Weak universality for a class of 3d stochastic reaction–diffusion models

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Abstract

We establish the large scale convergence of a class of stochastic weakly nonlinear reaction–diffusion models on a three dimensional periodic domain to the dynamic Φ^4_3 model within the framework of paracontrolled distributions. Our work extends previous results of Hairer and Xu to nonlinearities with a finite amount of smoothness (C^9 is enough). We use the Malliavin calculus to perform a partial chaos expansion of the stochastic terms and control their L^p norms in terms of the graphs of the standard Φ^4_3 stochastic terms.

Keywords: weak universality, paracontrolled distributions, stochastic quantisation equation, Malliavin calculus, partial chaos expansion.

Consider a family of stochastic reaction–diffusion equation in a *weakly nonlinear regime*:

$$\mathscr{L} u(t,x) = -\varepsilon^\alpha F_\varepsilon(u(t,x)) + \eta(t,x) \qquad (t,x) \in [0, T/\varepsilon^2] \times (\mathbb{T}/\varepsilon)^3 \tag{1}$$

with $\varepsilon \in (0,1]$, initial condition $\bar{u}_{0,\varepsilon}\colon (\mathbb{T}/\varepsilon)^3 \to \mathbb{R}$, $F_\varepsilon \in C^9(\mathbb{R})$ with exponential growth at infinity, $\alpha > 0$ and $\mathscr{L} := (\partial_t - \Delta)$ the heat flow operator and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Here η is a centered Gaussian noise with stationary covariance

$$\mathbb{E}(\eta(t,x)\eta(s,y)) = \tilde{C}^\varepsilon(t-s,x,y)$$

such that $\tilde{C}^\varepsilon(t-s,x,y) = \Sigma(t-s,x-y)$ if $\mathrm{dist}(x,y) \leqslant 1$ and 0 otherwhise where $\Sigma\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^+$ is a smooth, positive function compactly supported in $[0,1]\times B_{\mathbb{R}^3}(0,1)$.

We look for a large scale description of the solution to eq. (1) and we introduce the "mesoscopic" scale variable $u_\varepsilon(t,x) = \varepsilon^{-\beta} u(t/\varepsilon^2, x/\varepsilon)$ where $\beta > 0$. Substituting u_ε into (1) we get

$$\mathscr{L} u_\varepsilon(t,x) = -\varepsilon^{\alpha-2-\beta} F_\varepsilon(\varepsilon^\beta u_\varepsilon(t,x)) + \varepsilon^{-2-\beta}\eta\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\Big). \tag{2}$$

In order for the term $\varepsilon^{-2-\beta}\eta(t/\varepsilon^2, x/\varepsilon)$ to converge to a non–trivial random limit we need that $\beta = 1/2$. Indeed the Gaussian field $\eta_\varepsilon(t,x) := \varepsilon^{-5/2}\eta(t/\varepsilon^2, x/\varepsilon)$ has covariance $\tilde{C}_\varepsilon(t,x) = \varepsilon^{-5}\tilde{C}^\varepsilon(t/\varepsilon^2, x/\varepsilon)$ and converges in distribution to the space-time white noise on $\mathbb{R}\times\mathbb{T}^3$. For large values of α the non–linearity will be negligible with respect to the additive noise term. Heuristically, we can attempt an expansion of the reaction term around the stationary solution Y_ε to the linear equation

$$\mathscr{L} Y_\varepsilon = -Y_\varepsilon + \eta_\varepsilon.$$

Let us denote with $C_{Y,\varepsilon}$ the covariance of Y_ε. We approximate the reaction term as

$$\varepsilon^{\alpha-5/2} F_\varepsilon(\varepsilon^{1/2} u_\varepsilon(t,x)) \simeq \varepsilon^{\alpha-5/2} F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t,x)).$$

The Gaussian r.v. $\varepsilon^{1/2} Y_\varepsilon(t,x)$ has variance $\sigma^2_{Y,\varepsilon} = \varepsilon\mathbb{E}[(Y_\varepsilon(t,x))^2] = \varepsilon\mathbb{E}[(Y_\varepsilon(0,0))^2] = \varepsilon C_{Y,\varepsilon}(0,0)$ independent of (t,x). Therefore we can expand the r.v. $F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t,x))$ according to the chaos decomposition relative to $\varepsilon^{1/2} Y_\varepsilon(t,x)$ and obtain

$$F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t,x)) = \sum_{n\geqslant 0} f_{n,\varepsilon}\, H_n(\varepsilon^{1/2} Y_\varepsilon(t,x), \sigma^2_{Y,\varepsilon}),$$

where $H_n(x, \sigma^2_{Y,\varepsilon})$ are standard Hermite polynomials with variance $\sigma^2_{Y,\varepsilon}$. Note that also the coefficients $(f_{n,\varepsilon})_{n\geqslant 0}$ do not depend on (t,x) by stationarity of the law of $\varepsilon^{1/2} Y_\varepsilon(t,x)$ since they are given by the formula

$$f_{n,\varepsilon} = n!\,\mathbb{E}[F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t,x)) H_n(\varepsilon^{1/2} Y_\varepsilon(t,x), \sigma^2_{Y,\varepsilon})] = n!\,\mathbb{E}[F_\varepsilon(\sigma_{Y,\varepsilon} G) H_n(\sigma_{Y,\varepsilon} G, \sigma^2_{Y,\varepsilon})]$$

where G is a standard Gaussian variable of unit variance. Let X be the stationary solution to the equation

$$\mathscr{L} X = -X + \xi$$

with ξ the space–time white noise and denote by $[\![X^N]\!]$ is the generalized random fields given by the N-th Wick power of X which are well defined as random elements of $\mathcal{S}'(\mathbb{R}\times\mathbb{T}^3)$ as long as $N \leqslant 4$. Denote with C_X the covariance of X. The Gaussian analysis which we set up in this paper shows in particular the following convergence result.

Theorem 1. *Fix $N \leqslant 4$ and assume that $\varepsilon^{(n-N)/2} f_{n,\varepsilon} \to g_n$ for $0 \leqslant n \leqslant N$ as $\varepsilon \to 0$, that $(F_\varepsilon)_\varepsilon \subseteq C^{N+1}(\mathbb{R})$ and there exists constants $c, C > 0$ such that*

$$\sup_{\varepsilon,x} \sum_{k=0}^{N+1} |\partial_x^k F_\varepsilon(x)| \leqslant C e^{c|x|}.$$

Then the family of random fields

$$\mathbb{F}_\varepsilon^N : (t,x) \mapsto \varepsilon^{-N/2} F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon(t,x))$$

converges in law in $\mathcal{S}'(\mathbb{R}\times\mathbb{T}^3)$ as $\varepsilon \to 0$ to $\sum_{n=0}^N g_n [\![X^N]\!]$.

Now take the smallest n such that $f_{n,\varepsilon}$ converges to a finite limit as $\varepsilon \to 0$. Since $H_n(\varepsilon^{1/2} Y_\varepsilon, \sigma^2_{Y,\varepsilon}) = \varepsilon^{n/2} [\![Y_\varepsilon^n]\!]$, the n-th term in the expansion of $F_\varepsilon(\varepsilon^{1/2} Y_\varepsilon)$ is $f_{n,\varepsilon}\, \varepsilon^{\alpha+(n-5)/2} [\![Y_\varepsilon^n]\!]$. From Theorem 1 the equation yields a non-trivial limit only if $\alpha = (5-n)/2$. We are interested mainly in the case $n = 3 \Rightarrow \alpha = 1$ and $n = 1 \Rightarrow \alpha = 2$. The case $\alpha = 2$ gives rise to a Gaussian limit and its analysis its not very difficult. In the following we will concentrate in the analysis of the $\alpha = 1$ case where the limiting behaviour of the model is the most interesting and given by the Φ^4_3 family of singular SPDEs. In this case we obtain the family of models

$$\mathscr{L} u_\varepsilon(t,x) = -\varepsilon^{-\frac{3}{2}} F_\varepsilon(\varepsilon^{\frac{1}{2}} u_\varepsilon(t,x)) + \eta_\varepsilon(t,x) \tag{3}$$

with initial condition $u_{0,\varepsilon}(\cdot) := \varepsilon^{-\frac{1}{2}} \bar{u}_{0,\varepsilon}(\varepsilon^{-1}\cdot)$ where $\bar{u}_{0,\varepsilon}$ is the initial condition of the microscopic model (1).

Define for $m \geqslant 0$ and $\zeta = (t,x) \in \mathbb{R}^+ \times \mathbb{T}^3$

$$\Phi^{(m)}_\zeta := \varepsilon^{(m-3)/2} \tilde{F}^{(m)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon,\zeta}). \tag{4}$$

where $\tilde{F}_\varepsilon$ is the centered function

$$\tilde{F}_\varepsilon(x) := F_\varepsilon(x) - f_{0,\varepsilon} - f_{1,\varepsilon}\, x - f_{2,\varepsilon} H_2(x, \sigma^2_{Y,\varepsilon}) = \sum_{n \geqslant 3} f_{n,\varepsilon}\, H_n(x, \sigma^2_{Y,\varepsilon})$$

and $\sigma^2_{Y,\varepsilon}$ is the variance of the centered Gaussian process $\varepsilon^{1/2} Y_\varepsilon$. Note that $H_n(\varepsilon^{1/2} Y_\varepsilon(\cdot),\, \sigma^2_{Y,\varepsilon}) = \varepsilon^{n/2} [\![Y_\varepsilon^n]\!]$ and denote with $f_{n,\varepsilon}$ the coefficients in the chaos expansion of $F_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon,\zeta})$. Define also various ε–dependent constants

$$\begin{aligned}
d_\varepsilon^{\boxed{?}} &:= \frac{1}{9} \int_{s,x} P_s(x) \mathbb{E}[\Phi^{(1)}_0 \Phi^{(1)}_{(s,x)}], & \tilde{d}_\varepsilon^{\boxed{?}} &:= 2\, \varepsilon^{-1/2} f_{3,\varepsilon} f_{2,\varepsilon} \int_{s,x} P_s(x) [C_{Y,\varepsilon}(s,x)]^2, \\
d_\varepsilon^{\boxed{?}} &:= \frac{1}{6} \int_{s,x} P_s(x) \mathbb{E}[\Phi^{(0)}_0 \Phi^{(2)}_{(s,x)}], & \hat{d}_\varepsilon^{\boxed{?}} &:= \frac{1}{3} \int_{s,x} P_s(x) \mathbb{E}[\Phi^{(0)}_0 \Phi^{(1)}_{(s,x)}], \\
d_\varepsilon^{\boxed{?}} &:= 2\, d_\varepsilon^{\boxed{?}} + 3\, d_\varepsilon^{\boxed{?}}
\end{aligned} \tag{5}$$

where $P_s(x)$ is the heat kernel.

Assumption 2. *All along the paper we enforce the following assumptions:*

a) $(u_{0,\varepsilon})_\varepsilon$ *of eq. (3) converges to a limit u_0 in $\mathscr{C}^{-1/2-\kappa}$ and is independent of η;*

b) $(\bar{u}_{0,\varepsilon})_\varepsilon$ *is uniformly bounded in L^∞, i.e. $\exists C > 0$ such that $\forall \varepsilon \in (0,1]$ $\|\bar{u}_{0,\varepsilon}\|_{L^\infty((\mathbb{T}/\varepsilon)^3)} \leqslant C$;*

c) $(F_\varepsilon)_\varepsilon \subseteq C^9(\mathbb{R})$ *and there exists constants $c, C > 0$ such that*

$$\sup_{\varepsilon,x} \sum_{k=0}^{9} |\partial_x^k F_\varepsilon(x)| \leqslant C e^{c|x|}, \tag{6}$$

d) the vector $\lambda_\varepsilon = (\lambda_\varepsilon^{(0)}, \lambda_\varepsilon^{(1)}, \lambda_\varepsilon^{(2)}, \lambda_\varepsilon^{(3)}) \in \mathbb{R}^4$ *given by*

$$\begin{aligned} \lambda_\varepsilon^{(3)} &= f_{3,\varepsilon} & \lambda_\varepsilon^{(1)} &= \varepsilon^{-1} f_{1,\varepsilon} - 3d_\varepsilon^{\boxed{?}} \\ \lambda_\varepsilon^{(2)} &= \varepsilon^{-1/2} f_{2,\varepsilon} & \lambda_\varepsilon^{(0)} &= \varepsilon^{-3/2} f_{0,\varepsilon} - \varepsilon^{-1/2} f_{2,\varepsilon}\, d_\varepsilon^{\boxed{?}} - 3\tilde{d}_\varepsilon^{\boxed{?}} - 3\hat{d}_\varepsilon^{\boxed{?}} \end{aligned} \tag{7}$$

has a finite limit $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \in \mathbb{R}^4$ *as* $\varepsilon \to 0$.

We can now formulate our main result.

Theorem 3. *Under Assumptions 2 the random field* u_ε *given by the solution to eq. (3) converges in law and locally in time to a limiting random field* $u(\lambda)$ *in the space* $C_T\mathscr{C}^{-\alpha}(\mathbb{T}^3)$ *for every* $1/2 < \alpha < 2/3$. *The law of* $u(\lambda)$ *depends only on the value of* λ *and not on the other details of the nonlinearity or on the covariance of the noise term. We call this limit the dynamic* Φ^4_3 *model with parameter vector* $\lambda \in \mathbb{R}^4$.

Remark 4. In particular we can take

$$F_\varepsilon(x) = \sigma^{(3)} H_3(x, \sigma^2_{Y,\varepsilon}) + \varepsilon^{1/2}\sigma^{(2)} H_2(x, \sigma^2_{Y,\varepsilon}) + \varepsilon\big(\sigma^{(1)} + \gamma_\varepsilon^{(1)}\big) H_1(x, \sigma^2_{Y,\varepsilon}) + \varepsilon^{3/2}\big(\sigma^{(0)} + \gamma_\varepsilon^{(0)}\big)$$

so that

$$f_{3,\varepsilon} = \sigma^{(3)}, \quad \varepsilon^{-1/2} f_{2,\varepsilon} = \sigma^{(2)}, \quad \varepsilon^{-1} f_{1,\varepsilon} = \sigma^{(1)} + \gamma_\varepsilon^{(1)}, \quad \varepsilon^{-3/2} f_{0,\varepsilon} = \sigma^{(0)} + \gamma_\varepsilon^{(0)},$$

and

$$d_\varepsilon^{\boxed{?}} = 3(\sigma^{(3)})^2 L_\varepsilon, \quad \tilde{d}_\varepsilon^{\boxed{?}} = \sigma^{(3)}\sigma^{(2)} L_\varepsilon, \quad d_\varepsilon^{\boxed{?}} = \hat{d}_\varepsilon^{\boxed{?}} = 0,$$

where $L_\varepsilon := 2\int_{s,x} P_s(x)(C_{Y,\varepsilon}(s,x))^2$. Choosing

$$\gamma_\varepsilon^{(1)} := 3d_\varepsilon^{\boxed{?}} = 9(\sigma^{(3)})^2 L_\varepsilon, \qquad \gamma_\varepsilon^{(0)} := 3\tilde{d}_\varepsilon^{\boxed{?}} = 3\sigma^{(3)}\sigma^{(2)} L_\varepsilon,$$

we obtain $\lambda_\varepsilon \to (\sigma^{(0)}, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)})$. This shows that all the possible limits $\lambda \in \mathbb{R}^4$ are attainable. In this case (3) takes the form

$$\mathscr{L} u_\varepsilon = -\sigma^{(3)} u_\varepsilon^3 - \sigma^{(2)} u_\varepsilon^2 - [\sigma^{(1)} - 3\sigma^{(3)}\varepsilon^{-1}\sigma^2_{Y,\varepsilon} + 9(\sigma^{(3)})^2 L_\varepsilon] u_\varepsilon - \sigma^{(0)} + \sigma^{(2)}\sigma^2_{Y,\varepsilon} - 3\sigma^{(3)}\sigma^{(2)} L_\varepsilon + \eta_\varepsilon\,. \tag{8}$$

When the nonlinearity is given by a cubic polynomial like in (8) the corresponding limiting model is called dynamic Φ^4_3 equation or stochastic quantisation equation. In that case the kind of convergence results described above are originally due to Hairer [6, 7] and constitute one of the first groundbreaking applications of his theory of regularity structures. Similar results were later obtained by Catellier and Chouk [3] using the paracontrolled approach of Gubinelli, Imkeller and Perkowski [4]. Kupiainen [10] described a third approach using renormalization group ideas.

Weak universality is the observation that the same limiting object describes the large scale behaviour of the solutions of more general equations, in particular that of the many parameters present in a general model, only a finite number of their combinations survive in the limit to describe the limiting object. The adjective "weak" is related to the fact that in order to control the large scale limit the non-linearity has to be very small in the microscopic scale. This sets up a perturbative regime which is well suited to the analysis via regularity structures or paracontrolled distributions.

The first result of weak universality for a singular stochastic PDE has been given by Hairer and Quastel [8] in the context the Kardar–Parisi–Zhang equation. Using the machinery developed there Hairer and Wu [9] proved a weak universality result for three dimensional reaction–diffusion equations in the case of Gaussian noise and a polynomial non–linearity, within the context of regularity structures. Weak universality for reaction–diffusion equations driven by non Gaussian noise is analysed in Shen and Wu [19]. Important results concerning the stochastic quantisation equation we obtained by Mourrat and Weber. In particular the convergence to the dynamic Φ^4_2 model for a class of Markovian dynamics of discrete spin systems [12] and also the global wellposedness of Φ^4_2 in space and time [13] and in time for Φ^4_3 [14].

The present work is the first which considers in detail the weak universality problem in the context of paracontrolled distributions, showing that on the analytic side the apriori estimates can be obtained via standard arguments and that all the work is related to showing the convergence of a finite number of random fields to universal limiting objects. The main point of our analysis is our use of the Malliavin calculus [18, 17] to perform the analysis of these stochastic terms without requiring polynomial non–linearity as in the previous works cited above. In particular we were inspired by the computations in [16] and in general by the use of the Malliavin calculus to establish normal approximations [17]. The main technical results of this part of our paper is not particularly linked to paracontrolled distributions. A similar analysis could be possible for the stochastic models in regularity structures. Moreover the same tools can also allow to prove similar non-polynomial weak universality statements for the KPZ along the lines of the present analysis. This is the subject of ongoing work.

1 Analysis of the mesoscopic model

In this section we describe the paracontrolled approach [4] to a solution theory for eq. (3) along the lines of the Catellier–Chouk [3] analysis. The fundamental results of paracontrolled calculus we use in this section are listed in Appendix A.

The continuity of the solution map for a paracontrolled equation (already established in [3] and recalled here) allows us to prove convergence of the solution u_ε (Theorem 3) by showing the convergence of the enhanced noise $\mathbb{Y}_\varepsilon$ and the remainder R_ε in the appropriate space. Finally, in Theorem 7 we identify the limiting process as the solution of a paracontrolled equation.

1.1 Paracontrolled structure

Write $u_\varepsilon = Y_\varepsilon + v_\varepsilon$, and perform a Taylor expansion of the reaction term $\tilde{F}_\varepsilon(\varepsilon^{1/2}Y_\varepsilon + \varepsilon^{1/2}v_\varepsilon)$ in (3) around $\varepsilon^{1/2}Y_\varepsilon$ up to the third order to get

$$\begin{aligned}\mathscr{L}\, u_\varepsilon \;=\;& \eta_\varepsilon - \Phi^{(0)} - \Phi^{(1)}v_\varepsilon - \frac{1}{2}\Phi^{(2)}v_\varepsilon^2 - \frac{1}{6}\Phi^{(3)}v_\varepsilon^3 - R_\varepsilon(v_\varepsilon)\\ & -\varepsilon^{-3/2}f_{0,\varepsilon} - \varepsilon^{-1}f_{1,\varepsilon}(Y_\varepsilon + v_\varepsilon) - \varepsilon^{-1/2}f_{2,\varepsilon}(\llbracket Y_\varepsilon^2\rrbracket + 2\,v_\varepsilon\,Y_\varepsilon + v_\varepsilon^2).\end{aligned} \tag{9}$$

with $\Phi_\zeta^{(m)}$ defined in (4) $\forall \zeta \in \mathbb{R}\times\mathbb{T}^3$ and $R_\varepsilon(v_\varepsilon)$ the remainder of the Taylor series.

Define the following random fields:

$$\begin{aligned}
\mathscr{L}\, Y_\varepsilon &:= \eta_\varepsilon & \tilde{Y}_\varepsilon^{\boxed{?}} &:= \varepsilon^{-1/2} f_{2,\varepsilon}\,\llbracket Y_\varepsilon^2\rrbracket\\
\mathscr{L}\, Y_\varepsilon^{\boxed{?}} &:= \Phi^{(0)}, & Y_\varepsilon^{\boxed{?}} &:= Y_\varepsilon^{\boxed{?}} \circ Y_\varepsilon^{\boxed{\cdot}} - d_\varepsilon^{\boxed{?}}\\
Y_\varepsilon^{\boxed{?}} &:= \frac{1}{3}\Phi^{(1)} & Y_\varepsilon^{\boxed{?}} &:= Y_\varepsilon^{\boxed{?}} \circ Y_\varepsilon^{\boxed{?}} - d_\varepsilon^{\boxed{?}} Y_\varepsilon - \hat{d}_\varepsilon^{\boxed{?}}\\
\mathscr{L}\, Y_\varepsilon^{\boxed{?}} &:= Y_\varepsilon^{\boxed{?}}, & Y_\varepsilon^{\boxed{?}} &:= Y_\varepsilon^{\boxed{?}} \circ Y_\varepsilon^{\boxed{?}} - d_\varepsilon^{\boxed{?}}\\
Y_\varepsilon^{\boxed{\cdot}} &:= \frac{1}{6}\Phi^{(2)} & \tilde{Y}_\varepsilon^{\boxed{?}} &:= \tilde{Y}_\varepsilon^{\boxed{?}} \circ Y_\varepsilon^{\boxed{?}} - \tilde{d}_\varepsilon^{\boxed{?}}\\
Y_\varepsilon^{\varnothing} &:= \frac{1}{6}\Phi^{(3)}
\end{aligned} \tag{10}$$

with $\Phi^{(m)}$ defined in (4) and Y_ε, $Y_\varepsilon^{\boxed{?}}$, $Y_\varepsilon^{\boxed{?}}$ stationary solutions. In the scope of this section we can take any constants d_ε^τ that satisfy

$$d_\varepsilon^{\boxed{?}} \;=\; 2\,d_\varepsilon^{\boxed{?}} + 3\,d_\varepsilon^{\boxed{?}}. \tag{11}$$

Equation (9) takes the form

$$\begin{aligned}\mathscr{L}\, v_\varepsilon \;=\;& Y_\varepsilon - Y_\varepsilon^{\boxed{?}} - \tilde{Y}_\varepsilon^{\boxed{?}} - 3Y_\varepsilon^{\boxed{?}}v_\varepsilon - 3Y_\varepsilon^{\boxed{\cdot}}v_\varepsilon^2 - Y_\varepsilon^{\varnothing}v_\varepsilon^3\\ & -\varepsilon^{-3/2}f_{0,\varepsilon} - \varepsilon^{-1}f_{1,\varepsilon}\,(Y_\varepsilon + v_\varepsilon) - \varepsilon^{-1/2}f_{2,\varepsilon}\,(2Y_\varepsilon\,v_\varepsilon + v_\varepsilon^2) - R_\varepsilon(v_\varepsilon).\end{aligned} \tag{12}$$

In this expression the products $Y_\varepsilon^{?}\, v_\varepsilon$, $Y_\varepsilon^{\square}\, v_\varepsilon^2$ and $Y_\varepsilon\, v_\varepsilon$ do not meet the conditions for continuity. In order to continue the analysis we pose the paracontrolled Ansatz

$$v_\varepsilon = -Y_\varepsilon^{?} - \tilde{Y}_\varepsilon^{?} - 3v_\varepsilon \prec\!\!\prec Y_\varepsilon^{?} + v_\varepsilon^\sharp, \tag{13}$$

and proceed to decompose the ill-defined products using the paracontrolled techniques recalled in Appendix A . We start with

$$v_\varepsilon Y_\varepsilon^{?} = v_\varepsilon \prec Y_\varepsilon^{?} + v_\varepsilon \succ Y_\varepsilon^{?} + v_\varepsilon \circ Y_\varepsilon^{?}.$$

The resonant term, together with Ansatz (13), yields:

$$\begin{aligned} v_\varepsilon \circ Y_\varepsilon^{?} \;=\; & -Y_\varepsilon^{?} \circ Y_\varepsilon^{?} - \tilde{Y}_\varepsilon^{?} \circ Y_\varepsilon^{?} - 3\, v_\varepsilon\, (Y_\varepsilon^{?} \circ Y_\varepsilon^{?}) \\ & -3\, \overline{\mathrm{com}}_1(v_\varepsilon, Y_\varepsilon^{?}, Y_\varepsilon^{?}) + v_\varepsilon^\sharp \circ Y_\varepsilon^{?}. \end{aligned}$$

So we let

$$\begin{aligned} Y_\varepsilon^{?} \hat{\diamond} v_\varepsilon \;:=\; & v_\varepsilon Y_\varepsilon^{?} - v_\varepsilon \prec Y_\varepsilon^{?} + (3\, v_\varepsilon\, d_\varepsilon^{?} + d_\varepsilon^{?} Y_\varepsilon + \hat{d}_\varepsilon^{?} + \tilde{d}_\varepsilon^{?}) \\ =\; & v_\varepsilon \succ Y_\varepsilon^{?} - \tilde{Y}_\varepsilon^{?} - Y_\varepsilon^{?} - 3\, v_\varepsilon\, Y_\varepsilon^{?} + v_\varepsilon^\sharp \circ Y_\varepsilon^{?} - 3\, \overline{\mathrm{com}}_1(v_\varepsilon, Y_\varepsilon^{?}, Y_\varepsilon^{?}) \end{aligned}$$

Moreover we have for $v_\varepsilon Y_\varepsilon$:

$$v_\varepsilon\, Y_\varepsilon \;=\; \varphi_\varepsilon Y_\varepsilon - Y_\varepsilon^{?} \prec Y_\varepsilon - Y_\varepsilon^{?} \succ Y_\varepsilon - Y_\varepsilon^{?} \circ Y_\varepsilon$$

where we introduced the shorthand $\varphi_\varepsilon = v_\varepsilon + Y_\varepsilon^{?}$. So we let

$$v_\varepsilon \diamond Y_\varepsilon := v_\varepsilon\, Y_\varepsilon + d_\varepsilon^{?} = \varphi_\varepsilon Y_\varepsilon - Y_\varepsilon^{?} \prec Y_\varepsilon - Y_\varepsilon^{?} \succ Y_\varepsilon - Y_\varepsilon^{?}$$

Finally to analyse the product $Y_\varepsilon^{\square} v_\varepsilon^2$ we write

$$Y_\varepsilon^{\square} v_\varepsilon^2 = Y_\varepsilon^{\square} (Y_\varepsilon^{?})^2 - 2 Y_\varepsilon^{\square} Y_\varepsilon^{?} \varphi_\varepsilon + Y_\varepsilon^{\square} \varphi_\varepsilon^2,$$

and consider the products involving only Y^τ factors: first

$$Y_\varepsilon^{\square} Y_\varepsilon^{?} = Y_\varepsilon^{\square} \succ Y_\varepsilon^{?} + Y_\varepsilon^{\square} \prec Y_\varepsilon^{?} + Y_\varepsilon^{?} + d_\varepsilon^{?} =: Y_\varepsilon^{\square} \diamond Y_\varepsilon^{?} + d_\varepsilon^{?},$$

and then we define the term $Y_\varepsilon^{\square} \diamond (Y_\varepsilon^{?})^2$ as follows:

$$\begin{aligned} Y_\varepsilon^{\square} \diamond (Y_\varepsilon^{?})^2 \;:=\; & Y_\varepsilon^{\square} (Y_\varepsilon^{?})^2 - 2 d_\varepsilon^{?} Y_\varepsilon^{?} \\ =\; & Y_\varepsilon^{\square} \prec (Y_\varepsilon^{?})^2 + Y_\varepsilon^{\square} \succ (Y_\varepsilon^{?})^2 + Y_\varepsilon^{\square} \circ (Y_\varepsilon^{?} \circ Y_\varepsilon^{?}) + 2\mathrm{com}_1(Y_\varepsilon^{?}, Y_\varepsilon^{?}, Y_\varepsilon^{?}) + 2 Y_\varepsilon^{?} Y_\varepsilon^{?}. \end{aligned}$$

so that

$$Y_\varepsilon^{\square} \diamond v_\varepsilon^2 := Y_\varepsilon^{\square} v_\varepsilon^2 + 2 d_\varepsilon^{?} v_\varepsilon = Y_\varepsilon^{\square} \diamond (Y_\varepsilon^{?})^2 - 2\, (Y_\varepsilon^{\square} \diamond Y_\varepsilon^{?}) \varphi_\varepsilon + Y_\varepsilon^{\square} \varphi_\varepsilon^2$$

We note also that

$$\mathscr{L}\, v_\varepsilon = -\mathscr{L}\, Y_\varepsilon^{?} - \mathscr{L}\, \tilde{Y}_\varepsilon^{?} + \mathscr{L}\, v_\varepsilon^\sharp - 3 v_\varepsilon \prec \mathscr{L}\, Y_\varepsilon^{?} - 3\mathrm{com}_3(v_\varepsilon, Y_\varepsilon^{?}) - 3\, \mathrm{com}_2(v_\varepsilon, Y_\varepsilon^{?}).$$

Substituting these renormalized products into (12) we obtain the following equation for $v_\varepsilon^\sharp$:

$$\begin{aligned} \mathscr{L}\, v_\varepsilon^\sharp \;=\; & 3\, \mathrm{com}_3(v_\varepsilon, Y_\varepsilon^{?}) + 3\, \mathrm{com}_2(v_\varepsilon, Y_\varepsilon^{?}) \\ & - Y_\varepsilon^{\varnothing} v_\varepsilon^3 - 3 Y_\varepsilon^{\square} \diamond v_\varepsilon^2 - 3 Y_\varepsilon^{?} \hat{\diamond}\, v_\varepsilon \\ & + Y_\varepsilon - \lambda_\varepsilon^{(2)} (2\, v_\varepsilon \diamond Y_\varepsilon + v_\varepsilon^2) \\ & - \lambda_\varepsilon^{(1)}\, (Y_\varepsilon + v_\varepsilon) + [9 d_\varepsilon^{?} + 6 d_\varepsilon^{?} - 3 d_\varepsilon^{?}] v_\varepsilon - \lambda_\varepsilon^{(0)} - R_\varepsilon(v_\varepsilon) \\ =\; & U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^\sharp) - R_\varepsilon(v_\varepsilon) \end{aligned} \tag{14}$$

with $R_\varepsilon(v_\varepsilon)$ the Taylor remainder which appears in (9) and $\lambda_\varepsilon = (\lambda_\varepsilon^{(0)}, \lambda_\varepsilon^{(1)}, \lambda_\varepsilon^{(2)}, \lambda_\varepsilon^{(3)}) \in \mathbb{R}^4$ given by eq. (7) and where we can use the constraint (11) to remove the term proportional to v_ε. The enhanced noise vector $\mathbb{Y}_\varepsilon$ is defined by

$$\begin{aligned} \mathbb{Y}_\varepsilon &:= (Y_\varepsilon^{\varnothing}, Y_\varepsilon^{\square}, Y_\varepsilon^{?}, \tilde{Y}_\varepsilon^{?}, Y_\varepsilon^{?}, Y_\varepsilon^{?}, Y_\varepsilon^{?}, \tilde{Y}_\varepsilon^{?}, Y_\varepsilon^{?}) \\ &\curlywedge \\ \mathcal{X}_T &:= C_T\mathscr{C}^{-\kappa} \times C_T\mathscr{C}^{-\frac{1}{2}-\kappa} \times (C_T\mathscr{C}^{-1-\kappa})^2 \times \mathscr{L}_T^{1/2-\kappa} \times (C_T\mathscr{C}^{-\kappa})^3 \times C_T\mathscr{C}^{-\frac{1}{2}-\kappa} \end{aligned} \tag{15}$$

for every $\kappa > 0$. We use the notation $\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} = \sum_\tau \|\mathbb{Y}_\varepsilon^\tau\|_{\mathcal{X}^\tau}$ for the associated norm where Y_ε^τ is a generic tree in $\mathbb{Y}_\varepsilon$. The homogeneities $|\tau| \in \mathbb{R}$ are given by

Y_ε^τ	$=$	$Y_\varepsilon^{\varnothing}$	$Y_\varepsilon^{\square}$	$Y_\varepsilon^{?}$	$\tilde{Y}_\varepsilon^{?}$	$Y_\varepsilon^{?}$	$Y_\varepsilon^{?}$	$Y_\varepsilon^{?}$	$\tilde{Y}_\varepsilon^{?}$	$Y_\varepsilon^{?}$
$\lvert\tau\rvert$	$=$	0	$-\frac{1}{2}$	-1	-1	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$

Notice that for every $\varepsilon > 0$ eq. (14) is equivalent to (3) together with Ansatz (13).

1.2 A-priori estimates

In this section we show uniform a-priori estimates for the pair $(v_\varepsilon, v_\varepsilon^\flat)$ which solves the following system of equations

$$\begin{cases} v_\varepsilon &= -Y_\varepsilon^{?} - \tilde{Y}_\varepsilon^{?} - 3v_\varepsilon \prec Y_\varepsilon^{?} + v_\varepsilon^\flat + v_\varepsilon^\natural \\ \mathscr{L} v_\varepsilon^\flat &= U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^\flat + v_\varepsilon^\natural) - R_\varepsilon(v_\varepsilon) \\ v_\varepsilon^\flat(0) &= Y_\varepsilon^{?}(0) + \tilde{Y}_\varepsilon^{?}(0) + 3v_{\varepsilon,0} \prec Y_\varepsilon^{?}(0) \end{cases} \tag{16}$$

with $v_\varepsilon^\natural(t) := P_t\, v_{\varepsilon,0}$ and $v_{\varepsilon,0} := u_{0,\varepsilon} - Y_\varepsilon(0) \in \mathscr{C}^{-1/2-}$. U is given in (14). It is easy to see, by taking $v_\varepsilon^\sharp = v_\varepsilon^\flat + v_\varepsilon^\natural$, that this is equivalent to eq. (14) together with Ansatz (13) on v_ε. We consider the spaces

$$\mathcal{V}_T^\flat := \mathscr{L}_T^{2\kappa} \cap \mathscr{L}_T^{1/4,1/2+2\kappa} \cap \mathscr{L}_T^{1/2,1+2\kappa}, \quad \mathcal{V}_T := \mathscr{L}_T^{1/2,1/2-\kappa} \cap \mathscr{L}_T^{1/4+3\kappa/2,2\kappa},$$

with the corresponding norms

$$\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat} := \|v_\varepsilon^\flat\|_{\mathscr{L}_T^{2\kappa}} + \|v_\varepsilon^\flat\|_{\mathscr{L}_T^{1/4,1/2+2\kappa}} + \|v_\varepsilon^\flat\|_{\mathscr{L}_T^{1/2,1+2\kappa}}, \tag{17}$$

$$\|v_\varepsilon\|_{\mathcal{V}_T} := \|v_\varepsilon\|_{\mathscr{L}^{1/2,1/2-\kappa}} + \|v_\varepsilon\|_{\mathscr{L}^{1/4+3\kappa/2,2\kappa}}. \tag{18}$$

Define also the quantity

$$M_\varepsilon(Y_\varepsilon, u_{0,\varepsilon}) := \|\varepsilon^{\delta/2} e^{c\varepsilon^{1/2}|Y_\varepsilon| + c\varepsilon^{1/2}|P.v_{\varepsilon,0}|}\|_{L^p[0,T]L^p(\mathbb{T}^3)}. \tag{19}$$

which will be used to control the remainder R_ε. The main result of this section is the following lemma.

Lemma 5. *There exists a time $T_\star = T_\star(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}, \|u_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}}, |\lambda_\varepsilon|) \in (0, T]$ depending only on $\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}$, $\|u_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}}$ and $|\lambda_\varepsilon|$, a constant $M_\varepsilon = M_\varepsilon(Y_\varepsilon, u_{0,\varepsilon}) > 0$ defined by (19), and a universal constant $C > 0$ such that, whenever $M_\varepsilon \leqslant T_\star^{\kappa/2}$ we have*

$$\begin{aligned} \|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_{T_\star}^\flat} &\leqslant C(1 + |\lambda_{\varepsilon_n}|)\,(1 + \|\mathbb{Y}_{\varepsilon_n}\|_{\mathcal{X}_T})^3 (1 + \|u_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})^3, \\ \|v_\varepsilon\|_{\mathcal{V}_{T_\star}} &\leqslant C\big(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|u_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_{T_\star}^\flat}\big). \end{aligned}$$

Proof. Define $v_\varepsilon^\square := v_\varepsilon - v_\varepsilon^\natural$ such that

$$v_\varepsilon^\square = -Y_\varepsilon^{?} - \tilde{Y}_\varepsilon^{?} - 3(v_\varepsilon^\square + v_\varepsilon^\natural) \prec Y_\varepsilon^{?} + v_\varepsilon^\flat,$$

and $v_\varepsilon^\square(0) = 0$. Note also $v_\varepsilon^\boxtimes := v_\varepsilon^\square + Y_\varepsilon^{?}$. Using Lemma 15 (and the fact that $\|f\|_{\mathscr{L}_T^{\kappa,\alpha}} \lesssim T^\kappa \|f\|_{\mathscr{L}_T^\alpha}$) we obtain for $\kappa, \theta > 0$ small enough

$$\|If\|_{\mathscr{L}_T^{-\theta+2\kappa,2\kappa}} + \|If\|_{\mathscr{L}_T^{1/4-\theta+2\kappa,1/2+2\kappa}} + \|If\|_{\mathscr{L}_T^{1/2-\theta+2\kappa,1+2\kappa}} \lesssim T^{\frac{\kappa}{2}} (\|f\|_{\mathcal{M}_T^{1-\theta}\mathscr{C}^{-\kappa}} + \|f\|_{\mathcal{M}^{1/2+2\kappa}\mathscr{C}^{-1/2-2\kappa}}). \tag{20}$$

We choose $\theta > 2\kappa$ small enough so that

$$\mathscr{L}_T^{-\theta+3\kappa/2,2\kappa} \cap \mathscr{L}_T^{1/4-\theta+3\kappa/2,1/2+2\kappa} \cap \mathscr{L}_T^{1/2-\theta+3\kappa/2,1+2\kappa} \subseteq \mathcal{V}_T^\flat$$

We define also the norm

$$\|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} := \|v^\square\|_{\mathscr{L}_T^{\frac{2}{T}\kappa}} + \|v^\square\|_{\mathcal{M}_T^{1/4}\mathscr{C}^{1/2+2\kappa}}.$$

Now

$$\begin{aligned}
\|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} &\lesssim \|Y_\varepsilon^{\boxed{?}} + \tilde{Y}_\varepsilon^{\boxed{?}}\|_{\mathcal{V}_T^\square} + \|v_\varepsilon^\square\|_{C_T L^\infty}(\|Y_\varepsilon^{\boxed{?}}\|_{C_T\mathscr{C}^{1-\kappa}} + \|Y_\varepsilon^{\boxed{?}}\|_{C_T\mathscr{C}^{-1-\kappa}}) \\
&\quad + \big(\|v_\varepsilon^\natural\|_{C_T\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\natural\|_{\mathcal{M}_T^{1/4}\mathscr{C}^{-\kappa}}\big)(\|Y_\varepsilon^{\boxed{?}}\|_{C_T\mathscr{C}^{1-\kappa}} + \|Y_\varepsilon^{\boxed{?}}\|_{C_T\mathscr{C}^{-1-\kappa}}) + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\square} \\
&\lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + T^\kappa \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} + \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}
\end{aligned}$$

where we used that $v^\square(0) = 0$ and as a consequence that $\|v_\varepsilon^\square\|_{C_T L^\infty} \leqslant T^\kappa \|v_\varepsilon^\square\|_{C_T^\kappa L^\infty} \leqslant T^\kappa \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square}$ to gain a small power of T. So provided T is small enough (depending only on $\mathbb{Y}_\varepsilon$) this yields the following a-priori estimation on $v_\varepsilon^\square$:

$$\|v_\varepsilon^\square\|_{C_T L^\infty} \lesssim \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} \lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}.$$

Therefore we have an estimation on v_ε:

$$\|v_\varepsilon\|_{\mathcal{V}_T} \leqslant \|v_\varepsilon^\natural\|_{\mathcal{V}_T} + \|v^\square\|_{\mathcal{V}_T} \lesssim \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\square\|_{\mathcal{V}_T^\square} \lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}.$$

In order to estimate terms in $U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^\flat + v_\varepsilon^\natural)$ we decompose the renormalised products as

$$\begin{aligned}
Y_\varepsilon^{\boxed{?}} \hat{\diamond} v_\varepsilon &= v_\varepsilon \succ Y_\varepsilon^{\boxed{?}} - \tilde{Y}_\varepsilon^{\boxed{?}} - Y_\varepsilon^{\boxed{?}} - 3\, v_\varepsilon\, Y_\varepsilon^{\boxed{?}} + v_\varepsilon^\flat \circ Y_\varepsilon^{\boxed{?}} + v_\varepsilon^\natural \circ Y_\varepsilon^{\boxed{?}} - 3\, \overline{\mathrm{com}}_1(v_\varepsilon, Y_\varepsilon^{\boxed{?}}, Y_\varepsilon^{\boxed{?}}) \\
v_\varepsilon \diamond Y_\varepsilon &= -\tilde{Y}_\varepsilon^{\boxed{?}} Y_\varepsilon - 3(v_\varepsilon \prec\!\!\prec Y_\varepsilon^{\boxed{?}}) Y_\varepsilon + Y_\varepsilon \preccurlyeq (v_\varepsilon^\flat + v_\varepsilon^\natural) + Y_\varepsilon \succ (v_\varepsilon^\flat + v_\varepsilon^\natural) \\
&\quad - Y_\varepsilon^{\boxed{?}} \prec Y_\varepsilon - Y_\varepsilon^{\boxed{?}} \succ Y_\varepsilon - Y_\varepsilon^{\boxed{?}} \\
Y_\varepsilon^{\boxed{\cdot}} \diamond v_\varepsilon^2 &= Y_\varepsilon^{\boxed{\cdot}} \diamond (Y_\varepsilon^{\boxed{?}})^2 + 2(Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}})(\tilde{Y}_\varepsilon^{\boxed{?}} + 3 v_\varepsilon \prec\!\!\prec Y_\varepsilon^{\boxed{?}}) - 2\,(Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}}) \preccurlyeq (v_\varepsilon^\flat + v_\varepsilon^\natural) \\
&\quad + 2\,(Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}}) \succ (v_\varepsilon^\flat + v_\varepsilon^\natural) + Y_\varepsilon^{\boxed{\cdot}} \preccurlyeq (v_\varepsilon^{\boxed{\varnothing}} + v^\natural)^2 + Y_\varepsilon^{\boxed{\cdot}} \succ (v_\varepsilon^{\boxed{\varnothing}} + v^\natural)^2.
\end{aligned}$$

We have $U(\lambda_\varepsilon, \mathbb{Y}_\varepsilon; v_\varepsilon, v_\varepsilon^\flat + v_\varepsilon^\natural) = Q_{-1/2}(\lambda_\varepsilon, \mathbb{Y}_\varepsilon, v_{0,\varepsilon}, v_\varepsilon, v_\varepsilon^\flat) + Q_0(\lambda_\varepsilon, \mathbb{Y}_\varepsilon, v_{0,\varepsilon}, v_\varepsilon, v_\varepsilon^\flat) + Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon}$ with the definitions

$$\begin{aligned}
Q_{-1/2} &:= -3[v_\varepsilon \succ Y_\varepsilon^{\boxed{?}} - 3\,\overline{\mathrm{com}}_1(v_\varepsilon, Y_\varepsilon^{\boxed{?}}, Y_\varepsilon^{\boxed{?}}) + Y_\varepsilon^{\boxed{\cdot}} \succ (v_\varepsilon^{\boxed{\varnothing}} + v^\natural)^2] \\
&\quad -6[(Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}})(3 v_\varepsilon \prec\!\!\prec Y_\varepsilon^{\boxed{?}}) + (Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}}) \succ (v_\varepsilon^\flat + v_\varepsilon^\natural)] \\
&\quad +2\lambda_\varepsilon^{(2)}(3(v_\varepsilon \prec\!\!\prec Y_\varepsilon^{\boxed{?}}) Y_\varepsilon - Y_\varepsilon \succ (v_\varepsilon^\flat + v_\varepsilon^\natural)) + 3\,\mathrm{com}_3(v_\varepsilon, Y_\varepsilon^{\boxed{?}}) + 3\,\mathrm{com}_2(v_\varepsilon, Y_\varepsilon^{\boxed{?}}) \\
Q_0 &:= 3[3 v_\varepsilon\, Y_\varepsilon^{\boxed{?}} - v_\varepsilon^\flat \circ Y_\varepsilon^{\boxed{?}} - v_\varepsilon^\natural \circ Y_\varepsilon^{\boxed{?}} + 2\,(Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}}) \preccurlyeq (v_\varepsilon^\flat + v_\varepsilon^\natural) - Y_\varepsilon^{\boxed{\cdot}} \preccurlyeq (v_\varepsilon^{\boxed{\varnothing}} + v^\natural)^2] \\
&\quad -Y_\varepsilon^{\boxed{\varnothing}} v_\varepsilon^3 - \lambda_\varepsilon^{(2)}[v_\varepsilon^2 + 2 Y_\varepsilon \preccurlyeq (v_\varepsilon^\flat + v_\varepsilon^\natural)] \\
Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon} &:= \big(1 - \lambda_\varepsilon^{(1)}\big) Y_\varepsilon - \lambda_\varepsilon^{(0)} + 3\,[\tilde{Y}_\varepsilon^{\boxed{?}} + Y_\varepsilon^{\boxed{?}} - Y_\varepsilon^{\boxed{\cdot}} \diamond (Y_\varepsilon^{\boxed{?}})^2 - 2(Y_\varepsilon^{\boxed{\cdot}} \diamond Y_\varepsilon^{\boxed{?}}) \tilde{Y}_\varepsilon^{\boxed{?}}] \\
&\quad +2\lambda_\varepsilon^{(2)}(\tilde{Y}_\varepsilon^{\boxed{?}} Y_\varepsilon + Y_\varepsilon^{\boxed{?}} \prec Y_\varepsilon + Y_\varepsilon^{\boxed{?}} \succ Y_\varepsilon + Y_\varepsilon^{\boxed{?}})
\end{aligned}$$

With the same technique we used above for $v_\varepsilon^\square$, we obtain the following estimate on $v_\varepsilon^{\boxed{\varnothing}}$

$$\|v_\varepsilon^{\boxed{\varnothing}}\|_{\mathscr{L}_T^{1/2+3\kappa/2,1/2+2\kappa}} + \|v_\varepsilon^{\boxed{\varnothing}}\|_{\mathscr{L}_T^{1/4+\kappa,\kappa}} \lesssim \|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}$$

and this yields $\|(v_\varepsilon^{\boxed{\varnothing}})^2\|_{\mathscr{L}_T^{3/4+5\kappa/2,1/2+2\kappa}} + \|(v_\varepsilon^{\boxed{\varnothing}})^2\|_{\mathscr{L}_T^{1/2+2\kappa,\kappa}} \lesssim (\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T} + \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat})^2$.

We obtain, using the results of Appendix A

$$\begin{aligned}
\|Q_{-1/2}\|_{\mathcal{M}^{1/2+2\kappa}\mathscr{C}^{-1/2-2\kappa}} + \|Q_0\|_{\mathcal{M}_T^{1-\theta}\mathscr{C}^{-\kappa}} &\lesssim (1+|\lambda_\varepsilon|)(1+\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3(1 + \|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}} + \|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat})^3 \\
\|Q_{\lambda_\varepsilon, \mathbb{Y}_\varepsilon}\|_{C_T\mathscr{C}^{-1/2-\kappa}} &\lesssim (1+|\lambda_\varepsilon|)\,(1+\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3
\end{aligned}$$

In order to conclude the estimation of $\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}$ we have to bound $\|IR_\varepsilon(v_\varepsilon)\|_{\mathcal{V}_T^\flat}$. By Lemma 6 $\forall\delta\in(0,1)$, $\forall\theta>0$ such that $\frac{1-\theta}{3+\delta}>\frac14+\frac{3\kappa}{2}$ we have $\|t\mapsto t^{1-\theta}IR_\varepsilon(v_\varepsilon)(t,x)\|_{L^p[0,T]L^p(\mathbb{T}^3)}\lesssim M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})\|v_\varepsilon\|_{\mathcal{V}_T}^{3+\delta}e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{\mathcal{V}^\square}}$. By Lemma 16 together with (50) we obtain then

$$\|IR_\varepsilon(v_\varepsilon)\|_{\mathcal{V}_T^\flat}\lesssim M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})\|v_\varepsilon\|_{\mathcal{V}_T}^{3+\delta}e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{\mathcal{V}^\square}}.$$

Noticing that $\|Pv_\varepsilon^\flat(0)\|_{\mathcal{V}_T^\flat}\lesssim\|v_\varepsilon^\flat(0)\|_{C_T\mathscr{C}^{1/2-2\kappa}}\lesssim(1+\|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}$ we obtain that $\exists C'>0$ such that

$$\begin{aligned}\|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_T^\flat}&\leqslant C'(1+|\lambda_{\varepsilon_n}|)\,(1+\|\mathbb{Y}_{\varepsilon_n}\|_{\mathcal{X}_T})^3(1+\|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})^3+C'T^{\kappa/2}(1+|\lambda_\varepsilon|)(1+\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}^3\\&\quad+C'\,M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})e^{c\varepsilon^{1/2}(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}+\|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})}e^{c\varepsilon^{1/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}}\|v_\varepsilon\|_{\mathcal{V}_T}^{3+\delta}\\&\leqslant D+C\,M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}}+CT^{\kappa/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}^3+C\,M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}}\|v_\varepsilon^\flat\|_{\mathcal{V}_T^\flat}^{3+\delta}\end{aligned}$$

with $C:=C'[(1+|\lambda_\varepsilon|)(1+\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T})^3+e^{c\varepsilon^{1/2}(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}+\|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})}(1+(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}+\|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})^{3+\delta})]$ and $D:=C'(1+|\lambda_{\varepsilon_n}|)\,(1+\|\mathbb{Y}_{\varepsilon_n}\|_{\mathcal{X}_T})^3(1+\|v_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}})^3$.

Let $T_\star\in(0,T]$ such that:

$$CT_\star^{\kappa/2}[(5\,C)^2+e^{c\varepsilon^{1/2}(5C)}(5\,C)^{2+\delta}]\leqslant\frac12$$

and

$$CT_\star^{\kappa/2}e^{c\varepsilon^{1/2}(5C)}\leqslant D.$$

Assume that $M_\varepsilon\leqslant T_\star^{\kappa/2}$. Define a closed interval $[0,S]=\{t\in[0,T_\star]\colon\|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_t^\flat}\leqslant4D\}\subseteq[0,T_\star]$. This interval is well defined and non–empty since $t\mapsto\|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_t^\flat}$ is continuous and nondecreasing and $\|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_0^\flat}\leqslant4D$. Let us assume that $S<T_\star$, then we can take $\epsilon>0$ small enough such that $S+\epsilon<T_\star$ and by continuity $\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}\leqslant5C$, then

$$\begin{aligned}\|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}&\leqslant D+C\,M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}}+C\,(S+\epsilon)^{\kappa/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}^3+C\,M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})e^{c\varepsilon^{1/2}\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}}\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}^{3+\delta}\\&\leqslant D+C\,M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})e^{c\varepsilon^{1/2}(5C)}+CT_\star^{\kappa/2}(5\,C)^2\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}+CT_\star^{\kappa/2}e^{c\varepsilon^{1/2}(5C)}(5C)^{2+\delta}\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}\\&\leqslant 2D+\frac12\|v_\varepsilon^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}\end{aligned}$$

which gives $\|v_{\varepsilon_n}^\flat\|_{\mathcal{V}_{S+\epsilon}^\flat}\leqslant4D$. This implies $S=T_\star$ (by contradiction). □

Lemma 6. *For every $\gamma\in(0,1)$, $\delta\in[0,1]$ we have*

$$\|t\mapsto t^\gamma R_\varepsilon(v_\varepsilon,v_\varepsilon^\flat,v_\varepsilon^\natural)(t,x)\|_{L^p[0,T]L^p(\mathbb{T}^3)}\lesssim M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})\|v_\varepsilon\|_{\mathcal{M}^{\gamma/(3+\delta)}L^\infty}^{3+\delta}e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{C_TL^\infty}}$$

with $v_\varepsilon^\square:=-Y_\varepsilon^{\boxed{?}}-\tilde{Y}_\varepsilon^{\boxed{?}}-3v_\varepsilon\prec Y_\varepsilon^{\boxed{?}}+v_\varepsilon^\flat$.

Proof. We can write the remainder in two ways:

$$R_\varepsilon(v_\varepsilon)=v_\varepsilon^3\int_0^1[F_\varepsilon^{(3)}(\varepsilon^{\frac12}Y_\varepsilon+\tau\varepsilon^{\frac12}v_\varepsilon)-F_\varepsilon^{(3)}(\varepsilon^{\frac12}Y_\varepsilon)]\frac{(1-\tau)^2}{2!}\mathrm{d}\tau=\varepsilon^{\frac12}v_\varepsilon^4\int_0^1F_\varepsilon^{(4)}(\varepsilon^{\frac12}Y_\varepsilon+\tau\varepsilon^{\frac12}v_\varepsilon)\frac{(1-\tau)^3}{3!}\mathrm{d}\tau.$$

From assumption (6) on F we obtain by interpolation of these two expressions, $\forall\delta\in[0,1]$, $\forall t\geqslant0,x\in\mathbb{T}^3$,

$$|R_\varepsilon(v_\varepsilon)(t,x)|\lesssim\varepsilon^{\delta/2}|v_\varepsilon(t,x)|^{3+\delta}e^{c\varepsilon^{\frac12}|Y_\varepsilon(t,x)|+c\varepsilon^{\frac12}|v_\varepsilon^\natural(t,x)|+c|\varepsilon^{\frac12}v_\varepsilon^\square(t,x)|},$$

and we estimate, $\forall\gamma\in[0,1)$,

$$\|t\mapsto t^\gamma R_\varepsilon(v_\varepsilon)(t,x)\|_{L^p[0,T]L^p(\mathbb{T}^3)}\lesssim\|t^{\frac{\gamma}{3+\delta}}\,v_\varepsilon(t)\|_{C_TL^\infty}^{3+\delta}e^{c\varepsilon^{1/2}\|v_\varepsilon^\square\|_{C_TL^\infty}}\Big\|\varepsilon^{\frac\delta2}e^{c\varepsilon^{\frac12}|Y_\varepsilon(t,x)|+c\varepsilon^{\frac12}|v_\varepsilon^\natural(t,x)|}\Big\|_{L^p[0,T]L^p(\mathbb{T}^3)}.$$ □

1.3 Identification of the limit

In order to identify interesting limits for equation (3), we introduce the enhanced universal noise $\mathbb{X}$, defined as

$$\mathbb{X}=(X,X^{\boxed{?}},X^{\boxed{?}},X^{\boxed{?}},X^{\boxed{?}},X^{\boxed{?}}),$$

where X is the stationary solution to to the linear equation $\mathscr{L}X=-X+\xi$ and ξ is the time-space white noise on $\mathbb{R}\times\mathbb{T}^3$. We define

$$\begin{aligned}
X^{\textcolor{red}{?}} &:= [\![X^3]\!],\\
X^{\textcolor{red}{?}} &:= [\![X^2]\!],\\
\Delta_q X^{\textcolor{red}{?}} &:= \Delta_q(X^{\textcolor{red}{?}}\circ X)=\int_{\zeta_1,\zeta_2}[\![X^3_{\zeta_1}]\!]X_{\zeta_2}\,\mu_{\zeta_1,\zeta_2},\\
\Delta_q X^{\textcolor{red}{?}} &:= \Delta_q(1-J_0)(X^{\textcolor{red}{?}}\circ X^{\textcolor{red}{?}})=\int_{\zeta_1,\zeta_2}(1-J_0)([\![X^2_{\zeta_1}]\!][\![X^2_{\zeta_2}]\!])\,\mu_{\zeta_1,\zeta_2},\\
\Delta_q X^{\textcolor{red}{?}} &:= \int_{\zeta_1,\zeta_2}(1-J_1)([\![X^3_{\zeta_1}]\!][\![X^2_{\zeta_2}]\!])\,\mu_{\zeta_1,\zeta_2}+6\int_{s,x}[\Delta_q X(t+s,\bar{x}-x)-\Delta_q X(t,\bar{x})]P_s(x)\,[C_X(s,x)]^2,
\end{aligned}\tag{21}$$

where as before $[\![\cdot]\!]$ stands for the Wick product, $\zeta_i=(x_i,s_i)\in\mathbb{R}\times\mathbb{T}^3$, $C_X(t,x)$ is the covariance of X and μ_{ζ_1,ζ_2} is defined as

$$\mu_{\zeta_1,\zeta_2}:=[\int_{x,y}K_{q,\bar{x}}(x)\sum_{i\sim j}K_{i,x}(y)K_{j,x}(x_2)P_{t-s_1}(y-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2.$$

Standard computations (see e.g. [3] or [15]) show that, for any $T>0$, $0<\kappa<\kappa'$

$$\mathbb{X}\in C_T^\kappa\mathscr{C}^{-\frac{1}{2}-2\kappa'}\times(C_T^\kappa\mathscr{C}^{-1-2\kappa'})^2\times C_T^\kappa\mathscr{C}^{\frac{1}{2}-2\kappa'}\times(C_T^\kappa\mathscr{C}^{0-2\kappa'})^2\times C_T^\kappa\mathscr{C}_T^{-\frac{1}{2}-2\kappa'},$$

almost surely. Finally, for every $\lambda=(\lambda^{(0)},\lambda^{(1)},\lambda^{(2)},\lambda^{(3)})\in\mathbb{R}^4$ we define

$$\mathbb{Y}(\lambda):=(\lambda^{(3)},\lambda^{(3)}X,\lambda^{(3)}X^{\textcolor{red}{?}},\lambda^{(2)}X^{\textcolor{red}{?}},\lambda^{(3)}X^{\textcolor{red}{?}},(\lambda^{(3)})^2X^{\textcolor{red}{?}},(\lambda^{(3)})^2X^{\textcolor{red}{?}},\lambda^{(3)}\lambda^{(2)}X^{\textcolor{red}{?}},(\lambda^{(3)})^2X^{\textcolor{red}{?}}).\tag{22}$$

Using the paracontrolled structure we developed in the preceding sections and its continuity with respect to $\mathbb{Y}_\varepsilon$, we can state the convergence of the solution of the mesoscopic equation, under the hypothesis that $\mathbb{Y}_\varepsilon$ and M_ε converge (this will is shown in Theorem-9 and Lemma 8).

Theorem 7. *The family of random fields u_ε given by the solutions to eq. (3) converges in law and locally in time to a limiting random field $u(\lambda)$ in the space $C_T\mathscr{C}^{-\alpha}(\mathbb{T}^3)$ for every $1/2<\alpha<2/3$. The limiting random field $u(\lambda)$ solves the paracontrolled equation*

$$\begin{cases}
u(\lambda) &= X+v(\lambda)\\
v(\lambda) &= X-\lambda^{(3)}X^{\textcolor{red}{?}}-\lambda^{(2)}X^{\textcolor{red}{?}}-3\lambda^{(3)}v(\lambda)\prec\!\!\prec X^{\textcolor{red}{?}}+v^\sharp(\lambda)\\
\mathscr{L}v^\sharp(\lambda) &= U(\lambda,\mathbb{Y}(\lambda);v(\lambda),v^\sharp(\lambda))\\
v^\sharp(\lambda)(t=0) &= v_0+\lambda^{(3)}X^{\textcolor{red}{?}}(t=0)+\lambda^{(2)}X^{\textcolor{red}{?}}(t=0)+3\lambda^{(3)}v_{\varepsilon,0}\prec X^{\textcolor{red}{?}}(t=0)
\end{cases}\tag{23}$$

with U defined in (14) and $v_0=u_0-X(t=0)$.

Proof. Fix $T>0$. Let us denote via Γ the solution map for (16) so that $u_\varepsilon=\Gamma(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,\lambda_\varepsilon,R_\varepsilon(v_\varepsilon))$. Denote by $u_\varepsilon^\bullet=\Gamma_\bullet(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,\lambda_\varepsilon,R_\varepsilon(v_\varepsilon))$ the process u_ε stopped at the time $T_\star(\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T},\|u_{\varepsilon,0}\|_{\mathscr{C}^{-1/2-\kappa}},|\lambda_\varepsilon|)$ and $\Gamma_\bullet$ the corresponding solution map. Note that u solves the same equation with $\mathbb{Y}_\varepsilon$ replaced by $\mathbb{Y}(\lambda)$, $u_{\varepsilon,0}$ with u_0, λ_ε replaced by λ and $R_\varepsilon=0$. So $u(\lambda)=u^\bullet(\lambda)=\Gamma_\bullet(u_0,\mathbb{Y}(\lambda),\lambda,0)$ up to time $T_\star(\|\mathbb{Y}(\lambda)\|_{\mathcal{X}_T},\|u_0\|_{\mathscr{C}^{-1/2-\kappa}},|\lambda|)$. Let us introduce the random field $\tilde{u}_\varepsilon^\bullet=\Gamma_\bullet(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,\lambda_\varepsilon,0)$ which solves the paracontrolled equation (16) but with remainder $R_\varepsilon=0$. Consider the n-uple of random variables $(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,u_\varepsilon^\bullet,\tilde{u}_\varepsilon^\bullet)$ and let μ_ε be its law on $\mathcal{Z}=\mathscr{C}^{-\alpha}\times\mathcal{X}_T\times(C_T\mathscr{C}^{-\alpha})^2$ conditionally on $\mathcal{E}_\varepsilon:=\{M_\varepsilon\leqslant T_\star^{\kappa/2}\}$. Note that we know that $\mathbb{P}(\mathcal{E}_\varepsilon)\to1$ from Lemma 8. By the apriori bounds of Lemma 5 we have tightness of the family $(\mu_\varepsilon)_\varepsilon$. By standard arguments it is easy to obtain continuity of the map $\Gamma_\bullet$ and also to observe that for any $\delta>0$, $\mu_\varepsilon(\|u_\varepsilon^\bullet-\tilde{u}_\varepsilon^\bullet\|>\delta)\to0$ as $\varepsilon\to0$ since $M_\varepsilon\to0$ in probability. This shows that μ_ε concentrates on $\mathscr{C}^{-\alpha}\times\mathcal{X}_T\times\{(z,z)\in C_T\mathscr{C}^{-\alpha}\}$. Let μ any accumulation point of $(\mu_\varepsilon)_\varepsilon$. Then $\mu(\mathscr{C}^{-\alpha}\times\mathcal{X}_T\times\{(z,z)\in C_T\mathscr{C}^{-\alpha}\})=1$. Moreover along subsequences we have that for any bounded continuous function

$$\mathbb{E}(\varphi(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,\tilde{u}_\varepsilon^\bullet))=\mathbb{E}(\varphi(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,\Gamma_\bullet(u_{\varepsilon,0},\mathbb{Y}_\varepsilon,\lambda_\varepsilon,0)))\to\mathbb{E}(\varphi(u_0,\mathbb{Y}(\lambda),\Gamma_\bullet(u_0,\mathbb{Y}(\lambda),\lambda,0)))$$

since by Theorem 9 the vector $\mathbb{Y}_\varepsilon$ converges in law to $\mathbb{Y}(\lambda)$ and $u_{\varepsilon,0}$ to u_0 and $\Gamma_\bullet$ is a continuous function. We deduce that, still along subsequences, for any test function φ,

$$\int_{\mathcal{Z}}\varphi(x,y,z,t)\mathrm{d}\mu_\varepsilon(x,y,z,t)\to\int_{\mathcal{Z}}\varphi(x,y,t,t)\mathrm{d}\mu(x,y,z,t)=\int_{\mathcal{Z}}\varphi(x,y,\Gamma_\bullet(x,y,\lambda,0),\Gamma_\bullet(x,y,\lambda,0))\mathrm{d}\mu(x,y,z,t)$$

but we know also that since $\mathbb{P}(\mathcal{E}_\varepsilon)\to 1$ we have

$$\mathbb{E}[\psi(u_{\varepsilon,0},\mathbb{Y}_\varepsilon)|\mathcal{E}_\varepsilon]=\frac{\mathbb{E}[\psi(u_{\varepsilon,0},\mathbb{Y}_\varepsilon)\mathbb{I}_{\mathcal{E}_\varepsilon}]}{\mathbb{P}(\mathcal{E}_\varepsilon)}\to\mathbb{E}[\psi(u_0,\mathbb{Y}(\lambda))],$$

for any test function ψ. So the first two marginals of μ have the law of $(u_0,\mathbb{Y}(\lambda))$ and they are independent since $(u_{\varepsilon,0},\mathbb{Y}_\varepsilon)$ are independent for any ε. Calling ν the law of $(u_0,\mathbb{Y}(\lambda))$ we have that

$$\int_{\mathcal{Z}}\varphi(x,y,z,t)\mathrm{d}\mu(x,y,z,t)=\int_{\mathscr{C}^{-\alpha}\times\mathcal{X}_T}\varphi(x,y,\Gamma_\bullet(x,y,\lambda,0),\Gamma_\bullet(x,y,\lambda,0))\mathrm{d}\nu(x,y)$$

which implies that μ is unique and that the whole family $(\mu_\varepsilon)_\varepsilon$ converges to μ. □

2 Convergence of random fields

In this section we prove the convergence of the random fields $\mathbb{Y}_\varepsilon$ and M_ε. The convergence in probability of M_ε is easily obtained as we show in the following lemma.

Lemma 8. *Under Assumptions 2 the random variable $M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})$ defined in (19) converges to zero in probability.*

Proof. Recalling that $v_{\varepsilon,0}:=u_{0,\varepsilon}-Y_\varepsilon(0)$ we can use Young's inequality estimate $M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})$ for some $c'>0$ as

$$\begin{aligned}M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})\ \lesssim\ &\varepsilon^{\delta/2}\|e^{c'\varepsilon^{1/2}|Y_\varepsilon|}\|_{L^p[0,T]L^p(\mathbb{T}^3)}+\varepsilon^{\delta/2}\|e^{c'\varepsilon^{1/2}|P_\cdot Y_\varepsilon(0)|}\|_{L^p[0,T]L^p(\mathbb{T}^3)}\\&+\varepsilon^{\delta/2}T^{1/p}e^{c'\|\varepsilon^{1/2}u_{0,\varepsilon}\|_{L^\infty}}.\end{aligned}$$

Under Assumptions 2 the term $\|\varepsilon^{1/2}u_{0,\varepsilon}\|_{L^\infty(\mathbb{T}^3)}$ is uniformly bounded, so the third term above converges to zero almost surely. Note that $\varepsilon^{1/2}Y_\varepsilon(t,x)$ and $P_t\varepsilon^{1/2}Y_\varepsilon(t=0)$ are centered Gaussian random variables, and then both $\mathbb{E}\|e^{c'\varepsilon^{1/2}|Y_\varepsilon|}\|^p_{L^p[0,T]L^p(\mathbb{T}^3)}$ and $\mathbb{E}\|e^{c'\varepsilon^{1/2}|P_\cdot Y_\varepsilon(0)|}\|^p_{L^p[0,T]L^p(\mathbb{T}^3)}$ are uniformly bounded in $\varepsilon>0$. This yields the convergence in probability of $M_\varepsilon(Y_\varepsilon,u_{0,\varepsilon})$ by Markov inequality. □

The central result of this paper is the convergence of the enhanced noises (or *trees*) Y^τ in law, and their uniform boundedness.

Theorem 9. *Under Assumptions 2 there exists $C>0$ such that for any $p\in[2,\infty)$ we have $\|\mathbb{Y}_\varepsilon\|_{\mathcal{X}_T}<C$ in $L^p(\mathbb{P})$. Moreover, $\mathbb{Y}_\varepsilon\to\mathbb{Y}(\lambda)\in\mathcal{X}_T$ in law.*

Strategy of the proof

The strategy of proof is the following. Denote $\mathbb{X}=(X^\tau)_\tau$, $\mathbb{Y}(\lambda)=(Y^\tau(\lambda))_\tau$ and let K^τ the measurable function of the Gaussian process $X\in C_T\mathscr{C}^{-1/2-\kappa}$ such that $X^\tau=K^\tau(X)$ and $Y^\tau(\lambda)=f_\tau(\lambda)K^\tau(X)$ with $f_\tau(\lambda)$ suitable deterministic functions of λ. For each τ we will show that Y^τ_ε can be decomposed as

$$Y^\tau_\varepsilon=f_\tau(\lambda_\varepsilon)K^\tau(Y_\varepsilon)+\hat{Y}^\tau_\varepsilon \tag{24}$$

where $\hat{Y}^\tau_\varepsilon$ are suitable remainder terms. For all $p\geqslant 2$ it is well-known (see [3],[6]) that the term $f_\tau(\lambda_\varepsilon)K^\tau(Y_\varepsilon)$ is uniformly bounded in $L^p(\Omega;\mathcal{X}^\tau)$ (with $\mathcal{X}^\tau$ given by (15)), then we will just prove that $\hat{Y}^\tau_\varepsilon$ converges to zero in $L^p(\Omega;\mathcal{X}^\tau)$. This con be done by showing that, by Besov embedding, for $1\leqslant p<+\infty$ and $\forall\alpha<|\tau|$ we have

$$\mathbb{E}(\|\hat{Y}^\tau_\varepsilon(t)\|^p_{\mathscr{C}^{\alpha-3/p}})\lesssim\mathbb{E}(\|\hat{Y}^\tau_\varepsilon(t)\|^p_{B^\alpha_{p,p}})\leqslant\sum_q 2^{\alpha pq}\int_{\mathbb{T}^3}\|\Delta_q\hat{Y}^\tau_\varepsilon(t,x)\|^p_{L^p(\Omega)}\mathrm{d}x\leqslant C_\varepsilon\to 0 \tag{25}$$

thanks to the stationarity of the process $Y(t,x)$. For this it suffices to show

$$\sum_q 2^{\alpha pq} \sup_x \|\Delta_q \hat{Y}_\varepsilon^\tau(t,x)\|_{L^p(\Omega)}^p \to 0 \quad \text{as } \varepsilon \to 0 \tag{26}$$

In order to conclude uniform convergence for $t \in [0,T]$ it suffices to show that for $\sigma \in [0,1/2]$, $q \geqslant -1$:

$$\sup_x \|\Delta_q \hat{Y}_\varepsilon^\tau(t,x) - \Delta_q \hat{Y}_\varepsilon^\tau(t,x)\|_{L^p(\Omega)}^p \leqslant C_\varepsilon |t-s|^{\sigma p} 2^{-(\alpha-2\sigma)pq} \quad \text{with } C_\varepsilon \to 0. \tag{27}$$

Indeed, by the Garsia-Rodemich-Rumsey inequality we obtain for $\delta > 0$ small enough and p large enough

$$\begin{aligned} \sup_\varepsilon \mathbb{E}\Big(\|\hat{Y}_\varepsilon^\tau\|^p_{C_T^{\sigma-2/p} B^{\alpha-2\sigma-\delta}_{p,p}} \Big) &\leqslant T^2 \sum_q 2^{(\alpha-2\sigma-\delta)pq} \sup_{s<t\in[0,T]} \sup_x \frac{\|\Delta_q \hat{Y}_\varepsilon^\tau(t,x) - \Delta_q \hat{Y}_\varepsilon^\tau(t,x)\|^p_{L^p(\Omega)}}{|t-s|^{\sigma p}} \\ &\leqslant C_\varepsilon T^2 \sum_q 2^{-\delta pq} \end{aligned}$$

which by Besov embedding yields an estimation on $\mathbb{E}\big(\|Y_\varepsilon^\tau\|_{C_T^{\sigma-\kappa/2} \mathscr{C}^{\alpha-2\sigma-\kappa}}\big)$ for $\kappa > 0$ small enough.

This gives us the necessary tightness to claim that $\mathbb{Y}_\varepsilon$ has weak limits along subsequences. The only thing left to prove is that for each τ we have $K^\tau(Y_\varepsilon) \to K^\tau(X)$ in law. However this is clear since we can introduce a convolution regularisation of X called X_ε which has the same law of Y_ε for any $\varepsilon > 0$. At this point an approximation argument gives that $K^\tau(Y_\varepsilon)$ has the same law of $K^\tau(X_\varepsilon)$. Transposing the regularisation to the kernels of the chaos expansion we can write $K^\tau(X_\varepsilon) = K_\varepsilon^\tau(X)$ and now it is easy to check that $K_\varepsilon^\tau(X) \to K^\tau(X)$ in probability (as done e.g. in [3],[6]). We can then conclude that $K^\tau(Y_\varepsilon) \to K^\tau(X)$ in law for any τ.

Details of the proof

Let us now give the details of the the decomposition (24) and the convergence to zero of the remainder $\hat{Y}_\varepsilon^\tau$ in $L^p(\Omega; \mathcal{X}^\tau)$. We need to introduce some notations based on the results of Appendix C. Looking at the definitions of trees listed in (10), it is clear that $\Delta_q Y_\varepsilon$ can be written in the form

$$\frac{(3-j)!}{3!} \int_\zeta \Phi_\zeta^{(j)} \, \mu_\zeta \qquad \text{or} \qquad \frac{(3-j)!(3-k)!}{3!\,3!} \int_{\zeta_1,\zeta_2} \Phi_{\zeta_1}^{(j)} \Phi_{\zeta_2}^{(k)} \, \mu_{\zeta_1,\zeta_2} - [\text{renormalisation}]$$

for $\zeta, \zeta_1, \zeta_2 \in \mathbb{R} \times \mathbb{T}^3$, $0 \leqslant j,k \leqslant 3$ and some measures μ_ζ and μ_{ζ_1,ζ_2}. Note that the k-th Malliavin derivative of $\Phi_\zeta^{(m)}$, namely $\mathrm{D}^k \Phi_\zeta^{(m)}$ is $\Phi_\zeta^{(m+k)} h_\zeta^{\otimes k}$. Then expansion (69) of Appendix C takes a more explicit form $\forall n \geqslant 1$:

$$\begin{aligned} \Phi_\zeta^{(m)} &= \sum_{k=0}^{n-1} \frac{\mathbb{E}\big(\Phi_\zeta^{(m+k)}\big)}{k!} [\![Y_{\varepsilon,\zeta}^k]\!] + \delta^n\big(Q_1^n \Phi_\zeta^{(m+n)} h_\zeta^{\otimes n} \big) \\ &= \sum_{k=0}^{n-1} \varepsilon^{(m+k-3)/2} \frac{(m+k)!}{k!} \tilde{f}_{m+k,\varepsilon} [\![Y_{\varepsilon,\zeta}^k]\!] + \delta^n\big(Q_1^n \Phi_\zeta^{(m+n)} h_\zeta^{\otimes n} \big) \end{aligned} \tag{28}$$

with $Q_n^m := \prod_{k=n}^m (k-L)^{-1}$ and $Y_{\varepsilon,\zeta} := Y_\varepsilon(t,x)$, $\zeta = (t,x) \in \mathbb{R} \times \mathbb{T}^3$. Here we used the fact that $\delta^n(h_\zeta^{\otimes n}) = [\![Y_\zeta^n]\!]$ (see Remark 39) and that by the definition of $\Phi_\zeta^{(m)}$ (4) we obtain $\forall \zeta \in \mathbb{R} \times \mathbb{T}^3$

$$\mathbb{E}\big(\Phi_\zeta^{(m+k)}\big) = \varepsilon^{(m+k-3)/2} \, (m+k)! \, \tilde{f}_{m+k,\varepsilon}$$

where $\tilde{f}_{n,\varepsilon}$ is the n-th coefficient of the chaos expansion of $\tilde{F}_\varepsilon(\varepsilon^{1/2} Y_\varepsilon)$ relative to $\varepsilon^{1/2} Y_\varepsilon(t,x)$, so that $\tilde{f}_{n,\varepsilon} = 0$ for $n < 3$ and $\tilde{f}_{n,\varepsilon} = f_{n,\varepsilon}$ for $n \geqslant 3$. Choosing $n = 4-m$ in eq. (28) we obtain

$$\begin{aligned} \Phi_\zeta^{(m)} &= \frac{3!}{(3-m)!} f_{3,\varepsilon} [\![Y_{\varepsilon,\zeta}^{3-m}]\!] + \delta^{4-m}\big(Q_1^{4-m} \Phi_\zeta^{(4)} h_\zeta^{\otimes 4-m} \big) \\ &= \frac{3!}{(3-m)!} f_{3,\varepsilon} [\![Y_{\varepsilon,\zeta}^{3-m}]\!] + \hat{\Phi}_\zeta^{(m)} \end{aligned} \tag{29}$$

with

$$\hat{\Phi}_\zeta^{(m)} := \delta^{4-m}(Q_1^{4-m} \Phi_\zeta^{(4)} h_\zeta^{\otimes 4-m}). \tag{30}$$

This yields:

$$\begin{aligned}
\Phi^{(0)}_\zeta &= f_{3,\varepsilon}[\![Y^3_{\varepsilon,\zeta}]\!] + \delta^4(Q_1^4\Phi^{(4)}_\zeta h_\zeta^{\otimes 4}),\\
\Phi^{(1)}_\zeta &= 3f_{3,\varepsilon}[\![Y^2_{\varepsilon,\zeta}]\!] + \delta^3(Q_1^3\Phi^{(4)}_\zeta h_\zeta^{\otimes 3}),\\
\Phi^{(2)}_\zeta &= 6f_{3,\varepsilon}Y_{\varepsilon,\zeta} + \delta^2(Q_1^2\Phi^{(4)}_\zeta h_\zeta^{\otimes 2}),\\
\Phi^{(3)}_\zeta &= 6f_{3,\varepsilon} + \delta(Q_1\Phi^{(4)}_\zeta h_\zeta).
\end{aligned}$$

It suffices to substitute this decomposition in (10) to identify the remainder terms $\hat{Y}^\tau_\varepsilon$ for every tree Y^τ_ε. In the next two sections we will consider separately *first order trees*, which are defined a function of $\Phi^{(m)}_\zeta$ and *second order trees*, which in turn are obtained by multiplying first order trees (and renormalising). We will show that each of these remainder terms satisfy (26) and (27).

Remark 10. Notice that for $m \geqslant 3$ we can easily estimate terms of the form $\varepsilon^{-(m-3)/2}\Phi^{(m)}_\zeta$ $\forall \zeta \in \mathbb{R}\times\mathbb{T}^3$. We have

$$\left\|\varepsilon^{-\frac{m-3}{2}}\Phi^{(m)}_\zeta\right\|^p_{L^p} = \left\|F^{(m)}_\varepsilon(\varepsilon^{1/2}Y_{\varepsilon,\zeta})\right\|^p_{L^p} = \int_{\mathbb{R}}\left|F^{(m)}_\varepsilon(x)\right|^p\gamma(\mathrm{d}x)$$

where $\gamma(\mathrm{d}x)$ is the density of a centered Gaussian with variance $\sigma^2_{Y,\varepsilon}$. The integral is finite by Assumption 2.

2.1 First-order trees

First of all note that the term $\tilde{Y}_\varepsilon^{?}$ has no remainder and then it can be shown to converge in law to $\lambda^{(2)}X^{?}$ by usual techniques. We start with the bound (26) for $\Delta_q Y_\varepsilon^{?}, \Delta_q Y_\varepsilon^{?}, \Delta_q Y_\varepsilon^{\bullet}, \Delta_q Y_\varepsilon^{\varnothing}$. We obtain from (29) that

$$\begin{aligned}
\Delta_q Y^\tau_\varepsilon(t,\bar{x}) &:= \frac{(3-m)!}{3!}\int_\zeta \Phi^{(m)}_\zeta \mu_\zeta\\
&= f_{3,\varepsilon}\int_\zeta [\![Y^{(3-m)}_{\varepsilon,\zeta}]\!]\mu_\zeta + \frac{(3-m)!}{3!}\int_\zeta \hat{\Phi}^{(m)}_\zeta\mu_\zeta\\
&= f_\tau(\lambda_\varepsilon)\Delta_q K^\tau(Y_\varepsilon)(t,\bar{x}) + \Delta_q\hat{Y}^\tau_\varepsilon(t,\bar{x}).
\end{aligned}$$

As said before, $f_{3,\varepsilon}\int_\zeta [\![Y^{(3-m)}_{\varepsilon,\zeta}]\!]\mu_\zeta$ converges in law in L^p for every $2 \leqslant p < +\infty$ to $\lambda^{(3)}\int_\zeta [\![X^{(3-m)}_\zeta]\!]\mu_\zeta$ since $f_{3,\varepsilon}\to\lambda^{(3)}$. We can bound the remainder term $\int_\zeta \hat{\Phi}^{(m)}_\zeta\mu_\zeta$ in $L^p(\Omega)$ using Lemma 32 and Lemma 35 to obtain

$$\begin{aligned}
\left\|\int_\zeta \hat{\Phi}^{(m)}_\zeta\mu_\zeta\right\|_{L^p(\Omega)} &= \left\|\delta^{4-m}\int_\zeta Q_1^{4-m}\Phi^{(4)}_\zeta h_\zeta^{\otimes 4-m}\mu_\zeta\right\|_{L^p(\Omega)} \leqslant \left\|Q_1^{4-m}\int_\zeta \Phi^{(4)}_\zeta h_\zeta^{\otimes 4-m}\mu_\zeta\right\|_{\mathbb{D}^{4-m,p}}\\
&\lesssim \sum_{k=0}^{4-m}\left\|\mathrm{D}^k Q_1^{4-m}\int_\zeta \Phi^{(4)}_\zeta h_\zeta^{\otimes 4-m}\mu_\zeta\right\|_{L^p(\Omega)} \lesssim \left\|\left\|\int_\zeta \Phi^{(4)}_\zeta h_\zeta^{\otimes 4-m}\mu_\zeta\right\|^2_{H^{\otimes 4-m}}\right\|^{1/2}_{L^{p/2}(\Omega)}\\
&\lesssim \left\|\int_\zeta \Phi^{(4)}_\zeta\Phi^{(4)}_{\zeta'}\langle h_\zeta^{\otimes 4-m}, h_{\zeta'}^{\otimes 4-m}\rangle_{H^{\otimes 4-m}}\mu_\zeta\mu_{\zeta'}\right\|^{1/2}_{L^{p/2}(\Omega)}\\
&\lesssim \left[\int_{\zeta,\zeta'}\left\|\Phi^{(4)}_\zeta\Phi^{(4)}_{\zeta'}\right\|_{L^{p/2}(\Omega)}|\langle h_\zeta, h_{\zeta'}\rangle|^{4-m}\,|\mu_\zeta\mu_{\zeta'}|\right]^{1/2}\\
&\lesssim \left[\varepsilon\int_{\zeta,\zeta'}\left\|\varepsilon^{-\frac12}\Phi^{(4)}_\zeta\right\|_{L^p(\Omega)}\left\|\varepsilon^{-\frac12}\Phi^{(4)}_{\zeta'}\right\|_{L^p(\Omega)}|\langle h_\zeta, h_{\zeta'}\rangle|^{4-m}|\mu_\zeta\mu_{\zeta'}|\right]^{\frac12}\\
&\lesssim \left[\varepsilon^\delta\int_{\zeta,\zeta'}\left\|\varepsilon^{-\frac12}\Phi^{(4)}_\zeta\right\|_{L^p(\Omega)}\left\|\varepsilon^{-\frac12}\Phi^{(4)}_{\zeta'}\right\|_{L^p(\Omega)}|\langle h_\zeta, h_{\zeta'}\rangle|^{3-m+\delta}|\mu_\zeta\mu_{\zeta'}|\right]^{\frac12},
\end{aligned}$$

for every $\delta > 0$, where we used the estimation of Lemma 25. Now using Remark 10 and the fact that $\langle h_\zeta, h_{\zeta'}\rangle_H = C_{Y,\varepsilon}(\zeta-\zeta')$ we obtain as a final estimation

$$\left\|\int_\zeta \hat{\Phi}^{(m)}_\zeta\mu_\zeta\right\|_{L^p(\Omega)} \lesssim \varepsilon^{\frac{\delta}{2}}\left[\int |C_{Y,\varepsilon}(\zeta-\zeta')|^{3-m+\delta}\,|\mu_\zeta\mu_{\zeta'}|\right]^{1/2}. \tag{31}$$

Remark 11. These last computations are one of the key observations of this paper, exploiting the properties of Malliavin calculus to replace hypercontractivity in the estimation of L^p norms with arbitrarily large p without resorting to explicit expansions.

The measure $\mu_{\zeta=(s,y)}$ being either $[\int_x K_{q,\bar{x}}(x)P_{t-s}(x-y)]\mathrm{d}\zeta$ for $\Delta_q Y_{\varepsilon,\zeta}^{\textcolor{red}{?}}$ or $K_{q,\bar{x}}(y)\delta(t-s)\mathrm{d}\zeta$ for the other trees, the l.h.s of (31) can be estimated with Lemma 27 to obtain for every $\bar{x}\in\mathbb{T}^3$, $q>0$:

$$\begin{aligned}
\Big\|\Delta_q\hat{Y}_\varepsilon^{\textcolor{red}{?}}(t,\bar{x})\Big\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}}2^{-\frac{1-\delta}{2}q} \qquad \|\Delta_q\hat{Y}_\varepsilon^{\textcolor{red}{\square}}(t,\bar{x})\|_{L^p(\Omega)} \lesssim \varepsilon^{\frac{\delta}{2}}2^{\frac{1+\delta}{2}q}\\
\Big\|\Delta_q\hat{Y}_\varepsilon^{\textcolor{red}{?}}(t,\bar{x})\Big\|_{L^p(\Omega)} &\lesssim \varepsilon^{\frac{\delta}{2}}2^{\frac{2+\delta}{2}q} \qquad \|\Delta_q\hat{Y}_\varepsilon^{\varnothing}(t,\bar{x})\|_{L^p(\Omega)} \lesssim \varepsilon^{\frac{\delta}{2}}2^{\frac{\delta}{2}q}
\end{aligned}$$

The time regularity of trees

We want to show (27). In order to do that, we compute

$$\begin{aligned}
\Big\|\int_\zeta\big(\hat{\Phi}^{(m)}_{t,x}-\hat{\Phi}^{(m)}_{s,x}\big)\mu_\zeta\Big\|_{L^p(\Omega)} &\lesssim \Big\|\delta^{4-m}\int_\zeta Q_1^{4-m}\big(\Phi^{(4)}_{t,x}h_{t,x}^{\otimes 4-m}-\Phi^{(4)}_{s,x}h_{s,x}^{\otimes 4-m}\big)\mu_\zeta\Big\|_{L^p(\Omega)}\\
&\lesssim \Big\|\Big\|\int_\zeta\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big)h_{s,x}^{\otimes 4-m}\mu_\zeta\Big\|^2_{H^{\otimes 4-m}}\Big\|^{1/2}_{L^{p/2}(\Omega)}\\
&\quad+\Big\|\Big\|\int_\zeta\Phi^{(4)}_{s,x}(h_{t,x}^{\otimes 4-m}-h_{s,x}^{\otimes 4-m})\mu_\zeta\Big\|^2_{H^{\otimes 4-m}}\Big\|^{1/2}_{L^{p/2}(\Omega)}.
\end{aligned}$$

We focus on the first term above to obtain that it is bounded by

$$\begin{aligned}
&\Big\|\int_\zeta\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big)\big(\Phi^{(4)}_{t,x'}-\Phi^{(4)}_{s,x'}\big)\langle h_{s,x}^{\otimes 4-m},h_{s,x'}^{\otimes 4-m}\rangle_{H^{\otimes 4-m}}\mu_\zeta\mu_{\zeta'}\Big\|^{1/2}_{L^{p/2}(\Omega)}\\
\lesssim&\Big[\int_{\zeta,\zeta'}\big\|\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big)\big(\Phi^{(4)}_{t,x'}-\Phi^{(4)}_{s,x'}\big)\big\|_{L^{p/2}(\Omega)}|\langle h_{s,x},h_{s,x'}\rangle|^{4-m}\,|\mu_\zeta\mu_{\zeta'}|\Big]^{1/2}\\
\lesssim&\Big[\varepsilon\int_{\zeta,\zeta'}\big\|\varepsilon^{-1}\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big)\big(\Phi^{(4)}_{t,x'}-\Phi^{(4)}_{s,x'}\big)\big\|_{L^{p/2}(\Omega)}|\langle h_{s,x},h_{s,x'}\rangle|^{4-m}|\mu_\zeta\mu_{\zeta'}|\Big]^{\frac12}\\
\lesssim&\Big[\varepsilon^{\delta}\int_{\zeta,\zeta'}\big\|\varepsilon^{-1}\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big)\big(\Phi^{(4)}_{t,x'}-\Phi^{(4)}_{s,x'}\big)\big\|_{L^{p/2}(\Omega)}|\langle h_{s,x},h_{s,x'}\rangle|^{3-m+\delta}|\mu_\zeta\mu_{\zeta'}|\Big]^{\frac12}
\end{aligned}$$

now notice that

$$\begin{aligned}
\varepsilon^{-\frac12}\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big) &= F^{(4)}(\varepsilon^{\frac12}Y_\varepsilon(t,x))-F^{(4)}(\varepsilon^{\frac12}Y_\varepsilon(s,x))\\
&= \varepsilon^{\frac12}\int_0^1 F^{(5)}[\varepsilon^{\frac12}Y_\varepsilon(s,x)+\tau\varepsilon^{\frac12}(Y_\varepsilon(t,x)-Y_\varepsilon(s,x))]\,(Y_\varepsilon(t,x)-Y_\varepsilon(s,x))
\end{aligned}$$

and we can estimate $\Big\|\varepsilon^{-\frac12}\big(\Phi^{(4)}_{t,x}-\Phi^{(4)}_{s,x}\big)\Big\|_{L^p(\Omega)}$ by hypercontractivity and using Lemma 28 as

$$\begin{aligned}
\lesssim_p\ & \varepsilon^{1/2}\Big\|\int_0^1 F^{(5)}[\varepsilon^{\frac12}Y_\varepsilon(s,x)+\tau\varepsilon^{\frac12}(Y_\varepsilon(t,x)-Y_\varepsilon(s,x))]\Big\|_{L^{2p}(\Omega)}\|Y_\varepsilon(t,x)-Y_\varepsilon(s,x)\|_{L^2(\Omega)}\\
\lesssim\ & \varepsilon^{1/2}[C_{Y,\varepsilon}(0,0)-C_{Y,\varepsilon}(t-s,0)]^{1/2}\\
\lesssim\ & \varepsilon^{-2\sigma}|t-s|^\sigma,
\end{aligned}$$

for any $\sigma\in[0,1/2]$. The other term can be estimated more easily by

$$\begin{aligned}
&\Big[\varepsilon^{\delta}\int_{\zeta,\zeta'}|\langle h_{s,x},h_{s,x'}\rangle|^{2-m+\delta}|\langle h_{t,x}-h_{s,x},h_{t,x'}-h_{s,x'}\rangle||\mu_\zeta\mu_{\zeta'}|\Big]^{\frac12}\\
\lesssim\ & \varepsilon^{-2\kappa}|t-s|^\sigma\Big[\varepsilon^{\delta}\int_{\zeta,\zeta'}|\langle h_{s,x},h_{s,x'}\rangle|^{3-m+\delta+2\sigma}|\mu_\zeta\mu_{\zeta'}|\Big]^{\frac12},
\end{aligned}$$

and finally obtain

$$\Big\|\int_\zeta\big(\hat{\Phi}^{(m)}_{t,x}-\hat{\Phi}^{(m)}_{s,x}\big)\mu_\zeta\Big\|_{L^p(\Omega)}\lesssim\varepsilon^{\delta/2-2\kappa}|t-s|^\sigma\Big[\int_{\zeta,\zeta'}|\langle h_{s,x},h_{s,x'}\rangle|^{3-m+\delta+2\sigma}|\mu_\zeta\mu_{\zeta'}|\Big]^{\frac12}.$$

Which yields estimation (27) by applying Lemma 27 as before.

2.2 Second-order trees

In this section we show the decomposition (24) and the bound (26) for the trees $Y_\varepsilon^{\boxed{?}}$, $Y_\varepsilon^{\boxed{?}}$, $\tilde{Y}_\varepsilon^{\boxed{?}}$, $Y_\varepsilon^{\boxed{?}}$. The time regularity (27) of Y_ε^τ can be obtained with the same technique as in the previous section assuming that $(F_\varepsilon)_\varepsilon \subseteq C^9(\mathbb{R})$, and we do not repeat the argument here. Looking at the definitions in (10) it is clear that we can write the Littlewood-Paley blocks of $Y_\varepsilon^{\boxed{?}}$, $Y_\varepsilon^{\boxed{?}}$, $\tilde{Y}_\varepsilon^{\boxed{?}}$ and $Y_\varepsilon^{\boxed{?}}$ $\forall \varepsilon > 0$ as:

$$\begin{aligned}
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= \frac{1}{6}\int_{\zeta_1,\zeta_2} \Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}\,\mu_{\zeta_1,\zeta_2} - d_\varepsilon^{\boxed{?}}\Delta_q(1)(\bar\zeta),\\
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= \frac{1}{9}\int_{\zeta_1,\zeta_2} \Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\,\mu_{\zeta_1,\zeta_2} - d_\varepsilon^{\boxed{?}}\Delta_q(1)(\bar\zeta),\\
\Delta_q \tilde{Y}_\varepsilon^{\boxed{?}}(\bar\zeta) &= \frac{1}{3}\int_{\zeta_1,\zeta_2} \tilde\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\,\mu_{\zeta_1,\zeta_2} - \tilde d_\varepsilon^{\boxed{?}}\Delta_q(1)(\bar\zeta),\\
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= \frac{1}{3}\int_{\zeta_1,\zeta_2} \Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\,\mu_{\zeta_1,\zeta_2} - d_\varepsilon^{\boxed{?}}\Delta_q Y_\varepsilon(\bar\zeta) - \hat d_\varepsilon^{\boxed{?}}\Delta_q(1)(\bar\zeta),
\end{aligned} \tag{32}$$

for $\bar\zeta = (t,\bar x)$ where $\Phi^{(m)}_\zeta$ is defined in (4), $\tilde\Phi^{(1)}_{\zeta_1} := \varepsilon^{-1/2} f_{2,\varepsilon}[\![Y^2_\varepsilon(\zeta_1)]\!]$ and the measure μ_{ζ_1,ζ_2} on $(\mathbb{R}\times\mathbb{T}^3)^2$ is given by

$$\mu_{\zeta_1,\zeta_2} := [\int_{x,y} K_{q,\bar x}(x)\sum_{i\sim j} K_{i,x}(y)K_{j,x}(x_2)P_{t-s_1}(y-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2$$

with $\zeta_i = (s_i, x_i)$ $i = 1,2$.

The first step for decomposing (32) is to expand them using the partial chaos expansion (69) to obtain

$$\begin{aligned}
\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2} &= \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}\big] + \delta Q_1\mathrm{D}\,(\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}),\\
\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2} &= \mathbb{E}\big[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + \delta Q_1\mathrm{D}\,(\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}),\\
\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2} &= \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + \delta\big[J_0\mathrm{D}\big(\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big)\big] + \delta^2 Q_1^2\mathrm{D}^2(\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2})\\
&= \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + Y_\varepsilon(\zeta_1)\mathbb{E}\big[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + Y_\varepsilon(\zeta_2)\mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}\big] + \delta^2 Q_1^2\mathrm{D}^2(\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}).
\end{aligned} \tag{33}$$

Like the trees appearing in the Φ^4_3 model, we expect second-order trees to require a further renormalisation, on top of the Wick ordering.

2.2.1 Renormalisation of second-order trees

In this section we show how to renormalise (32) by estimating the terms of the type $\mathbb{E}\big[\Phi^{(m)}_{\zeta_1}\Phi^{(n)}_{\zeta_2}\big]$ in expansion (33). We are going to need the following result:

Lemma 12. *We have*

$$\int_{\zeta_1,\zeta_2} Y_\varepsilon(\zeta_1)\mathbb{E}[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}]\mu_{\zeta_1,\zeta_2} = \int_{s,x} \Delta_q Y_\varepsilon(s,\bar x - x)\,G(t-s,x).$$

and

$$\int_{\zeta_1,\zeta_2} Y_\varepsilon(\zeta_2)\mathbb{E}[\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}]\mu_{\zeta_1,\zeta_2} = \int_{x} \Delta_q Y_\varepsilon(t,\bar x - x)H(t,x),$$

where

$$\begin{aligned}
G(t-s,x) &:= \int_{x_1',x_2}\sum_{i\sim j} K_{i,x}(x_1')K_{j,x}(x_2)P_{t-s}(x_1')\mathbb{E}\big[\Phi^{(1)}_0\Phi^{(1)}_{(t-s,x_2)}\big],\\
H(t,x) &:= \int_{s,x_1,x_1'}\sum_{i\sim j} K_{i,x}(x_1')K_{j,x}(0)P_{t-s}(x_1'-x_1)\mathbb{E}\big[\Phi^{(0)}_0\Phi^{(2)}_{(t-s,-x_1)}\big].
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\int_{\zeta_1,\zeta_2} Y_\varepsilon(\zeta_1)\mathbb{E}\big[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big]\mu_{\zeta_1,\zeta_2}\\
&= \int_{s_1,x_1,x_2,x,x_1'} K_{q,\bar x}(x)\sum_{i\sim j} K_{i,x}(x_1')K_{j,x}(x_2)P_{t-s_1}(x_1'-x_1)Y_\varepsilon(s_1,x_1)\mathbb{E}\big[\Phi^{(1)}_0\Phi^{(1)}_{(t-s_1,x_2-x_1)}\big]
\end{aligned}$$

and by change of variables, exploiting the translation invariance of the problem we obtain:

$$= \int_{s_1,x_1,x} K_{q,\bar{x}}(x+x_1) Y(s_1,x_1) \int_{x_1',x_2} \sum_{i\sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s_1}(x_1') \mathbb{E}\big[\Phi_0^{(1)} \Phi^{(1)}_{(t-s_1,x_2)}\big].$$

Using the definition of K_q we have

$$= \int_{s_1,x} \Delta_q Y_\varepsilon(s_1, \bar{x}-x) \int_{x_1',x_2} \sum_{i\sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s_1}(x_1') \mathbb{E}\big[\Phi_0^{(1)} \Phi^{(1)}_{(t-s_1,x_2)}\big].$$

Finally we can write

$$\int_{\zeta_1,\zeta_2} Y_\varepsilon(\zeta_1) \mathbb{E}\big[\Phi^{(1)}_{\zeta_1} \Phi^{(1)}_{\zeta_2}\big] \mu_{\zeta_1,\zeta_2} = \int_{s_1,x} \Delta_q Y_\varepsilon(s_1, \bar{x}-x)\, G(t-s_1,x).$$

Similar computations holds for the other term, indeed

$$\begin{aligned}
&\int_{\zeta_1,\zeta_2} Y_\varepsilon(\zeta_2) \mathbb{E}\big[\Phi^{(0)}_{\zeta_1} \Phi^{(2)}_{\zeta_2}\big] \mu_{\zeta_1,\zeta_2} \\
&= \int_{s_1,x_1,x_2,x,x_1'} K_{q,\bar{x}}(x) \sum_{i\sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s_1}(x_1'-x_1) Y_\varepsilon(t,x_2) \mathbb{E}\big[\Phi_0^{(0)} \Phi^{(2)}_{(t-s_1,x_2-x_1)}\big] \\
&= \int_{x_2} K_{q,\bar{x}}(x+x_2) Y_\varepsilon(t,x_2) \int_{s_1,x_1,x,x_1'} \sum_{i\sim j} K_{i,x}(x_1') K_{j,x}(0) P_{t-s_1}(x_1'-x_1) \mathbb{E}\big[\Phi_0^{(0)} \Phi^{(2)}_{(t-s_1,-x_1)}\big] \\
&= \int_x \Delta_q Y_\varepsilon(t,\bar{x}-x) \int_{s_1,x_1,x_1'} \sum_{i\sim j} K_{i,x}(x_1') K_{j,x}(0) P_{t-s_1}(x_1'-x_1) \mathbb{E}\big[\Phi_0^{(0)} \Phi^{(2)}_{(t-s_1,-x_1)}\big] \\
&= \int_x \Delta_q Y_\varepsilon(t,\bar{x}-x) H(t,x)
\end{aligned}$$

□

Using the lemma above and the partial chaos expansion (33), we can write (32) as:

$$\begin{aligned}
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar{\zeta}) &= \frac{1}{9} \int_{\zeta_1,\zeta_2} \delta Q_1 \mathrm{D}\left(\Phi^{(1)}_{\zeta_1} \Phi^{(1)}_{\zeta_2}\right) \mu_{\zeta_1,\zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{9} \int_{s,x} G(t-s,x) - d_\varepsilon^{\boxed{?}}\right] \\
\Delta_q \tilde{Y}_\varepsilon^{\boxed{?}}(\bar{\zeta}) &= \frac{1}{3} \int_{\zeta_1,\zeta_2} \delta Q_1 \mathrm{D}\left(\tilde{\Phi}^{(1)}_{\zeta_1} \Phi^{(1)}_{\zeta_2}\right) \mu_{\zeta_1,\zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{3} \int_{s,x} \tilde{G}(t-s,x) - \tilde{d}_\varepsilon^{\boxed{?}}\right] \\
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar{\zeta}) &= \frac{1}{6} \int_{\zeta_1,\zeta_2} \delta Q_1 \mathrm{D}\left(\Phi^{(0)}_{\zeta_1} \Phi^{(2)}_{\zeta_2}\right) \mu_{\zeta_1,\zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{6} \int_x H(t,x) - d_\varepsilon^{\boxed{?}}\right] \\
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar{\zeta}) &= \frac{1}{3} \int_{\zeta_1,\zeta_2} \delta^2 Q_1^2 \mathrm{D}^2(\Phi^{(0)}_{\zeta_1} \Phi^{(1)}_{\zeta_2}) \mu_{\zeta_1,\zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[\frac{1}{3} \int_{\zeta_1,\zeta_2} \mathbb{E}\big[\Phi^{(0)}_{\zeta_1} \Phi^{(1)}_{\zeta_2}\big] \mu_{\zeta_1,\zeta_2} - \hat{d}_\varepsilon^{\boxed{?}}\right] \\
&\quad + \Delta_q Y_\varepsilon(\bar{\zeta}) \left[\frac{1}{3} \int_{s,x} G(t-s,x) + \frac{1}{3} \int_x H(t,x) - d_\varepsilon^{\boxed{?}}\right] \\
&\quad + \frac{1}{3} \Delta_q R_\varepsilon^{\boxed{?}}(\bar{\zeta}) + \frac{1}{3} \Delta_q R_\varepsilon^{\boxed{?}}(\bar{\zeta})
\end{aligned}$$

with the additional definitions

$$\begin{aligned}
\tilde{G}(t-s,x) &:= \int_{x_1,x_1'} \sum_{i\sim j} K_{i,x}(x_1') K_{j,x}(0) P_{t-s_1}(x_1'-x_1) \mathbb{E}\big[\tilde{\Phi}^{(1)}_0 \Phi^{(1)}_{(t-s,-x_1)}\big], \\
\Delta_q R_\varepsilon^{\boxed{?}}(\bar{\zeta}) &:= \int_{s,x} \left[\Delta_q Y_\varepsilon(s,\bar{x}-x) - \Delta_q Y_\varepsilon(t,\bar{x})\right] G(t-s,x), \\
\Delta_q R_\varepsilon^{\boxed{?}}(\bar{\zeta}) &:= \int_x \left[\Delta_q Y_\varepsilon(t,\bar{x}-x) - \Delta_q Y_\varepsilon(t,\bar{x})\right] H(t,x).
\end{aligned}$$

To proceed further in the estimation of these integrals, we need to characterise the local behaviour of $\mathbb{E}[\Phi^{(m)}_{\zeta_1}\Phi^{(n)}_{\zeta_2}]$. From the decomposition (29) we can write

$$\begin{aligned}\mathbb{E}\big[\Phi^{(m)}_{\zeta_1}\Phi^{(n)}_{\zeta_2}\big] &= \frac{3!^2}{(3-m)!(3-n)!}(f_{3,\varepsilon})^2\,\mathbb{E}[[\![Y^{3-m}_{\varepsilon,\zeta_1}]\!][\![Y^{3-n}_{\varepsilon,\zeta_2}]\!]] + \frac{3!}{(3-m)!}f_{3,\varepsilon}\,\mathbb{E}\big[[\![Y^{3-m}_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(n)}_{\zeta_2}\big]\\ &\quad+\frac{3!}{(3-n)!}f_{3,\varepsilon}\,\mathbb{E}\big[[\![Y^{3-n}_{\varepsilon,\zeta_2}]\!]\hat\Phi^{(m)}_{\zeta_1}\big] + \mathbb{E}\big[\hat\Phi^{(m)}_{\zeta_1}\hat\Phi^{(n)}_{\zeta_2}\big],\end{aligned}$$

where $\mathbb{E}[[\![Y^{3-m}_{\varepsilon,\zeta_1}]\!][\![Y^{3-n}_{\varepsilon,\zeta_2}]\!]] = (3-m)!\delta(3-m,3-n)C_{Y,\varepsilon}(\zeta_1-\zeta_2)^{3-n}$ and to bound all other terms we introduce the following result.

Lemma 13. *Under Assumption 2 we have, for every $0\leqslant m,n\leqslant 3$ and $m\leqslant n$:*

$$\big|\mathbb{E}\big[\hat\Phi^{(m)}_{\zeta_1}\hat\Phi^{(n)}_{\zeta_2}\big]\big| \lesssim \sum_{i=0}^{4-n}\varepsilon^{1+\frac{n-m}{2}+i}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{4-m+i} \lesssim \varepsilon^{\delta}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{3-\frac{m+n}{2}+\delta}, \qquad \forall\delta\in[0,1].$$

Moreover for every $0\leqslant m,n\leqslant 3$,

$$\begin{aligned}\big|\mathbb{E}\big[[\![Y^m_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(n)}_{\zeta_2}\big]\big| &\lesssim \varepsilon^{\frac{m+n-3}{2}}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^m \quad \text{if } m\geqslant 4-n,\\ \mathbb{E}\big[[\![Y^m_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(n)}_{\zeta_2}\big] &= 0 \qquad\qquad\qquad\quad \text{if } m<4-n.\end{aligned}$$

Proof. Using the integration by parts formula (72) we decompose

$$\begin{aligned}\mathbb{E}\big[\hat\Phi^{(m)}_{\zeta_1}\hat\Phi^{(n)}_{\zeta_2}\big] &= \mathbb{E}\big[\delta^{4-m}(Q_1^{4-m}\Phi^{(4)}_{\zeta_1}h^{\otimes 4-m}_{\zeta_1})\delta^{4-n}(Q_1^{4-n}\Phi^{(4)}_{\zeta_2}h^{\otimes 4-n}_{\zeta_2})\big]\\ &= \sum_{i=0}^{4-n}\binom{4-m}{i}\binom{4-n}{i}i!\,\mathbb{E}\big(Q^{8-m-n-i}_{5-n-i}\Phi^{(8-n-i)}_{\zeta_1}\,Q^{8-m-n-i}_{5-m-i}\Phi^{(8-m-i)}_{\zeta_2}\big)\langle h_{\zeta_1},h_{\zeta_2}\rangle^{8-m-n-i}.\end{aligned}$$

We can bound the term

$$\varepsilon^{\frac{m+n}{2}+i-5}\mathbb{E}\big(Q^{8-m-n-i}_{5-n-i}\Phi^{(8-n-i)}_{\zeta_1}\,Q^{8-m-n-i}_{5-m-i}\Phi^{(8-m-i)}_{\zeta_2}\big)\lesssim\Big\|\varepsilon^{\frac{n+i-5}{2}}\Phi^{(8-n-i)}_{\zeta_1}\Big\|_{L^2}\Big\|\varepsilon^{\frac{m+i-5}{2}}\Phi^{(8-m-i)}_{\zeta_2}\Big\|_{L^2}$$

(see Remark 10) and therefore, using the bound $\varepsilon|\langle h_{\zeta_1},h_{\zeta_2}\rangle| = \varepsilon C_{Y,\varepsilon}(\zeta_1-\zeta_2)\lesssim 1$,

$$\big|\mathbb{E}\big[\hat\Phi^{(m)}_{\zeta_1}\hat\Phi^{(n)}_{\zeta_2}\big]\big|\lesssim\sum_{i=0}^{4-n}\varepsilon^{1+\frac{n-m}{2}+i}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{4-m+i}\lesssim\varepsilon^{\delta}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{3-\frac{m+n}{2}+\delta}.$$

For the second bound we compute

$$\begin{aligned}\mathbb{E}\big[[\![Y^m_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(n)}_{\zeta_2\ \zeta_2}\big] &= \mathbb{E}\big[\delta^m(h^{\otimes m}_{\zeta_1})\delta^{4-n}(Q_1^{4-n}\Phi^{(4)}_{\zeta_2}h^{\otimes 4-n}_{\zeta_2})\big]\\ &= \sum_{i=0}^{m\wedge 4-n}\binom{m}{i}\binom{4-n}{i}i!\,\mathbb{E}\big(\big\langle \mathrm{D}^{4-n-i}(h^{\otimes m}_{\zeta_1}),Q^{m+4-n-i}_{m+1-i}\Phi^{(4+m-i)}_{\zeta_2}h^{\otimes m+4-n-i}_{\zeta_2}\big\rangle_{H^{\otimes m+4-n-i}}\big).\end{aligned}$$

Since $\mathrm{D}h^{\otimes m}_{\zeta_1}=0$ we obtain $\mathbb{E}\big[[\![Y^m_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(n)}_{\zeta_2}\big]=0$ if $m<4-n$ and

$$\big|\mathbb{E}\big[[\![Y^m_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(n)}_{\zeta_2}\big]\big| \lesssim \varepsilon^{\frac{m+n-3}{2}}\mathbb{E}[\varepsilon^{-\frac{3-m-n}{2}}Q^m_{m+n-3}\Phi^{(m+n)}_{\zeta_2}]|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^m$$

if $m\geqslant 4-n$, with

$$\mathbb{E}[\varepsilon^{-\frac{3-m-n}{2}}Q^m_{m+n-3}\Phi^{(m+n)}_{\zeta_2}]\lesssim 1. \qquad \square$$

Using Lemma 13 we obtain

$$\mathbb{E}\big[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] = 9\,\mathbb{E}\big[(f_{3,\varepsilon}[\![Y^2_{\varepsilon,\zeta_1}]\!]+\hat\Phi^{(1)}_{\varepsilon,\zeta_1})(f_{3,\varepsilon}[\![Y^2_{\varepsilon,\zeta_2}]\!]+\hat\Phi^{(1)}_{\zeta_2})\big] = 18\,(f_{3,\varepsilon})^2\,[C_{Y,\varepsilon}(\zeta_1-\zeta_2)]^2 + \mathbb{E}\big[\hat\Phi^{(1)}_{\zeta_1}\hat\Phi^{(1)}_{\zeta_2}\big]$$

and thus $G(t-s,x) = 18(f_{3,\varepsilon})^2\int_{x_1',x_2}\sum_{i\sim j}K_{i,x}(x_1')K_{j,x}(x_2)P_{t-s}(x_1')[C_{Y,\varepsilon}(\zeta_1-\zeta_2)]^2 + \hat G(t-s,x)$ with

$$\hat G(t-s,x) := \int_{x_1',x_2}\sum_{i\sim j}K_{i,x}(x_1')K_{j,x}(x_2)P_{t-s}(x_1')\mathbb{E}\big[\hat\Phi^{(1)}_0\hat\Phi^{(1)}_{(t-s,x_2)}\big].$$

We have the estimation

$$\big|\mathbb{E}\big[\hat{\Phi}^{(1)}_{\zeta_1}\hat{\Phi}^{(1)}_{\zeta_2}\big]\big| \lesssim \varepsilon^{\delta} C_{Y,\varepsilon}(\zeta_1-\zeta_2)^{2+\delta}. \tag{34}$$

Similarly

$$\mathbb{E}\big[\tilde{\Phi}^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] \;=\; 3\,\varepsilon^{-1/2} f_{2,\varepsilon}\,\mathbb{E}\big[\llbracket Y^2_{\varepsilon,\zeta_1}\rrbracket\big(f_{3,\varepsilon}\llbracket Y^2_{\varepsilon,\zeta_2}\rrbracket + \hat{\Phi}^{(1)}_{\zeta_2}\big)\big] = 6\,\varepsilon^{-1/2} f_{2,\varepsilon} f_{3,\varepsilon}[C_{Y,\varepsilon}(\zeta_1-\zeta_2)]^2,$$

and

$$\begin{aligned}\big|\mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}\big]\big| \;&=\; \big|\mathbb{E}\big[\big(f_{3,\varepsilon}\llbracket Y^3_{\varepsilon,\zeta_1}\rrbracket + \hat{\Phi}^{(0)}_{\zeta_1}\big)\big(6 f_{3,\varepsilon} Y_{\varepsilon,\zeta_2} + \hat{\Phi}^{(2)}_{\zeta_2}\big)\big]\big| \lesssim |f_{3,\varepsilon}|\big|\mathbb{E}\big[\llbracket Y^3_{\varepsilon,\zeta_1}\rrbracket\hat{\Phi}^{(2)}_{\zeta_2}\big]\big| + \big|\mathbb{E}\big[\hat{\Phi}^{(0)}_{\zeta_1}\hat{\Phi}^{(2)}_{\zeta_2}\big]\big| \\ &\lesssim\; \varepsilon^{\delta}(|f_{3,\varepsilon}|+1) C_{Y,\varepsilon}(\zeta_1-\zeta_2)^{2+\delta},\end{aligned} \tag{35}$$

and

$$\begin{aligned}\big|\mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big]\big| \;&=\; |\mathbb{E}[\Phi^{(0)}(3 f_{3,\varepsilon}\llbracket Y^2_{\varepsilon,\zeta_2}\rrbracket + \hat{\Phi}^{(1)})]| \lesssim |f_{3,\varepsilon}|\mathbb{E}\big[\llbracket Y^3_{\varepsilon,\zeta_1}\rrbracket\hat{\Phi}^{(1)}_{\zeta_2}\big] + \mathbb{E}\big[\hat{\Phi}^{(0)}_{\zeta_1}\hat{\Phi}^{(1)}_{\zeta_2}\big] \\ &\lesssim\; \varepsilon^{1/2}(|f_{3,\varepsilon}|+1) C_{Y,\varepsilon}(\zeta_1-\zeta_2)^{3}.\end{aligned} \tag{36}$$

We have by Lemma 29 that for all $\delta \in (0,1)$ $|\hat{G}(t-s,x)| \lesssim \varepsilon^{\delta}(|t-s|^{1/2}+|x|)^{-5-\delta}$. Using estimate (64) together with Lemma 23, we have that for all $\delta \in (0,1)$, $\delta' \in (0,\delta)$ that $|H(t,x)| \lesssim \varepsilon^{\delta'}(|t-s|^{1/2}+|x|)^{-\delta}$. Furthermore, letting

$$\Delta_q \hat{R}_\varepsilon^{\boxed{?}} = \int_{s,x} [\Delta_q Y_\varepsilon(t,\bar{x}-x) - \Delta_q Y_\varepsilon(t,\bar{x})]\,\hat{G}(t-s,x),$$

we have

$$\frac{1}{3}\Delta_q R_\varepsilon^{\boxed{?}} = 6(f_{3,\varepsilon})^2 \int_{s,x} [\Delta_q Y_\varepsilon(t+s,\bar{x}-x) - \Delta_q Y_\varepsilon(t,\bar{x})] P_s(x)[C_{Y,\varepsilon}(s,x)]^2 + \frac{1}{3}\Delta_q \hat{R}_\varepsilon^{\boxed{?}}.$$

The term

$$6(f_{3,\varepsilon})^2 \int_{s,x} [\Delta_q Y_\varepsilon(t+s,\bar{x}-x) - \Delta_q Y_\varepsilon(t,\bar{x})] P_s(x)[C_{Y,\varepsilon}(s,x)]^2$$

can be shown to converge in law to $6\int_{s,x}[\Delta_q X(t+s,\bar{x}-x) - \Delta_q X(t,\bar{x})]P_s(x)\,[C_X(s,x)]^2$ in $C^\kappa_T\mathscr{C}^{-1/2-2\kappa}$ with the standard techniques used in the analysis of the Φ^4_3 model. For all $\delta>0$ sufficiently small we have the bounds,

$$\begin{aligned}\Big\|\Delta_q R_\varepsilon^{\boxed{?}}\Big\|_{L^\infty} + \Big\|\Delta_q \hat{R}_\varepsilon^{\boxed{?}}\Big\|_{L^\infty} \;&\leqslant\; \varepsilon^{\delta}\|Y_\varepsilon\|_{C^\kappa_T\mathscr{C}^{-1/2-2\kappa}} 2^{q(1/2+2\kappa+2\delta)} \int_{s,x} (|x|+|t-s|^{1/2})^{\delta-5} \\ &\lesssim\; \varepsilon^{\delta}\|Y_\varepsilon\|_{C^\kappa_T\mathscr{C}^{-1/2-2\kappa}} 2^{q(1/2+2\kappa+2\delta)},\end{aligned}$$

which shows that these remainders go to zero in $\mathscr{C}^{-1/2-2\kappa}$ as $\varepsilon\to 0$, since $\|Y_\varepsilon\|_{C^\kappa_T\mathscr{C}^{-1/2-2\kappa}}$ is bounded in $L^p(\Omega)$. Moreover, it is easy to see that $\Delta_q R_\varepsilon^{\boxed{?}} - \Delta_q \hat{R}_\varepsilon^{\boxed{?}}$ is bounded in $L^p(\Omega,\mathscr{C}^{-1/2})$. Note that

$$\int_{s,x} G(t-s,x) = \int_{s,x} P_s(x)\mathbb{E}\big[\Phi^{(1)}_0\Phi^{(1)}_{(s,x)}\big] = 18(f_{3,\varepsilon})^2\int_{s,x} P_s(x)[C_{Y,\varepsilon}(s,x)]^2 + \int_{s,x} P_s(x)\mathbb{E}\big[\hat{\Phi}^{(1)}_0\hat{\Phi}^{(1)}_{(s,x)}\big],$$

$$\int_x H(t,x) = \int_{s,x} P_s(x)\mathbb{E}\big[\Phi^{(0)}_0\Phi^{(2)}_{(s,x)}\big] = \int_{s,x} P_s(x)\mathbb{E}\big[\Phi^{(0)}_0\hat{\Phi}^{(2)}_{(s,x)}\big].$$

Here we used the fact that

$$\int_x \sum_{i\sim j} K_{i,x}(x_1')K_{j,x}(0) = \int_x \sum_{i,j} K_{i,x}(x_1')K_{j,x}(0) = \delta(x_1'),$$

since $\int_x K_{i,x}(x_1')K_{j,x}(0) = 0$, where $|i-j|>1$. This is readily seen in Fourier space taking into account the support properties of the Littlewood-Paley blocks. Now,

$$\int_{s,x} P_s(x)\mathbb{E}\big[\hat{\Phi}^{(1)}_0\hat{\Phi}^{(1)}_{(s,x)}\big], \qquad \int_{s,x} P_s(x)\mathbb{E}\big[\Phi^{(0)}_0\hat{\Phi}^{(2)}_{(s,x)}\big],$$

converge to finite constants due to the bounds (34) and (35) and by Lemma 26 $\int_{s,x} P_s(x)[C_{Y,\varepsilon}(s,x)]^2 \lesssim |\log\varepsilon|$.

Finally, from (36) we have

$$\int_{\zeta_1,\zeta_2} \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big]\,\mu_{\zeta_1,\zeta_2} = \int_{s,x} P_s(x)\mathbb{E}\big[\Phi^{(0)}_0\Phi^{(1)}_{(s,x)}\big] = O(\varepsilon^{-1/2}).$$

Indeed Lemma 26 again yields $\varepsilon\int_{s,x}P_s(x)C_{Y,\varepsilon}(s,x)^3\lesssim 1$.

Thus $\int_{\zeta_1,\zeta_2}\mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big]\,\mu_{\zeta_1,\zeta_2}$ gives a diverging constant which depends on all the $(f_{n,\varepsilon})_n$. Making the choice to define the renormalisation constants d^τ as in eq. (5) we cancel exactly these contributions which are either $(F_\varepsilon)_\varepsilon$ dependent and/or diverging. In particular we verify that we can satisfy the constraint (11).

Finally, noting that $\delta Q_1\mathrm{D}=(1-J_0)$ and $\delta^2Q_1^2\mathrm{D}^2=(1-J_0-J_1)$ we can write the trees of (32) as

$$\begin{aligned}
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= (f_{3,\varepsilon})^2\int_{\zeta_1,\zeta_2}(1-J_0)([\![Y^2_{\varepsilon,\zeta_1}]\!][\![Y^2_{\varepsilon,\zeta_2}]\!])\,\mu_{\zeta_1,\zeta_2}\\
&\quad+\frac{f_{3,\varepsilon}}{3}\int_{\zeta_1,\zeta_2}\delta Q_1\mathrm{D}\,(\hat\Phi^{(1)}_{\zeta_1}[\![Y^2_{\varepsilon,\zeta_2}]\!]+[\![Y^2_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(1)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2}+\frac19\int_{\zeta_1,\zeta_2}\delta Q_1\mathrm{D}\,(\hat\Phi^{(1)}_{\zeta_1}\hat\Phi^{(1)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2},\\
\Delta_q \tilde Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= \varepsilon^{-\frac12}f_{2,\varepsilon}\,f_{3,\varepsilon}\int_{\zeta_1,\zeta_2}(1-J_0)([\![Y^2_{\varepsilon,\zeta_1}]\!][\![Y^2_{\varepsilon,\zeta_2}]\!])\,\mu_{\zeta_1,\zeta_2}+\frac13\int_{\zeta_1,\zeta_2}\delta Q_1\mathrm{D}\,(\tilde\Phi^{(1)}_{\zeta_1}\hat\Phi^{(1)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2},\\
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= (f_{3,\varepsilon})^2\int_{\zeta_1,\zeta_2}[\![Y^3_{\varepsilon,\zeta_1}]\!]Y_{\zeta_2}\,\mu_{\zeta_1,\zeta_2}\\
&\quad+\frac{f_{3,\varepsilon}}{6}\int_{\zeta_1,\zeta_2}\delta Q_1\mathrm{D}\,(6\hat\Phi^{(0)}_{\zeta_1}Y_{\varepsilon,\zeta_2}+[\![Y^3_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(2)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2}+\frac16\int_{\zeta_1,\zeta_2}\delta Q_1\mathrm{D}\,(\hat\Phi^{(0)}_{\zeta_1}\hat\Phi^{(2)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2},\\
\Delta_q Y_\varepsilon^{\boxed{?}}(\bar\zeta) &= (f_{3,\varepsilon})^2\int_{\zeta_1,\zeta_2}(1-J_1)([\![Y^3_{\varepsilon,\zeta_1}]\!][\![Y^2_{\varepsilon,\zeta_2}]\!])\,\mu_{\zeta_1,\zeta_2}+\frac13\Delta_q R_\varepsilon^{\boxed{?}}(\bar\zeta)\\
&\quad+6(f_{3,\varepsilon})^2\int_{s,x}[\Delta_qY_\varepsilon(t+s,\bar x-x)-\Delta_qY_\varepsilon(t,\bar x)]P_s(x)[C_{Y,\varepsilon}(s,x)]^2+\frac13\Delta_q\hat R_\varepsilon^{\boxed{?}}+\\
&\quad+\frac13\int_{\zeta_1,\zeta_2}\delta^2Q_1^2\mathrm{D}^2(3\hat\Phi^{(0)}_{\zeta_1}[\![Y^2_{\varepsilon,\zeta_2}]\!]+[\![Y^3_{\varepsilon,\zeta_1}]\!]\hat\Phi^{(1)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2}+\frac13\int_{\zeta_1,\zeta_2}\delta^2Q_1^2\mathrm{D}^2(\hat\Phi^{(0)}_{\zeta_1}\hat\Phi^{(1)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2},
\end{aligned}\tag{37}$$

with $\frac13\Delta_qR_\varepsilon^{\boxed{?}}(\bar\zeta)\sim O_{L^\infty}(2^{q(1/2+2\kappa+2\delta)})$ and $\frac13\Delta_q\hat R_\varepsilon^{\boxed{?}}(\bar\zeta)+\frac13\Delta_qR_\varepsilon^{\boxed{?}}(\bar\zeta)\sim O_{L^\infty}(\varepsilon^\delta 2^{q(1/2+2\kappa+2\delta)})$.

Comparing (37) with the canonical trees in (21) we can identify the remainder terms $\Delta_q\hat Y_\varepsilon^\tau$ that need to converge to zero in order for $\Delta_qY_\varepsilon^\tau$ to converge to Δ_qX^τ.

2.2.2 Estimation of renormalised second-order trees

In this section we show that the remainder terms identified in (37) converge to zero in probability. First notice that we can bound (37) using Lemmas 32,34 and 35 as

$$\Big\|\delta^nQ_1^n\int_{\zeta_1,\zeta_2}\mathrm{D}^k\,(\Phi^{(i)}_{\zeta_1}\Phi^{(j)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2}\Big\|_{L^p}\lesssim\Big\|\int_{\zeta_1,\zeta_2}\mathrm{D}^k\,(\Phi^{(i)}_{\zeta_1}\Phi^{(j)}_{\zeta_2})\,\mu_{\zeta_1,\zeta_2}\Big\|_{L^p}.$$

Therefore, taking the derivatives in (37) we see that it suffices to bound in $L^p(H^{\otimes k+\ell})$ the term

$$\Phi^{(4-m)}_{\zeta_1}\Phi^{(4-n)}_{\zeta_2}h_{\zeta_1}^{\otimes k}\otimes h_{\zeta_2}^{\otimes\ell}=\Big[\frac{3!f_{3,\varepsilon}}{(3-m)!}[\![Y^{m-1}_{\zeta_1}]\!]+\hat\Phi^{(4-m)}_{\zeta_1}\Big]\Big[\frac{3!f_{3,\varepsilon}}{(3-n)!}[\![Y^{n-1}_{\zeta_2}]\!]+\hat\Phi^{(4-n)}_{\zeta_2}\Big]h_{\zeta_1}^{\otimes k}\otimes h_{\zeta_2}^{\otimes\ell}\tag{38}$$

for $m+n=5$ and $0\leqslant k+\ell\leqslant 2$. This yields some constraints on the number of branches of trees:

$$\begin{aligned}
\Delta_qY_\varepsilon^{\boxed{?}} &\leftrightarrow\ m+k=3,\quad n+\ell=3\\
\Delta_q\tilde Y_\varepsilon^{\boxed{?}} &\leftrightarrow\ m=3,k=0,n=2,\ell=1\\
\Delta_qY_\varepsilon^{\boxed{?}} &\leftrightarrow\ m+k=4,\quad n+\ell=2\\
\Delta_qY_\varepsilon^{\boxed{?}} &\leftrightarrow\ m+k=4,\quad n+\ell=3.
\end{aligned}\tag{39}$$

In (37), the terms proportional to $(f_{3,\varepsilon})^2$ will generate finite contributions in the limit. In particular it is easy to see that they converge respectively to $(\lambda^{(3)})^2X^{\boxed{?}}$, $(\lambda^{(3)})^2X^{\boxed{?}}$, $\lambda^{(3)}\lambda^{(2)}X^{\boxed{?}}$, $(\lambda^{(3)})^2X^{\boxed{?}}$. All other terms will vanish in probability, verifying (26), due to estimates we are going to establish now.

We consider the terms proportional to $\hat\Phi^{(4-m)}_{\zeta_1}\hat\Phi^{(4-n)}_{\zeta_2}$, all the other similar terms featuring at least one remainder $\hat\Phi^{(m)}_{\zeta}$ can be estimated with exactly the same technique, and are easily shown to be vanishing in the appropriate topology. One of the key observations of this paper, Lemma 36, allows us to rewrite products of divergencies in the form $\delta^m(u)\delta^n(v)$ as a sum of divergences $\delta^\ell(w)$, which are then easy to estimate in L^p using Lemma 32. We obtain

$$\begin{aligned}\hat\Phi^{(4-m)}_{\zeta_1}\hat\Phi^{(4-n)}_{\zeta_2} &= \delta^m(Q_1^m\Phi^{(4)}_{\zeta_1}h_{\zeta_1}^{\otimes m})\delta^n\big(Q_1^n\Phi^{(4)}_{\zeta_2}h_{\zeta_2}^{\otimes n}\big)\\ &= \sum_{(q,r,i)\in I} C_{_{q,r,i}}\delta^{m+n-q-r}\big(\langle \mathrm{D}^{r-i}Q_1^m\Phi^{(4)}_{\zeta_1}h_{\zeta_1}^{\otimes m},\mathrm{D}^{q-i}Q_1^n\Phi^{(4)}_{\zeta_2}h_{\zeta_2}^{\otimes n}\rangle_{H^{\otimes q+r-i}}\big)\\ &= \sum_{(q,r,i)\in I} C_{_{q,r,i}}\varepsilon^{1+\frac{r+q}{2}-i}\delta^{m+n-q-r}(\langle\Theta^{m+r-i}_{1+r-i}(\zeta_1)h_{\zeta_1}^{\otimes m+r-i},\Theta^{n+q-i}_{1+q-i}(\zeta_2)h_{\zeta_2}^{\otimes n+q-i}\rangle_{H^{\otimes q+r-i}})\end{aligned}$$

with $I=\{(q,r,i)\in\mathbb{N}^3\colon 0\leqslant q\leqslant m, 0\leqslant r\leqslant n, 0\leqslant i\leqslant q\wedge r\}$ and

$$\Theta^j_i(\zeta):=\varepsilon^{-\frac{i}{2}}Q_i^j\,\Phi^{(3+i)}_{\zeta}.$$

By Remark 33, for every $n,m\geqslant 1$ and $\Psi\in\operatorname{Dom}\delta^n$ we can write $\delta^n(\Psi)h^{\otimes m}=\delta^n(\Psi\otimes h^{\otimes m})$, and therefore

$$\int\hat\Phi^{(4-m)}_{\zeta_1}\hat\Phi^{(4-n)}_{\zeta_2}h_{\zeta_1}^{\otimes k}\otimes h_{\zeta_2}^{\otimes\ell}\,\mu_{\zeta_1,\zeta_2}=$$

$$=\sum_I C_{_{q,r,i}}\varepsilon^{\frac{2+q+r-2i}{2}}\delta^{m+n-q-r}\int\Theta^{m+r-i}_{1+r-i}(\zeta_1)\Theta^{n+q-i}_{1+q-i}(\zeta_2)\,h_{\zeta_1}^{\otimes m-q}\otimes h_{\zeta_2}^{\otimes n-r}\otimes h_{\zeta_1}^{\otimes k}\otimes h_{\zeta_2}^{\otimes\ell}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}\mu_{\zeta_1,\zeta_2}.$$

The following result allows us to estimate the quantity above in $L^p(H^{\otimes k+\ell})$.

Lemma 14. *Under Assumption 2 we have the bound*

$$\begin{aligned}&\Big\|\delta^{m+n-q-r}\int\Theta^{m+r-i}_{1+r-i}(\zeta_1)\Theta^{n+q-i}_{1+q-i}(\zeta_2)\,h_{\zeta_1}^{\otimes m-q}\otimes h_{\zeta_2}^{\otimes n-r}\otimes h_{\zeta_1}^{\otimes k}\otimes h_{\zeta_2}^{\otimes\ell}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}\mu_{\zeta_1,\zeta_2}\Big\|^2_{L^p(H^{\otimes k+\ell})}\\ &\lesssim\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{q+r-i}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|.\end{aligned}$$

Proof. Thanks to Lemma 32 the integral can be estimated with

$$\sum_{j=0,h\leqslant j}^{m+n-q-r}\Big\|\int\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1)\,\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2)\,h_{\zeta_1}^{\otimes m-q}\otimes h_{\zeta_2}^{\otimes n-r}\otimes h_{\zeta_1}^{\otimes k}\otimes h^{\otimes\ell}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}\mu_{\zeta_1,\zeta_2}\Big\|^2_{L^p(V)},$$

with $V=H^{\otimes m+k+n+\ell-q-r+j}$. We have that $\|\cdot\|^2_{L^p(H^{\otimes k+\ell})}=\|\|\cdot\|^2_{H^{\otimes k+\ell}}\|^{1/2}_{L^{p/2}}$ and therefore we can bound each term in the sum above as

$$\begin{aligned}\lesssim\ &\Big(\int\|\langle\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1)\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2),\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1')\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2')\rangle_{H^{\otimes j}}\|_{L^{p/2}}\times\\ &\times|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{q+r-i}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\Big)^{1/2}\end{aligned}$$

Using Hölder's inequality we get the estimate

$$\begin{aligned}&\|\langle\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1)\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2),\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1')\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2')\rangle_{H^{\otimes j}}\|_{L^{p/2}}\\ \lesssim\ &\|\langle\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1),\mathrm{D}^h\Theta^{m+r-i}_{1+r-i}(\zeta_1')\rangle_{H^{\otimes h}}\|_{L^p}\|\langle\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2),\mathrm{D}^{j-h}\Theta^{n+q-i}_{1+q-i}(\zeta_2')\rangle_{H^{\otimes j-h}}\|_{L^p}\end{aligned}$$

Now to bound terms of the type $\|\langle\mathrm{D}^h\Theta^{m+a}_{1+a}(\zeta),\mathrm{D}^h\Theta^{m+a}_{1+a}(\zeta')\rangle_{H^{\otimes h}}\|_{L^p}$ we consider the cases $h\leqslant m$ and $h>m$. In the first region we use Lemma 35 to estimate

$$\begin{aligned}\|\langle\mathrm{D}^h\Theta^{m+a}_{1+a}(\zeta),\mathrm{D}^h\Theta^{m+a}_{1+a}(\zeta')\rangle_{H^{\otimes h}}\|_{L^p}\ &\lesssim\ \Big\|\mathrm{D}^hQ^{m+a}_{1+a}\,\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1}\Big\|^2_{L^{4p}(H^{\otimes h})}\Big\|\mathrm{D}^hQ^{m+a}_{1+a}\,\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1'}\Big\|^2_{L^{4p}(H^{\otimes h})}\\ &\lesssim\ \Big\|\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1}\Big\|^2_{L^{4p}}\Big\|\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1'}\Big\|^2_{L^{4p}}.\end{aligned}$$

If $h>m$ we first commute $h-m$ derivatives in the expression $\mathrm{D}^hQ^{m+a}_{1+a}$ using formula (68) and then apply Lemma 35 to obtain the bound

$$\Big\|\mathrm{D}^hQ^{m+a}_{1+a}\,\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1}\Big\|^2_{L^{4p}(H^{\otimes h})}\lesssim\Big\|\mathrm{D}^{h-m}\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1}\Big\|^2_{L^{4p}}\lesssim\Big\|\varepsilon^{-\frac{1+a}{2}}\Phi^{(4+a)}_{\zeta_1}\Big\|_{\mathbb{D}^{h-m,4p}}.$$

Thus, we need to know $\Phi_\zeta^{(N)}$ up to the order $(4+r-i+h-m)\vee(4+q-i+j-h-n)\leqslant(4+n)\vee(4+m)$ to perform this estimates. □

From Lemma 14 we obtain $\forall\delta\in[0,1/2)$:

$$\left\|\varepsilon^{\frac{2+q+r-2i}{2}}\delta^{m+n-q-r}\int\Theta_{1+r-i}^{m+r-i}(\zeta_1)\Theta_{1+q-i}^{n+q-i}(\zeta_2)\,h_{\zeta_1}^{\otimes m-q}\otimes h_{\zeta_2}^{\otimes n-r}\otimes h_{\zeta_1}^{\otimes k}\otimes h_{\zeta_2}^{\otimes\ell}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}\mu_{\zeta_1,\zeta_2}\right\|_{L^p(H^{\otimes k+\ell})}$$

$$\lesssim\varepsilon^{\delta}\left(\varepsilon^{2+q+r-2i-\delta}\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{q+r-i}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\right)^{\frac12}$$

$$:=\varepsilon^{\frac{\delta}{2}}(\mathfrak{I})^{\frac12}.$$

Our aim now is to estimate the quantity $\mathfrak{I}$. The idea is to use the bound $\varepsilon|\langle h_\zeta,h_{\zeta'}\rangle|=\varepsilon C_{Y,\varepsilon}(\zeta-\zeta')\lesssim1$ to cancel strategically some of the covariances $|\langle h_\zeta,h_{\zeta'}\rangle|$. We will consider three regions:

If $q+r\leqslant2$ we use the bounds

$$\varepsilon^{q+r-2i}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{q+r-i}\lesssim\varepsilon^2|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{r}$$

and then (we suppose $r<2$)

$$\varepsilon^{2-r-\delta}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}\lesssim|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-2+\delta}$$

to obtain

$$\begin{aligned}\mathfrak{I}&\lesssim\varepsilon^{r-\delta}\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-2}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{r}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\\&\lesssim\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-2+\delta}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|.\end{aligned}\tag{40}$$

(If vice-versa $q<2$ it suffices to put δ on the term $|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+\delta}$.) Notice that in this case $m+k-q>0$.

In the case $q+r=3$ if $m+k-q\geqslant2$ we estimate like before to obtain

$$\begin{aligned}\mathfrak{I}&\lesssim\varepsilon^{2-\delta}\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{\frac{q+r}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{\frac{q+r}{2}}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\\&\lesssim\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{1+\delta}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|.\end{aligned}\tag{41}$$

Note that $m+k-q+\delta-1>0$ and $m+k-q+2\delta-3>-1$ here. If $m+k-q=1$ we bound

$$\mathfrak{I}\lesssim\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r-2}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{\frac{3+\delta}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{\frac{3+\delta}{2}}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\tag{42}$$

and note that $m+k-q-1/2+\delta/2>0$, $m+k-q-1+\delta>0$, $n+\ell-r-2\geqslant0$. Finally if $m+k-q=0$ we can only have $m+k=3$, $q=3$, $r=0$, $i=0$ and thus

$$\begin{aligned}\mathfrak{I}&\lesssim\varepsilon^{3-2\delta}\int|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\delta}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\delta}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\\&\lesssim\int|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell+m+k-6}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\delta}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\delta}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\end{aligned}\tag{43}$$

If $q+r\geqslant4$ we bound first

$$\varepsilon^{2q+2r-2i+\delta-4}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{q+r-i}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{q+r-i}\lesssim|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\frac{\delta}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\frac{\delta}{2}}$$

(note that $2q+2r-2i+\delta-4\geqslant\delta$) to obtain:

$$\mathfrak{I}\lesssim\varepsilon^{6-q-r-\delta}\int|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\frac{\delta}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\frac{\delta}{2}}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|$$

Now in the cases $m+k=3, n+\ell=3$ and $m+k=4, n+\ell=2$ we can just write $\varepsilon^{6-q-r-\delta}=\varepsilon^{m+k-q}\varepsilon^{6-m-k-r-\delta}$ and cancel the corresponding number of covariances to obtain

$$\mathfrak{I}\lesssim\int|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{\delta}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\frac{\delta}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\frac{\delta}{2}}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|\tag{44}$$

while for the case $m+k=4, n+\ell=3$ we have either $\ell\geqslant 1$ or $k\geqslant 1$ and therefore with one of the following bounds

$$\begin{aligned}\varepsilon^{m+k-1-q}\varepsilon^{n+\ell-r-\delta}|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r} &\lesssim |\langle h_{\zeta_1},h_{\zeta_1'}\rangle||\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{\delta}\\ \varepsilon^{m+k-q}\varepsilon^{n+\ell-1-r-\delta}|\langle h_{\zeta_1},h_{\zeta_1'}\rangle|^{m+k-q}|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{n+\ell-r} &\lesssim |\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{1+\delta}\end{aligned}$$

we obtain the estimates

$$\mathfrak{I} \lesssim \int |\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{1+\delta}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\frac{\delta}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\frac{\delta}{2}}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}| \tag{45}$$

$$\mathfrak{I} \lesssim \int |\langle h_{\zeta_1},h_{\zeta_1'}\rangle||\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^{\delta}|\langle h_{\zeta_1},h_{\zeta_2}\rangle|^{2-\frac{\delta}{2}}|\langle h_{\zeta_1'},h_{\zeta_2'}\rangle|^{2-\frac{\delta}{2}}|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta_1',\zeta_2'}|. \tag{46}$$

We can use directly Lemma 30 to obtain a final estimate of (40), (41), (42), (45). For (43), (44) and (46) notice that the integral over ζ_1,ζ_1' is finite and thus the whole quantity is proportional to $|\langle h_{\zeta_2},h_{\zeta_2'}\rangle|^n$. Globally, we have

$$\mathfrak{I}\lesssim 2^{(m+k+n+\ell-6)q}$$

as needed to prove (26). Lastly, by taking one more derivative of F_ε as done in Section 2.1, we can show (27) for $Y^\tau = Y_\varepsilon^{\boxed{?}}, Y_\varepsilon^{\boxed{?}}, \tilde{Y}_\varepsilon^{\boxed{?}}, Y_\varepsilon^{\boxed{?}}$, thus proving that $\hat{Y}^\tau\to 0$ in $C_T^{\kappa/2}\mathscr{C}^{\alpha-\kappa}$ in probability $\forall\alpha<|\tau|$.

Appendix A Basics of paracontrolled analysis

In this section we recall the notations and the basic results of paracontrolled calculus introduced in [4] without proofs. For more details on Besov spaces, Littlewood–Paley theory, and Bony's paraproduct the reader can refer to the nice recent monograph [1].

A.1 Notation and conventions.

Throughout the paper, we use the notation $a\lesssim b$ if there exists a constant $c>0$, independent of the variables under consideration, such that $a\leqslant c\cdot b$, and we write $a\simeq b$ if $a\lesssim b$ and $b\lesssim a$. If we want to emphasize the dependence of c on the variable x, then we write $a(x)\lesssim_x b(x)$. For index variables i and j of Littlewood-Paley decompositions (see below) we write $i\lesssim j$ if there exists $N\in\mathbb{N}$, independent of j, such that $i\leqslant j+N$, and we write $i\sim j$ if $i\lesssim j$ and $j\lesssim i$.

An *annulus* is a set of the form $\mathscr{A}=\{x\in\mathbb{R}^3\colon a\leqslant|x|\leqslant b\}$ for some $0<a<b$. A *ball* is a set of the form $\mathscr{B}=\{x\in\mathbb{R}^3\colon |x|\leqslant b\}$. If f is a map from $A\subset\mathbb{R}$ to the linear space Y, then we write $f_{s,t}=f(t)-f(s)$. For $f\in L^p(\mathbb{T}^d)$ we write $\|f(x)\|_{L^p_x(\mathbb{T}^3)}^p:=\int_{\mathbb{T}^3}|f(x)|^p\mathrm{d}x$.

Given a Banach space X with norm $\|\cdot\|_X$ and $T>0$, we write $C_TX=C([0,T],X)$ for the space of continuous maps from $[0,T]$ to X, equipped with the supremum norm $\|\cdot\|_{C_TX}$, and we set $CX=C(\mathbb{R}_+,X)$. For $\alpha\in(0,1)$ we also define $C_T^\alpha X$ as the space of α-Hölder continuous functions from $[0,T]$ to X, endowed with the seminorm $\|f\|_{C_T^\alpha X}=\sup_{0\leqslant s<t\leqslant T}\|f(t)-f(s)\|_X/|t-s|^\alpha$, and we write $C^\alpha_{\mathrm{loc}}X$ for the space of locally α-Hölder continuous functions from $\mathbb{R}_+$ to X. For $\gamma>0$, we define

$$\mathcal{M}_T^\gamma X=\{v\colon C((0,T],X)\colon \|v\|_{\mathcal{M}_T^\gamma X}=\|t\mapsto t^\gamma v(t)\|_{C_TX}<\infty\}.$$

The space of distributions on the torus is denoted by $\mathscr{D}'(\mathbb{T}^3)$ or $\mathscr{D}'$. The Fourier transform is defined with the normalization

$$\mathscr{F}u(k)=\hat{u}(k)=\int_{\mathbb{T}^d}e^{-\iota\langle k,x\rangle}u(x)\mathrm{d}x,\qquad k\in\mathbb{Z}^3,$$

so that the inverse Fourier transform is given by $\mathscr{F}^{-1}v(x)=(2\pi)^{-1}\sum_k e^{\iota\langle k,x\rangle}v(k)$. Throughout the paper, (χ,ρ) will denote a dyadic partition of unity such that $\mathrm{supp}(\rho(2^{-i}\cdot))\cap\mathrm{supp}(\rho(2^{-j}\cdot))=\emptyset$ for $|i-j|>1$. The family of operators $(\Delta_j)_{j\geq -1}$ will denote the Littlewood-Paley projections associated to this partition of unity, that is $\Delta_{-1}u=\mathscr{F}^{-1}(\chi\mathscr{F}u)$ and $\Delta_j=\mathscr{F}^{-1}(\rho(2^{-j}\cdot)\mathscr{F}u)$ for $j\geq 0$. We also use the notation $S_j=\sum_{i<j}\Delta_i$. The Hölder-Besov space $B^\alpha_{\infty,\infty}(\mathbb{T}^3,\mathbb{R})$ for $\alpha\in\mathbb{R}$ will be denoted by $\mathscr{C}^\alpha$ and equipped with the norm

$$\|f\|_\alpha=\|f\|_{B^\alpha_{\infty,\infty}}=\sup_{i\geqslant -1}(2^{i\alpha}\|\Delta_i f\|_{L^\infty(\mathbb{T}^3)}).$$

If f is in $\mathscr{C}^{\alpha-\varepsilon}$ for all $\varepsilon > 0$, then we write $f \in \mathscr{C}^{\alpha-}$. We let K_q the kernel of Δ_q so that $\Delta_q f(\bar{x}) = \int_{\mathbb{T}^3} K_{\bar{x},q}(x) f(x) \mathrm{d}x$.

A.2 Schauder estimates

For $\alpha \in (0,2)$, we define the space $\mathscr{L}^{\alpha}_T = C_T^{\alpha/2} L^\infty \cap C_T \mathscr{C}^\alpha$, equipped with the norm

$$\|f\|_{\mathscr{L}^{\alpha}_T} = \max\left\{\|f\|_{C_T^{\alpha/2}L^\infty}, \|f\|_{C_T\mathscr{C}^\alpha}\right\}.$$

The notation is chosen to be reminiscent of $\mathscr{L} = \partial_t - \Delta$, by which we will always denote the heat operator with periodic boundary conditions on $\mathbb{T}^d$. We also write $\mathscr{L}^\alpha = C^{\alpha/2}_{\mathrm{loc}} L^\infty \cap C\mathscr{C}^\alpha$. When working with irregular initial conditions, we will need to consider explosive spaces of parabolic type. For $\gamma \geqslant 0$, $\alpha \in (0,1)$, and $T > 0$ we define the norm

$$\|f\|_{\mathscr{L}^{\gamma,\alpha}_T} = \max\left\{\|t \mapsto t^\gamma f(t)\|_{C_T^{\alpha/2}L^\infty}, \|f\|_{\mathcal{M}^\gamma_T \mathscr{C}^\alpha}\right\}$$

and the space $\mathscr{L}^{\gamma,\alpha}_T = \{f\colon [0,T] \to \mathbb{R} \colon \|f\|_{\mathscr{L}^{\gamma,\alpha}_T} < \infty\}$. In particular, we have $\mathscr{L}^{0,\alpha}_T = \mathscr{L}^{\alpha}_T$. We introduce the linear operator $I\colon C(\mathbb{R}_+, \mathscr{D}'(\mathbb{T})) \to C(\mathbb{R}_+, \mathscr{D}'(\mathbb{T}))$ given by

$$If(t) = \int_0^t P_{t-s} f(s) \mathrm{d}s,$$

where $(P_t)_{t\geqslant 0}$ is the heat semigroup. Standard estimates in exposive spaces that are summarized in the following Lemma.

Lemma 15. *Let $\alpha \in (0,2)$ and $\gamma \in [0,1)$. Then*

$$\|If\|_{\mathscr{L}^{\gamma,\alpha}_t} \lesssim \|f\|_{\mathcal{M}^\gamma_t \mathscr{C}^{\alpha-2}} \tag{47}$$

for all $t > 0$. If further $\beta \geqslant -\alpha$, then

$$\|s \mapsto P_s u_0\|_{\mathscr{L}^{(\beta+\alpha)/2,\alpha}_t} \lesssim \|u_0\|_{\mathscr{C}^{-\beta}}. \tag{48}$$

For all $\alpha \in \mathbb{R}$, $\gamma \in [0,1)$, and $t > 0$ we have

$$\|If\|_{\mathcal{M}^\gamma_t \mathscr{C}^\alpha} \lesssim \|f\|_{\mathcal{M}^\gamma_t \mathscr{C}^{\alpha-2}} \tag{49}$$

For all $\alpha \in (0,2)$, $\gamma \in [0,1)$, $\varepsilon \in [0, \alpha \wedge 2\gamma)$, $t > 0$ and $f \in \mathscr{L}^{\gamma,\alpha}_t$ with $f(0) = 0$ we have

$$\|f\|_{\mathscr{L}^{\gamma-\varepsilon/2,\alpha-\varepsilon}_t} \lesssim \|f\|_{\mathscr{L}^{\gamma,\alpha}_t} \tag{50}$$

Proofs can be found in [5]. We need also some well known estimates for the solutions of the heat equation with sources in space–time Lebesgue spaces.

Lemma 16. *Let $\beta \in \mathbb{R}$ and $f \in L^p_T B^\beta_{p,\infty}$, then for every $\kappa \in [0,1]$ we have $If \in C^{\kappa/q}\mathcal{C}^{\beta+2(1-\kappa)-(2-2\kappa+d)/p}$ with*

$$\|If\|_{C_T^{\kappa/q}\mathcal{C}^{\beta+2(1-\kappa)-(2-2\kappa+d)/p}} \lesssim_T \|f\|_{L^p_T B^\beta_{p,\infty}},$$

with $\frac{1}{q} + \frac{1}{p} = 1$. Moreover, for every $\gamma < \gamma' < 1 - 1/p$ and every $\alpha < 2 - 5/p + \beta$ we have

$$\|If\|_{\mathscr{L}^{\gamma',\alpha}_T} \lesssim \|v \mapsto v^\gamma f(v)\|_{L^p_T B^\beta_{p,\infty}}$$

Proof. We only show the second inequality as the first one is easier and obtained with similar techniques. Let $u = If$, we have

$$\begin{aligned} t^\gamma \|\Delta_i u(t)\|_{L^\infty} &\leqslant t^{1/q} 2^{di/p} \left[\int_0^1 s^{-\gamma q} e^{-cq2^{2i}t(1-s)} \mathrm{d}s\right]^{1/q} \left[\int_0^t s^{\gamma p} \|\Delta_i f(s)\|^p_{L^p} \mathrm{d}\, s\right]^{1/p} \\ &\lesssim_{\gamma,q} 2^{id/p} 2^{-2i/q} \left[\int_0^t s^{\gamma p} \|\Delta_i f(s)\|^p_{L^p} \mathrm{d}\, s\right]^{1/p} \end{aligned}$$

which allows us to bound $\|If\|_{\mathcal{M}_T^\gamma \mathscr{C}^\alpha}$. In order to estimate $\|t \mapsto t^{\gamma'} If\|_{C_T^{\alpha/2} L^\infty}$ we write

$$\begin{aligned}\|t^{\gamma'}\Delta_i u(t) - s^{\gamma'}\Delta_i u(s)\|_{L^\infty} \lesssim & \int_s^t v^{\gamma'-1}\|\Delta_i u(v)\|_{L^\infty}\mathrm{d}v + |t-s|2^{i(d+2)/p}\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}} \\ & + \left\| \int_s^t v^{\gamma'}\Delta_i f(v)\,\mathrm{d}\,v \right\|_{L^\infty}\end{aligned}$$

We can estimate the first term as

$$\int_s^t v^{\gamma'-1}\|\Delta_i u(v)\|_{L^\infty}\mathrm{d}v \lesssim 2^{i(d+2)/p}\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}} \int_s^t v^{\gamma'-\gamma-1}\mathrm{d}v.$$

For the third term we have

$$\begin{aligned}\left\| \int_s^t v^{\gamma}\Delta_i f(v)\,\mathrm{d}\,v \right\|_{L^\infty} &\lesssim \left[\int_s^t \mathrm{d}v\right]^{1/q}\left[\int_s^t v^{\gamma p}\|\Delta_i f(s)\|_{L^\infty}^p\,\mathrm{d}v\right]^{1/p} \\ &\lesssim 2^{id/p}\,|t-s|^{1/q}\,\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}}\end{aligned}$$

We obtain then if $2^{2i}|t-s| \leqslant 1$

$$\|t^{\gamma'}\Delta_i u(t) - s^{\gamma'}\Delta_i u(s)\|_{L^\infty} \lesssim 2^{id/p}|t-s|^{1/q}\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}}$$

and if $2^{2i}|t-s| > 1$ we just use the trivial estimate

$$\|t^{\gamma'}\Delta_i u(t) - s^{\gamma'}\Delta_i u(s)\|_{L^\infty} \lesssim 2^{id/p}2^{-2i/q}\,\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}} \lesssim 2^{id/p}|t-s|^{1/q}\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}}.$$

Therefore, for every $\kappa \in [0,1]$:

$$\|t^{\gamma\prime}\Delta_i u(t) - s^{\gamma\prime}\Delta_i u(s)\|_{L^\infty} \lesssim 2^{(\frac{d+2}{p}-2)i}\,2^{2\kappa i/q}\,|t-s|^{\kappa/q}\,\|v \mapsto v^\gamma \Delta_i f(v)\|_{L^p_{t,x}}.$$

Choosing $\kappa/q = \alpha/2$ we obtain the desired estimate. □

A.3 Bony's paraproduct and some commutators

Paraproducts are bilinear operations introduced by Bony [2] in order to linearize a class of non-linear PDE problems. They appear naturally in the analysis of the product of two Besov distributions. In terms of Littlewood–Paley blocks, the product fg of two distributions f and g can be decomposed as

$$fg = f \prec g + f \succ g + f \circ g,$$

where

$$f \prec g = g \succ f := \sum_{j \geqslant -1}\sum_{i=-1}^{j-2} \Delta_i f \Delta_j g \qquad \text{and} \qquad f \circ g := \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g.$$

This decomposition behaves nicely with respect to Littlewood–Paley theory. We call $f \prec g$ and $f \succ g$ *paraproducts*, and $f \circ g$ the *resonant* term. We use the notation $f \preccurlyeq g = f \prec g + f \circ g$. The basic result about these bilinear operations is given by the following estimates, essentially due to Bony [2] and Meyer [11].

Lemma 17. (Bony's paraproduct estimates) *For any $\beta \in \mathbb{R}$ we have*

$$\|f \prec g\|_{\mathscr{C}^\beta} \lesssim_\beta \|f\|_{L^\infty}\|g\|_{\mathscr{C}^\beta}, \tag{51}$$

and for $\alpha < 0$ furthermore

$$\|f \prec g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|f\|_{\mathscr{C}^\alpha}\|g\|_{\mathscr{C}^\beta}. \tag{52}$$

For $\alpha + \beta > 0$ we have

$$\|f \circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|f\|_{\mathscr{C}^\alpha}\|g\|_{\mathscr{C}^\beta}. \tag{53}$$

A natural corollary is that the product fg of two elements $f \in \mathscr{C}^\alpha$ and $g \in \mathscr{C}^\beta$ is well defined as soon as $\alpha + \beta > 0$, and that it belongs to $\mathscr{C}^\gamma$, where $\gamma = \min\{\alpha, \beta, \alpha+\beta\}$.

We will also need the several commutator lemmas:

Lemma 18. (Bony's commutator estimate) *Let $\alpha > 0$, $\beta \in \mathbb{R}$, and let $f, g \in \mathscr{C}^\alpha$, and $h \in \mathscr{C}^\beta$. Then*

$$\|f \prec (g \prec h) - (fg) \prec h\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\alpha} \|h\|_{\mathscr{C}^\beta}.$$

When dealing with paraproducts in the context of parabolic equations it would be natural to introduce parabolic Besov spaces and related paraproducts. But to keep a simpler setting, we choose to work with space–time distributions belonging to the scale of spaces $(C_T\mathscr{C}^\alpha)_{\alpha\in\mathbb{R}}$ for some $T > 0$. To do so efficiently, we will use a modified paraproduct which introduces some smoothing in the time variable that is tuned to the parabolic scaling. Let therefore $\varphi \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ be nonnegative with compact support contained in $\mathbb{R}_+$ and with total mass 1, and define for all $i \geqslant -1$ the operator

$$Q_i \colon C\mathscr{C}^\beta \to C\mathscr{C}^\beta, \qquad Q_i f(t) = \int_0^\infty 2^{-2i} \varphi(2^{2i}(t-s)) f(s) \mathrm{d}s.$$

We will often apply Q_i and other operators on $C\mathscr{C}^\beta$ to functions $f \in C_T\mathscr{C}^\beta$ which we then simply extend from $[0, T]$ to $\mathbb{R}_+$ by considering $f(\cdot \wedge T)$. With the help of Q_i, we define a modified paraproduct

$$f \prec\!\!\prec g := \sum_i (Q_i S_{i-1} f) \Delta_i g$$

for $f, g \in C(\mathbb{R}_+, \mathscr{D}'(\mathbb{T}))$. We collect in the following lemma various estimates for the modified paraproduct, proofs are again in [5].

Lemma 19.

a) For any $\beta \in \mathbb{R}$ and $\gamma \in [0, 1)$ we have

$$t^\gamma \|f \prec\!\!\prec g(t)\|_{\mathscr{C}^\beta} \lesssim \|f\|_{\mathcal{M}_t^\gamma L^\infty} \|g(t)\|_{\mathscr{C}^\beta}, \tag{54}$$

for all $t > 0$, and for $\alpha < 0$ furthermore

$$t^\gamma \|f \prec\!\!\prec g(t)\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{M}_t^\gamma \mathscr{C}^\alpha} \|g(t)\|_{\mathscr{C}^\beta}. \tag{55}$$

b) Let $\alpha, \delta \in (0, 2)$, $\gamma \in [0, 1)$, $T > 0$, and let $f \in \mathscr{L}_T^{\gamma,\delta}$, $g \in C_T\mathscr{C}^\alpha$, and $\mathscr{L} g \in C_T\mathscr{C}^{\alpha-2}$. Then

$$\|f \prec\!\!\prec g\|_{\mathscr{L}_T^{\gamma,\alpha}} \lesssim \|f\|_{\mathscr{L}_T^{\gamma,\delta}} (\|g\|_{C_T\mathscr{C}^\alpha} + \|\mathscr{L} g\|_{C_T\mathscr{C}^{\alpha-2}}). \tag{56}$$

c) Let $\alpha \in (0, 2)$, $\gamma \in (0, 1)$, $T > 0$, and let $f \in \mathscr{L}_T^\alpha$. Then for all $\delta \in (0, \alpha]$ we have

$$\begin{aligned} \|f\|_{\mathscr{L}_T^\delta} &\lesssim \|f(0)\|_{\mathscr{C}^\delta} + T^{(\alpha-\delta)/2} \|f\|_{\mathscr{L}_T^\alpha}, \\ \|f\|_{\mathscr{L}_T^{\gamma,\delta}} &\lesssim T^{(\alpha-\delta)/2} \|f\|_{\mathscr{L}_T^{\gamma,\alpha}}. \end{aligned} \tag{57}$$

Finally we introduce various commutators which allow to control non-linear functions of paraproducs and also the interaction of the paraproducts with the heat kernel.

Lemma 20.

a) For $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta + \gamma > 0$ and $\alpha \in (0, 1)$ there exists bounded trilinear maps

$$\mathrm{com}_1, \overline{\mathrm{com}}_1 \colon \mathscr{C}^\alpha \times \mathscr{C}^\beta \times \mathscr{C}^\gamma \to \mathscr{C}^{\alpha+\beta+\gamma},$$

such that for smooth f, g, h they satisfy

$$\mathrm{com}_1(f, g, h) = (f \prec g) \circ h - f\,(g \circ h). \tag{58}$$

$$\overline{\mathrm{com}}_1(f, g, h) = (f \prec\!\!\prec g) \circ h - f\,(g \circ h). \tag{59}$$

b) Let $\alpha \in (0, 2)$, $\beta \in \mathbb{R}$, and $\gamma \in [0, 1)$. Then the bilinear maps

$$\mathrm{com}_2(f, g) := f \prec g - f \prec\!\!\prec g. \tag{60}$$

$$\mathrm{com}_3(f, g) := [\mathscr{L}, f \prec\!\!\prec] g := \mathscr{L}(f \prec\!\!\prec g) - f \prec\!\!\prec \mathscr{L} g. \tag{61}$$

have the bounds

$$t^\gamma \|\mathrm{com}_2(f, g)(t)\|_{\alpha+\beta} \lesssim \|f\|_{\mathscr{L}_t^{\gamma,\alpha}} \|g(t)\|_{\mathscr{C}^\beta}, \qquad t > 0. \tag{62}$$

as well as

$$t^{\gamma}\|\mathrm{com}_3(f,g)(t)\|_{\alpha+\beta-2}\lesssim \|f\|_{\mathscr{L}^{\gamma,\alpha}_t}\|g(t)\|_{\mathscr{C}^{\beta}},\qquad t>0. \tag{63}$$

Proofs can be found in [5].

Appendix B Estimation of the kernels

In this section we recall a few well-known results on convolution of functions with known singularities around zero. We remand to Section 10.3 of [6] for an extensive treatment of this subject. First of all we need to characterize the local behaviour of the heat kernel $P_t(x)$ and of the covariance $C_{Y,\varepsilon}(t,x)$ of the Gaussian field Y_ε.

Lemma 21. *The heat kernel $P\colon \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ defined by $P_t(x)=\frac{1}{(4\pi t)^{3/2}}e^{-\frac{|x|^2}{4t}}\mathbb{I}_{t\geqslant 0}$ has the bound*

$$|P(\zeta)|\lesssim (|t|^{1/2}+|x|)^{-3}.$$

Let $k\in\mathbb{N}^4$ a multi-index with $|k|=2k_1+k_2+\cdots+k_4$. Then for every multi-index $|k|\leqslant 2$ we have:

$$|D^kP_t(x)|\lesssim (|t|^{1/2}+|x|)^{-3-|k|}.$$

Remark 22. In this article we use a slightly different version of the heat kernel, namely

$$P_t(x)=\frac{1}{(4\pi t)^{3/2}}e^{-\frac{|x|^2}{4t}}e^{-t}\mathbb{I}_{t\geqslant 0}$$

in order to have that $X(t,x)=\int_{-\infty}^t\int_{\mathbb{T}^3}P_{t-s}(x-y)v(s,y)\mathrm{d}s\mathrm{d}y$ is the stationary solution to $\mathscr{L}X=-X+v$. However, every estimate remains trivially valid in this setting.

Proof.

$$|P_t(x)|(|t|^{1/2}+|x|)^3\lesssim[1+(|x||t|^{-1/2})^3]e^{-\frac{|x|^2}{4|t|}}=(1+|\alpha|^3)e^{-\frac{|\alpha|}{4}}<+\infty$$

In the same way we prove that $|\partial_tP_t(x)|\lesssim(|t|^{1/2}+|x|)^5$, $|\partial_{x_i}P_t(x)|\lesssim(|t|^{1/2}+|x|)^4$ and $|\partial_{x_i}\partial_{x_j}P_t(x)|\lesssim(|t|^{1/2}+|x|)^5$. □

We recall a special case of Lemma 10.14 of [6], which is enough for our purpose. We use the notation $|||\zeta|||:=(|t|^{1/2}+|x|)$ for $\zeta=(t,x)\in\mathbb{R}\times\mathbb{T}^3$.

Lemma 23. *Let f, $g\colon \mathbb{R}\times\mathbb{T}^3\setminus\{0\}\to\mathbb{R}$ smooth, integrable at infinity and such that $|f(\zeta)|\lesssim|||\zeta|||^{\alpha}$ and $|g(\zeta)|\lesssim|||\zeta|||^{\beta}$ in a ball $B=\{\zeta\in\mathbb{R}\times\mathbb{T}^3\colon|||\zeta|||<1,\zeta\neq 0\}$. Then if $\alpha,\beta\in(-5,0)$ and $\alpha+\beta+5<0$ we have*

$$|f*g(\zeta)|\lesssim|||\zeta|||^{\alpha+\beta+5}$$

in a ball centered in the origin.

Moreover, if $\alpha,\beta\in(-5,0)$ and $0<\alpha+\beta+5<1$ and for every multi-index $|k|\leqslant 2$ we have $|\mathrm{D}^kf(\zeta)|\lesssim|||\zeta|||^{\alpha-|k|}$ and $|\mathrm{D}^kg(\zeta)|\lesssim|||\zeta|||^{\beta-|k|}$, then

$$|f*g(\zeta)-f*g(0)|\lesssim|||\zeta|||^{\alpha+\beta+5}$$

in a ball centered in the origin.

Remark 24. The covariance $C_{Y,\varepsilon}$ of Y_ε can be written as $C_{Y,\varepsilon}=P*\tilde{C}_\varepsilon*P$ with $\tilde{C}_\varepsilon(t,x):=\mathbb{E}(\eta_\varepsilon(t,x)\eta_\varepsilon(0,0))$. Recall from the introduction that $\tilde{C}_\varepsilon(t,x)=\varepsilon^{-5}\tilde{C}^\varepsilon(\varepsilon^{-2}t,\varepsilon^{-1}x)$ where $\tilde{C}^\varepsilon$ is the covariance of the Gaussian process η defined on $\mathbb{R}\times(\mathbb{T}/\varepsilon)^3$, and $\tilde{C}^\varepsilon(t-s,x-y)=\Sigma(t-s,x-y)$ if $\mathrm{dist}(x,y)\leqslant 1$ and 0 otherwhise (so that the family of functions $\tilde{C}^\varepsilon$ is bounded uniformly on ε by a C^∞_c function). Then there exists a family of functions C^ε_Y defined on $\mathbb{R}\times(\mathbb{T}/\varepsilon)^3$ such that $C_{Y,\varepsilon}(t,x)=\varepsilon^{-1}C^\varepsilon_Y(\varepsilon^{-2}t,\varepsilon^{-1}x)$ and $C^\varepsilon_Y(t,x)=[P*\tilde{C}^\varepsilon*P](t,x)$.

Lemma 25. *The covariance $C_{Y,\varepsilon}$ has the bound, for every multi-index $|k|\leqslant 2$:*

$$|D^kC_{Y,\varepsilon}(t,x)|\lesssim(|t|^{1/2}+|x|)^{-1-|k|}.$$

Moreover, we have

$$\varepsilon^{|k|+1}|D^k C_{Y,\varepsilon}(t,x)| \lesssim 1$$

Proof. The first bound is obtaned directly from Lemma 21 and Lemma 23. Indeed, since by hypothesis C^ε has compact support, it is easy to see that $|\tilde{C}_\varepsilon(t,x)| \lesssim (|t|^{1/2}+|x|)^{-5}$. The second bound is obtained by a simple change of variables in the convolution defining $C_{Y,\varepsilon}$. □

Lemma 26. *We have* $\int_{s,x} P_s(x)[C_{Y,\varepsilon}(s,x)]^2 \lesssim |\log\varepsilon|$ *and for every* $n \geqslant 3$ $\varepsilon^{n-2}\int_{s,x} P_s(x)[C_{Y,\varepsilon}(s,x)]^n \lesssim 1$.

Proof. From the fact that $P_{\varepsilon^2 s}(\varepsilon x) = \varepsilon^{-3} P_s(x)$ together with Remark 24 we obtain

$$\int_{\mathbb{R}\times\mathbb{T}^3} P_s(x)[C_{Y,\varepsilon}(s,x)]^2 \mathrm{d}s\mathrm{d}x \lesssim \int_{B(0,\varepsilon^{-1})} P_s(x)[C^\varepsilon_Y(s,x)]^2 \mathrm{d}s\mathrm{d}x \lesssim |\log\varepsilon|$$

with $B(0,R) = \{\zeta \in \mathbb{R}^4 \colon |||\zeta||| < R,\ \zeta \neq 0\}$ a "parabolic" ball centered in the origin. The second estimation is obtained in the same way. □

Lemma 27. *For* $m \in (0,3)$, $n \in (3,5)$, *define for* $\zeta, \zeta' \in \mathbb{R}\times\mathbb{T}^3$

$$I_m := \int |C_{Y,\varepsilon}(\zeta-\zeta')|^m \, |\mu_\zeta \mu_{\zeta'}|, \qquad \tilde{I}_n := \int |C_{Y,\varepsilon}(\zeta-\zeta')|^n \, |\tilde{\mu}_\zeta \tilde{\mu}_{\zeta'}|$$

with $\mu_\zeta := K_{q,\bar{x}}(y)\delta(t-s)\mathrm{d}\zeta$ *and* $\tilde{\mu}_\zeta := [\int_x K_{q,\bar{x}}(x) P_{t-s}(x-y)]d\zeta$ *for* $\zeta = (s,y)$. *Then*

$$I_m \lesssim 2^{mq} \qquad \text{and} \qquad \tilde{I}_n \lesssim 2^{(n-4)q}.$$

Proof. The estimation of I_m is easily obtained by Lemma 25 and a change of variables. For $\tilde{I}_n$ observe that for every $q > 0$

$$\tilde{\mu}_\zeta = [\int_x K_{q,\bar{x}}(x)(P_{t-s}(x-y) - P_{t-s}(\bar{x}-y))]d\zeta$$

and then we can apply Lemma 23 to obtain the result. □

Lemma 28. *We have for every* $\sigma \in [0,1]$

$$\sup_{x\in\mathbb{T}^3} |C_{Y,\varepsilon}(t,x) - C_{Y,\varepsilon}(0,x)| \lesssim \varepsilon^{-1-2\sigma}|t|^\sigma$$

Proof. It is easy to obtain by interpolation knowing that $|\partial_t C_{Y,\varepsilon}(t,x)| \lesssim \varepsilon^{-3}$ from Lemma 25. □

Lemma 29. *We have for every* $\alpha < 3$

$$\sum_{i\sim j} \left| \int K_i(x-y) P_t(y)\mathrm{d}y \right| \int \frac{|K_j(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim \frac{1}{(|x|+t^{1/2})^{3+\alpha}}$$

Proof. We will show that

$$\left| \int K_i(x-y) P_t(y)\mathrm{d}y \right| \lesssim 2^{-i}(|x|+t^{1/2}+2^{-i})^{-4}, \tag{64}$$

and that

$$\int \frac{|K_i(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim (|x|+t^{1/2}+2^{-i})^{-\alpha}, \tag{65}$$

from which we deduce that

$$\sum_{i\sim j} \left| \int K_i(x-y) P_t(y)\mathrm{d}y \right| \int \frac{|K_j(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim \sum_i \frac{2^{-i}}{(|x|+t^{1/2}+2^{-i})^{4+\alpha}}.$$

Bounding the sum over i with an integral, we conclude

$$\int_0^1 \frac{\mathrm{d}\lambda}{\lambda} \frac{\lambda}{(|x|+t^{1/2}+\lambda)^{4+\alpha}} = \frac{1}{(|x|+t^{1/2})^{3+\alpha}} \int_0^{1/(|x|+t^{1/2})} \frac{\mathrm{d}\lambda}{(1+\lambda)^{4+\alpha}} \lesssim \frac{1}{(|x|+t^{1/2})^{3+\alpha}}.$$

Let us show (64). We want to estimate

$$\begin{aligned} I &= \int K_i(x-y)P_t(y)\mathrm{d}y = \int K_i(x-y)[P_t(y)-P_t(x)]\mathrm{d}y \\ &= \int_0^1 \mathrm{d}\tau \int K_i(x-y)[P_t'(x+\tau(y-x))(y-x)]\mathrm{d}y \\ |I| &\lesssim \int_0^1 \mathrm{d}\tau \int |(y-x)K_i(x-y)||P_t'(x+\tau(y-x))|\mathrm{d}y \lesssim 2^{-i}\int_0^1 \mathrm{d}\tau \int |yK_1(y)||P_t'(x+\tau 2^{-i}y)|\mathrm{d}y \\ &\lesssim 2^{-i}\int_0^1 \mathrm{d}\tau \int |yK_1(y)|\frac{e^{-c|x+\tau 2^{-i}y|^2/t}}{t^2}\mathrm{d}y \end{aligned}$$

where

$$|P_t'(z)| = \left| C\frac{e^{-|z|^2/t}}{t^{4/2}}\frac{z}{t^{1/2}}\right| \leqslant C\frac{e^{-c|z|^2/t}}{t^2}.$$

When $t^{1/2} \geqslant 2^{-i}, |x|$ we have

$$|I| \lesssim 2^{-i}t^{-2} \lesssim 2^{-i}(|x|+t^{1/2}+2^{-i})^{-4}.$$

When $2^{-i} \geqslant t^{1/2}, |x|$ we estimate simply

$$|I| \lesssim \int |K_i(x-y)|P_t(y)\mathrm{d}y \lesssim 2^{3i} \lesssim 2^{-i}(|x|+t^{1/2}+2^{-i})^{-4}.$$

When $|x| \geqslant 2^{-i}, t^{1/2}$ we have instead that either $|x| \geqslant 2\tau 2^{-i}|y|$ or $|x| < 2\tau 2^{-i}|y|$. In the first region $|x+\tau 2^{-i}y| \geqslant c|x|$ so

$$|I| \lesssim 2^{-i}\int_0^1 \mathrm{d}\tau \int |yK_1(y)|\frac{e^{-c'|x|^2/t}}{t^2}\mathrm{d}y \lesssim 2^{-i}\frac{e^{-c'|x|^2/t}}{t^2} \lesssim 2^{-i}|x|^{-4} \lesssim 2^{-i}(|x|+t^{1/2}+2^{-i})^{-4}.$$

while in the second region $|y| \geqslant 2^i|x|/(2\tau)$, then $|yK_1(y)| \leqslant |yK_1(y)|^{1/2}F(2^i|x|/(2\tau))$ where F is a rapidly decreasing function and in this region

$$\begin{aligned} |I| &\lesssim 2^{-i}\int_0^1 \mathrm{d}\tau F(2^i|x|/(2\tau)) \int \frac{e^{-c|x+\tau 2^{-i}y|^2/t}}{t^2}\mathrm{d}y \\ &\lesssim 2^{-i}\int_0^1 \mathrm{d}\tau F(2^i|x|/(2\tau)) \int \frac{e^{-c'|\tau 2^{-i}y|^2/t}}{t^{3/2}|\tau 2^{-i}y|}\mathrm{d}y \lesssim 2^{-i}\int_0^1 \mathrm{d}\tau F(2^i|x|/(2\tau))\frac{2^{3i}}{\tau^3|x|}\int \frac{e^{-c'|y|^2/t}}{t^{3/2}}\mathrm{d}y \\ &\lesssim \frac{2^{-i}}{|x|^4}\int_0^1 \mathrm{d}\tau F(2^i|x|/(2\tau))\frac{2^{3i}|x|^3}{\tau^3} \lesssim \frac{2^{-i}}{|x|^4} \lesssim 2^{-i}(|x|+t^{1/2}+2^{-i})^{-4}. \end{aligned}$$

So we conclude that (64) holds. Let us turn to (65). When $t^{1/2} \geqslant 2^{-i}, |x|$ we have

$$\int \frac{|K_i(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim \frac{1}{t^\alpha}\int |K_i(x-y)|\mathrm{d}y \lesssim \frac{1}{t^\alpha} \lesssim (|x|+t^{1/2}+2^{-i})^{-\alpha}.$$

When $2^{-i} \geqslant t^{1/2}, |x|$ we estimate

$$\int \frac{|K_i(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim 2^{\alpha i}\int \frac{|K_1(y)|}{|2^i x+y|^\alpha}\mathrm{d}y \lesssim 2^{2i}\sup_z \int \frac{|K_1(y)|}{|z+y|^\alpha}\mathrm{d}y \lesssim 2^{2i} \lesssim (|x|+t^{1/2}+2^{-i})^{-\alpha},$$

and finally when $|x| \geqslant 2^{-i}, t^{1/2}$ we have either $|x| \geqslant 2^{-i+1}|y|$ or $|x| < 2^{-i+1}|y|$. In the first region $|x+2^{-i}y| \geqslant c|x|$ so

$$\int \frac{|K_i(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim \int \frac{|K_1(y)|}{|x+2^{-i}y|^\alpha}\mathrm{d}y \lesssim |x|^{-\alpha} \lesssim (|x|+t^{1/2}+2^{-i})^{-\alpha},$$

while in the second $|y| \geqslant 2^i|x|/2$, then $|K_1(y)| \leqslant |K_1(y)|^{1/2}F(2^i|x|/2)$ where F is another rapidly decreasing function and in this region

$$\int \frac{|K_i(x-y)|}{(|y|+t^{1/2})^\alpha}\mathrm{d}y \lesssim F(2^i|x|/2)\int \frac{|K_1(y)|^{1/2}}{|2^{-i}y|^\alpha}\mathrm{d}y \lesssim 2^{\alpha i}F(2^i|x|/2) \lesssim |x|^{-\alpha} \lesssim (|x|+t^{1/2}+2^{-i})^{-\alpha},$$

concluding our argument. □

Lemma 30. *For $m,n\in(0,5)$, $k,\ell\in[0,2)$ define*

$$I_{k,m,n}:=\int_{\zeta_{1,2},\zeta'_{1,2}} C_{Y,\varepsilon}(\zeta_1-\zeta_2)^k C_{Y,\varepsilon}(\zeta_1-\zeta_2)^\ell C_{Y,\varepsilon}(\zeta_1-\zeta'_1)^m C_{Y,\varepsilon}(\zeta_2-\zeta'_2)^n|\mu_{\zeta_1,\zeta_2}||\mu_{\zeta'_1,\zeta'_2}|,$$

with μ_{ζ_1,ζ_2} for $\bar\zeta=(t,\bar x)$, $\zeta_i=(s_i,x_i)$ $i=1,2$ defined as

$$\mu_{\zeta_1,\zeta_2}:=[\int_{x,y} K_{q,\bar x}(x)\sum_{i\sim j} K_{i,x}(y)K_{j,x}(x_2)P_{t-s_1}(y-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2.$$

If $\ell=0$, $0<m+k-2<5$, $m+k-2\in(-1,5)$ and $k+m+n-4\in(0,5)$ we have the estimate

$$I_{k,m,n}\lesssim 2^{(k+m+n-4)q}.$$

If $(k+m-2),(\ell+m-2)\in(0,5)$, $k+m+\ell-4\in(0,5)$ and $k+\ell+m+n-4\in(0,5)$ we have the estimate

$$I_{k,m,n}\lesssim 2^{(k+\ell+m+n-4)q}.$$

Proof. Observe that

$$\begin{aligned}
\mu_{\zeta_1,\zeta_2} &= [\sum_{i\sim j}\int_{x,y} K_{q,\bar x}(x_2)K_{i,x}(y)K_{j,x}(x_2)(P_{t-s_1}(y-x_1)-P_{t-s_1}(\bar x-x_1))]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\
&\quad+[\sum_{i\sim j}\int_{x,y}(K_{q,\bar x}(x)-K_{q,\bar x}(x_2))K_{i,x}(y)K_{j,x}(x_2)P_{t-s_1}(y-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\
&= K_{q,\bar x}(x_2)[P_{t-s_1}(x_2-x_1)-P_{t-s_1}(\bar x-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\
&\quad+[\sum_{i\sim j}\int_{x,y}[K_{q,\bar x}(x)-K_{q,\bar x}(x_2)]K_{i,x}(y)K_{j,x}(x_2)P_{t-s_1}(y-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\
&= \bar\mu_{\zeta_1,\zeta_2}+\hat\mu_{\zeta_1,\zeta_2}
\end{aligned}$$

where in the first line we used $\int_y K_{i,x}(y)=0$ and the fact that $\int_x K_{i,x}(x'_1)K_{j,x}(x_2)=0$ if $|i-j|>1$ and $\sum_{i,j}K_{i,x}(y)K_{j,x}(x_2)=\delta(x_2-y)\delta(x_2-x)$. Now the estimation of the term

$$\bar I_{k,m,n}:=\int_{\zeta_{1,2},\zeta'_{1,2}} C_{Y,\varepsilon}(\zeta_1-\zeta_2)^k C_{Y,\varepsilon}(\zeta'_1-\zeta'_2)^\ell C_{Y,\varepsilon}(\zeta_1-\zeta'_1)^m C_{Y,\varepsilon}(\zeta_1-\zeta_2)^n|\bar\mu_{\zeta_1,\zeta_2}||\bar\mu_{\zeta'_1,\zeta'_2}|,$$

with $\bar\mu_{\zeta_1,\zeta_2}=K_{q,\bar x}(x_2)[P_{t-s_1}(x_2-x_1)-P_{t-s_1}(\bar x-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2$ can be done with Lemma 23 and gives the expected result. The integral

$$\hat I_{k,m,n}:=\int_{\zeta_{1,2},\zeta'_{1,2}} C_{Y,\varepsilon}(\zeta_1-\zeta_2)^k C_{Y,\varepsilon}(\zeta'_1-\zeta'_2)^\ell C_{Y,\varepsilon}(\zeta_1-\zeta'_1)^m C_{Y,\varepsilon}(\zeta_1-\zeta_2)^n|\hat\mu_{\zeta_1,\zeta_2}||\hat\mu_{\zeta'_1,\zeta'_2}|,$$

with $\hat\mu_{\zeta_1,\zeta_2}=[\sum_{i\sim j}\int_{x,y}[K_{q,\bar x}(x)-K_{q,\bar x}(x_2)]K_{i,x}(y)K_{j,x}(x_2)P_{t-s_1}(y-x_1)]\delta(t-s_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2$ can be estimated by multiple changes of variables. We have $K_{q,\bar x}(x)-K_{q,\bar x}(x_2)=2^{3q}(x_2-x)\int_0^1 K'(2^q(x_2-x)\tau-2^q(\bar x-x_2))\mathrm{d}\tau$, and by the scaling properties of $C_{Y,\varepsilon}$ and $P_{t,y}$, namely $C_{Y,\varepsilon}(2^{-2i}s,2^{-i}x)\lesssim 2^iC_{Y,\varepsilon}(s,x)$ and $P_{2^{-2i}s}(2^{-i}x)\lesssim 2^{3i}P_s(x)$ given by Lemma 21 and Lemma 25, we obtain easily the bound on $\hat I_{k,m,n}$ by rescaling the integral. □

Appendix C Some Malliavin calculus results

Let D be the Malliavin derivative, δ the divergence (defined as the adjoint of D) and P_t the Ornstein-Uhlenbeck semigroup. We refer to [18] for an extensive discussion on these operators. Call $\{W(h)\}_{h\in H}$ the isonormal Gaussian process indexed by H some real separable Hilbert space. For every $\Psi\in L^2(\Omega)$, P_t can be written via the well-known Mehler's formula for $\Psi=F(W(h))$:

$$P_t\Psi=P_tF(W(h))=\mathbb{E}_{Y_\varepsilon}[F(e^{-t}W(h)+(1-e^{-2t})^{1/2}W'(h))] \tag{66}$$

where $\{W'(h)\}_{h\in H}$ is an independent copy of W. In our case we will consider the Gaussian process Y_ε indexed by $h_{t,x}\ \forall(t,x)\in\mathbb{R}\times\mathbb{T}^3$, with $H=L^2(\mathbb{R}\times\mathbb{T}^3)$, with $Y_\varepsilon(t,x)$ defined as in Section 1. By a direct calculation we obtain

$$\mathrm{D}\,P_t\Psi=e^{-t}\,\mathbb{E}_W[F'(e^{-t}W(h)+(1-e^{-2t})^{1/2}\,W'(h))]h=e^{-t}P_t\,\mathrm{D}\,\Psi. \tag{67}$$

This gives a commutation result between the Malliavin derivative D and the generator of the Ornstein-Uhlenbeck semigroup L (defined by $P_t = e^{tL}$, recall that $L = -\delta \mathrm{D}$ ([18], Proposition 1.4.3)). Indeed, let Ψ such that $\mathbb{E}(\Psi) = 0$, then we have for every $\alpha > 0$ and every $j \geqslant 0$:

$$\mathrm{D}\,(j-L)^{-\alpha}\Psi = \mathrm{D}\,\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\,e^{-jt}e^{tL}\Psi\,\mathrm{d}t = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\,e^{-(j+1)t}P_t\,\mathrm{D}\,\Psi = (j+1-L)^{-\alpha}\mathrm{D}\,\Psi \tag{68}$$

and the same works for every Ψ (not necessarily centered) if $j > 0$. It is well-known (see [18]) that L acts on square integrable functions Ψ as

$$L\,\Psi = -\sum_{n=0}^{\infty} n\,J_n\Psi$$

where $J_n\Psi$ is the projection of Ψ on the n-th Wiener chaos. We can define $(j-L)^{-1}$ by its action on n-th order chaoses as $(j-L)^{-1}\,J_n\Psi = \frac{1}{j+n}J_n\,\Psi\ \forall n \geqslant 0,\, j \geqslant 1$. The results recalled above allow for the following partial chaos expansion :

Lemma 31. *Let $\Psi \in L^2(\Omega)$. Then for every $n \in \mathbb{N}\setminus\{0\}$:*

$$\Psi = \sum_{k=0}^{n-1}\frac{1}{k!}\,\delta^k J_0 D^k\Psi + \delta^n Q_1^n \mathrm{D}^n\Psi \tag{69}$$

with $Q_j^m = \prod_{k=j}^m (k-L)^{-1}$.

Proof. We have for any $\Psi \in L^2(\Omega)$:

$$\Psi - J_0\Psi = L\,L^{-1}(\Psi - J_0\Psi) = -\delta\,\mathrm{D}\,L^{-1}(\Psi - J_0\Psi) = \delta\,(1-L)^{-1}\mathrm{D}\,\Psi$$

where we used (68) and the fact that the Malliavin derivative of a constant is zero. This yields

$$\Psi = \mathbb{E}(\Psi) + \delta\,(1-L)^{-1}\mathrm{D}\,\Psi \tag{70}$$

Iterating this formula up to an order n and using the fact that $J_0(k-L)^{-1} = \frac{1}{k}J_0$ we obtain the result. □

Notice that the lemma above implies $\delta^n Q_1^n \mathrm{D}^n\Psi = (1 - J_0 \ldots - J_{n-1})\Psi$.

In order to have L^p estimations of the terms $\delta^n Q_1^n D^n\Psi$ generated by expansion (69), we will need the following lemmas:

Lemma 32. ([18], Proposition 1.5.7) *Let V be a real separable Hilbert space. For every $p > 1$ and every $q \in \mathbb{N}, k \geqslant q$ and every $u \in \mathbb{D}^{k,p}(H^q \otimes V)$ we have*

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}(V)} \lesssim_{k,p} \|u\|_{\mathbb{D}^{k,p}(H^q\otimes V)}$$

Remark 33. Using Lemma 32 we can state a simplified version of Lemma 38 in the case where $F \in V$ is deterministic. Let V be a real separable Hilbert space. For every $F \in V$ and every $u \in \mathbb{D}^{q,2}(H^{\otimes q})$ with $q \in \mathbb{N}$ we have $u \otimes F \in \mathrm{Dom}\,\delta^q$ and

$$\delta^q(u)F = \delta^q(u \otimes F).$$

We can prove this formula as follows. First notice that for every smooth $G \in \mathbb{D}^{q,2}(V)$ and every smooth $u \in \mathbb{D}^{q,2}(H^{\otimes q})$ we have

$$\mathbb{E}(\langle \delta^q(u\otimes F), G\rangle_V) = \mathbb{E}(\langle u\otimes F, \mathrm{D}^q G\rangle_{H^{\otimes q}\otimes V}) = \mathbb{E}(\langle \delta^q(u)F, G\rangle_V).$$

Now since $\mathrm{D}^q(u\otimes F) = \mathrm{D}^q u \otimes F$ and $u \in \mathbb{D}^{q,2}(H^{\otimes q})$ we have that $u\otimes F \in \mathbb{D}^{q,2}(H^{\otimes q}\otimes V)$. Lemma 32 yields the bound $\|\delta^q(u\otimes F)\|_{L^2(V)} \lesssim \|u\otimes F\|_{\mathbb{D}^{q,2}(H^{\otimes q}\otimes V)}$ which allows to pass to the limit for G and $\delta^q(u\otimes F)$ in $L^2(V)$.

Lemma 34. *For every $n, m \in \mathbb{N}\setminus\{0\}$, the operator $Q_n^m = \prod_{k=n}^m (k-L)^{-1}$ is bounded in L^p for every $1 < p < \infty$.*

Proof. We have $Q_j\,\Psi = (j-L)^{-1}\,\Psi = \sum_{n=0}^\infty (j+n)^{-1}J_n\Psi = (1/j)\mathbb{E}(\Psi) + T_\phi\Psi$ with $T_\phi\Psi = \sum_{n=0}^\infty \phi(n)J_n\Psi$ and $\phi(0) = 0$. Then the operator T_ϕ satisfies the hypotheses of Theorem 1.4.2 of [18], with $\phi(n) = h(n^{-1})$ and $h(x) = \frac{x}{j\,x+1}$ analytic in a neighbourhood of 0. Therefore

$$\|Q_j\Psi\|_{L^p} \leqslant \mathbb{E}(\Psi) + \|T_\phi\Psi\|_{L^p} \lesssim \|\Psi\|_{L^p}$$

and the result follows applying repeatedly this inequality. □

Lemma 35. *Let $j \in \mathbb{N}\setminus\{0\}$ and $p>1$. There exists a finite constant c_p such that for every $\Psi \in L^p$:*

$$\left\| \mathrm{D}\,(j-L)^{-\frac{1}{2}}\Psi \right\|_{L^p(H)} \leqslant c_p \|\Psi\|_{L^p}$$

(where the operator $\mathrm{D}(j-L)^{-\frac{1}{2}}$ is defined on every Ψ polynomial in $W(h_1),...,W(h_n)$ and can be extended by density on L^p).

Proof. First notice that we can suppose w.l.o.g. $\mathbb{E}(\Psi)=0$ thanks to the commutation (68). Therefore we can write $\mathrm{D}\,(j-L)^{-\frac{1}{2}}$ as

$$\mathrm{D}(j-L)^{-\frac{1}{2}} = \mathrm{D}(-C)^{-1}(-C)\,(j-L)^{-\frac{1}{2}}$$

with $C=-\sqrt{-L}$. We decompose the second part as:

$$-C\,(j-L)^{-\frac{1}{2}}\Psi = \sum_{n=1}^{\infty}\left(\frac{n}{j+n}\right)^{\frac{1}{2}} J_n\Psi = T_\phi\Psi$$

With $T_\phi\Psi := \sum_{n=0}^{\infty}\phi(n)J_n\Psi$. We can apply Theorem 1.4.2 of [18] to show that T_ϕ is bounded in L^p, indeed $\phi(n)=h(1\,/\,n)$ and $h(x)=(j\;x+1)^{-1/2}$ which is analytic in a neighbourhood of 0. Finally, we can apply Proposition 1.5.2 of [18] to show that DC^{-1} is bounded in L^p, thus concluding the proof. □

The following lemma is the most useful tool we used in the paper. It allows us to write products of decompositions of the type (69) as sums of iterated Skorohod integrals.

Lemma 36.
Let $u=F_u(W(h_u))h_u^{\otimes m}$, $v=F_v(W(h_v))h_v^{\otimes n}$ with F_u, $F_v\in C^{m+n}(\mathbb{R})$ such that $u\in\mathbb{D}^{m+n,2}(H^{\otimes m})$, $v\in\mathbb{D}^{m+n,2}(H^{\otimes n})$. Then:

$$\delta^m(u)\delta^n(v) = \sum_{i=0}^{m\wedge n}\sum_{q=0}^{m-i}\sum_{r=0}^{n-i}\binom{m}{q+i}\binom{q+i}{i}\binom{n}{r+i}\binom{r+i}{i} i!\,\delta^{m+n-q-r-2i}(\langle \mathrm{D}^r u, \mathrm{D}^q v\rangle_{H^{\otimes q+r+i}}). \tag{71}$$

And also

$$\delta^m(u)\delta^n(v) = \sum_{q=0}^{m}\sum_{r=0}^{n}\sum_{i=0}^{q\wedge r}\binom{m}{q}\binom{q}{i}\binom{n}{r}\binom{r}{i} i!\,\delta^{m+n-q-r}(\langle \mathrm{D}^{r-i} u, \mathrm{D}^{q-i} v\rangle_{H^{\otimes q+r-i}}). \tag{72}$$

Proof. We have using Cauchy-Schwarz inequality and Lemma 32 that $\langle D^r\delta^n(v),\delta^j(u)\rangle_{H^{\otimes r}}\in L^2(\Omega,H^{\otimes m-j-r})$ for every $0\leqslant r+j\leqslant m$. Then we can apply Lemma 38 to obtain

$$\delta^m(u)\delta^n(v) = \sum_{r=0}^{n}\binom{n}{r}\delta^{n-r}(\langle \mathrm{D}^r\,\delta^m(u), v\rangle_{H^{\otimes r}}).$$

Then using the commutation formula (73) we rewrite the r.h.s. as

$$\delta^m(u)\delta^n(v) = \sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{r\wedge m}\binom{r}{i}\binom{m}{i} i!\,\delta^{n-r}(\langle \delta^{m-i}(\mathrm{D}^{r-i}u), v\rangle_{H^{\otimes r}}).$$

We write $\langle \delta^{m-i}(\mathrm{D}^{r-i}u), v\rangle_{H^{\otimes r}} = \langle \delta^{m-i}(\mathrm{D}^{r-i}u), F_v(W(h_v))h_v^{\otimes r}\rangle_{H^{\otimes r}} h_v^{\otimes n-r}$ and verify in the same way as before that $\langle \delta^{m-i}(\mathrm{D}^{r-i}u), F_v(W(h_v))h_v^{\otimes r}\rangle_{H^{\otimes r}}$ satisfies the hypotheses of Lemma 40. We obtain

$$\begin{aligned}
\delta^m(u)\delta^n(v) &= \sum_{r=0}^{n}\sum_{i=0}^{r\wedge m}\binom{n}{r}\binom{r}{i}\binom{m}{i} i!\,\delta^{n-r}[\langle \delta^{m-i}(\mathrm{D}^{r-i}u), F_v(W(h_v))h_v^{\otimes r}\rangle_{H^{\otimes r}} h_v^{\otimes n-r}] \\
&= \sum_{r=0}^{n}\sum_{i=0}^{r\wedge m}\binom{n}{r}\binom{r}{i}\binom{m}{i} i!\sum_{\ell=0}^{m-i}\binom{m-i}{\ell}\delta^{n-r}[\delta^{m-i-\ell}(\langle \mathrm{D}^{r-i}u, \mathrm{D}^{\ell}F_v(W(h_v))h_v^{\otimes r}\rangle_{H^{\otimes r+\ell}})h_v^{\otimes n-r}] \\
&= \sum_{r=0}^{n}\sum_{i=0}^{r\wedge m}\sum_{\ell=0}^{m-i}\binom{n}{r}\binom{r}{i}\binom{m}{i}\binom{m-i}{\ell} i!\delta^{m+n-r-i-\ell}(\langle \mathrm{D}^{r-i}u, \mathrm{D}^{\ell}v\rangle_{H^{\otimes r+\ell}})
\end{aligned}$$

where we used the fact that $\delta^k(\Psi)h^{\otimes n-r}=\delta^k(\Psi\otimes h^{\otimes n-r})$ for $\Psi\in\operatorname{Dom}\delta^k$, as seen in Remark 33. Call $q=\ell+i$, then

$$\delta^m(u)\delta^n(v)=\sum_{i=0}^{m\wedge n}\sum_{q=i}^{m}\sum_{r=i}^{n}\binom{n}{r}\binom{r}{i}\binom{m}{i}\binom{m-i}{q-i}i!\delta^{m+n-q-r}(\langle \mathrm{D}^{r-i}u,\mathrm{D}^{q-i}v\rangle_{H^{\otimes q+r-i}}),$$

and noting that

$$\binom{m}{i}\binom{m-i}{q-i}=\frac{(m-i)!m!}{(q-i)!(m-q)!i!(m-i)!}=\frac{q!m!}{i!(q-i)!q!(m-q)!}=\binom{q}{i}\binom{m}{q},$$

we have the nicer symmetric expression

$$\delta^m(u)\delta^n(v)=\sum_{i=0}^{m\wedge n}\sum_{q=i}^{m}\sum_{r=i}^{n}\binom{m}{q}\binom{q}{i}\binom{n}{r}\binom{r}{i}i!\,\delta^{m+n-q-r}(\langle \mathrm{D}^{r-i}u,\mathrm{D}^{q-i}v\rangle_{H^{\otimes q+r-i}}).$$

Finally we perform the change of variables $q-i\to q$, $r-i\to r$ to get

$$\delta^m(u)\delta^n(v)=\sum_{i=0}^{m\wedge n}\sum_{q=0}^{m-i}\sum_{r=0}^{n-i}\binom{m}{q+i}\binom{q+i}{i}\binom{n}{r+i}\binom{r+i}{i}i!\,\delta^{m+n-q-r-2i}(\langle \mathrm{D}^{r}u,\mathrm{D}^{q}v\rangle_{H^{\otimes q+r+i}}).$$

The second formula is a straightforward change of indexes of the first one. □

Remark 37. Our choice of giving two distinct but closely related formulas in Lemma 36 is due to the fact that the first formula has a more evident "physical meaning". Indeed, vertices u and v (being non-polynomial) have an infinite chaos decomposition, which can be represented as having infinite "legs" in a Feynman-like diagram. It is apparent that the index i in first equation denotes contractions between the already existing legs of the vertices u,v and that r,q stay for new legs in each vertex created by the Malliavin derivatives which are then contracted with other legs from the other vertex. This leaves $m+n-r-q-2i$ legs overall uncontracted which are arguments to the iterated Skorokhod integral and would be contracted with other composite vertices in the L^p estimates. The second formula however, is more practical in the calculations.

We give below the results we used to prove Lemma 36.

Lemma 38. ([16], Lemma 2.1) *Let $q\geqslant 1$, $F\in\mathbb{D}^{q,2}$, $u\in\operatorname{Dom}(\delta^q)$ and symmetric. Assume also that $\forall 0\leqslant r+j\leqslant q$ $\langle D^rF,\delta^j(u)\rangle_{H^{\otimes r}}\in L^2(\Omega,H^{\otimes q-r-j})$. Then $\forall 0\leqslant r\leqslant q$ $\langle D^rF,u\rangle_r\in\operatorname{Dom}(\delta^{q-r})$ and*

$$F\delta^q(u)=\sum_{r=0}^{q}\binom{q}{r}\delta^{q-r}(\langle \mathrm{D}^rF,u\rangle_{H^{\otimes r}}).$$

Remark 39. Note that

$$\delta^k(h^{\otimes k})=[\![W^k(h)]\!]$$

where $[\![\cdot]\!]$ stands for the Wick product. Indeed $\forall F\in\mathbb{D}^{1,2}$ we know that $\mathbb{E}[\delta(h^{\otimes n})F]=\mathbb{E}[W(h)\ h^{\otimes n-1}]$ using the definition of δ and therefore $\delta^n(h^{\otimes n})=\delta^{n-1}(W(h)h^{\otimes n-1})$. We have also $\delta^{n-1}(W(h)h^{\otimes n-1})=\delta^{n-1}(h^{\otimes n-1})W(h)-(n-1)\langle h,h\rangle\delta^{n-2}(h^{\otimes n-2})$ using Lemma 38, and the result is proved by induction.

Lemma 40. *Let $\ell\in\mathbb{N}$, $F\in\mathbb{D}^{q,2}(H^{\otimes\ell})$, $u\in\operatorname{Dom}(\delta^q)$ with values in $H^{\otimes q+\ell}$ and symmetric. Assume also that $\langle\delta^j(u),D^rF\rangle_{H^{\otimes\ell+r}}\in L^2(\Omega,H^{\otimes q-r-j})$ $\forall 0\leqslant r+j\leqslant q$. Then $\forall 0\leqslant r\leqslant q$ $\langle u,D^rF\rangle_{H^{\otimes\ell+r}}\in\operatorname{Dom}(\delta^{q-r})$ and*

$$\langle\delta^q(u),F\rangle_{H^{\otimes\ell}}=\sum_{r=0}^{q}\binom{q}{r}\delta^{q-r}(\langle u,\mathrm{D}^rF\rangle_{H^{\otimes\ell+r}})$$

Proof. Let $q=1$. We have for smooth $G\in\mathbb{D}^{1,2}$, $F\in\mathbb{D}^{q,2}(H^{\otimes\ell})$ and $u\in\operatorname{Dom}(\delta^q)$:

$$\begin{aligned}\mathbb{E}(\langle\langle u,F\rangle_{H^{\otimes\ell}},\mathrm{D}G\rangle_H) &= \mathbb{E}(\langle u,\mathrm{D}G\otimes F\rangle_{H^{\otimes\ell+1}})\\ &= \mathbb{E}(\langle\delta(u),F\rangle_{H^{\otimes\ell}}G)-\mathbb{E}(\langle u,\mathrm{D}F\rangle_{H^{\otimes\ell+1}}G)\end{aligned}$$

where we used the fact that $\mathrm{D}(G\ F)=\mathrm{D}G\otimes F+G\ \mathrm{D}F$ for smooth functions. The equality $\langle\langle u,\ F\rangle_{H^{\otimes\ell}},\mathrm{D}G\rangle_H=\langle u,\mathrm{D}G\otimes F\rangle_{H^{\otimes\ell+1}}$ holds because u is symmetric.

We can pass to the limit thanks to the assumption $\langle \delta^j(u), D^r F\rangle_{H^{\otimes \ell+r}} \in L^2(\Omega, H^{\otimes q-r-j})$ and obtain

$$\delta(\langle u, F\rangle_{H^{\otimes \ell}}) = \langle \delta(u), F\rangle_{H^{\otimes \ell}} - \langle u, \mathrm{D}F\rangle_{H^{\otimes \ell+1}}.$$

Now suppose the statement true for $q-1$. We have that

$$\begin{aligned}
\langle \delta^q(u), F\rangle_{H^{\otimes \ell}} &= \sum_{r=0}^{q-1} \binom{q-1}{r} \delta^{q-1-r}(\langle \delta(u), \mathrm{D}^r F\rangle_{H^{\otimes \ell+r}}) \\
&= \sum_{r=0}^{q-1} \binom{q-1}{r} \delta^{q-r}(\langle u, D^r F\rangle_{H^{\otimes \ell+r}}) + \sum_{r=0}^{q-1} \binom{q-1}{r} \delta^{q-r-1}(\langle u, \mathrm{D}^{r+1} F\rangle_{H^{\otimes \ell+r+1}}) \\
&= \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}(\langle u, \mathrm{D}^r F\rangle_{H^{\otimes \ell+r}})
\end{aligned}$$

□

Lemma 41. *Let $j, k \in \mathbb{N}$, $u \in \mathbb{D}^{j+k,2}(H^{\otimes j})$ symmetric and such that all its derivatives are symmetric. We have*

$$\mathrm{D}^k \delta^j(u) = \sum_{i=0}^{k\wedge j} \binom{k}{i}\binom{j}{i} i!\, \delta^{j-i}(D^{k-i} u) \tag{73}$$

Proof. If $j = 0$, $k = 1$ or $k = 0$, $j = 1$ we have identities. Now let $j = k = 1$ and $u \in \mathbb{D}^{2,2}(H)$. We have that $\mathbb{D}^{1,2}(H) \subset \mathrm{Dom}(\delta)$ ([18], Proposition 1.3.1), and since $\mathrm{D}\, u \in \mathbb{D}^{1,2}(H^{\otimes 2})$ it is easy to see that $\forall h \in H$ $\langle \mathrm{D}\, u, h\rangle \in \mathbb{D}^{1,2}(H)$ by computing its norm. Then we can apply Proposition 1.3.2 of [18] to obtain $\forall h \in H$

$$\langle \mathrm{D}\delta(u), h\rangle = \langle u, h\rangle + \delta(\langle \mathrm{D}u, h\rangle)$$

and since by hypothesis $\mathrm{D}u$ is symmetric we have $\delta(\langle \mathrm{D}u,\, h\rangle) = \langle \delta \mathrm{D}u,\, h\rangle$ and then $\mathrm{D}\delta(u) = u + \delta \mathrm{D}u$. The proof by induction is easy noticing that δv is symmetric whenever v is symmetric, and using the fact that $\mathrm{D}\delta^j = \delta^j \mathrm{D} + j\delta^{j-1}$. □

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