\mathrm{ren}}}\right)\right]\right| \\ &\quad\le \mathrm{e}^{2T\|V\|_\infty}\int_{\mathbb{R}^3} dx|f(x)|\mathbb{E}^0\left[|h(B_{2T}+x)|\left(\mathrm{e}^{A_\varepsilon(x)+\alpha X_\varepsilon^T(x)} - \mathrm{e}^{A_0(x)+\alpha X_0^T(x)}\right)\right] \\ &\quad\le \mathrm{e}^{2T\|V\|_\infty}\|f\|\|h\| \sup_{x\in\mathbb{R}^3}\left(\mathbb{E}^0[|\mathrm{e}^{A_\varepsilon(x)} - \mathrm{e}^{A_0(x)}|^4]\mathbb{E}^0[\mathrm{e}^{4\alpha X_\varepsilon^T(x)}]\right)^{1/4} \\ &\qquad + \mathrm{e}^{2T\|V\|_\infty}\int_{\mathbb{R}^3} dx|f(x)|E_\varepsilon(x) \sup_{x\in\mathbb{R}^3}\left(\mathbb{E}^0[|\mathrm{e}^{4A_0}]\right)^{1/4},\end{aligned}$$

where $E_\varepsilon(x) = \left(\mathbb{E}^0[|h(B_{2T}+x)|^2]\right)^{1/2}\left(\mathbb{E}^0[(\mathrm{e}^{\alpha X_\varepsilon^T(x)} - \mathrm{e}^{\alpha X_0^T(x)})^4]\right)^{1/4}$. We show that each term at the right-hand side converges to zero. Note that both $\sup_{x\in\mathbb{R}^3}\mathbb{E}^0[\mathrm{e}^{4\alpha X_\varepsilon^T(x)}]$ and $\sup_{x\in\mathbb{R}^3}\mathbb{E}^0[\mathrm{e}^{4A_0}]$ are uniformly bounded with respect to ε by Lemma 2.11. In Lemma 2.14 it is shown that

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in\mathbb{R}^3}\mathbb{E}^0[|\mathrm{e}^{A_\varepsilon(x)} - \mathrm{e}^{A_0(x)}|^4] = 0.$$

Hence the first term converges to zero. Furthermore,

$$E_\varepsilon(x) \le \left(\mathbb{E}^0[|h(B_{2T}+x)|^2]\right)^{1/2}\sup_{x\in\mathbb{R}^3}\left(\mathbb{E}^0\left[(\mathrm{e}^{\alpha X_\varepsilon^T(x)} - \mathrm{e}^{\alpha X_0^T(x)})^4\right]\right)^{1/4} \in L^1(\mathbb{R}^{3N}).$$

By Lemma 2.13 we have that $\mathbb{E}^0[(\mathrm{e}^{\alpha X_\varepsilon^T(x)} - \mathrm{e}^{\alpha X_0^T(x)})^4] \to 0$ as $\varepsilon \downarrow 0$, for all $x \in \mathbb{R}^{3N}$. Hence the Lebesgue dominated convergence theorem implies that the second term converges to zero, and the lemma follows. □

Lemma 2.16 *It follows that*

$$\lim_{\varepsilon\downarrow 0}(f\otimes \mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes\mathbb{1}) = \int_{\mathbb{R}^3}\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}}\right]dx, \tag{2.53}$$

where

$$\begin{aligned} S_0^{\mathrm{ren}} &= 2\sum_{i\neq j}^N\int_{-T}^T\varphi_0(B_s^i-B_s^j,0)ds + 2\sum_{i,j=1}^N\int_{-T}^T ds\int_s^T\nabla\varphi_0(B_t^i-B_s^j,t-s)\cdot dB_t \\ &\quad - 2\sum_{i,j=1}^N\int_{-T}^T\varphi_0(B_T^i-B_s^j,T-s)ds, \end{aligned} \tag{2.54}$$

and integrands are given by $\varphi_0(X,t) = \int_{\mathbb{R}^3}\frac{e^{-ikX}e^{-|t|\omega(k)}}{2\omega(k)}\beta(k)\mathbb{1}_\Lambda^\perp(k)dk$ *and* $\nabla\varphi_0(X,t) = \int_{\mathbb{R}^3}\frac{-ike^{-ikX}e^{-|t|\omega(k)}}{2\omega(k)}\beta(k)\mathbb{1}_\Lambda^\perp(k)dk$.

Proof. We have

$$(f\otimes\mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes\mathbb{1}) = \int_{\mathbb{R}^3}\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_\varepsilon^{\mathrm{ren}}}\right]dx. \tag{2.55}$$

The right-hand side above converges to $\int_{\mathbb{R}^3}\mathbb{E}^x[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}}]dx$ as $\varepsilon\downarrow 0$. Thus (2.53) follows. We also see that

$$\begin{aligned} S_0^{\mathrm{ren}} &= 2\sum_{i\neq j}^N\int_{-T}^T\varphi_0(B_s^i-B_s^j,0)ds + 2\sum_{i,j=1}^N\int_{-T}^T ds\int_s^{[s+\tau]_T}\nabla\varphi_0(B_t^i-B_s^j,t-s)\cdot dB_t \\ &\quad - 2\sum_{i,j=1}^N\int_{-T}^T\varphi_0(B_{[s+\tau]_T}^i-B_s^j,[s+\tau]_T-s)ds. \end{aligned} \tag{2.56}$$

Taking $\tau = T$, we obtain (2.54). □

2.3.2 Extension beyond the vacuum vector

Now we extend the result in Section 2.3.1 from vectors of the form $f\otimes\mathbb{1}$ to more general vectors of the form $f\otimes F(\phi(f_1),\dots,\phi(f_n))\mathbb{1}$, with $F\in\mathscr{S}(\mathbb{R}^3)$, where $\phi(f)$ stands for a scalar field given by $\frac{1}{\sqrt{2}}(a^*(\hat{f})+a(\widetilde{\hat{f}}))$, where $\widetilde{\hat{f}}(k) = \hat{f}(-k)$. To do this we need a Feynman-Kac-type formula giving a representation of $\mathrm{e}^{-2TH_\varepsilon}$.

Denote

$$H_{-k}(\mathbb{R}^n) = \{f\in\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^n)\,|\,\widehat{f}\in L^1_{\mathrm{loc}}(\mathbb{R}^n), |\cdot|^{-k/2}\widehat{f}\in L^2(\mathbb{R}^n)\}$$

endowed with the norm $\|f\|^2_{H_{-k}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|\widehat{f}(x)|^2|x|^{-k}dx$. Recall that a Euclidean field is a family of Gaussian random variables $\{\phi_{\mathrm{E}}(F),\, F\in H_{-1}(\mathbb{R}^4)\}$ on a probability space

$(Q_{\mathrm{E}}, \Sigma_{\mathrm{E}}, \mu_{\mathrm{E}})$, such that the map $F \mapsto \phi_{\mathrm{E}}(F)$ is linear, and their mean and covariance are given by

$$\mathbb{E}_{\mu_{\mathrm{E}}}[\phi_{\mathrm{E}}(F)] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_{\mathrm{E}}}[\phi_{\mathrm{E}}(F)\phi_{\mathrm{E}}(G)] = \frac{1}{2}(F,G)_{H_{-1}(\mathbb{R}^4)}.$$

For the reader's convenience we summarize in Appendix B some basic facts on Euclidean fields and their representation in L^2 space (including the operators $J_t : \mathscr{F} \to L^2(Q_{\mathrm{E}}; \mu_{\mathrm{E}})$), which will be used here. In what follows, we identify $\mathscr{H}$ with the set of $\mathscr{F}$-valued L^2 functions $L^2(\mathbb{R}^{3N}; \mathscr{F})$, i.e., $F \in \mathscr{H}$ can be regarded as a function $\mathbb{R}^{3N} \ni x \mapsto F(x) \in \mathscr{F}$ such that $\int_{\mathbb{R}^{3N}} \|F(x)\|_{\mathscr{F}}^2 dx < \infty$.

Proposition 2.17 *Let $F, G \in \mathscr{H}$. Then*

$$\begin{aligned}&(F, \mathrm{e}^{-2TH_\varepsilon} G)\\ &= \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \mathbb{E}_{\mu_{\mathrm{E}}} \left[J_{-T} F(B_{-T}) \cdot \mathrm{e}^{-\phi_{\mathrm{E}}(\int_{-T}^{T} \sum_{j=1}^{N} \delta_s \otimes \tilde{\varphi}(\cdot - B_s^j) ds)} J_T G(B_T) \right] \right],\end{aligned} \tag{2.57}$$

where $\tilde{\varphi}_\varepsilon(x) = \left(\mathrm{e}^{-\varepsilon|\cdot|^2/2} \mathbb{1}_\Lambda^\perp / \sqrt{\omega} \right)^\vee (x)$, and $\delta_s(x) = \delta(x - s)$ is Dirac delta distribution with mass on s.

Proof. See [LHB11, Theorem 6.3]. □

Lemma 2.18 *Let $\rho_j \in H_{-1/2}(\mathbb{R}^3)$ for $j = 1, 2$, $f, h \in L^2(\mathbb{R}^{3N})$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\begin{aligned}&\lim_{\varepsilon \downarrow 0} (f \otimes \mathrm{e}^{\alpha\phi(\rho_1)} \mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon + g^2 N \varphi_\varepsilon(0,0))} h \otimes \mathrm{e}^{\beta\phi(\rho_2)} \mathbb{1})\\ &\qquad = \int_{\mathbb{R}^3} \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \mathrm{e}^{\frac{g^2}{2} S_0^{\mathrm{ren}} + \frac{1}{4}\xi} \right] dx,\end{aligned} \tag{2.58}$$

where

$$\begin{aligned}\xi = \xi(g) = {} & \bar{\alpha}^2 \|\rho_1/\sqrt{\omega}\|^2 + \beta^2 \|\rho_2/\sqrt{\omega}\|^2 + 2\bar{\alpha}\beta(\rho_1/\sqrt{\omega}, \mathrm{e}^{-2T\omega}\rho_2/\sqrt{\omega})\\ & + 2\bar{\alpha} g \sum_{j=1}^{N} \int_{-T}^{T} ds \int_{\mathbb{R}^3} dk \frac{\widehat{\rho}_1(k)}{\sqrt{\omega(k)}} \mathbb{1}_\Lambda^\perp(k) \mathrm{e}^{-|s-T|\omega(k)} \mathrm{e}^{-ikB_s^j}\\ & + 2\beta g \sum_{j=1}^{N} \int_{-T}^{T} ds \int_{\mathbb{R}^3} dk \frac{\widehat{\rho}_2(k)}{\sqrt{\omega(k)}} \mathbb{1}_\Lambda^\perp(k) \mathrm{e}^{-|s+T|\omega(k)} \mathrm{e}^{-ikB_s^j}.\end{aligned}$$

Proof. By the functional integral representation (2.57) we have

$$\begin{aligned}&(f \otimes \mathrm{e}^{\alpha\phi(\rho_1)} \mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon + g^2 N \varphi_\varepsilon(0,0))} h \otimes \mathrm{e}^{\beta\phi(\rho_2)} \mathbb{1}) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \Big[\bar{f}(B_{-T}) h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds}\\ &\times \mathbb{E}_{\mu_E} [\mathrm{e}^{\bar{\alpha}\phi_E(\delta_{-T} \otimes \rho_1)} \mathrm{e}^{\beta\phi_E(\delta_T \otimes \rho_2)} \mathrm{e}^{g\phi_E(-\sum_{j=1}^{N} \int_{-T}^{T} \delta_s \otimes \tilde{\varphi}_\varepsilon(\cdot - B_s^j) ds)}] \Big] \mathrm{e}^{-2Tg^2 N \varphi_\varepsilon(0,0)}.\end{aligned}$$

It can be directly seen that

$$\mathbb{E}_{\mu_E} \left[\mathrm{e}^{\bar{\alpha}\phi_E(\delta_{-T} \otimes \rho_1)} \mathrm{e}^{\beta\phi_E(\delta_T \otimes \rho_2)} \mathrm{e}^{g\phi_E(-\sum_{j=1}^{N} \int_{-T}^{T} \delta_s \otimes \tilde{\varphi}_\varepsilon(\cdot - B_s^j) ds)} \right] \mathrm{e}^{-2Tg^2 N \varphi_\varepsilon(0,0)} = \mathrm{e}^{\frac{g^2}{2} S_\varepsilon^{\mathrm{ren}} + \frac{1}{4}\xi_\varepsilon},$$

where ξ_ε is defined by ξ with $\mathbb{1}_\Lambda^\perp(k)$ replaced by $\mathbb{1}_\Lambda^\perp(k)\mathrm{e}^{-\varepsilon|k|^2/2}$. Thus

$$(f\otimes \mathrm{e}^{\alpha\phi(\rho_1)}\mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes \mathrm{e}^{\beta\phi(\rho_2)}\mathbb{1})$$
$$=\int_{\mathbb{R}^3}dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_\varepsilon^{\mathrm{ren}}+\frac{1}{4}\xi_\varepsilon}\right].$$

Notice that with a constant C we have $\xi_\varepsilon\le C$ uniformly in the paths and $\varepsilon\ge 0$. Hence we can complete the proof of the lemma in a similar way to Lemma 2.16. □

Consider the dense subspace $\mathscr{D}$ of $\mathscr{H}$ given by

$$\mathscr{D}=\mathrm{L.H.}\left\{f\otimes\mathbb{1}\,|\,f\in L^2(\mathbb{R}^{3N})\right\}\cup$$
$$\left\{f\otimes F(\phi(f_1),\dots,\phi(f_n))\mathbb{1}\,|\,F\in\mathscr{S}(\mathbb{R}^n),f_j\in C_0^\infty(\mathbb{R}^3),1\le j\le n,n\in\mathbb{N},f\in L^2(\mathbb{R}^{3N})\right\}.$$

By Lemma 2.18 the next result is immediate.

Lemma 2.19 *Let $\Phi=f\otimes F(\phi(u_1),\dots,\phi(u_n))\mathbb{1}$, $\Psi=h\otimes G(\phi(v_1),\dots,\phi(v_m))\mathbb{1}\in\mathscr{D}$. Then*

$$\lim_{\varepsilon\downarrow 0}(\Phi,\mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}\Psi)=(2\pi)^{-(n+m)/2}\int_{\mathbb{R}^{n+m}}dK_1dK_2\overline{\widehat{F}(K_1)}\widehat{G}(K_2)$$
$$\times\int_{\mathbb{R}^3}dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}+\frac{1}{4}\xi(K_1,K_2)}\right],\tag{2.59}$$

where

$$\xi(K_1,K_2)=-\|K_1\cdot u/\sqrt{\omega}\|^2-\|K_2\cdot v/\sqrt{\omega}\|^2-2(K_1\cdot u/\sqrt{\omega},\mathrm{e}^{-2T\omega}K_2\cdot v/\sqrt{\omega})$$
$$-2ig\sum_{j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3}dk\frac{K_1\cdot\widehat{u}(k)}{\sqrt{\omega(k)}}\mathbb{1}_\Lambda^\perp(k)\mathrm{e}^{-|s-T|\omega(k)}\mathrm{e}^{-ikB_s^j}$$
$$+2ig\sum_{j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3}dk\frac{K_2\cdot\widehat{v}(k)}{\sqrt{\omega(k)}}\mathbb{1}_\Lambda^\perp(k)\mathrm{e}^{-|s+T|\omega(k)}\mathrm{e}^{-ikB_s^j}$$

and $u=(u_1,...,u_n)$, $v=(v_1,...,v_m)$.

Proof. Notice that $F(\phi(f_1),\dots,\phi(f_n))\mathbb{1}=(2\pi)^{-n/2}\int_{\mathbb{R}^n}\widehat{F}(K)\mathrm{e}^{i\phi(K\cdot f)}\mathbb{1}dK$. Hence

$$(\Phi,\mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}\Psi)$$
$$=\frac{1}{(2\pi)^{(n+m)/2}}\int_{\mathbb{R}^{m+n}}dK_1dK_2\overline{\widehat{F}(K_1)}\widehat{G}(K_2)(f\otimes\mathrm{e}^{-i\phi(K_1\cdot f)}\mathbb{1},\mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes\mathrm{e}^{-i\phi(K_2\cdot h)}\mathbb{1}).$$

Thus the statement follows from Lemma 2.18. □

2.3.3 Uniform lower bound

In this section we show a crucial lower bound on $H_\varepsilon + g^2 N\varphi_\varepsilon(0,0)$ uniform in $\varepsilon > 0$, and give the proof of Theorem 2.2.

Corollary 2.20 *There exists a constant C independent of ε such that*

$$\int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^{T} V(B_s)ds}\mathrm{e}^{S_\varepsilon^{\mathrm{ren}}}] \le C\|f\|\|h\| \tag{2.60}$$

for every $f, h \in L^2(\mathbb{R}^3)$ and $\varepsilon \ge 0$, and

$$C \le \sup_{x\in\mathbb{R}^3} \left(\mathbb{E}^0[\mathrm{e}^{-2\int_0^{2T} V(B_s+x)ds}\mathrm{e}^{2(S_\varepsilon^{\mathrm{OD},T}(x)+X_\varepsilon^T(x)+Y_\varepsilon^T(x)+Z_\varepsilon^T(x))}]\right)^{1/2}. \tag{2.61}$$

Proof. This is directly derived from (2.44). □

Corollary 2.21 *There exists $C \in \mathbb{R}$ such that $H_\varepsilon + g^2 N\varphi_\varepsilon(0,0) > C$, uniformly in $\varepsilon > 0$.*

Proof. In this proof the positive constants a_j, b_j will be understood to be independent of both $\varepsilon \ge 0$ and T. It is straightforward to see that

$$\mathbb{E}^0[\mathrm{e}^{S_\varepsilon^{\mathrm{OD},T}(x)}] \le a_1\mathrm{e}^{b_1 T} \quad \text{and} \quad \mathbb{E}^0[\mathrm{e}^{Z_\varepsilon^T(x)}] \le a_2\mathrm{e}^{b_2 T}.$$

By (2.35) and (2.43) we also have

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}^0[\mathrm{e}^{X_\varepsilon^T(x)}] \le a_3\mathrm{e}^{b_3 T} \quad \text{and} \quad \sup_{x\in\mathbb{R}^3} \mathbb{E}^0[\mathrm{e}^{Y_\varepsilon^T(x)}] \le a_4\mathrm{e}^{b_4\tau T}.$$

Here we used Proposition A.3 in Appendix A. Since

$$\begin{aligned}&\mathbb{E}^0\left[\mathrm{e}^{2(S_\varepsilon^{\mathrm{OD},T}(x)+X_\varepsilon^T(x)+Y_\varepsilon^T(x)+Z_\varepsilon^T(x))}\right]\\ &\le \left(\mathbb{E}^0\left[\mathrm{e}^{4S_\varepsilon^{\mathrm{OD},T}(x)}\right]\right)^{1/2}\left(\mathbb{E}^0\left[\mathrm{e}^{8X_\varepsilon^T(x)}\right]\right)^{1/4}\left(\mathbb{E}^0\left[\mathrm{e}^{16Y_\varepsilon^T(x)}\right]\right)^{1/8}\left(\mathbb{E}^0\left[\mathrm{e}^{32Z_\varepsilon^T(x)}\right]\right)^{1/16},\end{aligned}$$

there exist constants a_5 and b_5 such that

$$\left(\mathbb{E}^0\left[\mathrm{e}^{2(S_\varepsilon^{\mathrm{OD},T}(x)+X_\varepsilon^T(x)+Y_\varepsilon^T(x)+Z_\varepsilon^T(x))}\right]\right)^{1/2} \le a_5\mathrm{e}^{b_5 T} \tag{2.62}$$

for all $T > 0$. Consider the function

$$W(x^1, ..., x^N) = \sum_{j=1}^{N} |x^j|^2.$$

We denote H_ε with V replaced by δW by $H_\varepsilon(\delta)$, for $\delta \ge 0$. Then $-\frac{1}{2}\sum_{j=1}^{N}\Delta_j + \delta W$, $\delta > 0$, has a compact resolvent, which implies that $H_\varepsilon(\delta)$ for $\delta > 0$ has the unique ground state $\Psi_{\mathrm{g}}(\delta)$ by [Spo98, Ger00], see Remark 2.22 below. By the Feynman-Kac formula (2.57) we see that $\mathrm{e}^{-TH_\varepsilon(\delta)}$ is positivity improving for $\varepsilon > 0$, i.e., $(F, \mathrm{e}^{-TH_\varepsilon(\delta)}G) > 0$ for $F, G \in \mathscr{H}$

such that $F, G \geq 0$. Hence it follows that $\Psi_{\mathrm{g}}(\delta) > 0$. In particular, $(f \otimes \mathbb{1}, \Psi_{\mathrm{g}}(\delta)) \neq 0$, for every $0 \leq f \in L^2(\mathbb{R}^{3N})$, where $f \not\equiv 0$. Thus

$$\inf \sigma \left(H_\varepsilon(\delta) + g^2 N \varphi_\varepsilon(0,0) \right) = - \lim_{T \to \infty} \frac{1}{T} \log (f \otimes \mathbb{1}, \mathrm{e}^{-T(H_\varepsilon(\delta) + g^2 N \varphi_\varepsilon(0,0))} f \otimes \mathbb{1}), \quad (2.63)$$

for every $0 \leq f \in L^2(\mathbb{R}^{3N})$. By (2.60) and (2.62),

$$(f \otimes \mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon(\delta) + g^2 N \varphi_\varepsilon(0,0))} f \otimes \mathbb{1}) = \int_{\mathbb{R}^3} dx \mathbb{E}^x [f(B_{-T}) f(B_T) \mathrm{e}^{-\int_{-T}^{T} \delta W(B_s) ds} \mathrm{e}^{S_\varepsilon^{\mathrm{ren}}}]$$

$$\leq \|f\|^2 \sup_{x \in \mathbb{R}^3} \mathbb{E}^0 \left(\left[\mathrm{e}^{2(S_\varepsilon^{\mathrm{OD},T}(x) + X_\varepsilon^T(x) + Y_\varepsilon^T(x) + Z_\varepsilon^T(x))} \right] \right)^{1/2} \leq \|f\|^2 a_5 \mathrm{e}^{b_5 T},$$

which implies, together with (2.63), that

$$\inf \sigma \left(H_\varepsilon(\delta) + g^2 N \varphi_\varepsilon(0,0) \right) + \frac{b_5}{2} \geq 0, \quad \delta > 0. \quad (2.64)$$

Note that b_5 is independent of δ. Thus

$$|(F, \mathrm{e}^{-2T(H_\varepsilon(\delta) + g^2 N \varphi_\varepsilon(0,0))} G)| \leq \|F\| \|G\| \mathrm{e}^{b_5 T} \quad (2.65)$$

follows for every $\delta > 0$. Let $F, G \in \mathscr{H}$. By the Feynman-Kac formula (2.57) we have

$$(F, \mathrm{e}^{-2TH_\varepsilon(\delta)} G)$$

$$= \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^{T} \delta W(B_s) ds} \mathbb{E}_{\mu_{\mathrm{E}}} \left[J_{-T} F(B_{-T}) \cdot \mathrm{e}^{-\phi_{\mathrm{E}}(\int_{-T}^{T} \sum_{j=1}^{N} \delta_s \otimes \tilde{\varphi}(\cdot - B_s^j) ds)} J_T G(B_T) \right] \right].$$

The Lebesgue dominated convergence theorem furthermore implies

$$\lim_{\delta \downarrow 0} (F, \mathrm{e}^{-2T(H_\varepsilon(\delta) + g^2 N \varphi_\varepsilon(0,0))} G) = (F, \mathrm{e}^{-2T(H_\varepsilon(0) + g^2 N \varphi_\varepsilon(0,0))} G).$$

Taking the limit $\delta \downarrow 0$ on both sides of (2.65), we have

$$|(F, \mathrm{e}^{-2T(H_\varepsilon(0) + g^2 N \varphi_\varepsilon(0,0))} G)| \leq \|F\| \|G\| \mathrm{e}^{b_5 T}. \quad (2.66)$$

This implies that (2.64) also holds for $\delta = 0$. Since $H_\varepsilon = H_\varepsilon(0) + V$ and V is bounded, we obtain

$$\inf \sigma(H_\varepsilon + g^2 N \varphi_\varepsilon(0,0)) + \frac{b_5}{2} + \|V\|_\infty \geq 0.$$

Setting $C = -\frac{b_5}{2} - \|V\|_\infty$ yields the corollary. □

Remark 2.22 Let Σ be the infimum of the essential spectrum of the self-ajoint operator $-\frac{1}{2} \sum_{j=1}^{N} \Delta_j + V$ in $L^2(\mathbb{R}^{3N})$ and $E = \inf \sigma(-\frac{1}{2} \sum_{j=1}^{N} \Delta_j + V)$. Then it can be shown in [Spo98] by using functional integrations that H_ε has a unique ground state if

$$\Sigma - E > \frac{N^2}{4} \int_{\mathbb{R}^3} \mathrm{e}^{-\varepsilon |k|^2} \beta(k) \mathbb{1}_\Lambda^\perp(k) dk$$

(see also [LHB11, Theorem 6.6]). In particular, in the case of $V(x^1, ..., x^N) = \delta \sum_{j=1}^{N} |x^j|^2$, the operator H_ε has a unique ground state for every $\varepsilon > 0$ and $\delta > 0$, since $\Sigma - E = \infty$.

Now we can complete the proof of the main theorem.

Proof of Theorem 2.2. Let $F, G \in \mathscr{H}$ and $C_\varepsilon(F,G) = (F, \mathrm{e}^{-t(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))}G)$. By Lemma 2.18 we obtain that $C_\varepsilon(F,G)$ is convergent as $\varepsilon \downarrow 0$, for every $F, G \in \mathscr{D}$. By the uniform bound

$$\|\mathrm{e}^{-t(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))}\| < \mathrm{e}^{-tC}$$

obtained from Corollary 2.21 and since $\mathscr{D}$ is dense in $\mathscr{H}$, it follows that $\{C_\varepsilon(F,G)\}_\varepsilon$ is a Cauchy sequence for $F, G \in \mathscr{H}$. Let $C_0(F,G) = \lim_{\varepsilon\downarrow 0} C_\varepsilon(F,G)$. Hence we get $|C_0(F,G)| \leq \mathrm{e}^{-tC}\|F\|\|G\|$. The Riesz theorem implies that there exists a bounded operator T_t such that

$$C_0(F,G) = (F, T_t G), \quad F, G \in \mathscr{H}.$$

Thus $\mathrm{s}-\lim_{\varepsilon\downarrow 0} \mathrm{e}^{-t(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))} = T_t$ follows. Furthermore, we also have that

$$\mathrm{s}-\lim_{\varepsilon\downarrow 0} \mathrm{e}^{-t(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))}\mathrm{e}^{-s(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))} = \mathrm{s}-\lim_{\varepsilon\downarrow 0} \mathrm{e}^{-(t+s)(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))} = T_{t+s}.$$

Since the left-hand side above is $T_t T_s$, the semigroup property of T_t follows. Since $\mathrm{e}^{-t(H_\varepsilon + g^2 N\varphi_\varepsilon(0,0))}$ is a symmetric semigroup, T_t is also symmetric. By the functional integral representation (2.59) the functional $(F, T_t G)$ is continuous at $t = 0$ for every $F, G \in \mathscr{D}$. Given that $\mathscr{D}$ is in $\mathscr{H}$ and $\|T_t\|$ is uniformly bounded in a neighborhood of $t = 0$, it also follows that T_t is strongly continuous at $t = 0$. Then the semigroup version of Stone's theorem [LHB11, Proposition 3.26] implies that there exists a self-adjoint operator H_{ren}, bounded from below, such that $T_t = \mathrm{e}^{-tH_{\mathrm{ren}}}$, $t \geq 0$. The proof is completed by setting $E_\varepsilon = -g^2 N\varphi_\varepsilon(0,0)$. □

We established the existence of the renormalized Hamiltonian H_{ren}. We can obtain explicitly the pair potential associated with H_{ren}.

Corollary 2.23 *The pair potential associated with H_{ren} is given by $\frac{g^2}{2} S_0^{\mathrm{ren}}$.*

Proof. By Lemma 2.16 we see that

$$(f \otimes \mathbb{1}, \mathrm{e}^{-2TH_{\mathrm{ren}}} h \otimes \mathbb{1}) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s)ds} \mathrm{e}^{\frac{g^2}{2} S_0^{\mathrm{ren}}} \right]. \tag{2.67}$$

□

3 Effective potential in the weak coupling limit

In this section we take the cutoff function to be

$$\varphi_\varepsilon(k) = (2\pi)^{-3/2} \mathrm{e}^{-\varepsilon|k|^2/2}.$$

We consider the following weak coupling limit of the Hamiltonian. In order to have a physically reasonable effective potential, we take the dispersion relation

$$\omega_\nu(k) = \sqrt{|k|^2 + \nu^2}$$

with positive mass $\nu > 0$ instead of $\omega(k) = |k|$, and set the IR cutoff to zero. Then the Hamiltonian is defined on $L^2(\mathbb{R}^{3N}) \otimes \mathscr{F}$ and given by

$$H_\varepsilon = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm I},$$

where $H_{\rm p} = \sum_{j=1}^N (-\frac{1}{2}\Delta_j) + V(x_1, ..., x_N)$ denotes the N-body Schrödinger operator, and

$$H_{\rm f} = \int_{\mathbb{R}^3} \omega_\nu(k) a^*(k) a(k) dk$$

is the free massive boson field. We scale the Hamiltonian by replacing the annihilation operator a and the creation operator a^* by κa and κa^*, respectively, where $\kappa > 0$ is the scaling parameter. Then H_ε changes to

$$H_\varepsilon(\kappa) = H_{\rm p} \otimes \mathbb{1} + \kappa^2 \mathbb{1} \otimes H_{\rm f} + \kappa H_{\rm I}. \tag{3.1}$$

This scaling also implies the transformations $\omega \mapsto \kappa^2 \omega$ and $\widehat{\varphi} \mapsto \kappa^2 \widehat{\varphi}$, while the energy renormalization term scales as

$$E_\varepsilon(\kappa) = -g^2 N \int_{\mathbb{R}^3} \frac{{\rm e}^{-\varepsilon|k|^2}}{2(2\pi)^3 \omega_\nu(k)} \frac{\kappa^2}{\kappa^2 \omega_\nu(k) + |k|^2/2} dk. \tag{3.2}$$

By Theorem 2.2 there exists a self-adjoint operator $H_{\rm ren}(\kappa)$ such that

$$\lim_{\varepsilon \downarrow 0} (f \otimes \mathbb{1}, {\rm e}^{-t(H_\varepsilon(\kappa) - E_\varepsilon(\kappa))} h \otimes \mathbb{1}) = (f \otimes \mathbb{1}, {\rm e}^{-tH_{\rm ren}(\kappa)} h \otimes \mathbb{1}). \tag{3.3}$$

The next proposition is established in [Dav79, Hir99].

Proposition 3.1 *We have*

$$\mathrm{s}\text{-}\lim_{\varepsilon \downarrow 0} \lim_{\kappa \to \infty} {\rm e}^{-t(H_\varepsilon(\kappa) - E_\varepsilon(\kappa))} = {\rm e}^{-th_{\rm eff}} \otimes P_\Omega,$$

where P_Ω denotes the projection to $\{z\mathbb{1} \,|\, z \in \mathbb{C}\} \subset \mathscr{F}$ and

$$h_{\rm eff} = -\frac{1}{2} \sum_{j=1}^N \Delta_j + V(x^1, ..., x^N) - \frac{g^2}{4\pi} \sum_{i<j} \frac{{\rm e}^{-\nu|x_i - x_j|}}{|x_i - x_j|}.$$

We are now interested in the scaling limit of $H_{\rm ren}(\kappa)$ when $\kappa \to \infty$. By Theorem 2.2 we see that

Lemma 3.2 *If $f, h \in L^2(\mathbb{R}^3)$, then*

$$\lim_{\kappa \to \infty} (f \otimes \mathbb{1}, {\rm e}^{-tH_{\rm ren}(\kappa)} h \otimes \mathbb{1}) = (f, {\rm e}^{-th_{\rm eff}} h). \tag{3.4}$$

Proof. By Lemma 2.18 we have

$$(f \otimes \mathbb{1}, {\rm e}^{-2TH_{\rm ren}(\kappa)} h \otimes \mathbb{1}) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) {\rm e}^{-\int_{-T}^T V(B_s) ds} {\rm e}^{\frac{g^2}{2} S_0^{\rm ren}(\kappa)} \right], \tag{3.5}$$

where

$$S_0^{\mathrm{ren}}(\kappa) = 2\sum_{i\neq j}^{N}\int_{-T}^{T}\varphi_0(B_s^i - B_s^j, 0, \kappa)ds + 2\sum_{i,j=1}^{N}\int_{-T}^{T}ds\int_s^T \nabla\varphi_0(B_t - B_s, t-s, \kappa)\cdot dB_t$$
$$- 2\sum_{i,j=1}^{N}\int_{-T}^{T}\varphi_0(B_T - B_s, T-s, \kappa)ds, \tag{3.6}$$

and

$$\varphi_0(x,t,\kappa) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-ik\cdot x}\mathrm{e}^{-\kappa^2\omega(k)|t|}}{2\omega(k)}\frac{\kappa^2}{\kappa^2\omega(k)+|k|^2/2}\mathbb{1}_\Lambda^\perp(k)dk. \tag{3.7}$$

In particular, for $t=0$ we have

$$g^2\sum_{i\neq j}^{N}\varphi_0(x^i - x^j, 0, \kappa)ds \;\to\; \frac{g^2}{4\pi}\sum_{i<j}\frac{\mathrm{e}^{-\nu|x^i-x^j|}}{|x^i-x^j|},$$

and for $t\neq 0$,

$$|\nabla\varphi_0(X,t,\kappa)|\to 0, \quad |\varphi_0(X,t,\kappa)|\to 0$$

pointwise as $\kappa\to\infty$. It can be shown in the same way as in Lemma 2.16 that

$$\lim_{\kappa\to\infty}\int_{\mathbb{R}^3}dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^{T}V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}(\kappa)}\right]$$
$$= \int_{\mathbb{R}^3}dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^{T}V(B_s)ds}\mathrm{e}^{\frac{g^2}{4\pi}\sum_{i<j}\int_{-T}^{T}\frac{\mathrm{e}^{-\nu|B_s^i-B_s^j|}}{|B_s^i-B_s^j|}ds}\right].$$

This completes the proof of the corollary. □

Corollary 3.3 *If $F, G\in\mathscr{D}$, then*

$$\lim_{\kappa\to\infty}(F, \mathrm{e}^{-tH_{\mathrm{ren}}(\kappa)}G) = (F, \mathrm{e}^{-th_{\mathrm{eff}}}\otimes P_\Omega G). \tag{3.8}$$

Proof. This follows from Lemmas 2.18 and 3.2. □

A Kato-class potentials

We say that $V\in\mathscr{K}_d$ whenever

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\mathbb{E}^x\left[\int_0^t|V(B_s)|ds\right] = 0. \tag{A.1}$$

The space $\mathscr{K}_d$ is Kato-class (relative to the Laplacian) of potentials $V:\mathbb{R}^d\to\mathbb{R}$. For details on Kato-class potentials we refer to [LHB11, Chapter 3.3] and [AS82, CFKS08].

Proposition A.1 *If $V \in \mathscr{K}_d$, then $W(x) = \sum_{i\neq j}^N V(x^i - x^j) \in \mathscr{K}_{dN}$, with the notation $x = (x^1, ..., x^N) \in \mathbb{R}^{dN}$.*

For a proof we refer e.g. to [CFKS08, p.7]. An equivalent characterization of Kato-class potentials is as follows. $V \in \mathscr{K}_d$ if and only if

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} |g(x-y)V(y)|dy = 0 \quad \text{with} \quad g(x) = \begin{cases} |x| & d = 1 \\ -\log|x| & d = 2 \\ |x|^{2-d} & d \geq 3. \end{cases} \tag{A.2}$$

From this characterization we have the following result.

Proposition A.2 *If $V \in \mathscr{K}_d$, then $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |g(x-y)V(y)|dy < \infty$.*

Examples of Kato-class potentials include (1) $|x|^{-(2-\varepsilon)}$ with $d = 3$ for any $\varepsilon > 0$, (2) $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $p = 1$ for $d = 1$, and $p > d/2$ for $d \geq 2$. It is also known that the function $\mathrm{e}^{\int_0^t V(B_s)ds}$ of Brownian motion is integrable with respect to Wiener measure for a Kato-class potential V.

Proposition A.3 *Let $0 \leq V \in \mathscr{K}_d$. Then there exist $\beta, \gamma > 0$ such that*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x[\mathrm{e}^{\int_0^t V(B_s)ds}] < \gamma \mathrm{e}^{t\beta}. \tag{A.3}$$

In particular, if $V \in L^p(\mathbb{R}^d)$ with $p = 1$ for $d = 1$, and $p > d/2$ for $d \geq 2$, then there exists C such that $\beta \leq C\|V\|_p$.

Proof. See [LHB11, Lemma 3.38]. □

B Schrödinger representation and Euclidean field

It is well known that the boson Fock space $\mathscr{F}$ is unitary equivalent to $L^2(Q, \mu)$, where this space consists of square integrable functions on a probability space (Q, Σ, μ). Consider the family of Gaussian random variables $\{\phi_0(f),\ f \in H_{-1/2}(\mathbb{R}^3)\}$ on (Q, Σ, μ) such that $\phi_0(f)$ is linear in $f \in H_{-1/2}(\mathbb{R}^3)$, and their mean and covariance are given by

$$\mathbb{E}_\mu[\phi_0(f)] = 0 \quad \text{and} \quad \mathbb{E}_\mu[\phi_0(f)\phi_0(g)] = \frac{1}{2}(f, g)_{H_{-1/2}(\mathbb{R}^3)}.$$

Given this space, the Fock vacuum $\mathbb{1}_{\mathscr{F}}$ is unitary equivalent to $\mathbb{1}_{L^2(Q)} \in L^2(Q)$, and the scalar field $\phi(f)$ is unitary equivalent to $\phi_0(f)$ as operators, i.e., $\phi_0(f)$ is regarded as multiplication by $\phi_0(f)$. Then the linear hull of the vectors given by the Wick products $:\prod_{j=1}^n \phi_0(f_j):$ is dense in $L^2(Q)$, where recall that Wick product is recursively defined by

$$\begin{aligned} &:\phi_0(f): \ = \phi_0(f) \\ &:\phi_0(f)\prod_{j=1}^n \phi_0(f_j): \ = \phi_0(f):\prod_{j=1}^n \phi_0(f_j): - \frac{1}{2}\sum_{i=1}^n (f, f_i)_{H_{-1/2}(\mathbb{R}^3)} :\prod_{j\neq i}^n \phi_0(f_j): \end{aligned}$$

This allows to identify $\mathscr{F}$ and $L^2(Q)$, which we have done in (2.57), i.e., $F \in \mathscr{H}$ can be regarded as a function $\mathbb{R}^{3N} \ni x \mapsto F(x) \in L^2(Q)$ such that $\int_{\mathbb{R}^{3N}} \|F(x)\|^2_{L^2(Q)} dx < \infty$.

To construct a Feynman-Kac-type representation we use a Euclidean field. Consider the family of Gaussian random variables $\{\phi_{\mathrm{E}}(F),\ F \in H_{-1}(\mathbb{R}^4)\}$ with mean and covariance

$$\mathbb{E}_{\mu_E}[\phi_E(F)] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_E}[\phi_E(F)\phi_E(G)] = \frac{1}{2}(F,G)_{H_{-1}(\mathbb{R}^4)}$$

on a chosen probability space (Q_E, Σ_E, μ_E). Note that for $f \in H_{-1/2}(\mathbb{R}^3)$ the relations

$$\delta_t \otimes f \in H_{-1}(\mathbb{R}^4) \quad \text{and} \quad \|\delta_t \otimes f\|_{H_{-1}(\mathbb{R}^4)} = \|f\|_{H_{-1/2}(\mathbb{R}^3)}$$

hold, where $\delta_t(x) = \delta(x-t)$ is Dirac delta distribution with mass on t. The family of identities used in (2.57) is then given by $J_t : L^2(Q) \to L^2(Q_{\mathrm{E}})$, $t \in \mathbb{R}$, defined by the relations

$$J_t \mathbb{1}_{L^2(Q)} = \mathbb{1}_{L^2(Q_{\mathrm{E}})} \quad \text{and} \quad J_t{:}\prod_{j=1}^m \phi(f_j) : \ = \ : \prod_{j=1}^m \phi_{\mathrm{E}}(\delta_t \otimes f_j) :$$

Under the identification $\mathscr{F} \cong L^2(Q)$ it follows that $(J_tF, J_sG)_{L^2(Q_{\mathrm{E}})} = (F, \mathrm{e}^{-|t-s|H_{\mathrm{f}}}G)_{\mathscr{F}}$ for $F, G \in \mathscr{F}$. For an extensive discussion of the details we refer to [LHB11, Chapter 5].

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