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GLOBAL SOLUTIONS OF THE RANDOM VORTEX FILAMENT EQUATION

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ABSTRACT. We prove the existence of a global solution for the filament equation with inital condition given by a geometric rough path in the sense of Lyons [17]. Our work gives a positive answer to a question left open in recent publications: Berselli and Gubinelli [5] showed the existence of global solution for a smooth initial condition while Bessaih, Gubinelli, Russo [6] proved the existence of a local solution for a general initial condition given by a rough path.

1. THE FILAMENT EQUATION

In this note we prove the existence of a global solution for the following rough filament equation

$$\begin{cases} \frac{d\gamma(t)(\xi)}{dt} = u^{\gamma(t)}(\gamma(t)(\xi)), \quad t \in [0,\infty), \xi \in [0,1] \\ \gamma(0) = \gamma_0 \end{cases} \tag{1.1}$$

where the initial condition $\gamma_0 : [0,1] \to \mathbb{R}^3$ is a geometric ν-rough path (for some $\nu \in (\frac{1}{3}, 1)$), see Assumption 2.6. Here $\gamma : [0,\infty) \to \mathcal{D}_{\gamma_0} \subset \mathcal{C}$ is some trajectory in the subset $\mathcal{D}_{\gamma_0}$ of the space $\mathcal{C} \subseteq C([0,1];\mathbb{R}^3)$ of continuous closed curves in $\mathbb{R}^3$ parametrized by the interval $[0,1]$. Finally u^γ, $\gamma \in \mathcal{D}_{\gamma_0}$ is a vector field given by

$$u^\gamma(x) = \int_0^1 \nabla\phi(x - \gamma(\xi)) \times d\gamma(\xi). \tag{1.2}$$

where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function which satisfies certain assumptions (see Hypothesis 3.2). The exact meaning of the line integral above and set $\mathcal{D}_{\gamma_0}$ we consider will be explained below. Equation (1.1) has its origin in the theory of three dimensional Euler equations. It is well known that for two dimensional Euler equations vorticity $\omega = \operatorname{curl} u$ is transported along the fluid flow. The situation changes drastically in three dimensional case. Additional "stretching" term in the equation defining vorticity leads to possibility of blow up of the vorticity. Furthermore, a result of Beale, Kato, and Majda [2] suggests that a possible singularity of Euler equations appears when the vorticity field of the fluid blows up. Consequently, understanding the behaviour of the vorticity of an ideal incompressible fluid is an important problem in fluid dynamics.

The properties of the motion of the vorticity has been studied for the last 150 years starting from the works of Helmholtz [15] and Kelvin [16]. It has been suggested by Kelvin to use Biot-Savart law

$$u(x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)dy, \qquad x \in \mathbb{R}^3$$

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where $\times$ denotes the vector product, combined with assumption that the vorticity is supported by a smooth closed curve evolving in time $\gamma : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^3$

$$\omega(x,t) = \Gamma \int_0^1 \delta(x - \gamma(t,\xi)) \frac{\partial \gamma(t,\xi)}{\partial \xi} d\xi, \qquad x \in \mathbb{R}^3, t \geq 0$$

and definition of the flow

$$\left\{ \begin{array}{lcl} \frac{dX_t(x)}{dt} & = & u(X_t(x),t), \quad t \geq 0, \\ X_0(x) & = & x. \end{array} \right. \tag{1.3}$$

to formally deduce the *filament equation*

$$\frac{\partial \gamma}{\partial t}(t,\xi) = -\frac{\Gamma}{4\pi} \int_0^1 \frac{\gamma(t,\xi) - \gamma(t,\eta)}{|\gamma(t,\xi) - \gamma(t,\eta)|^3} \times \frac{\partial \gamma(t,\eta)}{\partial \eta} d\eta. \tag{1.4}$$

The assumption that vorticity is supported by some curve is coherent with numerical simulations of 3D turbulent fluids which show that regions of large vorticity have a form of a "filament", see, for instance [3], [21].

Equation (1.4) has singularity when ξ and η are close together and the initial curve γ is smooth. As a consequence, the energy of the solution of this equation given by the formula

$$E(t) = \frac{\Gamma^2}{8\pi} \int_0^1 \int_0^1 \frac{1}{|\gamma(t,\xi) - \gamma(t,\eta)|} \frac{\partial \gamma(t,\eta)}{\partial \eta} \cdot \frac{\partial \gamma(t,\xi)}{\partial \xi} d\xi d\eta, \qquad t \geq 0$$

is infinite for any smooth curve $\gamma(t,\cdot)$. In order to avoid the singularity Rosenhead [20] introduced the regularized model

$$\frac{\partial \gamma}{\partial t}(t,\xi) = -\frac{\Gamma}{4\pi} \int_0^1 \frac{\gamma(t,\xi) - \gamma(t,\eta)}{(|\gamma(t,\xi) - \gamma(t,\eta)|^2 + \mu^2)^{3/2}} \times \frac{\partial \gamma(t,\eta)}{\partial \eta} d\eta \tag{1.5}$$

which is widely used in aereonautics to study the dynamics of trailing vortices at the tips of airplane wings.

The problem (1.1) has been studied by Berselli and Bessaih [4] and then by Berselli and Gubinelli [5]. It contains the equation (1.5) as a particular case when

$$\phi(x) = \frac{\Gamma}{(|x|^2 + \mu^2)^{\frac{1}{2}}}, \qquad x \in \mathbb{R}^3, \mu > 0.$$

Equation (1.1) is in fact a nonlinear PDE for a function $\gamma : [0,\infty) \times [0,1] \to \mathbb{R}^3$. A natural setting for the well-posedness of the Cauchy problem is obtained by requiring the vector field u to be well defined and Lipshitz in the space variable. To this effect the approach followed in [4] is to set up the equation as an evolution problem in the Sobolev space H^1 of closed curves in $\mathbb{R}^3$ with square integrable first derivative (with respect to the parameter). This approach implies that the vector field (once lifted to H^1) does not allow strong enough estimates to deduce global existence. This is ultimately due to the fact that in $3d$ incompressible flows vortices strech and undergo a complex dynamics and that a priori this could lead to a explosion of the H^1 norm. Such a difficulty should be compared to the more stable behavior of 2d vortex points which, under incompressible flows, are simply transported along the flow lines. Exploiting the conservation of the kinetic energy of the flow and a control of the velocity field generated by the vortex line via the associated kinetic energy Berselli and Gubinelli [5] showed the existence of global solution to equation (1.1) with initial conditions in H^1.

Bessaih, Gubinelli, Russo [6], partly motivated by the random filament models suggested by Gallavotti [11] and Chorin [8], considered the above evolution problem when the initial condition is a random closed curve. For definiteness they considered initial condition sampled from the law of a 3-dimensional Brownian loop (a Brownian motion starting at $0 \in \mathbb{R}^3$ and conditioned to return to $0 \in \mathbb{R}^3$ at time 1). In this case it is no more possible to set up the problem in the Sobolev space H^1 since Brownian trajectories almost surely do not belong to this space. A more serious problem is the meaning to give to the generalized Biot-Savart relation (1.2) since the line integral along a Brownian trajectory does not allow a straighforward definition. Moreover stochastic integration (à la Itô or Stratonovich) does not provide a good framework to study this problem since it does not posses a natural filtration on the parameter space stable under the time evolution. For these reasons [6] identified the spaces of controlled rough paths as a natural framework to have a well posed problem.

Rough path theory has been introduced by T. J. Lyons in the seminal paper [17] (see also [10, 18, 19]) as a way to overcome certain difficulties of stochastic integration theories and have a robust analytical framework to solve stochastic differential equations and similar problems involving integration of non-regular vector-fields. It turns out that rough paths theory and in particular the notion of *controlled paths* introduced in [12] allows to give a natural interpretation to the Biot-Savart relation (1.2) and obtain a well-posed problem. Using this approach Bessaih, Gubinelli, Russo [6] obtained existence of a local solution to eq. (1.1) when initial condition is a closed curve of Hölder class with exponent $\nu \in (1/3, 1]$ (suitably lifted to rough path space).

The aim of the present paper is to extend the energy method of Berselli and Gubinelli to the rough path setting and obtain a global existence result for the equation (1.1) when the initial condition is a geometric rough path, thus completing the analysis of [6].

Recently there have been some deep progresses in the study of evolution equations in the space of controlled rough paths. In particular Hairer [13] showed how to use controlled path theory to have well-posedness of a multidimensional Burgers type equation driven by additive space-time white noise and later [14] used similar ideas to tackle the longstanding open problem of the Kardar–Parisi–Zhang equation for which he described a notion of solution for which existence and uniqueness can be proven.. The key technical tool to obtain these results has been the observation that the more singular non-linear term involved in the fixed-point argument has the same structure of the Biot-Savart relation (1.2) and thus can be similarly handled via controlled paths techniques.

A major open problem which remains largely unexplored is the rigorous study of the un-regularized filament equation (1.4), whose singularity seems to defy any reasonable analytic approach. On this respect the very suggestive heuristic computations of Gallavotti [11] seems to hint to the fact that rapid oscillations of the curve γ could provide a natural regularization. Another very interesting problem is the study of the rough filament equation as an Hamiltonian system on the space of parametrization invariant paths on the lines of [1] and [7].

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2. Definition and Properties of Rough Path integrals

In this section we present the controlled path framework and we define rough path integral and state some of its properties. We mainly follow [12] and [6].

In what follows V, W will usually denote fixed Banach spaces and $L(V,W)$ the Banach space of bounded linear maps from V to W and $C(X,Y)$ the space of continuous function from X to Y. Moreover we let $\mathbb{T} = [0,1]$ and

$$C_n(V) \;=\; \{f \in C(\mathbb{T}^n, V) :\; f(t_1, t_2, \cdots, t_n) = 0 \text{ if } t_i = t_{i+1} \text{ for some } i = 1, .., n-1\}.$$

We will understand that $C(V) = C_1(V)$ and if $f \in C_1(V)$ we will consider the Hölder norm

$$|f|_\mu = \sup_{a\in\mathbb{T}} |f(a)| + \sup_{0\le a<b\le 1} \frac{|f(a)-f(b)|_V}{|a-b|^\mu};$$

while for $f \in C_2(V), g \in C_3(V)$ consider the norms

$$|f|_\mu = \sup_{0\le a<b\le 1} \frac{|f(a,b)|_V}{|a-b|^\mu}; \qquad |g|_\mu = \sup_{0\le a<b<c\le 1} \frac{|g(a,b,c)|_V}{|a-c|^\mu}$$

and the associated normed spaces

$$C^\mu(V) = C_1^\mu(V) = \{f \in C_1(V) :\; |f|_\mu < \infty\},$$

$$C_2^\mu(V) = \{f \in C_2(V) :\; |f|_\mu < \infty\}, \qquad C_3^\mu(V) = \{g \in C_3(V) :\; |g|_\mu < \infty\}.$$

For $n = 1, 2$ define the operators $\delta : C_n(V) \to C_{n+1}(V)$ as

$$\delta f(a,b) = f(b) - f(a), \qquad \delta g(a,b,c) = g(a,c) - g(a,b) - g(b,c)$$

for all $a, b, c \in \mathbb{T}$. They satisfy the following fundamental property $\delta\delta f = 0$, for all $f \in C_1(V)$. Let $\mathcal{Z}C_2^\mu(V) = C_3^\mu(V) \cap \ker\delta$ and $\mathcal{B}C_2^\mu(V) = C_3^\mu(V) \cap \operatorname{im}\delta$. If $g \in \mathcal{Z}C_2^\mu(V)$ then $g \in \mathcal{B}C_2^\mu(V)$, that is there exists $f \in C_1(V)$ such that $\delta f = g$. Then following proposition has been proved in [12]:

Proposition 2.1 (Sewing lemma)**.** *For any $\mu > 1$ there exists an unique linear map (Sewing map) $\Lambda : \mathcal{Z}C_3^\mu(V) \to C_2^\mu(V)$ such that*

$$\delta\Lambda = \mathrm{id}_{\mathcal{Z}C_3^{1+}(V)}.$$

This map is continuous and we have

$$\|\Lambda h\|_\mu \le \frac{1}{2^\mu - 2}\|h\|_\mu, \qquad h \in \mathcal{Z}C_3^{1+}(V).$$

Now, we define class of paths for which rough path integral will be defined.

Definition 2.2. *Assume that $\nu \in (0,1)$ and fix $X \in C^\nu(V)$. We say that a path $Y \in C(V)$ is weakly controlled by X if there exist a function $Z \in C^\nu(L(V,V))$ such that the quantity*

$$R(\xi,\eta) = Y(\xi) - Y(\eta) - Z(\eta)(X(\xi) - X(\eta)), \qquad \xi,\eta \in \mathbb{T}, \tag{2.1}$$

belongs to $C_2^{2\nu}(V)$. We let $\mathcal{D}_X$ the set of all the pairs (Y,Z) satisfying (2.1)*. This is a vector space which can be endowed with the semi-norm $\|\cdot\|_{\mathcal{D}_X}$ given by*

$$\|(Y,Z)\|_{\mathcal{D}_X} = |Z|_{C^\nu} + \|R\|_{C_2^{2\nu}}, \tag{2.2}$$

Furthermore, let us define a norm $\|\cdot\|^*_{\mathcal{D}_X}$ *in* $\mathcal{D}_X$ *by*

$$\|(Y,Z)\|^*_{\mathcal{D}_X} = \|Y\|_{\mathcal{D}_X} + \sup_{\xi\in\mathbb{T}}|Y(\xi)|. \tag{2.3}$$

With these definitions $(\mathcal{D}_X, \|\cdot\|^*_{\mathcal{D}_X})$ *is a Banach space. While not necessary for the integration theory it will convenient for our intended application to assume that all our controlled paths take values in* $V=\mathbb{R}^3$ *and are closed, that is* $Y(0)=Y(1)$.

In the present paper an element $(Y,Y')\in\mathcal{D}_X$ can always be canonically identified from its first component Y by the mean of standard procedures discussed below. In this context we denote R^Y the corresponding remainder: $R^Y=\delta Y - Y'\delta X$. Moreover we will often omit to specify Y' when it is clear from the context and write $\|Y\|_{\mathcal{D}_X}$ instead of $\|(Y,Y')\|_{\mathcal{D}_X}$.

We will repeatedly use the basic estimate

$$\|Y\|_{C^\nu} \le \|Y\|^*_{\mathcal{D}_X}(1+\|X\|_{C^\nu}). \tag{2.4}$$

Controlled paths are stable by mapping via sufficiently regular functions:

Lemma 2.3. *Let* $\phi\in C^2(\mathbb{R}^3,\mathbb{R}^3)$ *and* $(Y,Y')\in\mathcal{D}_X$. *Then*

$$(W,W') := (\phi(Y),\phi'(Y)Y')\in\mathcal{D}_X \tag{2.5}$$

and the remainder has the following representation

$$\begin{aligned} R^W(\xi,\eta) &= \phi'(Y(\xi))R^Y(\xi,\eta) + (Y(\eta)-Y(\xi))\times \\ &\quad\times\int_0^1 [\nabla\phi(Y(\xi)+r(Y(\eta)-Y(\xi))) - \nabla\phi(Y(\xi))]dr, \qquad \xi,\eta\in\mathbb{T}. \end{aligned} \tag{2.6}$$

Furthermore, there exists a constant $K\ge 1$ *such that*

$$\|\phi(Y)\|_{\mathcal{D}_X} \le K\|\nabla\phi\|_{C^1}\|Y\|_{\mathcal{D}_X}(1+\|Y\|_{\mathcal{D}_X})(1+\|X\|_\nu)^2. \tag{2.7}$$

Moreover, if $(\tilde Y,\tilde Y')\in\mathcal{D}_{\tilde X}$ *and* $(\tilde W,\tilde W') := (\phi(\tilde Y),\phi'(\tilde Y)\tilde Y')$ *then*

$$\begin{aligned} \|W'-\tilde W'\|_\nu + \|R^W - R^{\tilde W}\|_{2\nu} + \|W-\tilde W\|_\nu \le \\ C(\|X-\tilde X\|_\nu + \|Y'-\tilde Y'\|_\nu + \|R^Y - R^{\tilde Y}\|_{2\nu} + \|Y-\tilde Y\|_\nu) \end{aligned} \tag{2.8}$$

with

$$C = K\|\phi\|_{C^3}(1+\|X\|_{C^\nu}+\|\tilde X\|_{C^\nu})^3|(1+\|Y\|_{\mathcal{D}_X}+\|\tilde Y\|_{\mathcal{D}_{\tilde X}})^2. \tag{2.9}$$

When $X=\tilde X$ *we have*

$$\begin{aligned} \|\phi(Y)-\phi(\tilde Y)\|_{\mathcal{D}_X} \le & K\|\nabla\phi\|_{C^2}\|Y\|_{\mathcal{D}_X} \\ & (1+\|Y\|_{\mathcal{D}_X}+\|\tilde Y\|_{\mathcal{D}_X})^2(1+\|X\|_{C^\nu})^4\|Y-\tilde Y\|_{\mathcal{D}_X}. \end{aligned} \tag{2.10}$$

Proof. See [12], Proposition 4 for all statements of the Lemma, except (2.6) (which is actually also proven, though not stated explicitly). Let us show (2.6). Denote $y(r)=Y(\xi)+r(Y(\eta)-$

$Y(\xi)), r \in [0,1]$. Then

$$\phi(y(1)) - \phi(y(0)) = \int_0^1 \phi'(y(r))y'(r)dr \tag{2.11}$$

$$= \sum_k (Y^k(\eta) - Y^k(\xi)) \int_0^1 \frac{\partial \phi}{\partial x_k}(y(r))dr = \sum_k \frac{\partial \phi}{\partial x_k}(Y(\xi))(Y^k(\eta) - Y^k(\xi))$$

$$+ \sum_k (Y^k(\eta) - Y^k(\xi)) \int_0^1 [\frac{\partial \phi}{\partial x_k}(y(r)) - \frac{\partial \phi}{\partial x_k}(Y(\xi))]dr$$

$$= \sum_{k,l} \frac{\partial \phi}{\partial x_k}(Y(\xi))(Y')^{kl}(X^l(\eta) - X^l(\xi)) + \sum_k \frac{\partial \phi}{\partial x_k}(Y(\xi))(R^Y)^k(\xi,\eta)$$

$$+ \sum_k (Y^k(\eta) - Y^k(\xi)) \int_0^1 [\frac{\partial \phi}{\partial x_k}(y(r)) - \frac{\partial \phi}{\partial x_k}(Y(\xi))]dr,$$

and the result follows. □

In order to define the integral for paths controlled by X we need a further ingredient: the existence of basic integrals of X with respect to X itself. These objects are encoded into what is refereed to as a *rough path* as follows.

Definition 2.4. *Assume that $\nu > 1/3$. We say that couple $\mathbb{X} = (X, \mathbb{X}^2)$ where $X \in C^\nu(\mathbb{T}, \mathbb{R}^3)$ and $\mathbb{X}^2 \in C_2^{2\nu}(L(\mathbb{R}^3, \mathbb{R}^3))$ is a ν-rough path if the following condition is satisfied:*

$$\mathbb{X}^2(\xi,\rho) - \mathbb{X}^2(\xi,\eta) - \mathbb{X}^2(\eta,\rho) = (X(\xi) - X(\eta)) \otimes (X(\eta) - X(\rho)), \quad \xi,\eta,\rho \in \mathbb{T}. \tag{2.12}$$

Remark 2.5. If $\nu > 1/2$ the solution $\mathbb{X}^2$ to (2.12) is unique. Indeed, assume that there exists another $\mathbb{X}_1^2$ which satisfies definition 2.4. Put $G(\xi) = \mathbb{X}^2(\xi, 0) - \mathbb{X}_1^2(\xi, 0)$. Then by condition 2.12

$$\mathbb{X}^2(\xi,\rho) - \mathbb{X}_1^2(\xi,\rho) = G(\xi) - G(\rho),$$

and, since $\mathbb{X}^2 \in C_2^{2\nu}$, G is a Hölder function of order bigger than 1. Hence, $G = 0$. Therefore, $\mathbb{X}_1^2 = \mathbb{X}^2$.

If $\nu > 1$ and $\mathbb{X}$ is a ν-rough path, then $\mathbb{X}$ is given by the pair of constants $(X(0), 0)$. Indeed, in this case X is Hölder function with exponent more than 1 i.e. constant $X(0)$ and the only possible choice for the second component is $\mathbb{X}^2 = 0$.

If $\nu \in (1/2, 1]$ then $\mathbb{X}^2$ is uniquely determined by X. Indeed, if we let

$$\mathbb{X}^{2,ij}(\xi,\eta) = \int_\xi^\eta (X_\rho^i - X_\eta^i)dX_\rho^j, \xi,\eta \in \mathbb{T}\ i,j = 1,2,3, \tag{2.13}$$

where the integral is understood in the sense of Young (see [22]) it is possible to show that $\mathbb{X}^2 \in C_2^{2\nu}$ and that this definition satisfies conditions of Definition 2.4. Since $\nu > 1/2$ it is the only possible choice.

Definition 2.6. *We say that our ν-rough path $(X, \mathbb{X}^2)$ is an geometric ν-rough path if there exist a sequence $(X_n, \mathbb{X}_n^2)$ such that $X_n \in C(\mathbb{T}, \mathbb{R}^3)$ is piecewise smooth,*

$$\mathbb{X}_n^{2,ij}(\xi,\eta) = \int_\xi^\eta (X_n^i(\rho) - X_n^i(\eta))dX_n^j(\rho), \qquad \xi,\eta \in \mathbb{T}, i,j = 1,2,3,$$

and

$$\lim_{n\to\infty} \left[\|X_n - X\|_\nu + \|\mathbb{X}_n^2 - \mathbb{X}^2\|_{2\nu}\right] = 0. \tag{2.14}$$

Example 2.7. Let $\{B_t\}_{t\in[0,1]}$ be the standard 3-dimensional Brownian bridge such that $B_0 = B_1 = x_0$ and let $\mathbb{B}^{2,ij}$, $i,j = 1,2,3$, be the area processes defined by

$$\mathbb{B}^{2,ij}(\xi,\eta) = \int_\xi^\eta (B_\rho^i - B_\eta^i)dB_\rho^j,$$

where the integral can be understood either in the Stratonovich or in the Itô sense. Then, the couple $(B, \mathbb{B}^2)$ is a ν-rough path for all $\nu \in (1/3, 1/2)$ (see [6, p.1849]). Moreover, if the integral is understood in Stratonovich sense then it is also a geometric ν-rough path. Indeed, it follows from Theorem 3.1 in [9] that one can approximate X with piecewise linear dyadic $(X_n)_{n\geq 1}$ in the sense of Definition 2.6 where the limit is understood in the almost sure sense.

From now on we suppose that the geometric ν-rough path $\mathbb{X} = (X, \mathbb{X}^2)$ with $\nu > 1/3$ and the corresponding Banach space $\mathcal{D}_X$ are fixed. For a finite partition $\pi = \{\xi_0 = \xi < \xi_1 < \cdots < \xi_n = \eta\}$ be of the interval $[\xi,\eta]$, let $d(\pi) = \sup_i |\xi_{i+1} - \xi_i|$ denote the mesh size of the partition π.

Lemma 2.8. *Let If $Y, Z \in \mathcal{D}_X$ then the limit*

$$\lim_{d(\pi)\to 0} \sum_{i=0}^{n-1} [Y(\xi_i)(Z(\xi_{i+1}) - Z(\xi_i)) + Y'(\xi_i)Z'(\xi_i)\mathbb{X}^2(\xi_{i+1},\xi_i)] =: \int_\xi^\eta Y dZ \tag{2.15}$$

exists and defined the integral on the r.h.s.

Proof. This statement is a direct consequence of the existence and properties of the sewing map and of the definition of the controlled path space, see [12], Theorem 1. For the sake of the presentation we sketch a proof. Let

$$A(\xi',\eta') = Y(\xi')(Z(\eta') - Z(\xi')) + Y'(\xi')Z'(\xi')\mathbb{X}^2(\eta',\xi')$$

and note that, with the assumption of the theorem we have $\delta A \in C_3^{3\nu}$ so, provided that $3\nu > 1$ we can apply the sewing map and we have the decomposition $A = \delta I + \Lambda\delta A$ where $I \in C_1$. But this implies that

$$\lim_{d(\pi)\to 0} \sum_{i=0}^{n-1} A(\xi_i,\xi_{i+1}) = \lim_{d(\pi)\to 0} \sum_{i=0}^{n-1} (I(\xi_{i+1}) - I(\xi_i)) + \lim_{d(\pi)\to 0} \sum_{i=0}^{n-1} (\Lambda\delta A(\xi_i,\xi_{i+1})).$$

The first term on the r.h.s. telescopes and since $|\Lambda\delta A(\xi_i,\xi_{i+1})| \lesssim |\xi_{i+1}-\xi_i|^{3\nu}$ the second term on the r.h.s goes to zero proving the existence of the limit and moreove that

$$\lim_{d(\pi)\to 0}\sum_{i=0}^{n-1} A(\xi_i,\xi_{i+1}) = I(\eta)-I(\xi).$$

□

The explicit representation in terms of the sewing map can be used to prove the various estimates for the rough integral below.

Remark 2.9. In the case of $\nu > 1/2$ the line integral defined in the Lemma 2.8 is reduced to the Young definition of the line integral $\int YdZ$. Indeed, it is enough to notice that second term in formula (2.15) is of the order $O(|\xi_{i+1}-\xi_i|^{2\nu})$, $2\nu > 1$. In this case the line integral does not depend upon Y', Z'.

Integrals of controlled paths have very nice continuity property with respect to variations of the controlled path and also with respect variations of the rough path on which the controlled path space is modelled.

In the following $\mathbb{X} = (X, \mathbb{X}^2)$ and $\tilde{\mathbb{X}} = (\tilde{X}, \tilde{\mathbb{X}}^2)$ are two ν-rough paths and $Y \in \mathcal{D}_X$ and $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$ two path respectively controlled by X and $\tilde{X}$. In this case we introduce the following distance

$$D(Y,\tilde{Y}) = \|X-\tilde{X}\|_{C^\nu} + \|\mathbb{X}^2-\tilde{\mathbb{X}}^2\|_{C_2^{2\nu}} + \|Y'-\tilde{Y}'\|_{C^\nu} + \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\nu}} + \|Y-\tilde{Y}\|_{C^\nu}.$$

Lemma 2.10. *Assume $Y, W \in \mathcal{D}_X$, $\tilde{Y}, \tilde{W} \in \mathcal{D}_{\tilde{X}}$. Define maps $Q, \tilde{Q} : \mathbb{T}^2 \to \mathbb{R}$ by the following identities*

$$Q(\eta,\xi) := \int_\xi^\eta YdW - Y(\xi)(W(\eta)-W(\xi)) - Y'(\xi)W'(\xi)\mathbb{X}^2(\eta,\xi), \qquad \eta,\xi\in\mathbb{T}, \quad (2.16)$$

and a similar expression for $\tilde{Q}$ with $Y, Y', W, W', \mathbb{X}^2$ replaced by $\tilde{Y}, \tilde{Y}', \tilde{W}, \tilde{W}', \tilde{\mathbb{X}}^2$. Then $Q, \tilde{Q} \in C_2^{3\nu}$ and there exists a constant $C = C(\nu) > 0$ such that for all $Y, W \in \mathcal{D}_X$

$$\|Q\|_{C_2^{3\nu}} \le C(1+\|X\|_{C^\nu} + \|\mathbb{X}^2\|_{C_2^{2\nu}})\|Y\|_{\mathcal{D}_X}\|W\|_{\mathcal{D}_X}. \quad (2.17)$$

Furthermore,

$$\|Q-\tilde{Q}\|_{C_2^{3\nu}} \le C(1+\|X\|_{C^\nu} + \|\mathbb{X}^2\|_{C_2^{2\nu}}) \\ ((\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_W + (\|W\|_{\mathcal{D}_X} + \|\tilde{W}\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_Y + \varepsilon_X). \quad (2.18)$$

where

$$\varepsilon_Y = \|Y'-\tilde{Y}'\|_{C^\nu} + \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\nu}} + \|Y-\tilde{Y}\|_{C^\nu},$$

$$\varepsilon_W = \|W'-\tilde{W}'\|_{C^\nu} + \|R^W - R^{\tilde{W}}\|_{C_2^{2\nu}} + \|W-\tilde{W}\|_{C^\nu},$$

$$\varepsilon_X = (\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})(\|W\|_{\mathcal{D}_X} + \|\tilde{W}\|_{\mathcal{D}_{\tilde{X}}})(\|X-\tilde{X}\|_{C^\nu} + \|\mathbb{X}^2-\tilde{\mathbb{X}}^2\|_{C_2^{2\nu}}).$$

Proof. See [12], Theorem 1. For formula (2.18) see [12], p.104, formula (27). □

By Lemmas 2.8 and 2.3 for any $A \in C^2(\mathbb{R}^3, L(\mathbb{R}^3, \mathbb{R}^3)), Y \in \mathcal{D}_X$ we can a define a map $V^Y : \mathbb{R}^3 \to \mathbb{R}$ by the means of the rough path integral as follows

$$V^Y(x) := \int_0^1 A(x - Y(\xi))dY(\xi), \qquad x \in \mathbb{R}^3. \tag{2.19}$$

For maps $F : \mathbb{R}^3 \to W$ we denote by $\|F\|_{C^n}$ the usual norms

$$\|F\|_{C^n} = \sup_{x\in\mathbb{R}^3} \sup_{0\le k\le n} |\nabla^k F(x)|$$

for all $n \geq 0$ where $\nabla^k F$ denote the tensor of k-th order derivatives of F for $k \geq 1$ and $\nabla^0 F = F$ for convenience.

We have following bounds on the regularity of V^Y:

Lemma 2.11. *Let $Y \in \mathcal{D}_X$, $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$, then there exists $C_1 = C_1(\nu), C_2 = C_2(\mathbb{X})$ such that for any integer $n \geq 0$,*

$$\|\nabla^n V^Y\|_{L^\infty} \leq 4C_1C_2^3\|\nabla^{n+1}A\|_{C^1}\|Y\|^2_{\mathcal{D}_X}(1 + \|Y\|_{\mathcal{D}_X}) \tag{2.20}$$

and

$$\|\nabla^n V^Y - \nabla^n V^{\tilde{Y}}\|_{L^\infty} \leq C(\nu)\|A\|_{C^{n+3}}C_X^4(1 + \|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})^3 D(Y, \tilde{Y}) \tag{2.21}$$

where

$$C_X = 1 + \|X\|_{C^\nu} + \|\tilde{X}\|_{C^\nu} + \|\mathbb{X}^2\|_{C_2^{2\nu}} + \|\tilde{\mathbb{X}}^2\|_{C_2^{2\nu}}.$$

In the case of $X = \tilde{X}$, inequality (2.21) *can be rewritten as*

$$\|\nabla^n V^Y - \nabla^n V^{\tilde{Y}}\|_{L^\infty} \leq 16C_1C_2^3\|\nabla^{n+1}A\|_{C^2}\|Y\|_{\mathcal{D}_X}(1 + \|Y\|_{\mathcal{D}_X})^2\|Y - \tilde{Y}\|^*_{\mathcal{D}_X}. \tag{2.22}$$

Proof. By the results [6, Lemma 7] it is known that $V^Y \in C^n$ if $A \in C^{n+2}$ and alsto that eqns. (2.20) and (2.22) hold. Now we will show (2.21): it is enough to consider the case of $n = 0$. By eq. (2.17) we have

$$\begin{aligned} V^Y(x) - V^{\tilde{Y}}(x) \;=\; & A(x - Y(0))(Y(1) - Y(0)) - A(x - \tilde{Y}(0))(\tilde{Y}(1) - \tilde{Y}(0)) \\ & + (A(x - Y))'(0)Y'(0)\mathbb{X}^2(0, 1) - (A(x - \tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0, 1) \\ & + Q^x(0, 1) - \tilde{Q}^x(0, 1) \end{aligned}$$

where Q^x and $\tilde{Q}^x$ (given by the eq. (2.17)) satisfy inequality (2.18). Recall that $Y(1) = Y(0)$ and $\tilde{Y}(1) = \tilde{Y}(0)$, hence we have

$$\begin{aligned} |V^Y - V^{\tilde{Y}}|_{L^\infty} \leq & \sup_x |(A(x - Y))'(0)Y'(0)\mathbb{X}^2(0, 1) - (A(x - \tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0, 1)| \\ & + \sup_x |Q^x(0, 1) - \tilde{Q}^x(0, 1)|. \end{aligned} \tag{2.23}$$

For the first term on the r.h.s. we have

$$
\begin{aligned}
&|(A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1) - (A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1)| \\
&\quad \le |(\nabla A(x-Y(0))Y'(0)Y'(0) - \nabla A(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0))\mathbb{X}^2(0,1)| \\
&\quad + |\nabla A(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0)(\mathbb{X}^2(0,1) - \tilde{\mathbb{X}}^2(0,1))| \\
&\quad \le \|\mathbb{X}^2\|_{C_2^{2\nu}} |\nabla A(x-Y(0))Y'(0)Y'(0) - \nabla A(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0)| \\
&\quad + \|A\|_{C^2}\|Y'\|_{L^\infty}^2 \|\mathbb{X}^2 - \tilde{\mathbb{X}}^2\|_{C_2^{2\nu}} \\
&\quad \le \|\mathbb{X}^2\|_{C_2^{2\nu}} \|Y'\|_{L^\infty}^2 \|A\|_{C^2}\|Y' - \tilde{Y}'\|_{L^\infty} + \|\mathbb{X}^2\|_{C_2^{2\nu}} \|A\|_{C^1}(\|Y'\|_{L^\infty} + \|\tilde{Y}'\|_{L^\infty})\|Y' - \tilde{Y}'\|_{L^\infty} \\
&\quad + \|A\|_{C^2}\|Y'\|_{L^\infty}^2 \|\mathbb{X}^2 - \tilde{\mathbb{X}}^2\|_{C_2^{2\nu}}.
\end{aligned}
\tag{2.24}
$$

By (2.18) we can estimate second term as follows

$$
\begin{aligned}
|Q^x - \tilde{Q}^x|_{C_2^{3\nu}} \le C\Big[&(\|A(x-Y)\|_{\mathcal{D}_X} + \|A(x-\tilde{Y})\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_Y \\
&+ (\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})\varepsilon_A + \varepsilon_X\Big].
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_Y &= \|Y - \tilde{Y}\|_{C^\nu} + \|Y' - \tilde{Y}'\|_{C^\nu} + \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\nu}}, \\
\varepsilon_A &= \|A(x-Y) - A(x-\tilde{Y})\|_{C^\nu} + \|A(x-Y)' - A(x-\tilde{Y})'\|_{C^\nu} \\
&\quad + \|R^{A(x-Y)} - R^{A(x-\tilde{Y})}\|_{C_2^{2\nu}}, \\
\varepsilon_X &= (\|A(x-Y)\|_{\mathcal{D}_X} + \|A(x-\tilde{Y})\|_{\mathcal{D}_{\tilde{X}}}) \\
&\quad \times (\|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})(|X - \tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{C_2^{2\nu}}).
\end{aligned}
$$

By formula (2.8) we can estimate ε_A as follows

$$
\begin{aligned}
\varepsilon_A \le K\|A\|_{C^3}&(1 + \|X\|_{C^\nu} + \|\tilde{X}\|_{C^\nu})^3(1 + \|Y\|_{D_X} + \|\tilde{Y}\|_{D_{\tilde{X}}})^2 \\
&\times (\|X - \tilde{X}\|_{C^\nu} + \|Y - \tilde{Y}\|_{C^\nu} + \|Y' - \tilde{Y}'\|_{C^\nu} + \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\nu}}).
\end{aligned}
\tag{2.25}
$$

By inequality (2.7) we infer that

$$
\|A(x-Y)\|_{\mathcal{D}_X} \le K\|A\|_{C^2}\|Y\|_{D_X}(1 + \|Y\|_{D_X})(1 + \|X\|_{C^\nu})^2, \tag{2.26}
$$

and similarly,

$$
\|A(x-\tilde{Y})\|_{\mathcal{D}_{\tilde{X}}} \le K\|A\|_{C^2}\|\tilde{Y}\|_{D_{\tilde{X}}}(1 + \|\tilde{Y}\|_{D_{\tilde{X}}})(1 + \|\tilde{X}\|_{C^\nu})^2. \tag{2.27}
$$

Therefore, combining (2.25) with (2.25), (2.26) and (2.27) we get

$$
\begin{aligned}
\|Q^x - \tilde{Q}^x\|_{C_2^{3\nu}} &\le C(\nu)\|A\|_{C^{n+3}}(1 + \|X\|_{C^\nu} + \|\tilde{X}\|_{C^\nu})^4(1 + \|Y\|_{\mathcal{D}_X} + \|\tilde{Y}\|_{\mathcal{D}_{\tilde{X}}})^3 \\
&(\|X - \tilde{X}\|_{C^\nu} + \|\mathbb{X}^2 - \tilde{\mathbb{X}}^2\|_{C_2^{2\nu}} + \|Y' - \tilde{Y}'\|_{C^\nu} + \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\nu}} + \|Y - \tilde{Y}\|_{C^\nu}).
\end{aligned}
\tag{2.28}
$$

Hence, the result follows from (2.24) and (2.28). □

3. The Evolution Problem for a Rough Filament

Let $\mathcal{D}_{\mathbb{X},T} = C([0,T],\mathcal{D}_X)$, where has been defined after (2.1), be a vector space with the usual supremum norm

$$\|F\|_{\mathcal{D}_{\mathbb{X},T}} = \sup_{t\in[0,T]} \|F(t)\|^*_{\mathcal{D}_X}. \tag{3.1}$$

Obviously $\mathcal{D}_{\mathbb{X},T}$ is a Banach space. For an element $\gamma \in \mathcal{D}_{\mathbb{X},T}$ the time-dependent vector field

$$(t,x) \mapsto u^{\gamma(t)}(x) = \int_0^1 \nabla\phi(x-\gamma(t,\xi)) \times d\gamma(t,\xi)$$

is well defined by the rough integral and it is meaningful to consider the Cauchy problem (1.1) in $\mathcal{D}_{\mathbb{X},T}$. The following local existence and uniqueness result been proved in [6], see Theorem 3, p.1842.

Theorem 3.1. *Assume $\phi \in C^6(\mathbb{R}^3,\mathbb{R})$, $\nu \in (1/3,1)$, $\mathbb{X} = (X,\mathbb{X}^2)$ is a ν-rough path, $\gamma_0 \in \mathcal{D}_X$. Then there exists a time $T_0 = T_0(\nu, \|\phi\|_{C^5}, \mathbb{X}) > 0$ such that the problem* (1.1) *has unique solution in the space $\mathcal{D}_{\gamma_0,T_0} \subset \mathcal{D}_{\mathbb{X},T_0}$. Moveover if $\mathbb{X},\tilde{\mathbb{X}}$ are two ν-rough paths, $\gamma_0 \in \mathcal{D}_X$, $\tilde{\gamma}_0 \in \mathcal{D}_{\tilde{X}}$ and $\gamma,\tilde{\gamma}$ two solutions of* (1.1) *respectively in $\mathcal{D}_{\mathbb{X},T_0}$ and $\mathcal{D}_{\tilde{\mathbb{X}},\tilde{T}_0}$ then there exists a constant $C = C(\mathbb{X},\tilde{\mathbb{X}},\phi,\nu,\|\gamma_0\|^*_{\mathcal{D}_X},\|\tilde{\gamma}_0\|^*_{\mathcal{D}_{\tilde{X}}})$ such that*

$$\sup_{0\le t\le \min(T_0,\tilde{T}_0)} D(\gamma(t),\tilde{\gamma}(t)) \le C D(\gamma(0),\tilde{\gamma}(0)).$$

This result can be easily proven by a fixed-point argument in $\mathcal{D}_{\mathbb{X},T}$ for sufficiently small T. The requirements on ϕ can be easily understood as follows: in order for u^γ to be well defined the function $\nabla\phi$ has to be twice differentiable. Moreover in order to apply a fixed point argument the function $(t,\xi) \mapsto u^{\gamma(t)}(\gamma(t,\xi))$ has to belongs to $\mathcal{D}_{\mathbb{X},T}$ and this can enforced by requiring that the map $x \mapsto u^{\gamma(t)}(x)$ has to be C^2 in space, uniformly in time. This requires $\nabla\phi$ to be at least 4 times differentiable. The fixed-point argument further requires the Lipshitz continuity of the fixed-point map which holds at the price of an additional derivative of $\nabla\phi$ ending up with the requirement that ϕ has to be six times differentiable. It is not obvious to improve on this requirement for general rough initial condition and $\nu \in (1/3,1/2)$.

The arguments used in Theorem 3.1 does not allow to have a global solution. The aim of the present paper is to enforce further structure condition on the kernel ϕ in order to be able to use the energy method introduced in [5].

To prove a global existence result for the problem (1.1) we need the following additional hypothesis

Hypothesis 3.2. *Assume that the function ϕ appearing in eq.* (1.2) *has a Fourier transform $\hat{\phi} : \mathbb{R}^3 \to \mathbb{R}$ which is real, even and non-negative and satisfying the integrability condition*

$$\int_{\mathbb{R}^3} (1+|k|^2)^2 \hat{\phi}(k) dk < \infty.$$

Example 3.3. The function ϕ_μ, $\mu > 0$ defined by

$$\phi_\mu(\cdot) = \frac{1}{(|\cdot|^2+\mu^2)^{\frac{1}{2}}}$$

is smooth and satisfies Hypothesis 3.2, see p.6 of [5]. This function is also C^∞ so it satisfy also the assumptions of Theorem 3.1 and as a result is an explicit example which satisfy all the assumptions needed for the following global existence result.

Theorem 3.4. *Assume that γ_0 is a geometric ν-rough path, $\phi \in C^6(\mathbb{R}^3, \mathbb{R})$ satisfies Hypothesis 3.2 and $\nu \in (1/3, 1)$. Then for every $T > 0$, the problem* (1.1) *has unique solution in $\mathcal{D}_{\gamma_0,T}$.*

We will need the following definition.

Definition 3.5. *Let $\phi \in C^4(\mathbb{R}^3, \mathbb{R})$, $\gamma \in \mathcal{D}_X$. Let*

$$\psi^\gamma(x) = \int_0^1 \phi(x - \gamma(\eta))d\gamma(\eta).$$

and

$$\mathcal{H}_X^\phi(\gamma) = \frac{1}{2}\int_0^1 \psi^\gamma(\gamma(\xi)) \cdot d\gamma(\xi) = \frac{1}{2}\int_0^1 \int_0^1 \phi(\gamma(\xi) - \gamma(\eta))(d\gamma(\xi) \cdot d\gamma(\eta)). \tag{3.2}$$

The quantity $\mathcal{H}_X^\phi(\gamma)$ is called the energy of path γ. We will omit ϕ below.

Remark 3.6. Definition (3.2) is well posed. Indeed, by Lemma 2.11 $\psi^\gamma \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ and, therefore, it follows by Lemma 2.3 that $\psi^\gamma \circ \gamma \in \mathcal{D}_X$. Moreover if $\nu > 1/2$ and $\gamma \in C^1(\mathbb{T}, \mathbb{R}^3)$. Then by Remark 2.9 the line integrals in the definition of the energy are understood in the sense of Young.

Lemma 3.7. *Assume $\phi \in C^4(\mathbb{R}^3, \mathbb{R})$. Then there exists a constant $C = C(\nu, \mathbb{X})$ such that for all $\gamma \in \mathcal{D}_X$*

$$|\mathcal{H}_X(\gamma)| \le C\|\phi\|_{C^4}\|\gamma\|_{\mathcal{D}_X}^4(1 + \|\gamma\|_{\mathcal{D}_X})^2. \tag{3.3}$$

Furthermore, for any $R > 0$ there exists $C = C(R)$ such that for any $\gamma \in \mathcal{D}_X$, $\tilde\gamma \in \mathcal{D}_{\tilde X}$ satisfying

$$\|\gamma\|_{\mathcal{D}_X} \le R, \|\tilde\gamma\|_{\mathcal{D}_{\tilde X}} \le R,$$
$$C_{X,\tilde X} = \|X\|_{C^\nu} + \|\tilde X\|_{C^\nu} + \|\mathbb{X}^2\|_{C_2^{2\nu}} + \|\tilde{\mathbb{X}}^2\|_{C_2^{2\nu}} < R$$

we have

$$|\mathcal{H}_X(\gamma) - \mathcal{H}_{\tilde X}(\tilde\gamma)| \le C(R)D(\gamma, \tilde\gamma). \tag{3.4}$$

which implies in particular that the map $\mathcal{H}_X : \mathcal{D}_X \to \mathbb{R}$ is locally Lipshitz: i.e. for any $R > 0$ there exists $C = C(R)$ such that for any $\gamma, \tilde\gamma \in \mathcal{D}_X$, $\|\gamma\|_{\mathcal{D}_X} \le R$, $\|\tilde\gamma\|_{\mathcal{D}_X} \le R$ we have

$$|\mathcal{H}_X(\gamma) - \mathcal{H}_X(\tilde\gamma)| \le C(R)\|\gamma - \tilde\gamma\|_{\mathcal{D}_X}^*. \tag{3.5}$$

Proof. First we will show inequality (3.3). By representation (2.16) we have

$$\begin{aligned}\mathcal{H}_X(\gamma) &= \frac{1}{2}(\psi^\gamma(\gamma(0))(\gamma(1) - \gamma(0))) + \left[\nabla\psi^\gamma(\gamma(0))\gamma'(0)\right]\gamma'(0)\mathbb{X}^2(1,0) + Q(0,1)\\ &= I + II + III.\end{aligned} \tag{3.6}$$

Since $\gamma(1) = \gamma(0)$ we infer that $I = 0$. Concerning the second term by Lemma 2.11 we have the following estimate

$$|II| \le \|\mathbb{X}^2\|_{C_2^{2\nu}}\|\nabla\psi^\gamma\|_{L^\infty}\|\gamma'\|_{L^\infty}^2 \le C(\nu, \mathbb{X})\|\phi\|_{C^3}\|\gamma\|_{\mathcal{D}_X}^4(1 + \|\gamma\|_{\mathcal{D}_X}). \tag{3.7}$$

For third term we infer from inequality (2.17)

$$|III| \le \|Q\|_{C_2^{3\nu}} \le C(\nu, \mathbb{X})\|\psi^\gamma(\gamma)\|_{\mathcal{D}_X}\|\gamma\|_{\mathcal{D}_X}. \tag{3.8}$$

Then by Lemmas 2.3 and 2.11 we have

$$\begin{aligned}\|\psi^\gamma(\gamma)\|_{\mathcal{D}_X} &\leq C(\nu,\mathbb{X})\|\psi^\gamma\|_{C^2}\|\gamma\|_{\mathcal{D}_X}(1+\|\gamma\|_{\mathcal{D}_X})\\ &\leq C(\nu,\mathbb{X})\|\phi\|_{C^4}\|\gamma\|^3_{\mathcal{D}_X}(1+\|\gamma\|_{\mathcal{D}_X})^2. \end{aligned} \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) we get inequality (3.3). To prove the inequality (3.4) we start by formula (2.17) to get

$$\begin{aligned}\mathcal{H}_X(\gamma)-\mathcal{H}_{\tilde{X}}(\tilde{\gamma}) &= \frac{1}{2}\Big[(\nabla\psi^\gamma(\gamma(0))\gamma'(0)\gamma'(0)-\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0))\mathbb{X}^2(1,0)\\ &\quad+\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0)(\mathbb{X}^2(1,0)-\tilde{\mathbb{X}}^2(1,0))+Q(0,1)-\tilde{Q}(0,1)\Big]\\ &= I+II+III \end{aligned} \tag{3.10}$$

The first term in (3.10) can be represented as follows

$$\begin{aligned} I &= (\nabla\psi^\gamma(\gamma(0))\gamma'(0)\gamma'(0)-\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0))\mathbb{X}^2(1,0)\\ &= \big[(\nabla\psi^\gamma(\gamma(0))-\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0)))\gamma'(0)\gamma'(0)\\ &+ \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))(\gamma'(0)-\tilde{\gamma}(0))\gamma'(0)\\ &+ \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}(0)(\gamma'(0)-\tilde{\gamma}(0))\big]\,\mathbb{X}^2(1,0)=A+B+C, \end{aligned} \tag{3.11}$$

and the first term in (3.11) can be estimated as follows

$$\begin{aligned}|A| &= |(\nabla\psi^\gamma(\gamma(0))-\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0)))\gamma'(0)\gamma'(0)\mathbb{X}^2(1,0)| \\ &\leq \|\mathbb{X}^2\|_{C_2^{2\nu}}\|\gamma\|^2_{\mathcal{D}_X}(|\nabla\psi^\gamma(\gamma(0))\\ &- \nabla\psi^\gamma(\tilde{\gamma}(0))|+|\nabla\psi^\gamma(\tilde{\gamma}(0))-\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))|)\\ &\leq \|\mathbb{X}^2\|_{C_2^{2\nu}}\|\gamma\|^2_{\mathcal{D}_X}(\|\psi^\gamma\|_{C^2}|\gamma(0)-\tilde{\gamma}(0)|\\ &+ C_X^4\|\phi\|_{C^4}(1+\|\gamma\|_{\mathcal{D}_X}+\|\tilde{\gamma}\|_{\mathcal{D}_{\tilde{X}}})^3D(\gamma,\tilde{\gamma})\\ &\leq KC_X^4\|\phi\|_{C^4}(1+\|\gamma\|_{\mathcal{D}_X}+\|\tilde{\gamma}\|_{\mathcal{D}_{\tilde{X}}})^3D(\gamma,\tilde{\gamma}). \end{aligned} \tag{3.12}$$

Here the second inequality follows from inequality (2.21) and third one from inequality (2.20). For second term in (3.11) we have by inequality (2.20)

$$\begin{aligned}|B| &\leq C\|\mathbb{X}^2\|_{C_2^{2\nu}}\|\gamma\|_{\mathcal{D}_X}\|\phi\|_{C^3}\|\tilde{\gamma}\|^2_{\mathcal{D}_X}(1+\|\tilde{\gamma}\|_{\mathcal{D}_X})D(\gamma,\tilde{\gamma})\\ &\leq CC_X(1+\|\gamma\|_{\mathcal{D}_X}+\|\tilde{\gamma}\|_{\mathcal{D}_{\tilde{X}}})^3D(\gamma,\tilde{\gamma}). \end{aligned} \tag{3.13}$$

Similarly, we have for third term

$$|C|\leq C(\nu,\mathbb{X},\|\gamma\|_{\mathcal{D}_X},\|\tilde{\gamma}\|_{\mathcal{D}_X})D(\gamma,\tilde{\gamma}). \tag{3.14}$$

Going back to the term II in (3.10) we observe that it can be estimated as follows

$$|II|\leq\|\nabla\psi^{\tilde{\gamma}}\|_{L^\infty}\|\tilde{\gamma}\|^2_{\mathcal{D}_X}D(\gamma,\tilde{\gamma})\leq C_X^3\|\phi\|_{C^3}(1+\|\gamma\|_{\mathcal{D}_X}+\|\tilde{\gamma}\|_{\mathcal{D}_{\tilde{X}}})^3D(\gamma,\tilde{\gamma}).$$

Thus it remains to estimate third term of equality (3.10). By inequality (2.18) we have

$$\begin{aligned}|Q(0,1)-\tilde{Q}(0,1)| &\le \|Q-\tilde{Q}\|_{C_2^{3\nu}}\\ &\le C_X\Big[(\|\psi^{\tilde\gamma}(\tilde\gamma)\|_{\mathcal{D}_{\tilde X}}+\|\psi^{\gamma}(\gamma)\|_{\mathcal{D}_X})D(\gamma,\tilde\gamma)\\ &+(\|\tilde\gamma\|_{\mathcal{D}_{\tilde X}}+\|\tilde\gamma\|_{\mathcal{D}_{\tilde X}})D(\psi^{\tilde\gamma}(\tilde\gamma),\psi^{\gamma}(\gamma))\\ &+(\|\psi^{\tilde\gamma}(\tilde\gamma)\|_{\mathcal{D}_{\tilde X}}+\|\psi^{\gamma}(\gamma)\|_{\mathcal{D}_X})(\|\tilde\gamma\|_{\mathcal{D}_{\tilde X}}+\|\tilde\gamma\|_{\mathcal{D}_{\tilde X}})\\ &\times(\|X-\tilde X\|_{C^\nu}+\|\mathbb{X}^2-\tilde{\mathbb{X}}^2\|_{C_2^{2\nu}})\Big]\end{aligned}$$

By inequality (2.20), the term $\|\psi^{\tilde\gamma}(\tilde\gamma)\|_{\mathcal{D}_X}$ is bounded by the constant $C=C(\nu,\mathbb{X},\|\tilde\gamma\|_{\mathcal{D}_X})$. Therefore, to prove estimate (3.5) it is enough to show that there exists a constant $C=C(\nu,\mathbb{X},R)$ such that for $\gamma,\tilde\gamma\in\mathcal{D}_X$ with $\|\gamma\|_{\mathcal{D}_X},\|\tilde\gamma\|_{\mathcal{D}_X}\le R$

$$D(\psi^{\tilde\gamma}(\tilde\gamma),\psi^{\gamma}(\gamma))\le CD(\gamma,\tilde\gamma). \tag{3.15}$$

By the triangle inequality we have

$$\begin{aligned}D(\psi^{\tilde\gamma}(\tilde\gamma),\psi^{\gamma}(\gamma)) &\le D(\psi^{\tilde\gamma}(\tilde\gamma),\psi^{\tilde\gamma}(\gamma))+D(\psi^{\tilde\gamma}(\gamma),\psi^{\gamma}(\gamma))\\ &= I+II.\end{aligned} \tag{3.16}$$

The first term can be estimated by using inequality (2.8) as follows

$$|I|\le KC_X^3\|\psi^{\tilde\gamma}\|_{C^3}(1+\|\tilde\gamma\|_{\mathcal{D}_{\tilde X}}+\|\gamma\|_{\mathcal{D}_X})^2D(\gamma,\tilde\gamma). \tag{3.17}$$

By inequality (2.20) we have

$$\|\psi^{\tilde\gamma}\|_{C^3}\le C\|\phi\|_{C^5}\|\tilde\gamma\|_{\mathcal{D}_X}^2(1+\|\tilde\gamma\|_{\mathcal{D}_X}) \tag{3.18}$$

Combining (3.17) and (3.18) we obtain the necessary estimate for I. It remains to find an estimate for the term II. By inequalities (2.7) and (2.22) we have

$$\begin{aligned}II=D(\psi^{\tilde\gamma}(\gamma),\psi^{\gamma}(\gamma)) &\le (1+\|X\|_\nu)\|\psi^{\tilde\gamma}(\gamma)-\psi^{\gamma}(\gamma)\|_{\mathcal{D}_X}\\ &\le K\|\nabla\psi^{\tilde\gamma}-\nabla\psi^{\gamma}\|_{C^1}\|\gamma\|_{\mathcal{D}_X}(1+\|\gamma\|_{\mathcal{D}_X})(1+\|X\|_\nu)^3\\ &\le K\|\phi\|_{C^5}C_X^7(1+\|\gamma\|_{\mathcal{D}_X}+\|\tilde\gamma\|_{\mathcal{D}_{\tilde X}})^5D(\gamma,\tilde\gamma).\end{aligned} \tag{3.19}$$

Hence the inequality (3.4) follows. Finally the bound (3.5) is a consequence of eq. (3.4). □

Let us recall the definition (1.2) of the vector field u^γ generated by a controlled path γ:

$$u^\gamma(x)=\int_0^1\nabla\phi(x-\gamma(\xi))\times d\gamma(\xi),\qquad \gamma\in\mathcal{D}_X. \tag{3.20}$$

Now we will show that if energy functional of γ is bounded then the associated velocity field is a smooth function. We have

Lemma 3.8. *Assume $\phi\in C^4(\mathbb{R}^3,\mathbb{R})$ and Hypothesis 2.6. Then the energy function $\mathcal{H}_X:\mathcal{D}_X\to\mathbb{R}$ is continuous. Furthermore if γ is a geometric rough path then*

$$\mathcal{H}_X(\gamma)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\hat\phi(k)\Big|\int_0^1 e^{i(k,\gamma(\xi))}d\gamma(\xi)\Big|^2dk\ge 0 \tag{3.21}$$

and for any $n \in \mathbb{Z}, n \geq 0$, we have following bound

$$\|\nabla^n u^\gamma\|_{L^\infty}^2 \leq \frac{1}{(2\pi)^3}\left[\int_{\mathbb{R}^3} |k|^{2(1+n)}\hat{\phi}(k)dk\right]\mathcal{H}_X(\gamma), \qquad \gamma \in \mathcal{D}_X, \tag{3.22}$$

provided that the integral $\int_{\mathbb{R}^3} |k|^{2(1+n)}\hat{\phi}(k)dk$ is finite and $\phi \in C^{n+4}(\mathbb{R}^3, \mathbb{R}^3)$.

Proof. For a smooth curve γ Lemma 3.8 has been proved in [5], see Lemma 3. In the general case, when $\gamma \in \mathcal{D}_X$, it is enough to notice that both sides of eq. (3.21) and of inequality (3.22) are locally Lipshitz and therefore, continuous w.r.t. distance $D(Y, \tilde{Y})$, $Y \in \mathcal{D}_X$, $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$. Indeed the continuity of $\mathcal{H}_X$ and continuity of $\|\nabla^n u^\gamma\|_{L^\infty}$ readily follows from Lemma 2.11. □

Now we are going to show that energy is a local integral of motion for problem (1.1).

Lemma 3.9. *Let γ_0 a geometric ν-rough path and $\gamma \in \mathcal{D}_{\gamma_0, T_0}$ be a local solution of problem* (1.1) *(such a solution exists by Theorem 3.1). Then*

$$\frac{d\mathcal{H}_{\gamma_0}(\gamma(s))}{ds} = 0, \qquad s \in [0, T_0).$$

Proof. Since $\gamma(0) = \gamma_0 \in \mathcal{D}_{\gamma_0}$ is a geometric rough path (we will denote its area component by Γ_0) there exist sequence $\{\gamma_0^n\}_{n=1}^\infty$ of piecewise smooth closed curves in $\mathbb{R}^3$ such that if we denote by (γ_0^n, Γ_0^n) their canonical lift to the space of ν-rough paths we have

$$\|\gamma_0^n - \gamma_0\|_{C^\nu} + \|\Gamma_0^n - \Gamma_0\|_{C_2^{2\nu}} \to 0, \qquad n \to \infty.$$

Now observe that $\gamma_0^n \in \mathcal{D}_{\gamma_0^n}$, $\gamma_0 \in \mathcal{D}_{\gamma_0}$ since we can take $(\gamma_0^n)' = (\gamma_0)' = 1$ and $R^{\gamma_0^n} = R^{\gamma_0} = 0$. Hence we deduce that

$$D(\gamma_0^n, \gamma_0) \to 0, \qquad n \to \infty.$$

Denote by $\gamma^n \in C([0, \infty), \mathbf{H}^1(\mathbb{T}, \mathbb{R}^3))$ the global solution of problem (1.1) with initial condition γ_0^n. Existence of such solution has been proved in Theorem 2 of [5]. Moreover note that for smooth functions γ^n controlled by the rough path (γ_0^n, Γ_0^n) the integral defined via rough paths coincide with the standard Lebesgue integral and the solution of the Cauchy problem in $C([0, \infty), \mathbf{H}^1(\mathbb{T}, \mathbb{R}^3))$ with the rough solution whose local existence is stated in Theorem 3.1. The locally Lipshitz dependence of the rough solution from the initial condition stated in Theorem 3.1 implies that

$$\lim_{n\to\infty} \sup_{t\in[0,T_0]} D(\gamma^n(t), \gamma(t)) \lesssim \lim_{n\to\infty} D(\gamma^n(0), \gamma(0)) = 0$$

for a stricly positive T_0 which depends only on the rough path norm of γ_0. Therefore, by the continuity of the energy functional $\mathcal{H}_{\gamma_0}$ we have

$$\mathcal{H}_{\gamma_0}(\gamma(s)) = \lim_{n\to\infty} \mathcal{H}_{\gamma_0^n}(\gamma^n(s)), \qquad s \in [0, T_0]. \tag{3.23}$$

Furthermore, by Lemma 2 of [5], we have

$$\mathcal{H}_{\gamma_0^n}(\gamma^n(s)) = \mathcal{H}_{\gamma_0^n}(\gamma_0^n), \qquad s \in [0, T_0]. \tag{3.24}$$

As a result, combining (3.23) and (3.24) we get the statement of the Lemma. □

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. According to Theorem 3.1 there exists unique local solution of problem (1.1). Then, we can find $T^* > 0$ and a unique maximal local solution $\gamma : [0, T^*) \to \mathcal{D}_{\gamma_0}$ and

$$\lim_{t \nearrow T^*} \|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} = \infty. \tag{3.25}$$

We need to show that $T^* = \infty$. Therefore, it is enough to prove

$$\sup_{t \in [0,T^*)} \|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} < \infty.$$

Indeed, by contradiction with (3.25), the result will follow. In the rest of the proof we show such estimate. Notice that we will have

$$\mathcal{H}_{\gamma_0}(\gamma(s)) = \mathcal{H}_{\gamma_0}(\gamma_0), \qquad s \in [0, T^*), \tag{3.26}$$

and recall that

$$\gamma(t) = \gamma_0 + \int_0^t u^{\gamma(s)}(\gamma(s))ds. \tag{3.27}$$

Firstly we have

$$\begin{aligned} \|\gamma(t)\|_{L^\infty} &\le \|\gamma_0\|_{L^\infty} + \int_0^t \|u^{\gamma(s)}\|_{L^\infty} ds \le \|\gamma_0\|_{L^\infty} + C \int_0^t \mathcal{H}_{\gamma_0}^{1/2}(\gamma(s))ds \\ &\le \|\gamma_0\|_{L^\infty} + C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t, \qquad t \in [0, T^*). \end{aligned} \tag{3.28}$$

It follows from (3.27) that

$$\gamma'(t) = \gamma_0' + \int_0^t \nabla u^{\gamma(s)}(\gamma(s))\gamma'(s)ds, t \in [0, T^*). \tag{3.29}$$

Therefore, by Lemmas 3.9 and 3.8

$$\begin{aligned} \|\gamma'(t)\|_{L^\infty} &\le \|\gamma_0'\|_{L^\infty} + \int_0^t \|\nabla u^{\gamma(s)}\|_{L^\infty} \|\gamma'(s)\|_{L^\infty} ds \le \|\gamma_0'\|_{L^\infty} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma(s))\|\gamma'(s)\|_{L^\infty} ds \\ &= \|\gamma_0'\|_{L^\infty} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)\|\gamma'(s)\|_{L^\infty} ds, \qquad t \in [0, T^*). \end{aligned} \tag{3.30}$$

Then by the Gronwall inequality we infer our second estimate

$$\|\gamma'(t)\|_{L^\infty} \le \|\gamma_0'\|_{L^\infty} e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}, \qquad t \in [0, T^*). \tag{3.31}$$

We will need one auxiliary estimate. We have

$$\begin{aligned} \|\gamma(t)\|_{C^\nu} &\le \|\gamma_0\|_{C^\nu} + \int_0^t \|u^{\gamma(s)}(\gamma(s))\|_{C^\nu} \|ds \le \|\gamma_0\|_{C^\nu} + \int_0^t \|\nabla u^{\gamma(s)}\|_{L^\infty} \|\gamma(s)\|_{C^\nu} ds \\ &\le \|\gamma_0\|_{C^\nu} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma(s))\|\gamma(s)\|_{C^\nu} ds \\ &= \|\gamma_0\|_{C^\nu} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)\|\gamma(s)\|_{C^\nu} ds, \qquad t \in [0, T^*). \end{aligned} \tag{3.32}$$

Thus, by the Gronwall inequality we get

$$\|\gamma(t)\|_{C^\nu} \le \|\gamma_0\|_{C^\nu} e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}, t \in [0, T^*). \tag{3.33}$$

Now we can estimate C^ν norm of γ'. We have

$$\begin{aligned}\|\gamma'(t)\|_{C^\nu} &\le \|\gamma_0'\|_{C^\nu} + \int_0^t \|\nabla u^{\gamma(s)}(\gamma(s))\gamma'(s)\|_{C^\nu} ds \\ &\le \|\gamma_0'\|_{C^\nu} + \int_0^t (\|\nabla u^{\gamma(s)}\|_{L^\infty}\|\gamma'(s)\|_{C^\nu} + \|\gamma'(s)\|_{L^\infty}\|\nabla u^{\gamma(s)}(\gamma(s))\|_{C^\nu}) ds \\ &\le \|\gamma_0'\|_{C^\nu} + \int_0^t (\|\nabla u^{\gamma(s)}\|_{L^\infty}\|\gamma'(s)\|_{C^\nu} + \|\gamma'(s)\|_{L^\infty}\|\nabla^2 u^{\gamma(s)}\|_{L^\infty}\|\gamma(s)\|_{C^\nu}) ds \\ &\le \|\gamma_0'\|_{C^\nu} \\ &+ \int_0^t (C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)(\|\gamma'(s)\|_{C^\nu} + \|\gamma_0'\|_{L^\infty}\|\gamma_0\|_{C^\nu} e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)s})) ds, \qquad t \in [0, T^*),\end{aligned} \tag{3.34}$$

where last inequality follows from Lemmas 3.9 and 3.8. Then by the Gronwall inequality we get the third estimate

$$\|\gamma'(t)\|_{C^\nu} \le (\|\gamma_0'\|_{C^\nu} + \|\gamma_0'\|_{L^\infty}\|\gamma_0\|_{C^\nu}) e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}, \qquad t \in [0, T^*). \tag{3.35}$$

It remains to find an estimate for $\|R^{\gamma(t)}\|_{2\nu}$. We have

$$R^{\gamma(t)} = R^{\gamma_0} + \int_0^t R^{u^{\gamma(s)}(\gamma(s))} ds, \qquad t \in [0, T^*). \tag{3.36}$$

By identity (2.6) we have for $s \in [0, T^*)$

$$\begin{aligned}R^{u^{\gamma(s)}(\gamma(s))}(\xi, \eta) = \nabla u^{\gamma(s)}(\gamma(s,\xi)) R^{\gamma(s)}(\xi, \eta) + \sum_k (\gamma^k(s,\eta) - \gamma^k(s,\xi)) \times \\ \int_0^1 \left[\frac{\partial u^{\gamma(s)}}{\partial x_k}(\gamma(s,\xi) + r(\gamma(s,\eta) - \gamma(s,\xi))) - \frac{\partial u^{\gamma(s)}}{\partial x_k}(\gamma(s,\xi)) \right] dr.\end{aligned} \tag{3.37}$$

Therefore,

$$\|R^{u^{\gamma(s)}(\gamma(s))}\|_{C_2^{2\nu}} \le \|\nabla u^{\gamma(s)}\|_{L^\infty}\|R^{\gamma(s)}\|_{C_2^{2\nu}} + \frac{1}{2}\|\gamma(s)\|_{C^\nu}^2 \|\nabla^2 u^{\gamma(s)}\|_{L^\infty}, \quad s \in [0, T^*). \tag{3.38}$$

Thus, by inequalities (3.38) and (3.33) we have for $t \in [0, T^*)$

$$\begin{aligned}\|R^{\gamma(t)}\|_{C_2^{2\nu}} &\le \|R^{\gamma_0}\|_{C_2^{2\nu}} + \int_0^t (\|\nabla u^{\gamma(s)}\|_{L^\infty}\|R^{\gamma(s)}\|_{C_2^{2\nu}} + \frac{1}{2}\|\gamma(s)\|_{C^\nu}^2 \|\nabla^2 u^{\gamma(s)}\|_{L^\infty}) ds \\ &\le \|R^{\gamma_0}\|_{C_2^{2\nu}} + \int_0^t (\|\nabla u^{\gamma(s)}\|_{L^\infty}\|R^{\gamma(s)}\|_{C_2^{2\nu}} + \|\gamma_0\|_{C^\nu} e^{C\mathcal{H}_{\gamma_0}^{\frac{1}{2}}(\gamma_0)t}\|\nabla^2 u^{\gamma(s)}\|_{L^\infty}) ds \\ &\le \|R^{\gamma_0}\|_{C_2^{2\nu}} + C(\|\gamma_0\|_{C^\nu}, \mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)) e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t} \\ &+ \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)\|R^{\gamma(s)}\|_{C_2^{2\nu}} ds,\end{aligned} \tag{3.39}$$

where in the last inequality we used Lemmas 3.9 and 3.8. Hence, by the Gronwall Lemma we get

$$\|R^{\gamma(t)}\|_{C_2^{2\nu}} \leq (\|R^{\gamma_0}\|_{C_2^{2\nu}} + C(\|\gamma_0\|_{C^\nu}, \mathcal{H}_{\gamma_0}^{1/2}(\gamma_0))e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}))e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}, \qquad t \in [0, T^*), \tag{3.40}$$

and combining estimates (3.28), (3.31), (3.35), and (3.40) we prove following a-priori estimate

$$\|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} \leq K(1 + \mathcal{H}_{\gamma_0}^{1/2}(\gamma_0))(1 + \|\gamma_0\|_{\mathcal{D}_{\gamma_0}})\|\gamma_0\|_{\mathcal{D}_{\gamma_0}} e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}, \quad t \in [0, T^*), \tag{3.41}$$

and the result follows. □

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