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PARACONTROLLED QUASILINEAR SPDES

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Abstract. We introduce a non-linear paracontrolled calculus and use it to renormalise a class of singular SPDEs including certain quasilinear variants of the periodic two dimensional parabolic Anderson model.

1. Introduction

We show how to renormalise a class of general quasilinear equations of which one of the simplest examples is the following parabolic SPDE:

$$\partial_t u(t,x) - a(u(t,x))\Delta u(t,x) = \xi(x), \qquad u(0,x) = u_0(x), \quad x \in \mathbb{T}^2, t \geqslant 0, \tag{1}$$

with $a : \mathbb{R} \to [\lambda, 1]$ for $\lambda > 0$ a uniformly bounded C^4 diffusion matrix, and $\|a^{(k)}\|_{L^\infty} \leqslant 1$ for $k = 0, \dots, 4$. We assume that $\xi \in \mathscr{C}^{\alpha-2}(\mathbb{T}^2)$ with $2/3 < \alpha < 1$ where $\mathscr{C}^\alpha(\mathbb{T}^2)$ is the Besov space $B^\alpha_{\infty,\infty}(\mathbb{T}^2)$. This would apply to the space white noise on $\mathbb{T}^2$, for example. In this case we only expect that $u(t,\cdot) \in \mathscr{C}^\alpha(\mathbb{T}^2)$ and the term $a(u(t,\cdot))\Delta u(t,\cdot)$ is not well defined when $2\alpha - 2 < 0$. Eq. (1) is a quasilinear generalisation of the two–dimensional periodic parabolic Anderson model (PAM).

Let us remark from the start that the framework we will consider below allows to deal with a class of equations of the form

$$a_1(u(t,x))\partial_t u(t,x) - a_2(u(t,x))\Delta u(t,x) = \xi(a_3(u(t,x)), t, x), \qquad x \in \mathbb{T}^2, t \geqslant 0, \tag{2}$$

where a_1, a_2 are sufficiently smooth non-degenerate coefficients and $\xi(z,t,x)$ is a Gaussian process with covariance

$$\mathbb{E}[\xi(z,t,x)\xi(z',t',x')] = F(z,z')Q(t-t', x-x'), \qquad x, x' \in \mathbb{T}^2, t, t', z, z' \in \mathbb{R},$$

with F a smooth function and Q a distribution of parabolic regularity $\rho > -4/3$. This includes the space white noise discussed before or a time white noise with a regular dependence on the space variable or some noise mildly irregular in space and time.

Also the scalar character of the equation or of the non-linear diffusion coefficient will not play any specific role and we could consider vector–valued equations with general diffusion coefficients provided the template problem (7) below remains uniformly parabolic.

For the sake of clarity and simplicity we will discuss mainly the basic example (1) since this contains already most of the technical difficulties. The fact that one can handle models as general as (2) can be considered a direct byproduct of the techniques we will introduce below.

Recently Otto and Weber [32] and Bailleul, Debussche and Hofmanova [7] investigated quasilinear SPDEs in the context of pathwise methods and in a range of regularities compatible with the ones we will consider in this paper.

- In [32] the authors obtained a priori estimates for equations of the form

  $$\partial_t u(t,x) - a(t,x)\partial_x u(t,x) = f(u(t,x))\xi(t,x), \qquad (t,x) \in \mathbb{T}^2$$

  where both space and time variables take values in a one dimensional periodic domain and their noise can be white in time but colored in space, essentially behaving like a distribution of parabolic regularity in $(-4/3, 1)$. Their choice of noise simplify the problem since the non-linear term $a(t,x)\partial_2 u(t,x)$ does not require renormalization. Our method can be applied also in their setting and gives an explicit representation of the limiting problem, which seems to be lacking in their approach.

- More closely related to the quasilinear gPAM model we will discuss below, are the results in [7] where the authors consider the equation in divergence form:

$$\partial_t u(t,x) - \nabla[A(u(t,x))\nabla u(t,x)] = f(u(t,x))\xi(x). \qquad t \geqslant 0, x \in \mathbb{T}^2. \tag{3}$$

  In this case they are able to implement the renormalisation for the r.h.s but the divergence structure do not present any need of renormalisation in the l.h.s.. Indeed the main difficulty of eq. (1) is the fact that the product $a(u)\Delta u$ is not well defined by classical considerations and the PDE cannot be formulated in this way. Note that for scalar u and a we can always rewrite

$$a(u)\Delta u = \nabla(A(u)\nabla u)$$

  where A is such that $A' = a$.

Let us state one simple result which can be obtained via the theory developed in this paper:

Theorem 1. *Fix $2/3 < \alpha < 1$ and let $u_0 \in \mathscr{C}^\alpha$ an initial condition. For some $\alpha' \in (\alpha, 1)$ let $\xi \in \mathscr{C}^{\alpha'-2}(\mathbb{T}^2)$ be a space white noise with zero average on the torus. Let $(\xi_\varepsilon, u_{0,\varepsilon})_{\varepsilon>0}$ be a family of smooth approximations to ξ, u_0 obtained by convolution with a rescaled smoothing kernel and u_ε the classical solution to the Cauchy problem*

$$\partial_t u_\varepsilon - a(u_\varepsilon)\Delta u_\varepsilon = \xi_\varepsilon - \sigma_\varepsilon \frac{a'(u_\varepsilon)}{a(u_\varepsilon)^2}, \qquad u(0) = u_{0,\varepsilon}. \tag{4}$$

Then we can choose the constants $(\sigma_\varepsilon)_{\varepsilon>0}$ and a random time $T > 0$ in such a way that the family of r.v. $(u_\varepsilon)_{\varepsilon>0} \subseteq C\left([0,T]; \mathscr{C}^\alpha(\mathbb{T}^2)\right)$ almost surely converge as $\varepsilon \to 0$ to a random element $u \in C\left([0,T], \mathscr{C}^\alpha(\mathbb{T}^2)\right)$. This element can be characterised as the solution to a paracontrolled singular SPDE (see below for more details).

In order to devise a suitable formulation of eq. (1) and obtain a theory with $u \in \mathscr{C}^\alpha$ we decompose the non-linear diffusion term in the l.h.s. with help of Bony's paraproduct [30] and write

$$\partial_t u - a(u) \prec \Delta u = \xi + \Phi(u) \tag{5}$$

with

$$\Phi(u) := a(u) \circ \Delta u + a(u) \succ \Delta u \tag{6}$$

where $\prec, \succ$ are standard paraproducts and $\circ$ denotes the resonant product (see below for precise definitions). Now the l.h.s. is always well defined irrespective of the regularity of the function u and the problem becomes that of controlling the resonant product $a(u) \circ \Delta u$ appearing in the r.h.s. . A key point of the analysis put forward below is that this term can be expected to be of regularity $2\alpha - 2 > \alpha - 2$ so better than the leading term ξ.

Our approach can be described as follows. For an equation of the form

$$\partial_t u - a_1(u)\Delta u = a_2(u)\xi,$$

we consider at first a *parametric* "template" problem with constant coefficients

$$\partial_t \vartheta(\eta, t, x) - \eta_1 \Delta \vartheta(\eta, t, x) = \eta_2 \xi(t, x), \tag{7}$$

where now $\eta = (\eta_1, \eta_2)$ are fixed numbers. A *nonlinear* paraproduct $\Pi_{\prec\!\!\prec}$ will allows us to modulate the parametric solution ϑ with the coefficient $a(u) = (a_1(u), a_2(u))$ as to capture the most irregular part of the solution u itself. As a consequence, the paracontrolled Ansatz

$$u = \Pi_{\prec\!\!\prec}(a(u), \vartheta) + u^\sharp$$

will define a regular remainder term $u^\sharp$ which solves a standard PDE. With this decomposition the resonant products appearing in the equation can be estimated along the lines of the standard paracontrolled arguments introduced in [15] and all the arguments introduced there can be extended in a straighforward manner to the quasilinear setting.

Some remarks are in order:

a) this approach has been inspired by the parametric controlled Ansatz of Otto and Weber [32]. At variance with their approach we use the parametric Ansatz in the context of the paradifferential calculus and this allows us to obtain a precise description of the various terms in the equation and perform eventual renormalisations.
b) using the paraproduct decomposition, equation (3) can be rewritten as

$$\partial_t u - g'(u) \prec \Delta u = f(u)\xi - \nabla(g'(u) \circ \nabla u + g'(u) \succ \nabla u)$$

and treated with the same techniques as (5).

Usefulness of paraproducts in the analysis of non-linear PDEs is by now well established: see for example the seminal paper of Meyer [30], the early review of Bony [10], the recent books of Alinhac and Gérard [1] and Bahouri, Chemin, and Danchin [3]. Let us mention also the interesting paper of Hörmander [25] where paradifferential operators allows to bypass the Nash–Moser fixpoint theorem in some applications where the loss of regularity prevents straighforward use of standard Banach fixpoint theorem. The main observation in that paper is that, with the aid of paradifferential operators, it is possible to identify a "corrected" problem for which standard Banach fixpoint applies.

Paracontrolled calculus for singular SPDEs has beed introduced by Gubinelli, Imkeller and Perkowski [15] (see also the lecture notes [17]) and used to study various equations like the KPZ equation [16], the dynamic Φ^4_3 model [11] in $d = 3$ and its global well–posedness [31], the spectrum of the continuous Anderson Hamiltonian in $d = 2$ [2]. By using heat–semigroup techniques paracontrolled calculus has been extended to the manifold context by Bailleul and Bernicot [4].

Non–linear generalisation of the classic bilinear paraproducts already appeared in the notion of *paracomposition* introduced by Alinhac [10]. Non–linear versions of rough paths have been considered by one of the authors in order to study the Korteweg–de Vries equation [14]. Non–linear Young integrals were used by one of the authors in joint work with Catellier [12] to study the the regularising properties of sample paths of stochastic processes processes. See also the related work of Hu and Le [24] on differential systems in Hölder media. Relevant to this discussion of non-linear variants of rough paths is the work of Bailleul on rough flows [8] and their application to homogeneisation [6]. By looking at the composition $f(g(x))$ as the action of the distribution $\delta_{g(x)}$ on the function f, non–linear constructions can be linearised at the price of working in infinite–dimensional spaces: this is the approach chosen by Kelly and Melbourne to avoid non–linear generalisations of rough path theory in their study of homogeneisation of fast–slow system with random initial conditions [26].

Paracontrolled calculus is currently limited to "first order" computations. This limitation is also ubiquitous in the present work. Even if, in practice, this is not a big issue, and the calculus is still able to deal with a large class of problems, it makes the paracontrolled approach less appealing for a general theory of singular SPDEs. Let us remark that recently Bailleul and Bernicot [5] developed an higher order version of the paracontrolled calculus. However, apart from these recent development, whose impact is still to be assessed, the most general theory for singular SPDEs has been developed by Hairer [19, 20, 13] under the name of *regularity structures* theory. Regularity structures are a vast generalisation of Lyons' rough paths [27, 29, 28] which give effective tools to describe non-linear operations acting on certain spaces of distributions, their renormalization by subtraction of local singularities and their use to solve singular SPDE. Regularity structures have been successfully applied to all the models mentioned so far [19, 18], to other models like the Sine–Gordon model [22] (which however can also be handled via paracontrolled techniques) and to study weak universality conjectures [21, 23]. In their current instantiation it does not seem possible to solve quasilinear SPDEs via regularity structures. The results of the present paper hint to the fact that a non-linear version of regularity structures is conceivable, at least in principle. Indeed one can imagine models depending on additional parameters and modelled distributions acting as evaluations of the parametric models at certain space–time dependent values of the parameters. It would be interesting to pursue further this intuition.

The structure of the paper is the following. In Section 2 we introduce our basic tools: the non-linear paraproduct decomposition and some related commutation lemmas. In Section 3 we introduce the paracontrolled Ansatz which allows to transform the singular problem (1) into a well–behaved PDE. In Section 4 we discuss the apriori estimates, the uniqueness of the solution of the transformed PDE and its continuity w.r.t. the random data and the initial condition, we introduce also the algebraic structure which allows to renormalise the model. Section 5 deals with the renormalization of the stochastic data and the construction of the enhanced noise associated to white noise. Section 6 deals with the extension of the results to more general equations, in particular with non-linear r.h.s. or with noise whose law depends on the solution itself. Finally Appendix A reviews some reference material on Besov spaces and proves some technical lemmas.

Notations. We will denote $\mathscr{C}^\alpha := B^\alpha_{\infty,\infty}(\mathbb{T}^2)$ the Zygmund space of regularity $\alpha \in \mathbb{R}$ on the torus $\mathbb{T}^2$. See Appendix A for the definition of the general Besov spaces $B^\alpha_{p,q}$, the Littlewoord–Paley operators $(\Delta_i)_{i\geqslant -1}$ and the basic properties thereof needed in this paper. If V is a Banach space and $T > 0$, we denote $C^\alpha_T V$ the space of α–Hölder functions in $C_T V := C([0,T];V)$. We introduce parabolic spaces $\mathscr{L}^\alpha_T := C_T^{\alpha/2}\mathscr{C}^0 \cap C_T\mathscr{C}^\alpha$ with norm

$$\|f\|_{\mathscr{L}^\alpha_T} = \|f\|_{C_T^{\alpha/2}\mathscr{C}^0} + \|f\|_{C_T\mathscr{C}^\alpha}.$$

Moreover for convenience we denote $\mathscr{C}^\alpha_T := C_T\mathscr{C}^\alpha$. We will avoid to note explicitly the time span T whenever this does not cause ambiguities. We will need also spaces for functions of (η,t,x) where η is an additional parameter in $[\lambda,1]$ for $\lambda < 1$ which we denote $C^k_\eta V$ with norm

$$\|F\|_{C^k_\eta V} = \sup_{\eta\in[\lambda,1]}\sup_{n=0,\dots,k} \|\partial^n_\eta F(\eta,)\|_V,$$

where V is a Banach space of function on $[0,T]\times\mathbb{T}^2$, in our case $V = \mathscr{C}^\alpha_T$ or $V = \mathscr{L}^\alpha_T$.

We will denote by $K_{i,x}(y) = 2^{2i}K(2^i(x-y))$ the kernel of the Littlewood–Paley operator Δ_i and $Q_{i,t}(s) = Q_i(t-s) = 2^{2i}Q(2^{2i}(t-s))$ is a smoothing kernel at scale 2^{2i} in the time direction where Q is a smooth, positive function with compact support in $\mathbb{R}_+$. We introduce also the shortcut $P_i = K_{<i-1}$. Another notation shortcut widely used in this article is to write $\int_{x,y}$ for integrals on $\mathbb{T}^2$ or $\mathbb{R}$ with respect to the measures $\mathrm{d}x$ and $\mathrm{d}y$ without specifying the integration bounds, whenever this does not create ambiguity. Finally, we will note $\delta f^{tx}_{sy} = f(t,x) - f(s,y)$ and $\delta_\tau f^{tx}_{sy} = f(s,y) + \tau(f(t,x) - f(s,y))$ for $\tau \in [0,1]$.

2. Nonlinear paraproducts

In this section we introduce the nonlinear paraproduct and related results that will be used in Section 3 to analyse equation (1).

Let $g : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$, and $h : \mathbb{R}\times[0,T]\times\mathbb{R}^3 \to \mathbb{R}$ be smooth functions. We can decompose the composition $h(g(\cdot),\cdot)$ via *nonlinear* paraproducts as follows. Define

$$\Pi_\prec(g,h)(t,x) := \sum_q \int_{y,z} P_{q,x}(y)K_{q,x}(z)h(g(t,y),t,z) \tag{8}$$

$$\Pi_\circ(g,h)(t,x) := \sum_{k\sim q} \int_{y,z} K_{k,x}(y)K_{q,x}(z)h(g(t,y),t,z) \tag{9}$$

$$\Pi_\succ(g,h)(t,x) := \sum_k \int_{y,z} K_{k,x}(z)P_{k,x}(y)h(g(t,z),t,y). \tag{10}$$

This gives a map

$$(g,h) \mapsto \Pi_\diamondsuit(g,h) := \Pi_\prec(g,h) + \Pi_\circ(g,h) + \Pi_\succ(g,h) = h(g(\cdot),\cdot) \tag{11}$$

that can be uniquely extended to

$$\Pi_\diamondsuit : \mathscr{C}^\rho_T \times C^2_\eta\mathscr{C}^\gamma_T \to \mathscr{C}^{\gamma\wedge\rho}_T \qquad \rho\in(0,1), \gamma\in\mathbb{R}, \rho+\gamma>0,$$

thanks to the following bounds:

Lemma 2 (Nonlinear paraproduct estimates)**.** *Let $g \in \mathscr{C}_T^\rho$ for some $\rho \in (0,1)$, and $h \in C_\eta^2 \mathscr{C}_T^\gamma$ for any $\gamma \in \mathbb{R}$. Then*

$$\|\Pi_\prec(g,h)\|_{\mathscr{C}_T^\gamma} \lesssim \|h\|_{C_\eta \mathscr{C}_T^\gamma}, \qquad \|\Pi_\succ(g,h)\|_{\mathscr{C}_T^{\rho\wedge(\rho+\gamma)}} \lesssim \|g\|_{\mathscr{C}_T^\rho}\|h\|_{C_\eta^1 \mathscr{C}_T^\gamma},$$

and

$$\|\Pi_\prec(g_1,h) - \Pi_\prec(g_2,h)\|_{\mathscr{C}_T^\gamma} \lesssim \|g_1 - g_2\|_{C_T L^\infty}\|h\|_{C_\eta^1 \mathscr{C}_T^\gamma},$$

$$\begin{aligned}\|\Pi_\succ(g_1,h) - \Pi_\succ(g_2,h)\|_{\mathscr{C}_T^{\rho\wedge(\rho+\gamma)}} \lesssim\ & \|g_1 - g_2\|_{C_T L^\infty}\left(\|g_1\|_{\mathscr{C}_T^\rho} + \|g_2\|_{\mathscr{C}_T^\rho}\right)\|h\|_{C_\eta^2 \mathscr{C}_T^\gamma} \\ & + \|g_1 - g_2\|_{\mathscr{C}_T^\rho}\|h\|_{C_\eta^1 \mathscr{C}_T^\gamma}.\end{aligned}$$

Moreover if $\rho + \gamma > 0$ we have also

$$\|\Pi_\circ(g,h)\|_{\mathscr{C}_T^{\rho+\gamma}} \lesssim \|h\|_{C_\eta^1 \mathscr{C}_T^\gamma}\|g\|_{\mathscr{C}_T^\rho},$$

$$\begin{aligned}\|\Pi_\circ(g_1,h) - \Pi_\circ(g_2,h)\|_{\mathscr{C}_T^{\rho+\gamma}} \lesssim\ & \|g_1 - g_2\|_{C_T L^\infty}\left(\|g_1\|_{\mathscr{C}_T^\rho} + \|g_2\|_{\mathscr{C}_T^\rho}\right)\|h\|_{C_\eta^2 \mathscr{C}_T^\gamma} \\ & + \|g_1 - g_2\|_{\mathscr{C}_T^\rho}\|h\|_{C_\eta^1 \mathscr{C}_T^\gamma}.\end{aligned}$$

In particular if $\rho + \gamma > 0$ the composition $\Pi_\diamondsuit(g,h) = h(g(\cdot),\cdot)$ is linear in h and locally Lipshitz in g:

$$\|\Pi_\diamondsuit(g,h)\|_{\mathscr{C}_T^\gamma} \lesssim \|h\|_{C_\eta^1 \mathscr{C}_T^\gamma}\|g\|_{\mathscr{C}_T^\rho},$$

$$\|\Pi_\diamondsuit(g_1,h) - \Pi_\diamondsuit(g_2,h)\|_{\mathscr{C}_T^\gamma} \lesssim \|g_1 - g_2\|_{\mathscr{C}_T^\rho}\left(1 + \|g_1\|_{\mathscr{C}_T^\rho} + \|g_2\|_{\mathscr{C}_T^\rho}\right)\|h\|_{C_\eta^2 \mathscr{C}^\gamma}.$$

Proof. Using the fact that

$$\|\Delta_k h(g(t,y),t,\cdot)\|_{L^\infty} \lesssim 2^{-\gamma k}\|h\|_{C_\eta \mathscr{C}_T^\gamma},$$

$$\|\Delta_k h(g(t,y),t,\cdot) - \Delta_k h(g(t,y'),t,\cdot)\|_{L^\infty} \lesssim 2^{-\gamma k}\|h\|_{C_\eta^1 \mathscr{C}_T^\gamma}\|g\|_{C_T \mathscr{C}^\rho}|y - y'|^\rho,$$

and

$$\|\Delta_k h(g_1(t,y),t,\cdot) - \Delta_k h(g_2(t,y),t,\cdot)\|_{L^\infty} \lesssim 2^{-\gamma k}\|h\|_{C_\eta^1 \mathscr{C}_T^\gamma}\|g_1 - g_2\|_{C_T L^\infty},$$

we obtain the bounds on $\Pi_\prec(g,h)$, $\Pi_\succ(g,h)$, $\Pi_\circ(g,h)$ and $\Pi_\prec(g_1,h) - \Pi_\prec(g_2,h)$. We proceed to estimate the term $\Pi_\succ(g_1,h) - \Pi_\succ(g_2,h)$. We will use the following notation for brevity:

$$\delta g_{2z}^{1y} := g_1(t,y) - g_2(t,z) \quad \text{and} \quad \delta_\tau g_{2z}^{1y} := g_2(t,z) + \tau(g_1(t,y) - g_2(t,z)).$$

Then

$$\begin{aligned}&\left|\int_{y,z} K_{k,x}(z)P_{k,x}(y)[h(g_1(t,z),t,y) - h(g_2(t,z),t,y)]\right| \\ =\ & \left|\int_{y,z,\tau\in[0,1]} K_{k,x}(z)P_{k,x}(y)[\partial_a h(\delta_\tau g_{2z}^{1z},t,y)\delta g_{2z}^{1z} - \partial_a h(\delta_\tau g_{2x}^{1x},t,y)\delta g_{2x}^{1x}]\right| \\ \lesssim\ & \left|\int_{y,z,t\in[0,1],\sigma\in[0,1]} K_{k,x}(z)P_{k,x}(y)\partial_\eta^2 h(\delta_\sigma(\delta_\tau g_2^1)_x^z,t,y)(\delta g_{1x}^{1z} - \delta g_{2x}^{2z})\delta g_{2z}^{1z}\right| \\ & + \left|\int_{y,z,\tau\in[0,1]} K_{k,x}(z)P_{k,x}(y)\partial_\eta h(\delta_\tau g_{2x}^{1x},t,y)(\delta g_{2z}^{1z} - \delta g_{2x}^{1x})\right| \\ \lesssim\ & \|g_1 - g_2\|_{C_T L^\infty}\left(\|g_1\|_{C_T \mathscr{C}^\rho} + \|g_2\|_{C_T \mathscr{C}^\rho}\right)\|h\|_{C_\eta^2 C_T \mathscr{C}^\gamma} 2^{-\rho k}\sum_{q<k-1} 2^{-\gamma q} \\ & + \|g_1 - g_2\|_{C_T \mathscr{C}^\rho}\|h\|_{C_\eta^1 C_T \mathscr{C}^\gamma} 2^{-\rho k}\sum_{q<k-1} 2^{-\gamma q}.\end{aligned}$$

With the same reasoning we can bound the norm of $\Pi_\circ(g_1,h) - \Pi_\circ(g_2,h)$. □

Lemma 3. *Let $g \in \mathscr{C}_T^\rho$ with $\rho < 0$ and $h \in C_\eta^2 \mathscr{C}_T^\gamma$, $\gamma \in \mathbb{R}$. Then*

$$\|\Pi_\prec(g_1,h) - \Pi_\prec(g_2,h)\|_{\mathscr{C}_T^\gamma} \lesssim \|g_1 - g_2\|_{\mathscr{C}_T^\rho}\|h\|_{C_\eta^1 \mathscr{C}_T^{\gamma-\rho}}.$$

Proof.

$$\begin{aligned}
&\left|\sum_{\ell\lesssim q}\int_{y,z} K_{\ell,x}(y)[\Delta_q h(g_1(t,y),t,z)-\Delta_q h(g_2(t,y),t,z)]\right| \\
=\;&\left|\sum_{\ell\lesssim q}\sum_{j,k}\int_{y,z,\tau,w} K_{\ell,x}(y)K_{j,y}(w)\Delta_q\partial_\eta h(\delta_\tau g_2^1(t,w),t,z)\Delta_k\delta g_2^1(t,y)\right| \\
\lesssim\;&\sum_{\ell\lesssim q}\left(\sum_{j\lesssim k\sim\ell}2^{(\rho-\gamma)q}2^{-\rho j}+\sum_{j\sim\ell}2^{(\rho-\gamma)q}2^{-\rho j}\right)\|h(\cdot,t,\cdot)\|_{C_\eta^1\mathscr{C}^{\gamma-\rho}}\|g_1(t,\cdot)-g_2(t,\cdot)\|_{\mathscr{C}^\rho} \\
\lesssim\;&2^{-\gamma q}\|h(\cdot,t,\cdot)\|_{C_\eta^1\mathscr{C}^{\gamma-\rho}}\|g_1(t,\cdot)-g_2(t,\cdot)\|_{\mathscr{C}^\rho}
\end{aligned}$$

where we have used the bounds

$$\|K_{j,\cdot}(w)\Delta_q\partial_\eta h(\delta_\tau g_2^1(t,w),t,z)\|_{L^\infty}\lesssim\|\Delta_q\partial_\eta h(\delta_\tau g_2^1(t,\cdot),t,z)\|_{L^\infty}\lesssim\|\Delta_q h(t,z)\|_{C_\eta^1L^\infty},$$

$$\left\|\sum_{j\lesssim k}K_{j,\cdot}(w)\Delta_q\partial_\eta h(\delta_\tau g_2^1(t,w),t,z)\right\|_{L^\infty}\lesssim\|\Delta_q\partial_\eta h(\delta_\tau g_2^1(t,\cdot),t,z)\|_{L^\infty}\lesssim\|\Delta_q h(t,z)\|_{C_\eta^1L^\infty}.$$

□

We will need a time-smoothed nonlinear paraproduct, defined as

$$\Pi_{\prec}(g,h)(t,x):=\sum_i\int_{y,s}Q_{i,t}(s)P_{i,x}(y)(\Delta_i h(g(s,y),t,\cdot))(x),\tag{12}$$

where, by convention, a continuous function $t\mapsto g(t)$ on $\mathbb{R}_+$ is extended to $\mathbb{R}$ by defining $g(t)=g(0)$ for $t\leqslant 0$. This convention preserves the Hölder norms of index in $[0,1]$. This modified nonlinear paraproduct enjoys similar bounds to the regular nonlinear paraproduct.

Lemma 4. *Let $g\in C_TL^\infty$ and $h\in C_\eta^1\mathscr{L}^\gamma_T$ for $\gamma\in(0,2)$. Then*

$$\|\Pi_{\prec}(g,h)\|_{\mathscr{L}^\gamma}\lesssim\|h\|_{C_\eta\mathscr{L}^\gamma_T},$$

and

$$\|\Pi_{\prec}(g_1,h)-\Pi_{\prec}(g_2,h)\|_{\mathscr{L}^\gamma_T}\lesssim\|g_1-g_2\|_{C_TL^\infty}\|h\|_{C_\eta^1\mathscr{L}^\gamma_T}.$$

Proof. The norm $\|\Pi_{\prec}(g,h)\|_{C_T\mathscr{C}^\gamma}$ can be treated in the same way as in Lemma 2. We estimate $\|\Pi_{\prec}(g,h)\|_{C_T^{\gamma/2}\mathscr{C}^0}$ as follows:

$$\begin{aligned}
&\|\Delta_j\Pi_{\prec}(g,h)(t_1)-\Delta_j\Pi_{\prec}(g,h)(t_2)\|_{L^\infty} \\
\lesssim\;&\sup_x\left|\int_z K_{j,x}(z)\sum_{i\sim j}\int_{y,s}Q_{i,t_1}(s)P_{i,z}(y)[\Delta_i h(g(s,y),t_1,z)-\Delta_i h(g(s,y),t_2,z)]\right| \\
&+\sup_x\left|\int_z K_{j,x}(z)\sum_{i\sim j}\int_{y,s}[Q_{i,t_1}(s)-Q_{i,t_2}(s)]P_{i,z}(y)\Delta_i h(g(s,y),t_2,z)\right| \\
\lesssim\;&\|h(\cdot,t_1,\cdot)-h(\cdot,t_2,\cdot)\|_{C_\eta\mathscr{C}^0}+2^{2j}|t_1-t_2|^{\gamma/2}\|h\|_{C_\eta\mathscr{C}^\gamma_T}.
\end{aligned}$$

The second inequality can be obtained easily with the same techniques used so far. □

Lemma 5. *Let $\rho<0$ and $\gamma\in(0,2)$. We have*

$$\|\Pi_{\prec}(g_1,h)-\Pi_{\prec}(g_2,h)\|_{C_T^{\gamma/2}\mathscr{C}^0}\lesssim\|g_1-g_2\|_{\mathscr{C}^\rho_T}\left(\|h\|_{C_\eta^1C_T^{\gamma/2}\mathscr{C}^{-\rho}_T}+\|h\|_{C_\eta^1\mathscr{C}^{\gamma-\rho}_T}\right)$$

and

$$\|\Pi_{\prec}(g_1,h)-\Pi_{\prec}(g_2,h)\|_{\mathscr{L}^\gamma_T}\lesssim\|g_1-g_2\|_{\mathscr{C}^\rho_T}\left(\|h\|_{C_\eta^1C_T^{\gamma/2}\mathscr{C}^{-\rho}_T}+\|h\|_{C_\eta^1\mathscr{C}^{\gamma-\rho}_T}\right).$$

Proof. We suppose w.l.o.g. that $t_1 < t_2$ and write

$$\begin{aligned}
&\Delta_j\Pi_{\prec}(g_1,h)(t_2) - \Delta_j\Pi_{\prec}(g_1,h)(t_1) - \Delta_j\Pi_{\prec}(g_2,h)(t_2) + \Delta_j\Pi_{\prec}(g_2,h)(t_1)_{L^\infty}\\
&= \sum_{i\sim j}\int_{\substack{z,y\\ s,\tau}} K_{j,x}(z)Q_{i,t_1}(s)P_{i,z}(y)[\Delta_i\partial_\eta h(\delta_\tau g^1_{2,s,y},t_2,z) - \Delta_i\partial_\eta h(\delta_\tau g^1_{2,s,y},t_1,z)]\delta g^1_{2,s,y}\\
&\quad + \sum_{i\sim j}\int_{z,y,s,\tau} K_{j,x}(z)[Q_{i,t_2}(s) - Q_{i,t_1}(s)]P_{i,z}(y)\Delta_i\partial_\eta h(\delta_\tau g^1_2(s,y),t_2,z)\delta g^1_2(s,y)
\end{aligned}$$

As in the proof of Lemma 3 we can estimate

$$\begin{aligned}
&\sup_{s\in[0,t_1]}\left|P_{i,z}(y)\left[\Delta_i\partial_\eta h(\delta_\tau g^1_2(s,y),t_2,z)\delta g^1_2(s,y) - \Delta_i\partial_\eta h(\delta_\tau g^1_2(s,y),t_1,z)\delta g^1_2(s,y)\right]\right|\\
&\qquad\lesssim \sup_{s\in[0,t_1]}\|g_1(s,\cdot)-g_2(s,\cdot)\|_{\mathscr{C}^\rho}\|h(\cdot,t_2,\cdot)-h(\cdot,t_1,\cdot)\|_{C^1_\eta\mathscr{C}^{-\rho}}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{s\in[0,t_2]}|P_{i,z}(y)\Delta_i\partial_\eta h(\delta_\tau g^1_2(s,y),t_2,z)\delta_\tau g^1_2(s,y)|\\
&\lesssim 2^{-\gamma i}\sup_{s\in[0,t_2]}\|g_1(s,\cdot)-g_2(s,\cdot)\|_{\mathscr{C}^\rho}\|h(\cdot,t_2,\cdot)\|_{C^1_\eta\mathscr{C}^{\gamma-\rho}}
\end{aligned}$$

and this gives the first bound. Following the proof of Lemma 3 we obtain

$$|\Delta_j\Pi_{\prec}(g_1,h)(t) - \Delta_j\Pi_{\prec}(g_2,h)(t)| \lesssim 2^{-\gamma j}\|h(\cdot,t,\cdot)\|_{C^1_\eta\mathscr{C}^{\gamma-\rho}}\sup_{s\in[0,t]}\|g_1(t,\cdot)-g_2(t,\cdot)\|_{\mathscr{C}^\rho}$$

and this concludes the proof of the second bound. □

2.1. **Nonlinear commutator.** The next technical ingredient is a commutator lemma between the non-linear paraproduct of (12) and the standard resonant product. It will be needed below to analyse a term of the form $\Pi_{\prec}(g,h)\circ\Delta\Pi_{\prec}(g,h)$ so we will specialise our discussion to this specific structure. Notice that in the following the various space–time operators act pointwise in the parameter η, in the sense that, for example:

$$(h\circ\Delta h)(\eta,t,x) = (h(\eta,t,\cdot)\circ\Delta h(\eta,t,\cdot))(x).$$

Lemma 6. *We define the map* $\Lambda : C^\infty([0,T],\mathbb{T}^2)\times C^2_\eta C^\infty([0,T],\mathbb{T}^2)\to C^\infty([0,T],\mathbb{T}^2)$ *by*

$$\Lambda(g,h) := [\Pi_{\prec}(g,h)\circ\Delta\Pi_{\prec}(g,h)] - \Pi_\diamondsuit(g,h\circ\Delta h).$$

Then for all $\rho\in(0,1)$, $\gamma\in(0,2)$, $\varepsilon>0$ *such that* $2\gamma-2+\rho-\varepsilon>0$, *we have*

$$\|\Lambda(g,h)\|_{\mathscr{C}^{2\gamma-2+\rho-\varepsilon}_T} \lesssim \|g\|_{\mathscr{L}^\rho_T}\|h\|^2_{C^1_\eta\mathscr{L}^\gamma_T}$$

and

$$\begin{aligned}
\|\Lambda(g_1,h)-\Lambda(g_2,h)\|_{\mathscr{C}^{2\gamma-2+\rho-\varepsilon}_T} \lesssim\ & \|g_1-g_2\|_{C_TL^\infty}\left(\|g_1\|_{\mathscr{L}^\rho_T}+\|g_2\|_{\mathscr{L}^\rho_T}\right)\|h\|^2_{C^2_\eta\mathscr{L}^\gamma_T}\\
&+\|g_1-g_2\|_{\mathscr{L}^\rho_T}\|h\|^2_{C^1_\eta\mathscr{L}^\gamma_T}.
\end{aligned}$$

As a consequence Λ *can be uniquely extended to a locally Lipshitz function*

$$\Lambda : \mathscr{L}^\rho_T\times C^2_\eta\mathscr{L}^\gamma_T\to\mathscr{C}^{2\gamma-2+\rho-\varepsilon}_T.$$

Proof. We let $\eta(t,x) := \Pi_{\prec}(g,h)\circ\Delta\Pi_{\prec}(g,h)(t,x)$. Then

$$\begin{aligned}
\Delta_q\eta(t,x) &= \int_z K_{q,x}(z)(\Pi_{\prec}(g,h)\circ\Delta\Pi_{\prec}(g,h))(t,z)\\
&= \int_z K_{q,x}(z)(\Pi_{\prec}(g(t,z),h)\circ\Delta\Pi_{\prec}(g(t,z),h))(t,z)\\
&\quad+\int_z K_{q,x}(z)((\Pi_{\prec}(g,h)-\Pi_{\prec}(g(t,z),h))\circ\Delta\Pi_{\prec}(g(t,z),h))(t,z) \qquad (13)\\
&\quad+\int_z K_{q,x}(z)(\Pi_{\prec}(g,h)\circ\Delta(\Pi_{\prec}(g,h)-\Pi_{\prec}(g(t,z),h)))(t,z) \qquad (14)
\end{aligned}$$

Now notice that

$$\Delta_j[\Delta\Pi_{\prec}(g(t,z),h)](t,z) = \Delta_j\Pi_{\prec}(g(t,z),\Delta h)(t,z)$$

and that

$$\begin{aligned}(\Pi_{\prec}(g(t,z),h)\circ\Pi_{\prec}(g(t,z),\Delta h))(t,z) &= \sum_{i\sim j}\Delta_i h(g(t,z),t,\cdot)(z)\Delta_j[\Delta h(g(t,z),t,\cdot)](z)\\ &= \Pi_{\diamondsuit}(g,h\circ\Delta h)(t,z)\end{aligned}$$

We proceed therefore to estimate (13) and (14). We obtain

$$\int_z K_{q,x}(z)[(\Pi_{\prec}(g,h)-\Pi_{\prec}(g(t,z),h))\circ\Delta\Pi_{\prec}(g(t,z),h)](t,z)$$

$$=\int_z K_{q,x}(z)\sum_{i\sim j\gtrsim q}(\Delta_i\Pi_{\prec}(g,h)(t,z)-\Delta_i\Pi_{\prec}(g(t,z),h)(t,z))\Delta_j\Pi_{\prec}(g(t,z),\Delta h)(t,z).$$

Using Lemma 4 we have

$$|\Delta_j\Delta\Pi_{\prec}(g(t,z),h)(t,z)|\lesssim 2^{(2-\gamma)j}\|h\|_{C_\eta\mathscr{L}^\gamma_T}.$$

Lemma 7 gives

$$|\Delta_i(\Pi_{\prec}(g,h)-\Pi_{\prec}(g(t,z),h))(t,z)|\lesssim 2^{-(\gamma+\rho-\varepsilon)i}\|g\|_{\mathscr{L}^\rho_T}\|h\|_{C^1_\eta\mathscr{L}^\gamma_T},$$

and thus (13) is bounded by $2^{-(\gamma+\rho-2-\varepsilon)q}\|g\|_{\mathscr{L}^\rho_T}\|h\|^2_{\mathscr{L}^\gamma_T}$.

We can easily obtain (14), using again Lemma 4 and Lemma 7. This proves the first inequality. For the second inequality, Lemma 4 yields

$$|\Delta_j\Delta\Pi_{\prec}(g_1,h)(t,z)-\Delta_j\Delta\Pi_{\prec}(g_2,h)(t,z)|\lesssim 2^{(2-\gamma)j}\|g_1-g_2\|_{C_TL^\infty}\|h\|_{C^1_\eta\mathscr{L}^\gamma_T},$$

and using the second inequality of Lemma 7 we obtain the desired bound.

The extension of Λ to $\mathscr{L}^\rho_T\times C^2_\eta\mathscr{L}^\gamma_T$ is standard (see e.g. the proof of the commutator lemma in [15], Lemma 2.4). □

Lemma 7. *With the same assumptions of Lemma 6 we have*

$$|\Delta_i\Pi_{\prec}(g,h)(t,z)-\Delta_i\Pi_{\prec}(g(t,z),h)(t,z)|\lesssim 2^{(\varepsilon-\rho-\gamma)i}\|g\|_{\mathscr{L}^\rho_T}\|h\|_{C^1_\eta\mathscr{L}^\gamma_T}$$

and

$$\begin{aligned}&|\Delta_i\Pi_{\prec}(g_1,h)(t,z)-\Delta_i\Pi_{\prec}(g_1(t,z),h)(t,z)-\Delta_i\Pi_{\prec}(g_2,h)(t,z)+\Delta_i\Pi_{\prec}(g_2(t,z),h)(t,z)|\\ &\lesssim 2^{-(\rho+\gamma)i}\|g_1-g_2\|_{\mathscr{C}^\rho_T}\|h\|_{C^1_\eta\mathscr{L}^\gamma_T}+2^{(\varepsilon-\rho-\gamma)i}\|g_1-g_2\|_{C_TL^\infty}\left(\|g_1\|_{\mathscr{L}^\rho_T}+\|g_2\|_{\mathscr{L}^\rho_T}\right)\|h\|_{C^2_\eta\mathscr{L}^\gamma_T}\end{aligned}$$

Proof.

$$\begin{aligned}&[\Delta_i\Pi_{\prec}(g,h)-\Delta_i\Pi_{\prec}(g(t,z),h)](t,z)\\ &= \sum_{\substack{k\sim i\\ \ell<k-1}}\int_{x,y,s}K_{i,z}(x)Q_{k,t}(s)K_{\ell,x}(y)[\Delta_k h(g(s,y),t,x)-\Delta_k h(g(t,z),t,x)]\\ &= \sum_{k\sim i}\int_{x,y,s}K_{i,z}(x)Q_{k,t}(s)[\Delta_k h(g(s,x),t,x)-\Delta_k h(g(t,z),t,x)] \qquad (15)\\ &\quad -\sum_{\substack{k\sim i\\ \ell\geqslant k-1}}\int_{x,y,s}K_{i,z}(x)Q_{k,t}(s)K_{\ell,x}(y)[\Delta_k h(g(s,y),t,x)-\Delta_k h(g(t,z),t,x)] \qquad (16)\end{aligned}$$

We use the notation $\delta g^{sy}_{tz}=g(s,y)-g(t,z)$ and $\delta_\tau g^{sy}_{tz}=g(t,z)+\tau(g(s,y)-g(t,z))$ in the estimation of (15) as follows:

$$\left|\sum_{\substack{k\sim i\\ m<i-1}}\int_{w,x,y,s}K_{i,z}(x)K_{m,z}(w)Q_{k,t}(s)[\Delta_k h(g(s,x),t,\cdot)-\Delta_k h(g(t,z),t,\cdot)](x)\right|$$

$$
\begin{aligned}
&= \left| \sum_{\substack{k\sim i\\ m<i-1}} \int_{w,x,y,s,\tau} K_{i,z}(x)K_{m,z}(w)Q_{k,t}(s)\partial_\eta\Delta_k h(\delta_\tau g^{sx}_{tz},t,x)(\delta g^{sw}_{tw}+\delta g^{sz}_{sw}+\delta g^{tw}_{tz}+\delta g^{sx}_{sz}) \right| \\
&\lesssim \sum_{\substack{k\sim i\\ m<i-1}} \int_s |Q_{k,t}(s)||\Delta_m g(s,z)-\Delta_m g(t,z)|2^{-\gamma k}\|h\|_{C^1_\eta \mathscr{C}^\gamma_T} \\
&\quad + \left| \sum_{\substack{k\sim i\\ m\geqslant i-1}} \int_{w,x,y,s,\tau} K_{i,z}(x)K_{m,z}(w)Q_{k,t}(s)\partial_\eta\Delta_k h(\delta_\tau g^{sx}_{tz},t,x)(\delta g^{sz}_{sw}+\delta g^{tw}_{tz}) \right| \\
&\lesssim 2^{-\gamma i}\|h\|_{C^1_\eta \mathscr{C}^\gamma_T}\left(2^{-(\rho-\varepsilon)i}\|g\|_{C^{\rho/2}_T\mathscr{C}^0}+2^{-\rho i}\|g\|_{\mathscr{C}^\rho_T}\right) \lesssim 2^{(\varepsilon-\rho-\gamma)i}\|g\|_{\mathscr{L}^\rho_T} \\
&\lesssim 2^{(\varepsilon-\rho+\gamma)i}\|g\|_{\mathscr{L}^\rho_T}\|h\|_{C^1_\eta \mathscr{C}^\gamma_T} \qquad (17)
\end{aligned}
$$

The norm of (16) can be treated with the same techniques to obtain again the bound (17), and this proves the first inequality. The second one can be obtained in the same way with the techniques already used here and in Lemma 2. □

2.2. **Approximate paradifferential problem.** In this section we construct an approximate solution to the equation

$$(\partial_t - g \prec \Delta)u = f, \qquad u(0,\cdot)=0, \tag{18}$$

with data $f \in \mathscr{C}^{\gamma-2}$ and $g \in \mathscr{L}^\rho_T$, for some fixed $\rho,\gamma \in (0,1)$. The idea is to obtain it via a certain class of paradifferential operators. We introduce the operator $\mathscr{L}$ acting on functions of (η,t,x) by

$$(\mathscr{L}U)(\eta,t,x) := \partial_t U(\eta,t,x) - \eta\Delta U(\eta,t,x) \tag{19}$$

Observe that if u does not depend on η we can define

$$\Pi_\prec(g,\mathscr{L})u := \Pi_\prec(g,\mathscr{L}u). \tag{20}$$

and we obtain $\Pi_\prec(g,\mathscr{L})u = \partial_t u - g\prec\Delta u$.

We can describe the commutation between the differential $\mathscr{L}$ and the paraproduct $\Pi_\prec(g,\cdot)$ via the following estimate:

Lemma 8. *Let $\rho,\gamma\in(0,1)$. Let $U \in C^2_\eta\mathscr{L}^\gamma_T$ such that $\mathscr{L}U \in C^2_\eta\mathscr{C}^{\gamma-2}_T$ and $g\in\mathscr{L}^\rho_T$ such that $g\in[\lambda,1]$. Then*

$$\Psi(g,U) := \Pi_\prec(g,\mathscr{L}U) - \Pi_\prec(g,\mathscr{L})\Pi_\prec(g,U) \in C_T\mathscr{C}^{\rho+\gamma-2-\varepsilon}$$

for every $\varepsilon>0$, with

$$\|\Psi(g,U)\|_{\mathscr{C}^{\rho+\gamma-2-\varepsilon}_T} \lesssim (1+\|g\|_{C_TL^\infty})\|g\|_{\mathscr{L}^\rho_T}\|U\|_{C^1_\eta\mathscr{L}^\gamma_T}. \tag{21}$$

with $\left\|\mathscr{L}_{\eta'}U(\eta)\right\|_{C_{\eta'}C^1_\eta\mathscr{C}^{\gamma-2}} := \sup_{\eta'}\sup_{n=0,1}\sup_{\eta\in[\lambda,1]}\left\|\mathscr{L}_{\eta'}\partial^n_\eta U(\eta)\right\|_{\mathscr{C}^\gamma_T}$.

Moreover, $\Psi(g,U)$ is linear in U and

$$\|\Psi(g_1,U)-\Psi(g_2,U)\|_{\mathscr{C}^{\rho+\gamma-2-\varepsilon}_T} \lesssim \|g_1-g_2\|_{\mathscr{L}^\rho_T}\left(1+\|g_1\|_{\mathscr{L}^\rho_T}+\|g_2\|_{\mathscr{L}^\rho_T}\right)\|U\|_{C^2_\eta\mathscr{L}^\gamma_T}.$$

Proof. Notice that $\Pi_\prec\left(g(t,y),\mathscr{L}_{g(t,y)}U\right) = \mathscr{L}U(g(t,y))$. As a consequence, we can estimate

$$\Pi_\prec(g,\mathscr{L}U)(t,x) - \Pi_\prec(g,\mathscr{L}\Pi_\prec(g,U))(t,x) =$$

$$
\begin{aligned}
&= \Pi_{\prec}(g, \mathscr{L} U)(t,x) - \sum_k \int_y P_{k,x}(y) \left(\mathscr{L}_{g(t,y)} \Delta_k \Pi_{\prec}(g,U)\right)(t,x) \\
&= \Pi_{\prec}(g, \mathscr{L} U)(t,x) - \sum_k \int_y P_{k,x}(y) (\partial_t \Delta_k \Pi_{\prec}(g,U))(t,x) \\
&\quad + \sum_k \int_y P_{k,x}(y) g(t,y) (\Delta\Delta_k \Pi_{\prec}(g,U))(t,x) \\
&= \Pi_{\prec}(g, \mathscr{L} U - \partial_t U)(t,x) + \sum_k \int_y P_{k,x}(y) g(t,y) (\Delta_k \Pi_{\prec}(g, \Delta U))(t,x) \\
&\quad + \sum_k \int P_{k,x}(y) g(t,y) (\Delta_k [\Delta, \Pi_{\prec}(g,\cdot)]U)(t,x) - \sum_k \int P_{k,x}(y) (\Delta_k [\partial_t, \Pi_{\prec}(g,\cdot)]U)(t,x)
\end{aligned}
$$

Now we set

$$
\begin{aligned}
R_1(t,x) &:= \sum_k \int_y P_{k,x}(y) g(t,y) (\Delta_k \Pi_{\prec}(g,\Delta U))(t,x) + \Pi_{\prec}(g, \mathscr{L} U - \partial_t U)(t,x) \\
&= \sum_{k,i} \int_{\substack{y',s \\ y,z}} P_{k,x}(y) K_{k,x}(z) P_{i,z}(y') Q_{i,t}(s) [g(t,y) - g(s,y')] \Delta\Delta_i U(g(s,y'),t,z),
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(t,x) &:= \sum_k \int_y P_{k,x}(y) g(t,y) (\Delta_k [\Delta, \Pi_{\prec}(g,\cdot)]U)(t,x) \\
&\quad - \sum_k \int_y P_{k,x}(y) (\Delta_k [\partial_t, \Pi_{\prec}(g,\cdot)]U)(t,x).
\end{aligned}
$$

With the techniques already used in Lemma 7 we estimate

$$
|\Delta_q R_1(t,x)| \lesssim \sum_{k \sim q} 2^{-\rho k} \left(2^{\varepsilon k} \|g\|_{C_T^{\rho/2} \mathscr{C}^0} + \|g\|_{\mathscr{C}^\rho_T} \right) 2^{(2-\gamma)k} \|U\|_{C_\eta \mathscr{C}^\gamma_T}.
$$

Now notice that

$$
\begin{aligned}
([\partial_t, \Pi_{\prec}(g,\cdot)]U)(t,x) &= \sum_i \int_{y,s} (\partial_t Q_{i,t})(s) P_{i,x}(y) (\Delta_i U(g(s,y),t,x)), \\
([\Delta, \Pi_{\prec}(g,\cdot)]U)(t,x) &= \sum_i \int_{y,s} Q_{i,t}(s) \Delta P_{i,x}(y) (\Delta_i U(g(s,y),t,x)) \\
&\quad + 2 \sum_i \int_{y,s} Q_{i,t}(s) \nabla P_{i,x}(y) (\Delta_i \nabla U(g(s,y),t,x))
\end{aligned} \tag{22}
$$

so we have, by the spectral support properties of the commutators, that

$$
\begin{aligned}
\|[\Delta, \Pi_{\prec}(g,\cdot)]U\|_{\mathscr{C}^{\gamma+\rho-2}_T} &\lesssim \|g\|_{\mathscr{C}^\rho_T} \|U\|_{C^1_\eta \mathscr{C}^\gamma_T}, \\
\|\Delta_q [\partial_t, \Pi_{\prec}(g,\cdot)]U\|_{C_T L^\infty} &\lesssim \left(2^{(2+\varepsilon-\rho-\gamma)q} \|g\|_{C_T^{\rho/2} \mathscr{C}^0} + 2^{(2-\rho-\gamma)q} \|g\|_{\mathscr{C}^\rho_T} \right) \|U\|_{C^1_\eta \mathscr{C}^\gamma_T}.
\end{aligned}
$$

This yields

$$
\|R_2\|_{\mathscr{C}^{\gamma+\rho-2-\varepsilon}_T} \lesssim (1 + \|g\|_{C_T L^\infty}) \|g\|_{\mathscr{L}^\rho_T} \|U\|_{C^1_\eta \mathscr{C}^\gamma_T}.
$$

We completed the proof of (21). The local Lipshitz dependence on g can be obtained via similar computations. □

Remark 9. *If f does not depends on η we consider the parametric problem*

$$
(\partial_t - \eta\Delta) U_f(\eta,t) = f, \qquad U_f(\eta,0) = 0, \qquad \eta \in [\lambda, 1], \tag{23}
$$

which is solved by

$$
U_f(\eta,t) = \int_0^t e^{\eta\Delta(t-s)} f \mathrm{d}s.
$$

Remark that

$$\partial_\eta U_f(\eta,t)=\int_0^t e^{\eta\Delta(t-s)}(t-s)\Delta f\mathrm{d}s \quad \text{and} \quad \partial_\eta^2 U_f(\eta,t)=\int_0^t e^{a\Delta(t-s)}(t-s)^2\Delta^2 f\mathrm{d}s$$

so that we have (since $\eta \geqslant \lambda$):

$$\|U_f\|_{C_\eta^2\mathscr{L}_T^\gamma}:=\sup_{n=0,1,2}\sup_{\eta\in[\lambda,1]}\|\partial_\eta^n U_f(\eta)\|_{\mathscr{L}_T^\gamma}\lesssim\|f\|_{\mathscr{C}_T^{\gamma-2}} \tag{24}$$

We define then

$$u(t,x):=\Pi_{\prec\!\!\prec}(g,U_f)(t,x) \tag{25}$$

and observe that $u(t,x)$ is an approximate solution of equation (18), indeed

$$(\partial_t-g\prec\Delta)u=\Pi_\prec\left(g,\mathscr{L}\,\Pi_{\prec\!\!\prec}(g,U_f)\right)=\Pi_{\prec\!\!\prec}\left(g,\mathscr{L}\,U_f\right)-\Psi(g,U_f)=f-\Psi(g,U_f)$$

and the estimation in Lemma 8 together with the bound (24) yield immediately the following inequality:

$$\|\Psi(g,U_f)\|_{\mathscr{C}_T^{\rho+\gamma-2-\varepsilon}}\lesssim\|g\|_{\mathscr{L}_T^\rho}(1+\|g\|_{C_TL^\infty})\|f\|_{\mathscr{C}_T^{\gamma-2}}. \tag{26}$$

3. Paracontrolled Ansatz

In order to give a meaning to the PDE in (5) with initial condition $u_0\in\mathscr{C}^\alpha$, our initial goal will be to get informations on solutions $\theta=\theta(g)$ of the equation

$$\partial_t\theta-g\prec\Delta\theta=\xi,$$

for a fixed $g\in\mathscr{C}_T^\alpha$, $2/3<\alpha<1$. Using the results of Section 2.2, we consider to this effect the parametric problem

$$(\partial_t-\eta\Delta)\vartheta(\eta,t)=\xi,$$

for $\eta\in[\lambda,1]$. We will consider the stationary solution of this problem which has the form

$$\vartheta(\eta,x)=\int_0^\infty e^{-\eta\Delta s}\xi\mathrm{d}s=(-\eta\Delta)^{-1}\xi \tag{27}$$

and in order to have this well defined we impose that the noise ξ has zero mean on $\mathbb{T}^2$ (this is a simplifying assumption which can be easily removed, e.g. at the price of adding a linear term to the equation). It is easy to see that

$$\|\vartheta\|_{C_\eta^2\mathscr{L}_T^\alpha}=\|\vartheta\|_{C_\eta^2\mathscr{C}_T^\alpha}\lesssim\|\xi\|_{\mathscr{C}^{\alpha-2}}. \tag{28}$$

We define now for every $t\in[0,T]$

$$\theta(t,x):=\Pi_{\prec\!\!\prec}(a(u),\vartheta).$$

Thanks to Lemma 4 we have the bound $\|\theta\|_{\mathscr{L}_T^\alpha}\lesssim\|\vartheta\|_{C_\eta\mathscr{L}_T^\alpha}\lesssim\|\xi\|_{\mathscr{C}^{\alpha-2}}$. We observe that this definition together with Lemma 8 gives

$$\partial_t\theta-a(u)\prec\Delta\theta=\xi-\Psi(a(u),\vartheta)$$

with $\|\Psi(a(u),\vartheta)\|_{\mathscr{C}_T^{2\alpha-2-\varepsilon}}\lesssim\|a(u)\|^2_{\mathscr{L}_T^\alpha}\|\xi\|_{\mathscr{C}_T^{\alpha-2}}$. We expect then $\Psi(a(u),\vartheta)$ to be bounded in $\mathscr{C}_T^{2\alpha-2-\varepsilon}$ for any $\varepsilon>0$. At this point let us introduce the Ansatz

$$u=\theta+u^\sharp \tag{29}$$

and observe that

$$\partial_t u-a(u)\prec\Delta u=(\partial_t-a(u)\prec\Delta)\theta+(\partial_t-a(u)\prec\Delta)u^\sharp=\xi+(\partial_t-a(u)\prec\Delta)u^\sharp-\Psi(a(u),\vartheta).$$

Recalling (5), $u^\sharp$ must solve the equation

$$\begin{cases}(\partial_t-a(u)\prec\Delta)u^\sharp=\Phi(u)+\Psi(a(u),\vartheta)\\ u^\sharp(t=0)=u_0^\sharp:=u_0-\Pi_{\prec\!\!\prec}(a(u_0),\vartheta)(t=0)\in\mathscr{C}^\alpha\end{cases} \tag{30}$$

with $\Phi(u)=a(u)\circ\Delta u+a(u)\succ\Delta u$, and if we can make sense of the resonant term $a(u)\circ\Delta u$, it is reasonable to expect $u^\sharp(t,\cdot)\in\mathscr{C}^{2\alpha}\ \forall t\in(0,T]$. Indeed, take $U^\sharp:=U_Q$ to be the solution of

$$(\partial_t-\eta\Delta)U^\sharp(\eta,t)=Q\qquad U^\sharp(\eta,0)=0 \tag{31}$$

for some $Q = Q(u^\sharp)$ to be determined and $\eta \in [\lambda, 1]$. Using again Lemma 8 we have

$$(\partial_t - a(u) \prec \Delta)\Pi_{\prec}(a(u), U^\sharp) = Q(u^\sharp) - \Psi(a(u), U^\sharp).$$

For $\eta \in [\lambda, 1]$we define $\mathscr{P}_t u_0^\sharp(\eta) := e^{\eta \Delta t} u_0^\sharp$ so that $\mathscr{L}\left(\mathscr{P}_t u_0^\sharp\right) = 0$, with $\mathscr{L}$ as in (19).

We set

$$u^\sharp := \Pi_{\prec}(a(u), U^\sharp) + \Pi_{\prec}\left(a(u), \mathscr{P} u_0^\sharp\right). \tag{32}$$

Taking

$$Q(u^\sharp) := \Phi(u) + \Psi(a(u), \vartheta) + \Psi(a(u), U^\sharp) + \Psi\left(a(u), \mathscr{P} u_0^\sharp\right),$$

we obtain that $U^\sharp$ solves equation (31) if and only if $u^\sharp$ solves equation (30). As we will see, $Q(u^\sharp)(t)$ belongs to $\mathscr{C}^{2\alpha-2}$ $\forall t \in (0, T]$ but not uniformly as $t \to 0$. However it belongs to $\mathscr{C}^{\alpha-2}$ uniformly as $t \to 0$.

It remains to control the resonant term $a(u) \circ \Delta u$ appearing in $\Phi(u)$. We have

$$a(u) \circ \Delta u = a(u) \circ \Delta\theta + a(u) \circ \Delta u^\sharp.$$

By paralinearization (see Theorem 22) $a(u) = a'(u) \prec u + R_a(u)$ with $\|R_a(u)\|_{\mathscr{C}_T^{2\alpha}} \lesssim 1 + \|u\|^2_{\mathscr{C}_T^{\alpha}}$, and then

$$a(u) \circ \Delta\theta = (a'(u) \prec u) \circ \Delta\theta + R_a(u) \circ \Delta\theta.$$

In order to use the commutator lemma (Lemma 23) we can estimate $a'(u)$, recalling that $\alpha \in (0, 1)$, as

$$\|a'(u)\|_{\mathscr{C}_T^{\alpha}} \lesssim \|a''\|_{L^\infty} \|u\|_{\mathscr{C}_T^{\alpha}}$$

and write

$$a(u) \circ \Delta\theta = a'(u)(u \circ \Delta\theta) + C(a'(u), u, \Delta\theta) + R_a(u) \circ \Delta\theta.$$

Then, Ansatz (29) gives

$$a(u) \circ \Delta\theta = a'(u)(\theta \circ \Delta\theta) + a'(u)(u^\sharp \circ \Delta\theta) + C(a'(u), u, \Delta\theta) + R_a(u) \circ \Delta\theta.$$

Summarizing, we have:

$$\begin{aligned} \Phi(u) \;=\; & a'(u)(\theta \circ \Delta\theta) + a(u) \succ \Delta u + a'(u)(u^\sharp \circ \Delta\theta) \\ & + C(a'(u), u, \Delta\theta) + R_a(u) \circ \Delta\theta + a(u) \circ \Delta u^\sharp \end{aligned}$$

Thanks to the nonlinear commutator (Lemma 6), we can decompose the resonant term $\theta \circ \Delta\theta$ to obtain

$$\begin{aligned} \Phi(u) \;=\; & a(u) \succ \Delta u + a'(u)(u^\sharp \circ \Delta\theta) + C(a'(u), u, \Delta\theta) + R_a(u) \circ \Delta\theta + a(u) \circ \Delta u^\sharp \\ & + a'(u)\Lambda(a(u), \vartheta) + a'(u)\Pi_{\diamondsuit}(a(u), \Theta_2) \end{aligned}$$

and $\Lambda(a(u), \vartheta) \in \mathscr{C}_T^{3\alpha-2-\varepsilon}$ if $u \in \mathscr{L}_T^{\alpha}$. Here we defined

$$\Theta_2(\eta, x) := (\vartheta \circ \Delta\vartheta)(\eta, x) = \sum_{i \sim j} \Delta_i \vartheta(\eta, \cdot)(x) \Delta_j [\Delta\vartheta(\eta, \cdot)](x) \tag{33}$$

Finally, recalling the decomposition of $u^\sharp$ in two terms (32) we obtain

$$\Phi(u) = a'(u)\Pi_{\diamondsuit}(a(u), \Theta_2) + \Phi_1(u) + \Phi_2(u)$$

where

$$\begin{aligned} \Phi_1(u) \;:=\; & a(u) \succ \Delta u + C(a'(u), u, \Delta\theta) + R_a(u) \circ \Delta\theta + a'(u)\Lambda(a(u), \vartheta) \\ & + a'(u)(\Pi_{\prec}(a(u), U^\sharp) \circ \Delta\theta) + a(u) \circ \Delta\Pi_{\prec}(a(u), U^\sharp), \\ \Phi_2(u) \;:=\; & a'(u)\left(\Pi_{\prec}\left(a(u), \mathscr{P} u_0^\sharp\right) \circ \Delta\theta\right) + a(u) \circ \Delta\Pi_{\prec}\left(a(u), \mathscr{P} u_0^\sharp\right). \end{aligned}$$

Thanks to Lemma 2 the terms $a'(u)\Pi_{\diamondsuit}(a(u), \Theta_2)$ and $\Phi_1(u)$ can be estimated in $\mathscr{C}_T^{2\alpha-2}$, provided $\Theta_2 \in C^2_\eta \mathscr{C}_T^{2\alpha-2}$ (see Section 5). On the other hand the term $\Phi_2(u)(t)$ can be estimated in $\mathscr{C}^{2\alpha-2}$ only for strictly positive times $t > 0$ due to the lack of regularity of the initial condition $u_0^\sharp$ which a priori lives only in $\mathscr{C}^{\alpha}$.

Note moreover that the specific form of Φ allows to deduce that if we replace Θ_2 by $\tilde{\Theta}_2 = \Theta_2 + H$ with $H \in C_\eta^2 \mathscr{C}_T^{2\alpha-2}$ then this is equivalent to consider an equation for u of the form

$$\partial_t u(t,x) - a(u(t,x))\Delta u(t,x) = \xi(x) + a'(u(t,x))H(a(u(t,x)),t,x).$$

Let us resume this long discussion in the following theorem:

Theorem 10. *Assume that $\xi \in \mathscr{C}^0, u_0 \in \mathscr{C}^2, H \in C_\eta^2 \mathscr{C}_T^0$. $u \in C_T^1 \mathscr{C}^2$ is the classical solution to the equation*

$$\partial_t u - a(u)\Delta u = \xi + a'(u)h(a(u)), \qquad u(0) = u_0,$$

up to time $T > 0$ if and only if

$$u = \Pi_{\prec}\left(a(u), \vartheta + U^\sharp + \mathscr{P} u_0^\sharp\right),$$

where ϑ is the solution to eq. (27) and $U^\sharp$ is the solution to the PDE

$$(\partial_t - \eta\Delta)U^\sharp(\eta) = F(u, U^\sharp, u_0^\sharp) \qquad U^\sharp(\eta,0) = 0 \qquad \eta \in [\lambda, 1] \tag{34}$$

with

$$\begin{aligned} F(u, U^\sharp, u_0) &= a'(u)\Pi_\diamondsuit(a(u), \Theta_2) + \Phi_1(u) + \Phi_2(u) + \Psi(a(u), \vartheta) + \Psi(a(u), U^\sharp) \\ &\quad + \Psi\left(a(u), \mathscr{P} u_0^\sharp\right) \end{aligned}$$

and $\Theta_2 = \vartheta \circ \Delta\vartheta + H$.

Definition 11. *For any $\alpha \in \mathbb{R}$ we define $\mathcal{X}^\alpha \subseteq C_\eta^2 \mathscr{C}^\alpha \times C_\eta^2 \mathscr{C}^{2\alpha-2}$ the closure of the image of the map*

$$(\rho, H) \in C_\eta^2 \mathscr{C}^2 \times C_\eta^2 \mathscr{C}^0 \mapsto J(\rho, H) = (\rho, \rho \circ \Delta\rho + H) \in C_\eta^2 \mathscr{C}^2 \times C_\eta^2 \mathscr{C}^0$$

(in the topology of $C_\eta^2 \mathscr{C}^\alpha \times C_\eta^2 \mathscr{C}^{2\alpha-2}$).

We call the elements in $\mathcal{X}^\alpha$ *enhanced noises*. In the next section we will exploit the space $\mathcal{X}^{\alpha'}$ for $2/3 < \alpha < \alpha' < 1$ to solve equation (34).

4. Local wellposedness

The main result of this section is the local well–posedness of the PDE (34) when $(\vartheta, \Theta_2) \in \mathcal{X}^{\alpha'}$ and $u_0 \in \mathscr{C}^\alpha$ for $2/3 < \alpha < \alpha' < 1$.

Theorem 12. *Let $\alpha > 2/3$. Then for any $(\vartheta, \Theta_2) \in \mathcal{X}^{\alpha'}$ with $\alpha < \alpha' < 1$ and $u_0 \in \mathscr{C}^\alpha$ there exists a time $T > 0$ depending only on $\|(\vartheta, \Theta_2)\|_{\mathcal{X}^{\alpha'}}$ and $\|u_0\|_\alpha$ up to which equation (34) has a unique solution $(u, U^\sharp) \in \mathscr{L}_T^\alpha \times C_\eta^2 \mathscr{L}_T^{2\delta}$ for all $\delta < \alpha$ such that $2\delta + \alpha > 1$. For any fixed $\tau > 0$ there exist a ball $B_\tau \subseteq \mathscr{C}^\alpha \times \mathcal{X}^{\alpha'}$ such that the solution map*

$$\Sigma_\tau : (u_0, \vartheta, \Theta_2) \in B_\tau \mapsto (u, U^\sharp) \in \mathscr{L}_\tau^\alpha \times C_\eta^2 \mathscr{L}_\tau^{2\delta}$$

is well defined and Lipshitz continuous in the data.

Proof. Let $\mathcal{G}_T = \mathscr{L}_T^\alpha \times C_\eta^2 \mathscr{L}_T^{2\delta}$. We introduce the map

$$\Gamma : (u, U^\sharp) \in \mathcal{G}_T \mapsto (\Gamma_u(u, U^\sharp), \Gamma_{U^\sharp}(u, U^\sharp)) \in \mathcal{G}_T$$

by

$$\Gamma_u(u, U^\sharp) := \Pi_{\prec}(a(u), \vartheta) + \Pi_{\prec}(a(u), \Gamma_{U^\sharp}(u, U^\sharp)) + \Pi_{\prec}\left(a(u), \mathscr{P} u_0^\sharp\right)$$

and

$$(\partial_t - \eta\Delta)\Gamma_{U^\sharp}(u, U^\sharp)(\eta) = F(u, U^\sharp, u_0^\sharp), \qquad \Gamma_{U^\sharp}(u, U^\sharp)(\eta)(0) = 0, \qquad \eta \in [\lambda, 1],$$

where we understand that $u^\sharp = u^\sharp(u, U^\sharp, u_0^\sharp) = \Pi_{\prec}(a(u), U^\sharp) + \Pi_{\prec}\left(a(u), \mathscr{P} u_0^\sharp\right)$. We will establish that this map is a contraction in the space $\mathcal{G}_T$. First, we have to show that there exists a ball $B \subset \mathcal{G}_T$ such that $\Gamma(B) \subseteq B$. We have the bound $\left\|\mathscr{P} u_0^\sharp\right\|_{C_\eta^1 \mathscr{L}_T^\alpha} \lesssim \|u_0^\sharp\|_{\mathscr{C}_T^\alpha}$.

It is easy to obtain, using the estimates of Section 2:

$$\left\| \int_0^T e^{-\eta\Delta(t-s)} \left[\Phi_1(u) + \Psi(a(u),\vartheta) + \Psi(a(u),U^\sharp) + \Psi\left(a(u), \mathscr{P} u_0^\sharp\right) \right]_s \right\|_{C_\eta^2 \mathscr{L}_T^{2\delta}}$$
$$\lesssim T^\kappa \left(1 + \|u\|_{\mathscr{L}_T^\alpha}\right)^4 \left(1 + \|\xi\|_{\mathscr{C}^{\alpha'-2}}\right)^2 \|u_0^\sharp\|_{\mathscr{C}^\alpha} \left(1 + \|U^\sharp\|_{C_\eta^2 \mathscr{L}_T^{2\delta}}\right)$$

for some $\kappa > 0$.

By the assumption that $(\vartheta, \Theta_2) \in \mathcal{X}^{\alpha'}$ we deduce that there exists $M > 0$ such that $\|\Theta_2\|_{C_\eta^2 \mathscr{C}_T^{2\alpha'-2}} \leqslant M$. We have

$$\left\| \int_0^T e^{-\eta\Delta(t-s)} [a'(u)\Pi_\diamondsuit(a(u),\Theta_2)]_s \right\|_{C_\eta^2 \mathscr{L}_T^{2\delta}} \lesssim T^{\alpha-\delta} \left(1 + \|u\|_{\mathscr{C}_T^\alpha}\right)^2 \|\Theta_2\|_{C_\eta^2 \mathscr{C}_T^{2\alpha'-2}}.$$

To bound the term $\Phi_2(u)$ we observe that $\left\| \mathscr{P}_t u_0^\sharp \right\|_{C_\eta^2 \mathscr{C}^{2\alpha}} \lesssim t^{-\frac{\alpha}{2}} \|u_0^\sharp\|_{\mathscr{C}^\alpha}$. This gives, thanks to Lemma 19,

$$\left\| \int_0^T e^{-\eta\Delta(t-s)} \Phi_2(u)_s \right\|_{C_\eta^2 \mathscr{L}_T^{2\delta}} \lesssim T^{\alpha-\delta} \left(1 + \|u\|_{\mathscr{C}_T^\alpha}\right) \left(1 + \|\xi\|_{\mathscr{C}^{\alpha'-2}}\right) \|u_0^\sharp\|_{\mathscr{C}^\alpha}$$

and then $\Gamma_{U^\sharp}(u, U^\sharp)$ is bounded in $C_\eta^2 \mathscr{L}_T^{2\delta}$ for T small enough. We have also

$$\begin{aligned} \|\Gamma_u(u,U^\sharp)\|_{\mathscr{L}_T^\alpha} &\lesssim \|\xi\|_{\mathscr{C}^{\alpha'-2}} + \|u_0^\sharp\|_{\mathscr{C}^\alpha} + \|\Gamma_{U^\sharp}(u,U^\sharp)\|_{C_\eta \mathscr{L}_T^\alpha} \\ &\lesssim \|\xi\|_{\mathscr{C}^{\alpha'-2}} + \|u_0^\sharp\|_{\mathscr{C}^\alpha} + T^{\frac{2\delta-\alpha}{2}} \|\Gamma_{U^\sharp}(u,U^\sharp)\|_{C_\eta^2 \mathscr{L}_T^{2\delta}} \end{aligned}$$

and these bounds show that $\Gamma(B) \subseteq B$. The contractivity of $\Gamma_{U^\sharp}(u, U^\sharp)$ can be obtained in the same way. Now consider $\Gamma_u(u, U^\sharp)$: we have

$$\|\Pi_\prec(a(u_1), U_1^\sharp) - \Pi_\prec(a(u_2), U_2^\sharp)\|_{\mathscr{L}_T^\alpha}$$
$$\lesssim T^{\frac{2\delta-\alpha}{2}} \left(\|U_1^\sharp - U_2^\sharp\|_{C_\eta \mathscr{L}^{2\delta}} + \|u_1 - u_2\|_{C_T L^\infty} \|U_2^\sharp\|_{C_\eta^1 \mathscr{L}^{2\delta}} \right)$$

while for the other terms in $\Gamma_u(u_1, U_1^\sharp) - \Gamma_u(u_2, U_2^\sharp)$ we remark that

$$\sup_{s\in[0,t]} \|u_{1,s} - u_0 - u_{2,s} + u_0\|_{\mathscr{C}^0} \lesssim t^{\alpha/2} \|u_1 - u_2\|_{C_{[0,t]}^{\alpha/2} \mathscr{C}^0}$$

and use Lemma 5 to estimate

$$\begin{aligned} \|\delta[\Pi_\prec(a(u),\vartheta)]_2^1\|_{\mathscr{L}_T^\alpha} &\lesssim \|a(u_1) - a(u_2)\|_{\mathscr{C}_T^0} \left(\|\vartheta\|_{C_\eta^1 C_T^{\alpha/2} \mathscr{C}^{\alpha'-\alpha}} + \|\vartheta\|_{C_\eta^1 \mathscr{C}_T^{\alpha'}} \right) \\ &\lesssim M \|u_1 - u_2\|_{\mathscr{C}_T^0} \|\xi\|_{\mathscr{C}^{\alpha'-2}} \\ &\lesssim T^{\alpha/2} M \|u_1 - u_2\|_{C_T^{\alpha/2} \mathscr{C}^0} \|\xi\|_{\mathscr{C}^{\alpha'-2}}. \end{aligned}$$

where $M := \left(1 + \|u_1\|_{\mathscr{C}_T^\alpha} + \|u_2\|_{\mathscr{C}_T^\alpha}\right)$. With the same reasoning we estimate

$$\begin{aligned} \left\| \delta \left[\Pi_\prec \left(a(u), \mathscr{P} u_0^\sharp \right) \right]_2^1 \right\|_{\mathscr{C}_T^\alpha} &\lesssim \sup_{t\in[0,T]} \left\| \mathscr{P}_t u_0^\sharp \right\|_{C_1^\eta \mathscr{C}^{\alpha'}} \sup_{s\in[0,t]} \|a(u_{1,s}) - a(u_{2,s})\|_{\mathscr{C}^0} \\ &\lesssim M \|u_0^\sharp\|_{\mathscr{C}^\alpha} \sup_{t\in[0,T]} t^{\frac{\alpha-\alpha'}{2}} \sup_{s\in[0,t]} \|u_{1,s} - u_{2,s}\|_{\mathscr{C}^0} \\ &\lesssim T^{\frac{2\alpha-\alpha'}{2}} M \|u_0^\sharp\|_{\mathscr{C}^\alpha} \|u_1 - u_2\|_{C_T^{\alpha/2} \mathscr{C}^0} \end{aligned}$$

and for $\left\| \delta \left[\Pi_\prec \left(a(u), \mathscr{P} u_0^\sharp \right) \right]_2^1 \right\|_{C_T^{\alpha/2} \mathscr{C}^0}$ we take w.l.o.g. $t_1 < t_2$ and proceed as in the proof of Lemma 5 to obtain, for $0 < \varepsilon < \alpha$:

$$\left\| \delta \left[\Pi_\prec \left(a(u), \mathscr{P} u_0^\sharp \right) \right]_2^1 (t_2) - \delta \left[\Pi_\prec \left(a(u), \mathscr{P} u_0^\sharp \right) \right]_2^1 (t_1) \right\|_{\mathscr{C}^0}$$

$$
\begin{aligned}
\lesssim\ & \left\| \mathscr{P}_{t_2} u_0^\sharp - \mathscr{P}_{t_1} u_0^\sharp \right\|_{C_\eta^1 \mathscr{C}^{\,\varepsilon}} \sup_{s\in[0,t_1]} \|a(u_{1,s}) - a(u_{2,s})\|_{\mathscr{C}^0} \\
& + |t_2 - t_1|^{\alpha/2} \left\| \mathscr{P}_{t_2} u_0^\sharp \right\|_{C_\eta^1 \mathscr{C}^{\,\alpha+\varepsilon}} \sup_{s\in[0,t_2]} \|a(u_{1,s}) - a(u_{2,s})\|_{\mathscr{C}^0} \\
\lesssim\ & \left(1 + \|u_1\|_{\mathscr{C}^{\,\alpha}_{[0,t_1]}} + \|u_2\|_{\mathscr{C}^{\,\alpha}_{[0,t_1]}}\right) |t_2 - t_1|^{\alpha/2} t_1^{\alpha/2-\varepsilon/2} \|u_0^\sharp\|_{C_\eta^1 \mathscr{C}^{\,\alpha}} \|u_1 - u_2\|_{C^{\alpha/2}_{[0,t_1]} \mathscr{C}^0} \\
& + \left(1 + \|u_1\|_{\mathscr{C}^{\,\alpha}_{[0,t_2]}} + \|u_2\|_{\mathscr{C}^{\,\alpha}_{[0,t_2]}}\right) |t_2 - t_1|^{\alpha/2} t_2^{\alpha/2-\varepsilon} \|u_0^\sharp\|_{C_\eta^1 \mathscr{C}^{\,\alpha}} \|u_1 - u_2\|_{C^{\alpha/2}_{[0,t_2]} \mathscr{C}^0}.
\end{aligned}
$$

This yields

$$
\left\| \delta \left[\Pi_{\prec} \left(a(u), \mathscr{P} u_0^\sharp \right) \right]_2^1 \right\|_{C_T^{\alpha/2} \mathscr{C}^0} \lesssim T^\kappa \left(1 + \|u_1\|_{\mathscr{C}^{\,\alpha}_T} + \|u_2\|_{\mathscr{C}^{\,\alpha}_T}\right) \|u_0^\sharp\|_{\mathscr{C}^\alpha} \|u_1 - u_2\|_{C_T^{\alpha/2} \mathscr{C}^0},
$$

for some $\kappa > 0$. We have shown that Γ is a contraction for small times.

The uniqueness of the solution $(u, U^\sharp) \in \mathscr{L}^{\,\alpha}_T \times C^2_\eta \mathscr{L}^{\,2\delta}_T$ and the Lipshitz continuity of the localized solution map Σ_τ can be proved along the same lines via standard arguments. □

5. Renormalization

At this point we want to construct *an* enhanced noise Ξ associated to the white noise ξ. Already in the standard setting of the generalised PAM model with constant diffusion matrix, the construction of the enhancement requires a renormalization since the resonant product $\vartheta \circ \Delta\vartheta$ is not well defined.

Let $\psi \in \mathcal{S}(\mathbb{T}^2)$ be a cutoff function and let $\psi_\varepsilon(x) = \varepsilon^{-2}\psi(x/\varepsilon)$. Then define a regularised noise by $\xi_\varepsilon = \psi_\varepsilon * \xi$ and let $\vartheta_\varepsilon = (-\eta\Delta)^{-1}\xi_\varepsilon$. Notice that

$$
H_\varepsilon(\eta) := \mathbb{E}[\vartheta_\varepsilon(\eta, x) \circ \Delta\vartheta_\varepsilon(\eta, x)] = \mathbb{E}[\vartheta_\varepsilon(\eta, x)\Delta\vartheta_\varepsilon(\eta, x)] = -\sum_{k\in\mathbb{Z}^2} \frac{\hat\psi_\varepsilon(k)^2}{\eta^2 k^2} = -\frac{\sigma_\varepsilon}{\eta^2}
$$

where

$$
\sigma_\varepsilon := \sum_{k\in\mathbb{Z}^2} \frac{\hat\psi_\varepsilon(k)^2}{k^2} \simeq |\log\varepsilon|
$$

as $\varepsilon \to 0$. Subtracting this diverging quantity to $\vartheta_\varepsilon \circ \Delta\vartheta_\varepsilon$ and then taking the limit as $\varepsilon \to 0$ delivers a finite result.

Theorem 13. *Take $\alpha < 1$ and let $\Xi_\varepsilon = (\xi_\varepsilon, \Xi_{2,\varepsilon}) := (\xi_\varepsilon, \vartheta_\varepsilon \circ \Delta\vartheta_\varepsilon - H_\varepsilon)$. Then the family $(\Xi_\varepsilon)_\varepsilon \subseteq \mathcal{X}^\alpha$ converges a.s. and in L^p to a random element $\Xi = (\xi, \Xi_2) \in \mathcal{X}^\alpha$.*

Proof. The proof is a mild modification of the proof for PAM [15]. In order to establish the required $C^2_\eta \mathscr{C}^{\,2\alpha-2}_T$ regularity for Ξ_2 we follow the computations for the case where the diffusion coefficient is constant. We have only to discuss the additional regularity in the η parameter. In order to do so observe that

$$
\Xi_{2,\varepsilon}(\eta) = \sum_{i\sim j} [\![\Delta_i \vartheta_\varepsilon(\eta) \Delta_j \Delta\vartheta_\varepsilon(\eta)]\!]
$$

where $[\![\,]\!]$ denotes the Wick product with respect to the Gaussian structure of ξ. Then we have

$$
\partial_\eta \Xi_{2,\varepsilon}(\eta) = \sum_{i\sim j} [\![\Delta_i \partial_\eta\vartheta_\varepsilon(\eta) \Delta_j \Delta\vartheta_\varepsilon(\eta)]\!] + \sum_{i\sim j} [\![\Delta_i \vartheta_\varepsilon(\eta) \Delta_j \Delta\partial_\eta\vartheta_\varepsilon(\eta)]\!],
$$

and

$$
\begin{aligned}
\partial_\eta^2 \Xi_{2,\varepsilon}(\eta) = & \sum_{i\sim j} [\![\Delta_i \partial_\eta^2\vartheta_\varepsilon(\eta) \Delta_j \Delta\vartheta_\varepsilon(\eta)]\!] + \sum_{i\sim j} [\![\Delta_i \vartheta_\varepsilon(\eta) \Delta_j \Delta\partial_\eta^2\vartheta_\varepsilon(\eta)]\!] \\
& + \sum_{i\sim j} 2 [\![\Delta_i \partial_\eta\vartheta_\varepsilon(\eta) \Delta_j \Delta\partial_\eta\vartheta_\varepsilon(\eta)]\!].
\end{aligned}
$$

Now the computations relative to the regularities of these additional stochastic objects are equivalent to those for the term $\Xi_{2,\varepsilon}$ where one or two instances of $\vartheta_\varepsilon(\eta)$ are replaced by

Gaussian fields of similar regularities of the form $\partial_\eta \vartheta_\varepsilon(\eta)$ and $\partial_\eta^2 \vartheta_\varepsilon(\eta)$, a direct inspection of the proof allows us to deduce that we have almost sure $\mathscr{C}^{2\alpha-2}$ regularity for these terms and also for random fields $\partial_\eta^n \Xi_{2,\varepsilon}$ for any finite n. This allows also to deduce that the random field is a.s. smooth in the parameter η. Similar computations allows to prove continuity in ε for $\varepsilon > 0$. The rest of the proof is standard. □

In conclusion we see that in order to be able to use this convergence result we need to modify our approximate PDE and consider instead

$$\partial_t u_\varepsilon - a(u_\varepsilon)\Delta u_\varepsilon = \xi_\varepsilon - a'(u_\varepsilon)H_\varepsilon(a(u_\varepsilon)).$$

Our well–posedness results for the paracontrolled formulation of this equation together with the convergence result in Theorem 13 allows to deduce that $u_\varepsilon \to u$ in $\mathscr{C}_T^\delta$ for any $2/3 < \delta < \alpha < 1$ and that the limiting process u satisfies a modified version of eq. (1), namely

$$\partial_t u - a(u) \diamond \Delta u = \xi, \qquad u(0) = u_0,$$

where $a(u) \diamond \Delta u$ denotes a *renormalized* diffusion term given by

$$a(u) \diamond \Delta u := a(u) \prec \Delta u + a'(u)\Pi_\diamond(a(u), \Xi_2) + \Phi_1(u) + \Phi_2(u). \tag{35}$$

6. Nonlinear source terms

Let us start by discussing the presence of a u dependent r.h.s. in eq. (1). We want to solve

$$\partial_t u - a_1(u)\Delta u = a_2(u)\xi$$

where a_1 is a non-linear diffusion coefficient as before and $a_2 : \mathbb{R} \to \mathbb{R}$ is another bounded function with sufficiently many bounded derivatives. We rewrite this equation as

$$\Pi_\prec\left(a(u), \mathscr{L}\right) u = a_2(u) \prec \xi + a_1(u) \circ \Delta u + a_2(u) \circ \xi + a_1(u) \succ \Delta u + a_2(u) \succ \xi$$

where now $a(u) = (a_1(u), a_2(u))$ is a vector valued non-linearity. Since we don't need u to depend on any parameter $\eta = (\eta_1, \eta_2)$, we have defined $\mathscr{L}$ as

$$\mathscr{L}(\eta) := \partial_t - \eta_1 \Delta$$

and used the identity $\Pi_\prec\left(a(u), \mathscr{L}\right) u = (\partial_t - a_1(u) \prec \Delta)u$, similarly to what we have done in (20).

Notice that the non–linear paraproduct can be extended trivially to the vector valued case in such a way that, for example,

$$\Pi_{\prec\!\!\prec}((g_1, g_2), h)(t, x) = \sum_i \int_{y,s} Q_{i,t}(s) P_{i,x}(y) (\Delta_i h((g_1(s, y), g_2(s, y)), t, \cdot))(x).$$

As before we make the Ansatz

$$u = \Pi_{\prec\!\!\prec}(a(u), \vartheta) + u^\sharp$$

where now ϑ solves

$$\mathscr{L}(\eta)\vartheta(\eta, x) = (\partial_t - \eta_1\Delta)\vartheta(\eta, x) = \eta_2 \xi(x),$$

for $\eta = (\eta_1, \eta_2) \in [\lambda, 1] \times [-L, L]$ where L is a large but fixed constant. The bounded domain is important to be able to have uniform estimates and reuse the estimates proved above in the simple situation of $\eta_2 = 1$. The solution of this equation is

$$\vartheta(\eta, \cdot) = \eta_2 \int_0^\infty e^{-\eta_1 \Delta} \xi \mathrm{d}s.$$

Observe that

$$\Pi_\prec\left(a(u), \mathscr{L}\right) u = \Pi_\prec\left(a(u), \mathscr{L}\right) \Pi_{\prec\!\!\prec}(a(u), \vartheta) + \Pi_\prec\left(a(u), \mathscr{L}\right) u^\sharp$$

and recall that (Lemma 8) $\Pi_\prec\left(a(u), \mathscr{L}\right) \Pi_{\prec\!\!\prec}(a(u), \vartheta) = \Pi_{\prec\!\!\prec}\left(a(u), \mathscr{L}\,\vartheta\right) + \Psi(a(u), \vartheta)$. Now

$$\left(\mathscr{L}\,\vartheta\right)(\eta) = (\partial_t - \eta_1\Delta)\vartheta(\eta, t, x) = \Xi(\eta), \qquad \eta = (\eta_1, \eta_2) \in [\lambda, 1] \times [-L, L]$$

with $\Xi(\eta)(t, x) = \eta_2 \xi(x)$ and then

$$\Pi_{\prec\!\!\prec}\left(a(u), \mathscr{L}\,\vartheta\right) = \Pi_{\prec\!\!\prec}(a(u), \Xi) = a_2(u) \prec\!\!\prec \xi.$$

In conclusion

$$\Pi_{\prec}\left(a(u),\mathscr{L}\,\right)\Pi_{\prec\!\!\prec}(a(u),\vartheta)=a_2(u)\prec\!\!\prec\xi+\Psi(a(u),\vartheta)$$

and the equation for $u^\sharp$ reads

$$\begin{aligned}\Pi_{\prec}\left(a(u),\mathscr{L}\,u^\sharp\right) &= a_1(u)\circ\Delta u+a_2(u)\circ\xi+[a_2(u)\prec\xi-a_2(u)\prec\!\!\prec\xi]\\ &\quad+a_1(u)\succ\Delta u+a_2(u)\succ\xi\\ &\quad-\Psi(a(u),\vartheta)\end{aligned}$$

where now all the terms on the r.h.s. can be treated as remainder terms. Let us just remark that the commutation term $a_2(u)\prec\xi-a_2(u)\prec\!\!\prec\xi$ has a standard treatment via Lemma 24. Of course, the first two terms require to be treated as resonant terms. Note that, modulo terms of order $\mathscr{C}_T^{3\alpha-2}$ (or $\mathcal{E}^{\alpha/2}\mathscr{C}^{2\alpha-2}$) the terms $a_1(u)\circ\Delta u+a_2(u)\circ\xi$ are equivalent to

$$a_1'(u)\Pi_{\diamondsuit}(a(u),\vartheta\circ\Delta\vartheta)+a_2'(u)(\Pi_{\prec\!\!\prec}(a(u),\vartheta)\circ\xi)$$

and that by computations similar to those of the previous sections one can prove that

$$(\Pi_{\prec\!\!\prec}(a(u),\vartheta)\circ\xi)=\Pi_{\diamondsuit}(a(u),\vartheta\circ\xi)+\mathscr{C}_T^{3\alpha-2}$$

so the resonant terms are comparable to the sum of the two terms

$$a_1'(u)\Pi_{\diamondsuit}(a(u),\vartheta\circ\Delta\vartheta)+a_2'(u)\Pi_{\diamondsuit}(a(u),\vartheta\circ\xi)$$

which require renormalization of the form

$$\frac{a_1'(u)a_2(u)^2}{a_1(u)^2}\sigma_\varepsilon+\frac{a_2'(u)a_2(u)}{a_1(u)}\sigma_\varepsilon. \tag{36}$$

We remark that the structure of the second renormalisation, which is due to the r.h.s. in the equation is the same of that found by Bailleul, Debussche and Hofmanova [7].

7. Full generality

Within the framework of the present work we are actually able to treat equations of the form

$$a_1(u(t,x))\partial_t u(t,x)-a_2(u(t,x))\Delta u(t,x)=\xi(a_3(u(t,x)),x) \tag{37}$$

where $\xi(\eta_3,t,x)$ is a Gaussian process with covariance

$$\mathbb{E}[\xi(\eta_3,x)\xi(\eta_3',x')]=F(\eta_3,\eta_3')\delta(x-x')$$

where F is a smooth covariance function. In this case we can take as skeleton equation

$$\mathscr{L}\,(\eta)\vartheta:=\eta_1\partial_t\vartheta(\eta,t,x)-\eta_2\Delta\vartheta(\eta,t,x)=\xi(\eta_3,x)$$

whose solution ϑ is a Gaussian process smooth in the variable $\eta=(\eta_1,\eta_2,\eta_3)$ which we assume taking value in a compact subset of $\mathbb{R}^3$ for which $\eta_1\geqslant\lambda>0$ and $\eta_2\geqslant\lambda>0$ with fixed λ. Letting $a(r)=(a_1(r),a_2(r),a_3(r))$ we can rewrite the l.h.s. of eq. (37) in the form

$$a_1(u)\partial_t u-a_2(u)\Delta u=\Pi_{\diamondsuit}\left(a(u),\mathscr{L}\,u\right)$$

and the r.h.s. as

$$\xi(a_3(u(t,x)),x)=\Pi_{\diamondsuit}(a(u),\Xi)$$

where $\Xi(\eta,x)=\xi(\eta_3,x)$. Now we perform the paraproduct decomposition to get

$$\Pi_{\prec}\left(a(u),\mathscr{L}\,u\right)-\Pi_{\prec}(a(u),\Xi)=\Pi_{\circ}(a(u),\Xi)-\Pi_{\circ}\left(a(u),\mathscr{L}\,u\right)+\Pi_{\succ}(a(u),\Xi)-\Pi_{\succ}\left(a(u),\mathscr{L}\,u\right).$$

Let

$$\mathscr{P}_t(\eta):=e^{t(\eta_2/\eta_1)\Delta}$$

and invoke the paracontrolled Ansatz in the form used before $u=\Pi_{\prec\!\!\prec}\left(a(u),\vartheta+U^\sharp+\mathscr{P}u_0\right)$. Using that

$$\begin{aligned}\Pi_{\prec}\left(a(u),\mathscr{L}\,\Pi_{\prec\!\!\prec}\left(a(u),\vartheta+U^\sharp+\mathscr{P}u_0\right)\right) &= \Pi_{\prec\!\!\prec}\left(a(u),\mathscr{L}\,\left(\vartheta+U^\sharp+\mathscr{P}u_0\right)\right)\\ &\quad+\Psi\left(a(u),\vartheta+U^\sharp+\mathscr{P}u_0\right)\end{aligned}$$

and observing that we can take $\mathscr{L}\,\vartheta = \Xi$ and that $\mathscr{L}\,\mathscr{P}u_0 = 0$ to get

$$\Pi_{\prec\!\prec}\left(a(u), \mathscr{L}\,U^\sharp\right) = F(u, U^\sharp)$$

where

$$\begin{aligned} F(u,U^\sharp) &= \Pi_\circ(a(u),\Xi) - \Pi_\circ\left(a(u), \mathscr{L}\,u\right) + \Pi_\succ(a(u),\Xi) - \Pi_\succ\left(a(u), \mathscr{L}\,u\right) \\ &\quad + [\Pi_\prec(a(u),\Xi) - \Pi_{\prec\!\prec}(a(u),\Xi)] - \Psi\left(a(u), \vartheta + U^\sharp + \mathscr{P}u_0\right) \end{aligned}$$

which is solved by $U^\sharp$ satisfying

$$\mathscr{L}\,U^\sharp = F(u, U^\sharp)$$

indeed $\Pi_\prec(a(u), F(u,U^\sharp)) = F(u,U^\sharp)$ since $F(u,U^\sharp)$ does not depends on the additional parameter. Remark that the term $\Pi_\prec(a(u),\Xi) - \Pi_{\prec\!\prec}(a(u),\Xi)$, which does not appear in the simpler case, can be treated with Lemma 25.

It remains now to discuss the handling of the resonant products under the paracontrolled assumption, namely $\Pi_\circ(a(u),\Xi)$ and $\Pi_\circ\left(a(u), \mathscr{L}\,u\right)$. Next lemma is a paralinearization result adapted to our non-linear context.

Lemma 14. *Assume that $u \in \mathscr{C}_T^\rho$ and $Z \in C_\eta^2 \mathscr{C}_T^\gamma$ then if $\gamma + 2\rho > 0$ we have*

$$C(u,\Xi) := \Pi_\circ(a(u), Z) - u \circ \Pi_\prec((a(u), a'(u)), \mathcal{D}Z) \in \mathscr{C}_T^{\gamma+2\rho}$$

where $\mathcal{D}Z((\eta,\eta'),t,x) := \sum_i \eta_i' \partial_{\eta_i} Z(\eta,t,x)$.

Proof.

$$\Pi_\circ(a(u),Z)(t,x) = \sum_{i\sim j} \int_{y,z} K_{i,x}(y) K_{j,x}(z) Z(a(u(t,y)),t,z)$$

$$= \sum_{i\sim j}\sum_k \int_{y,z,z',z''} K_{i,x}(y)K_{j,x}(z)P_{k,z}(z'')K_{k,z}(z')Z(a(u(t,y)),t,z')$$

$$= \sum_{i\sim j}\sum_k \int_{y,z,z',z''} K_{i,x}(y)K_{j,x}(z)P_{k,z}(z'')K_{k,z}(z')[Z(a(u(t,y)),t,z') - Z(a(u(t,z'')),t,z')]$$

$$= \sum_{i\sim j}\sum_k \int_{y,z,z',z''} K_{i,x}(y)K_{j,x}(z)P_{k,z}(z'')K_{k,z}(z') \times$$

$$\times \left[\sum_\ell a_\ell'(u(t,z''))(u(t,y) - u(t,z''))\partial_{a_\ell} Z(a(u(t,z'')),t,z')\right]$$

$$+ \sum_{i\sim j}\sum_{k\sim j} \int_{y,z,z',z''} K_{i,x}(y)K_{j,x}(z)P_{k,z}(z'')K_{k,z}(z') O((u(t,y)-u(t,z''))^2)\partial_\eta^2 Z(a(u(t,y)),t,z')$$

and observe that the first term is equal to $u \circ \Pi_\prec((a(u),a'(u)),\mathcal{D}Z)$ while the second term can be easily estimated in $\mathscr{C}_T^{\gamma+2\rho}$. □

Using this result and Lemma 25 we can expand

$$\begin{aligned} \Pi_\circ(a(u),\Xi) &= u \circ \Pi_\prec((a(u),a'(u)),\mathcal{D}\Xi) + \mathscr{C}_T^{\gamma+2\rho} \\ &= \Pi_{\prec\!\prec}(a(u),\vartheta) \circ \Pi_\prec((a(u),a'(u)),\mathcal{D}\Xi) + \mathscr{C}_T^{\gamma+2\rho} \\ &= \Pi_\diamondsuit((a(u),a'(u)),\vartheta \circ \mathcal{D}\Xi) + \mathscr{C}_T^{\gamma+2\rho} \end{aligned}$$

and similarly, noting that

$$\begin{aligned} \Pi_\prec\left((a(u),a'(u)),(\mathcal{D}\mathscr{L})\,u\right) &= \Pi_\prec\left((a(u),a'(u)),(\mathcal{D}\mathscr{L})\,\Pi_{\prec\!\prec}(a(u),\vartheta)\right) + \mathscr{C}_T^{3\rho-2} \\ &= \Pi_\prec\left((a(u),a'(u)),(\mathcal{D}\mathscr{L})\,\vartheta\right) + \mathscr{C}_T^{3\rho-2} \end{aligned}$$

where $(\mathcal{D}\mathscr{L})\,(\eta,\eta') = \eta_1'\partial_t + \eta_2'\Delta$, we have

$$\begin{aligned} \Pi_\circ\left(a(u), \mathscr{L}\,u\right) &= u \circ \Pi_\prec\left((a(u),a'(u)),(\mathcal{D}\mathscr{L})\,u\right) + \mathscr{C}_T^{3\rho-2} \\ &= \Pi_{\prec\!\prec}(a(u),\vartheta) \circ \Pi_\prec\left((a(u),a'(u)),(\mathcal{D}\mathscr{L})\,\vartheta\right) + \mathscr{C}_T^{3\rho-2} \end{aligned}$$

$$= \Pi_\diamondsuit \left((a(u), a'(u)), \vartheta \circ (\mathcal{D}\mathscr{L})\,\vartheta\right) + \mathscr{C}_T^{3\rho-2}$$

Finally the equation for $U^\sharp$ reads

$$\mathscr{L}\, U^\sharp = \Pi_\diamondsuit \left((a(u), a'(u)), \vartheta \circ \mathcal{D}\Xi + \vartheta \circ (\mathcal{D}\mathscr{L})\,\vartheta\right) + \mathscr{C}_T^{3\rho-2}.$$

which can be solved essentially as we did in the simpler context. We see that the general enhancement has the form

$$(\xi, \vartheta \circ \mathcal{D}\Xi + \vartheta \circ (\mathcal{D}\mathscr{L})\,\vartheta)$$

which of course will require renormalization like we did before. In particular

$$(\vartheta \circ \mathcal{D}\Xi + \vartheta \circ (\mathcal{D}\mathscr{L})\,\vartheta)\,(\eta, \eta') = \eta_3' \vartheta(\eta) \circ \partial_\eta \xi(\eta_3, \cdot, \cdot) + \vartheta(\eta) \circ (\eta_1' \partial_t \vartheta(\eta) + \eta_2' \Delta \vartheta(\eta))$$

$$= \frac{\eta_3'}{\eta_2} (\Delta^{-1} \xi(\eta_3, \cdot)) \circ \partial_{\eta_3} \xi(\eta_3, \cdot) + \frac{\eta_2'}{\eta_2^2} (\Delta^{-1} \xi(\eta_3, \cdot)) \circ \xi(\eta_3, \cdot)$$

where we used that $\partial_t \vartheta(\eta) = 0$ and that $\eta_2 \Delta \vartheta(\eta) = \xi(\eta_3, \cdot)$. Now observe that

$$\mathbb{E}[(\Delta^{-1} \xi_\varepsilon(\eta_3, \cdot)) \circ \xi_\varepsilon(\eta_3, \cdot)] = F(\eta_3, \eta_3)\sigma_\varepsilon$$

and that

$$\mathbb{E}[(\Delta^{-1} \xi(\eta_3, \cdot)) \circ \partial_{\eta_3} \xi(\eta_3, \cdot)] = (\partial_1 F)(\eta_3, \eta_3)\sigma_\varepsilon.$$

In the end the renormalized enhanced noise is obtained as the limit in $\mathcal{X}^\alpha$ of $(\xi_\varepsilon, \Xi_{2,\varepsilon})$ where

$$\Xi_{2,\varepsilon}(\eta, \eta') = \frac{\eta_3'}{\eta_2} (\Delta^{-1} \xi(\eta_3, \cdot)) \circ \partial_{\eta_3} \xi(\eta_3, \cdot) + \frac{\eta_2'}{\eta_2^2} (\Delta^{-1} \xi(\eta_3, \cdot)) \circ \xi(\eta_3, \cdot) - H_\varepsilon(\eta, \eta')$$

with

$$H_\varepsilon(\eta, \eta') = \frac{\eta_3'}{\eta_2} (\partial_1 F)(\eta_3, \eta_3)\sigma_\varepsilon + \frac{\eta_2'}{\eta_2^2} F(\eta_3, \eta_3)\sigma_\varepsilon.$$

We remark that if we take $F(\eta_3, \eta_3') = \eta_3 \eta_3'$ and $\eta_1 = 0$ we reobtain the situation treated in Section 6, indeed in this case

$$\Pi_\diamondsuit((a(u), a'(u)), H_\varepsilon) = \frac{a_3'(u) a_3(u)}{a_2(u)} \sigma_\varepsilon + \frac{a_2'(u) a_3(u)^2}{a_2(u)^2} \sigma_\varepsilon.$$

which coincides with (36) modulo the shift in notations.

Remark 15. *Consider the more general situation where the noise depends explicitly on time, e.g. with a covariance*

$$\mathbb{E}[\xi(\eta, t, x) \xi(\eta', t', x')] = F(\eta, \eta') Q(t - t', x - x')$$

with F a smooth function and Q a distribution of parabolic regularity $\rho > -4/3$. In this case the framework of this paper will apply, provided we consider space–time paraproducts instead of paraproducts which acts only on the space variable. This can be done exactly following the lines of the paper [15] *where time paraproducts were considered in the paracontrolled approach to solutions to SDE driven by gaussian signals. The constraint of regularity $\rho > -4/3$ does allow to treat a noise which is white in time and smooth in space, but not a space–time white noise. It is well known that the first order paracontrolled approach on which the present paper is based on does not allow to treat this kind of irregular signals in full generality.*

Appendix A. Besov spaces

In this Appendix we collect some classical results from harmonic analysis needed in the paper. For a gentle introduction to Littlewood-Paley theory and Besov spaces see the recent monograph [3], where most of our results are taken from. There the case of tempered distributions on $\mathbb{R}^d$ is considered. The Schauder estimates for the heat semigroup are classical and can be found in [15, 17].

Fix $d \in \mathbb{N}$ and denote by $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ the d-dimensional torus. We focus here on distributions and SPDEs on the torus, but everything in this Appendix applies mutatis mutandis

on the full space $\mathbb{R}^d$, see [15]. The space of distributions $\mathscr{D}' = \mathscr{D}'(\mathbb{T}^d)$ is defined as the set of linear maps f from $C^\infty = C^\infty(\mathbb{T}^d, \mathbb{C})$ to $\mathbb{C}$, such that there exist $k \in \mathbb{N}$ and $C > 0$ with

$$|\langle f, \varphi\rangle| := |f(\varphi)| \leqslant C \sup_{|\mu|\leqslant k} \|\partial^\mu \varphi\|_{L^\infty(\mathbb{T}^d)}$$

for all $\varphi \in C^\infty$. In particular, the Fourier transform $\mathscr{F} f : \mathbb{Z}^d \to \mathbb{C}$, $\mathscr{F} f(k) = \langle f, e^{-ik\cdot}\rangle$, is defined for all $f \in \mathscr{D}'$, and it satisfies $|\mathscr{F} f(k)| \leqslant |P(k)|$ for a suitable polynomial P. We will also write $\hat f(k) = \mathscr{F} f(k)$. Conversely, if $(g(k))_{k\in\mathbb{Z}^d}$ is at most of polynomial growth, then its inverse Fourier transform

$$\mathscr{F}^{-1} g = (2\pi)^{-d} \sum_{k\in\mathbb{Z}^d} e^{i\langle k,\cdot\rangle} g(k)$$

defines a distribution, and we have $\mathscr{F}^{-1}\mathscr{F} f = f$ as well as $\mathscr{F}\mathscr{F}^{-1} g = g$. To see this, it suffices to note that the Fourier transform of $\varphi \in C^\infty$ decays faster than any rational function (we say that it is of *rapid decay*). Indeed, for $\mu \in \mathbb{N}_0^d$ we have $|k^\mu \hat g(k)| = |\mathscr{F}(\partial^\mu g)(k)| \leqslant \|\partial^\mu g\|_{L^1(\mathbb{T}^d)}$ for all $k \in \mathbb{Z}^d$. As a consequence we get the Parseval formula $\langle f, \overline{\varphi}\rangle = (2\pi)^{-d} \sum_k \hat f(k)\overline{\hat\varphi(k)}$ for $f \in \mathscr{D}'$ and $\varphi \in C^\infty$.

Linear maps on $\mathscr{D}'$ can be defined by duality: if $A : C^\infty \to C^\infty$ is such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $C > 0$ with $\sup_{|\mu|\leqslant k} \|\partial^\mu(A\varphi)\|_{L^\infty} \leqslant C \sup_{|\mu|\leqslant n} \|\partial^\mu \varphi\|$, then we set $\langle {}^t A f, \varphi\rangle = \langle f, A\varphi\rangle$. Differential operators are defined by $\langle \partial^\mu f, \varphi\rangle = (-1)^{|\mu|}\langle f, \partial^\mu \varphi\rangle$. If $\varphi : \mathbb{Z}^d \to \mathbb{C}$ grows at most polynomially, then it defines a *Fourier multiplier*

$$\varphi(\mathrm{D}) f = \mathscr{F}^{-1}(\varphi \mathscr{F} f),$$

which gives us a distribution $\varphi(\mathrm{D}) f \in \mathscr{D}'$ for every $f \in \mathscr{D}'$.

Littlewood-Paley blocks give a decomposition of any distribution on $\mathscr{D}'$ into an infinite series of smooth functions.

Definition 16. *A dyadic partition of unity consists of two nonnegative radial functions $\chi, \rho \in C^\infty(\mathbb{R}^d, \mathbb{R})$, where ρ is supported in a ball $\mathscr{B} = \{|x| \leqslant c\}$ and ρ is supported in an annulus $\mathscr{A} = \{a \leqslant |x| \leqslant b\}$ for suitable $a, b, c > 0$, such that*

(1) $\chi + \sum_{j\geqslant 0} \rho(2^{-j}\cdot) \equiv 1$ and

(2) $\chi\rho(2^{-j}\cdot) \equiv 0$ for $j \geqslant 1$ and $\rho(2^{-i}\cdot)\rho(2^{-j}\cdot) \equiv 0$ for all $i, j \geqslant 0$ with $|i - j| > 1$.

We will often write $\rho_{-1} = \chi$ and $\rho_j = \rho(2^{-j}\cdot)$ for $j \geqslant 0$.

Dyadic partitions of unity exist, see [3]. The reason for considering smooth partitions rather than indicator functions is that indicator functions do not have good Fourier properties. We fix a dyadic partition of unity (χ, ρ) and define the dyadic blocks

$$\Delta_j f = \rho_j(\mathrm{D}) f = \mathscr{F}^{-1}(\rho_j \hat f), \quad j \geqslant -1.$$

We also use the notation

$$S_j f = \sum_{i<j-1} \Delta_i f$$

and notice that

$$\Delta_j f(x) = \int K_{i,x}(y) f(y)\mathrm{d}y, \qquad S_j f(x) = \int P_{i,x}(y) f(y)\mathrm{d}y$$

with $K_{i,x}(y) = 2^{di} K(2^i(x - y))$, $P_{i,x}(y) = \sum_{j<i-1} K_{j,x}(y)$, K radial with zero mean.

Every dyadic block has a compactly supported Fourier transform and it belongs therefore to C^∞. It is easy to see that $f = \sum_{j\geqslant -1} \Delta_j f = \lim_{j\to\infty} S_j f$ for all $f \in \mathscr{D}'$.

For $\alpha \in \mathbb{R}$, the Hölder-Besov space $\mathscr{C}^\alpha$ is given by $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d, \mathbb{R})$, where for $p, q \in [1, \infty]$ we define

$$B^\alpha_{p,q} = B^\alpha_{p,q}(\mathbb{T}^d, \mathbb{R}) = \left\{ f \in \mathscr{D}' : \|f\|_{B^\alpha_{p,q}} = \left(\sum_{j\geqslant -1} (2^{j\alpha}\|\Delta_j f\|_{L^p})^q \right)^{1/q} < \infty \right\},$$

with the usual interpretation as ℓ^∞ norm in case $q = \infty$. Then $B^\alpha_{p,q}$ is a Banach space and while the norm $\|\cdot\|_{B^\alpha_{p,q}}$ depends on (χ, ρ), the space $B^\alpha_{p,q}$ does not, and any other dyadic partition of unity corresponds to an equivalent norm.

If $\alpha \in (0,\infty) \setminus \mathbb{N}$, then $\mathscr{C}^\alpha$ is the space of $\lfloor\alpha\rfloor$ times differentiable functions whose partial derivatives of order $\lfloor\alpha\rfloor$ are $(\alpha - \lfloor\alpha\rfloor)$-Hölder continuous (see page 99 of [3]). Note however, that for $k \in \mathbb{N}$ the space $\mathscr{C}^k$ is strictly larger than C^k, the space of k times continuously differentiable functions.

The following lemma gives useful characterization of Besov regularity for functions that can be decomposed into pieces which are localized in Fourier space.

Lemma 17.

1. *Let $\mathscr{A}$ be an annulus, let $\alpha \in \mathbb{R}$, and let (u_j) be a sequence of smooth functions such that $\mathscr{F}u_j$ has its support in $2^j\mathscr{A}$, and such that $\|u_j\|_{L^\infty} \lesssim 2^{-j\alpha}$ for all j. Then*

$$u = \sum_{j\geqslant -1} u_j \in \mathscr{C}^\alpha \qquad \text{and} \qquad \|u\|_\alpha \lesssim \sup_{j\geqslant -1}\{2^{j\alpha}\|u_j\|_{L^\infty}\}.$$

2. *Let $\mathscr{B}$ be a ball, let $\alpha > 0$, and let (u_j) be a sequence of smooth functions such that $\mathscr{F}u_j$ has its support in $2^j\mathscr{B}$, and such that $\|u_j\|_{L^\infty} \lesssim 2^{-j\alpha}$ for all j. Then*

$$u = \sum_{j\geqslant -1} u_j \in \mathscr{C}^\alpha \qquad \text{and} \qquad \|u\|_\alpha \lesssim \sup_{j\geqslant -1}\{2^{j\alpha}\|u_j\|_{L^\infty}\}.$$

The Bernstein inequalities of the next lemma are extremely useful when dealing with functions with compactly supported Fourier transform.

Lemma 18. *Let $\mathscr{A}$ be an annulus and let $\mathscr{B}$ be a ball. For any $k \in \mathbb{N}_0$, $\lambda > 0$, and $1 \leqslant p \leqslant q \leqslant \infty$ we have that*

1. *if $u \in L^p$ is such that $\operatorname{supp}(\mathscr{F}u) \subseteq \lambda\mathscr{B}$, then*

$$\max_{\mu\in\mathbb{N}^d:|\mu|=k} \|\partial^\mu u\|_{L^q} \lesssim_k \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p};$$

2. *if $u \in L^p$ is such that $\operatorname{supp}(\mathscr{F}u) \subseteq \lambda\mathscr{A}$, then*

$$\lambda^k\|u\|_{L^p} \lesssim_k \max_{\mu\in\mathbb{N}^d:|\mu|=k} \|\partial^\mu u\|_{L^p}.$$

Lemma 19 (Schauder estimates)**.** *Let $V_t = \int_0^t e^{a(t-s)\Delta}v_s\mathrm{d}s$ with $a \geqslant \lambda$. We define $\mathscr{L}^\alpha_T = \mathscr{C}^\alpha_T \cap C^{\alpha/2}_T\mathscr{C}^0$ and introduce the norm*

$$\|v\|_{\mathcal{E}^\delta_T\mathscr{C}^\alpha} = \sup_{t\in[0,T]} t^\delta\|v(t,\cdot)\|_{\mathscr{C}^\alpha}.$$

Then for any $\gamma \in [0,1)$ and $\alpha \in \mathbb{R}$:

$$\sup_{t\in[0,T]} t^\gamma\|V_t\|_{\mathscr{C}^{\alpha-2\beta}} \lesssim \sup_{t\in[0,T]} t^{\gamma+\beta}\|v_t\|_{\mathscr{C}^{\alpha-2}} \quad \forall\beta\in[0,1)$$

$$\|V\|_{\mathscr{L}^{\alpha-2\beta}_T} \lesssim T^\beta\|v\|_{\mathscr{C}^{\alpha-2}_T} \quad \forall\beta\in[0,1)$$

$$\|V\|_{\mathscr{L}^{\alpha-2\beta}_T} \lesssim T^{\beta-\delta}\|v\|_{\mathcal{E}^\delta_T\mathscr{C}^{\alpha-2}} \quad \forall\beta\in[0,1), \forall\delta\in[0,\beta]$$

$$\|V\|_{\mathscr{L}^\gamma_T} \lesssim T^{\frac{\rho-\gamma}{2}+1-\delta}\|v\|_{\mathcal{E}^\delta_T\mathscr{C}^\rho} \quad \forall\rho\in[\gamma-2,\gamma), \forall\delta\in\left[0,\frac{\rho-\gamma}{2}+1\right]$$

Terms of the type $\|a(u(t,x))\|_{\mathscr{C}^0}$ with $a : \mathbb{R} \to \mathbb{R}$ cannot be estimated directly with their Hölder norm. In the following lemma we note some bounds used in Section 4.

Lemma 20. *Let $a \in C_b^3$ uniformly bounded and $u \in \mathscr{L}^{\alpha}_T = \mathscr{C}^{\alpha}_T \cap C_T^{\alpha/2}\mathscr{C}^0$, then*

$$\begin{aligned}\|a(u)\|_{\mathscr{L}^{\alpha}_T} &\lesssim \|a\|_{L^\infty} + \|a'\|_{L^\infty}\left(\|u\|_{C_T^{\alpha/2}\mathscr{C}^0} + \|u\|_{\mathscr{C}^{\alpha}_T}\right) + \|a''\|_{L^\infty}\|u\|_{C_T^{\alpha/2}\mathscr{C}^0}\|u\|_{\mathscr{C}^{\alpha}_T}\\ &\lesssim 1 + \|u\|_{\mathscr{L}^{\alpha}_T} + \|u\|^2_{\mathscr{L}^{\alpha}_T},\end{aligned}$$

$$\|a(u_1) - a(u_2)\|_{\mathscr{L}^{\alpha}_T} \lesssim \|u_1 - u_2\|_{\mathscr{L}^{\alpha}_T}\left(1 + \|u_1\|_{\mathscr{C}^{\alpha}_T} + \|u_2\|_{\mathscr{C}^{\alpha}_T}\right)^2,$$

$$\|a(u_1) - a(u_2)\|_{\mathscr{C}^0_T} \lesssim \|u_1 - u_2\|_{\mathscr{C}^0_T}\left(1 + \|u_1\|_{\mathscr{C}^{\alpha}_T} + \|u_2\|_{\mathscr{C}^{\alpha}_T}\right).$$

Proof. The bound on $\|a(u)\|_{\mathscr{C}^{\alpha}_T}$ is trivial. We estimate $\|a(u_t) - a(u_s)\|_{\mathscr{C}^0}$ as

$$\left|\int_z K_{i,x}(z)[a(u(t,z)) - a(u(s,z))]\right| = \left|\int_{z,\tau\in[0,1]} K_{i,x}(z)a'(\delta_\tau u^{tz}_{sz})[u(t,z) - u(s,z)]\right|$$

$$\begin{aligned}\leqslant\ & \left|\int_{z,w,\tau\in[0,1]} K_{i,x}(z)\sum_{j\lesssim k\sim i}[\Delta_j u(t,z) - \Delta_j u(s,z)]K_{k,z}(w)a'(\delta_\tau u^{tw}_{sw})\right|\\ &+\left|\int_{z,w,\tau\in[0,1]} K_{i,x}(z)\sum_{k\lesssim j\sim i}[\Delta_j u(t,z) - \Delta_j u(s,z)]K_{k,z}(w)a'(\delta_\tau u^{tw}_{sw})\right|\\ &+\left|\int_{z,w,\tau\in[0,1]} K_{i,x}(z)\sum_{k\sim j\gtrsim i}[\Delta_j u(t,z) - \Delta_j u(s,z)]K_{k,z}(w)a'(\delta_\tau u^{tw}_{sw})\right|.\end{aligned}$$

If $k > -1$ we have

$$\int_w K_{k,z}(w)a'(\delta_\tau u^{tw}_{sw}) = \int_w K_{k,z}(w)[a'(\delta_\tau u^{tw}_{sw}) - a'(\delta_\tau u^{tz}_{sz})]$$

and then the first term above becomes

$$\left|\int_{z,w,\tau,\sigma} K_{i,x}(z)\sum_{j\lesssim k\sim i}[\Delta_j u(t,z) - \Delta_j u(s,z)]K_{k,z}(w)a''(\delta_\sigma(\delta_\tau u^t_s)^w_z)[\delta u^{tw}_{tz} - \delta u^{sw}_{sz}]\right|$$

$$\lesssim \|a''\|_{L^\infty}\|u_t - u_s\|_{\mathscr{C}^0}\sum_{j\lesssim k\sim i}\int_w |K_{k,z}(w)||w-z|^\alpha\|u\|_{\mathscr{C}^{\alpha}_T} \lesssim i2^{-\alpha i}\|a''\|_{L^\infty}\|u_t - u_s\|_{\mathscr{C}^0}\|u\|_{\mathscr{C}^{\alpha}_T}.$$

The second term is

$$\left|\int_{z,w,\tau\in[0,1]} K_{i,x}(z)\sum_{j\sim i}[\Delta_j u(t,\cdot) - \Delta_j u(s,\cdot)](z)P_{j,z}(w)a'(\delta_\tau u^{tw}_{sw})\right|$$

$$\lesssim \|u_t - u_s\|_{\mathscr{C}^0}\|P_{j,z}(w)a'(\delta_\tau u^{tw}_{sw})\|_{L^\infty} \lesssim \|u_t - u_s\|_{\mathscr{C}^0}\|a'\|_{L^\infty}$$

The third term can be estimated as the first one when $k > -1$. Otherwise we just bound it as

$$\left|\int_{z,w,\tau\in[0,1]} K_{i,x}(z)\sum_{j\leqslant N}[\Delta_j u(t,\cdot) - \Delta_j u(s,\cdot)](z)K_{-1,z}(w)a'(\delta_\tau u^{tw}_{sw})\right| \lesssim \|u_t - u_s\|_{\mathscr{C}^0}\|a'\|_{L^\infty}.$$

For the three terms together we have the bound

$$\|a(u_t) - a(u_s)\|_{\mathscr{C}^0} \lesssim \|u_t - u_s\|_{\mathscr{C}^0}\left(\|a'\|_{L^\infty} + \|a''\|_{L^\infty}\|u\|_{\mathscr{C}^{\alpha}_T}\right)$$

With the same technique we obtain

$$\begin{aligned}\|a(u_1) - a(u_2)\|_{C_T^{\alpha/2}\mathscr{C}^0} \lesssim\ & \|a'\|_{L^\infty}\|u_1 - u_2\|_{C_T^{\alpha/2}\mathscr{C}^0}\\ &+\|a''\|_{L^\infty}\|u_1 - u_2\|_{C_T^{\alpha/2}\mathscr{C}^0}\|u_1 - u_2\|_{\mathscr{C}^{\alpha}_T}\\ &+\|a'''\|_{L^\infty}\|u_1 - u_2\|_{C_T^{\alpha/2}\mathscr{C}^0}\|u_1 - u_2\|^2_{\mathscr{C}^{\alpha}_T}\end{aligned}$$

and this gives the second estimate. The third one can be obtained easily. □

A.1. **Bony's paraproduct.**

In terms of Littlewood–Paley blocks, the product fg can be decomposed as

$$fg = \sum_{j\geqslant -1}\sum_{i\geqslant -1} \Delta_i f \Delta_j g = f \prec g + f \succ g + f \circ g.$$

Where the *paraproducts* $f \prec g$ and $f \succ g$ and the *resonant product* $f \circ g$ are defined as

$$f \prec g = g \succ f := \sum_{j\geqslant -1}\sum_{i=-1}^{j-2} \Delta_i f \Delta_j g \qquad \text{and} \qquad f \circ g := \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g.$$

We will often use the shortcuts $\sum_{i\sim j}$ for $\sum_{|i-j|\leqslant 1}$ and $\sum_{i\lesssim j}$ for $\sum_{i<j-1}$. Of course, the decomposition depends on the dyadic partition of unity used to define the blocks Δ_j, and also on the particular choice of the pairs (i,j) in the diagonal part. The choice of taking all (i,j) with $|i-j|\leqslant 1$ into the diagonal part corresponds to property 2 in our definition of dyadic partitions of unity.

Bony's crucial observation [9, 30] is that the paraproduct $f \prec g$ (and thus $f \succ g$) is always a well-defined distribution. Heuristically, $f \prec g$ behaves at large frequencies like g (and thus retains the same regularity), and f provides only a frequency modulation of g. The only difficulty in constructing fg for arbitrary distributions lies in handling the resonant product $f \circ g$. The basic result about these bilinear operations is given by the following estimates.

Theorem 21. *(Paraproduct estimates) For any $\beta \in \mathbb{R}$ and $f, g \in \mathscr{D}'$ we have*

$$\|f \prec g\|_{\mathscr{C}^\beta} \lesssim_\beta \|f\|_{L^\infty} \|g\|_{\mathscr{C}^\beta}, \tag{38}$$

and for $\alpha < 0$ furthermore

$$\|f \prec g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}. \tag{39}$$

For $\alpha + \beta > 0$ we have

$$\|f \circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}. \tag{40}$$

Bony proved also a basic *paralinearisation* result, soon after improved by Meyer. We give here a particular version suited to our purposes.

Theorem 22. *Let $\alpha \in (0,1)$, $f \in (\mathscr{C}^\alpha)^k$ and $F \in C^3(\mathbb{R}^k;\mathbb{R})$ then*

$$R_F(f) := F(f) - F'(f) \prec f \in \mathscr{C}^{2\alpha}$$

with

$$\|R_F(f)\|_{\mathscr{C}^{2\alpha}} \lesssim \|F\|_{C^2} \left(1 + \|f\|_{\mathscr{C}^\alpha}\right)^2.$$

Moreover the map $f \mapsto R_F(f)$ is locally Lipshitz and

$$\|R_F(f) - R_F(\tilde f)\|_{\mathscr{C}^{2\alpha}} \lesssim \|F\|_{C^3} \left(1 + \|f\|_{\mathscr{C}^\alpha} + \|\tilde f\|_{\mathscr{C}^\alpha}\right)^2 \|\tilde f - f\|_{\mathscr{C}^\alpha}.$$

The additional key ingredient at the core of the paracontrolled approach is the following commutation result proved in [15], Lemma 2.4:

Lemma 23. *Assume that $\alpha, \beta, \gamma \in \mathbb{R}$ are such that $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$. Then for $f, g, h \in C^\infty$ the trilinear operator*

$$C(f,g,h) := ((f \prec g) \circ h) - f(g \circ h)$$

allows for the bound

$$\|C(f,g,h)\|_{\mathscr{C}^{\beta+\gamma}} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta} \|h\|_{\mathscr{C}^\gamma}, \tag{41}$$

and can thus be uniquely extended to a bounded trilinear operator

$$C : \mathscr{C}^\alpha \times \mathscr{C}^\beta \times \mathscr{C}^\alpha \to \mathscr{C}^{\beta+\gamma}.$$

We will need the following two lemmas to compare standard and time-smoothed paraproducts. The first one has essentially the same proof as [15], Lemma 5.1.

Lemma 24. *Let $\rho \in (0,2)$, $\gamma \in \mathbb{R}$. Then for every $\varepsilon > 0$ we have the bound*

$$\|g \prec h - g \prec\!\!\!\prec h\|_{\mathscr{C}_T^{\rho+\gamma-\varepsilon}} \lesssim \|g\|_{C_T^{\rho/2}\mathscr{C}^0} \|h\|_{\mathscr{C}_T^{\gamma}}.$$

The second lemma has a standard proof.

Lemma 25. *Let $g \in \mathscr{L}_T^{\rho}$, $h \in C_\eta^1 \mathscr{C}_T^{\gamma}$ with $\rho \in (0,1)$, $\gamma \in \mathbb{R}$. We have, $\forall \varepsilon > 0$*

$$\|\Pi_\prec(g,h) - \Pi_{\prec\!\!\!\prec}(g,h)\|_{\mathscr{C}_T^{\rho+\gamma-\varepsilon}} \lesssim \|g\|_{\mathscr{L}_T^{\rho}} \|h\|_{C_\eta^1 \mathscr{C}_T^{\gamma}}.$$

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