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GIBBS MEASURES FOR SELF-INTERACTING WIENER PATHS

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Abstract. In this note we study a class of specifications over d-dimensional Wiener measure which are invariant under uniform translation of the paths. This degeneracy is removed by restricting the measure to the σ-algebra generated by the increments of the coordinate process. We address the problem of existence and uniqueness of Gibbs measures and prove a central limit theorem for the rescaled increments. These results apply to the study of the ground state of the Nelson model of a quantum particle interacting with a scalar boson field.

1. Introduction

The theory of Gibbs measures for lattice spin models or continuous point processes is by now a well established subject of probability theory (see e.g. the book of Georgii [11], the early monograph by Preston [13] or the very readable and concise introduction of Föllmer [10]). Gibbs measures on path spaces are a more unexplored domain and only recently a series of works started a systematic study of a class of Gibbs measures on paths motivated by applications in Quantum Mechanics [4, 2, 3, 1]. However, part of these results are obtained through functional analytic approaches which does not help to fully understand the probabilistic structure of these models.

An interesting class of Gibbs measures is obtained by perturbing the Wiener measure $\mathcal{W}$ on $C(\mathbb{R}, \mathbb{R}^d)$ by the exponential of a (finite-volume) energy of the form

$$H_T(x) = \int_{-T}^{T} V(x_t)dt + \int_{-T}^{T} dt \int_{-T}^{T} dsW(x_t, x_s, t-s)$$

where x is the path and the functions V and W are interpreted as interactions potentials. In this way we obtain finite-volume measures μ_T given by

$$\mu_T(dx) = \frac{e^{-H_T(x)}}{Z_T}\mathcal{W}(dx) \tag{1}$$

The study of these measures in the limit $T \to \infty$ has been addressed in the works cited above and a series of conditions on V and W have been found which are sufficient for the existence of the limit (in the topology of local weak convergence) and for its uniqueness.

Here we are interested in a class of models whose energy function enjoy an invariance under shift of the paths $x \to x + c$ where c is a fixed vector in $\mathbb{R}^d$. In particular

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we take $V = 0$ and let $W(x, y, t) = W(x - y, t)$ so that our finite-volume energy reads

$$H_T(x) = \int_{-T}^{T} dt \int_{-T}^{T} ds\, W(x_t - x_s, t - s).$$

A relevant model for which we will prove existence of a unique Gibbs measure for small coupling is the ($d = 3$, UV regularized) Nelson model corresponding to the interaction energy given by the function

$$W(\xi, t) = -\frac{1}{1 + |\xi|^2 + |t|^2}.$$

Nelson model is a member of a wide class of potentials justified by applications to Quantum Mechanics which are in the form

$$W(\xi, t) = \int_{\mathbb{R}^d} \frac{dk}{2\omega(k)} |\rho(k)|^2 e^{-ik\cdot\xi - \omega(k)t} \tag{2}$$

where $\omega(k) : \mathbb{R}^d \to \mathbb{R}_+$ is the dispersion law of a scalar boson field. Nelson model corresponds to the case $\omega(k) = |k|$ and $\rho(k)$ with a fast decay at infinity (ultraviolet cutoff) and $|\rho(0)| > 0$ (no infrared cutoff).

In [5] Betz and Spohn prove that if ρ satsify the follwing integrability conditions:

$$\int_{\mathbb{R}^d} dk |\rho(k)|^2 \left(\omega(k)^{-1} + \omega(k)^{-2} + \omega(k)^{-3}\right) < \infty \tag{3}$$

then the rescaled coordinate process $X_t^{(\varepsilon)} = \varepsilon^{-1/2} X_{t/\varepsilon}$ weakly converge as $\varepsilon \to 0$ to a d-dimensional Brownian motion. Note that this result does not cover the $d = 3$ Nelson model since eq. (3) is not satisfied. Moreover they proved that if

$$\int_{\mathbb{R}^d} dk |\rho(k)|^2 |k|^2 \left(\omega(k)^{-2} + \omega(k)^{-4}\right) < \infty,$$

the limiting Brownian motion has a non-trivial diffusion constant (i.e. different from zero). Their approach is based on the use of an auxiliary Gaussian field which allow to "linearize" the interaction and to see the process X as the projection of a Markov process on a larger state space. The functional central limit theorem then follows by using a technique due to Kipnis and Varadhan.

In [14] it is pointed out that this class of models can be naturally recast as models of spins on $\mathbb{Z}$ with single spin space $C([0, 1], \mathbb{R}^d)$. This is obtained by cutting the path into pieces and considering each piece as an independent spin. In this representation the interaction becomes multi-body and under natural conditions on the time decay of W, this multi-body interaction is long range with a power decay. Techniques form the theory of one dimensional spin systems can then be applied which will give also estimates on the decay of correlation and on the strong mixing coefficients. Then the central limit theorem (CLT) can be proved using standard results on strongly mixing processes. However our conditions for the CLT does *not* cover the 3d Nelson model.

The shift invariance of the Hamiltonian prevents to have tightness of the family of measures $\{\mu_T\}_{T>0}$. A natural approach is then to consider *increments* of the Wiener path as the basic variables. No relevant information is lost on the (local

or global) behavior of the path. The advantage is that in this way we can fully characterize the stationary behavior of the "diffusion" process associated with the infinite volume limit measure.

Plan of the paper. In Sec. 2 we introduce the space of increments and the reference Gaussian measure induced by the Wiener measure on the paths. In Sec. 3 we prove existence of the Gibbs measures under very mild conditions and in Sec. 4 we give some more restrictive conditions under which uniqueness can be proved. Sec. 5 is devoted to the proof of uniform lower bounds for the diffusion constant under the Gibbs measures. Finally in Sec. 6 we prove the CLT for a suitable class of interactions.

2. Brownian increments

Fixed a finite union of intervals $I \subseteq \mathbb{R}$ consider the linear subspace $\mathcal{X}_I \subset C(I^2, \mathbb{R}^d)$ such that $x \in \mathcal{X}$ iff $x_{tt} = 0$ for any $t \in I$ and satisfy the cocycle condition

$$x_{su} + x_{ut} = x_{st}, \qquad s, t, u \in I \tag{4}$$

Let $\mathcal{X} := \mathcal{X}_{\mathbb{R}}$ and $\mathcal{X}_T := \mathcal{X}_{[-T,T]}$ for each $T > 0$.

On $\mathcal{X}$ consider the Gaussian measure μ such that, if $\{X_{st}\}_{s,t\in\mathbb{R}}$ is the coordinate process:

$$\mathbb{E}^\mu[X_{st}] = 0, \qquad \mathbb{E}^\mu[X_{st}^2] = |s - t|$$

and X_{st}, X_{uv} are independent iff (s, t) and (u, v) are disjoint intervals. This process can be easily understood as deriving from a d-dimensional Brownian motion B by setting $X_{st} = B_t - B_s$. Take $a \le b$ and let $\mathcal{F}_{[a,b]} = \sigma(X_{st} : a \le s \le t \le b)$ the σ-field generated by the increments in the interval $[a, b]$ and let $\mathcal{F}^{T,c}_{[a,b]} = \sigma(X_{st} : [s, t] \subset [-T, T]\backslash[a, b])$. In general, for a finite union of intervals $I \subset \mathbb{R}$ we let $\mathcal{F}_I = \sigma(X_{ts} : [t, s] \in I)$.

3. Gibbs measures

The specification Π is the family of proper probability kernels $\pi_I(\cdot|\cdot)$ with I running on the set $\mathcal{I}$ of all intervals of $\mathbb{R}$ such that

$$\pi_I(dx|y) = Z_I^{-1}(y) e^{-\lambda U_I(x) - \lambda V_I(x,y)} \mu_I \times \delta_{I^c,y}(dx) \tag{5}$$

for any $I \in \mathcal{I}$, where $\delta_{I^c,y}$ is the Dirac measure on $\mathcal{X}_{I^c}$ concentrated on the path $y|_{I^c\times I^c}$, U_I is an $\mathcal{F}_I$ measurable function and V_I is $\mathcal{F}_I \times \mathcal{F}_{I^c}$ measurable given by

$$U_I(x) = \int_{I\times I} dtds\, W(x_{st}, t - s) \tag{6}$$

$$V_I(x, y) = \int_{J(I)} dtds\, W((x \otimes_I y)_{st}, t - s) \tag{7}$$

with $W \in C(\mathbb{R}^d \times \mathbb{R}; \mathbb{R})$ and $z = (x \otimes_I y)_{st}$ the unique path in $\mathcal{X}$ such that $z|_{I\times I} = x|_{I\times I}$ and $z|_{I^c\times I^c} = y|_{I^c\times I^c}$ and

$$J(I) = (I^+ \times I^-) \cup (I^- \times I^+) \cup (I \times I^c) \cup (I^c \times I)$$

with I^+ and I^- the positive and negative half lines determined by I ($\mathbb{R} = I^- \cup I \cup I^+$). Note that $J(I)$ is the "cone" of influence of the increments in the interval I. That is, increments outside $J(I)$ are independent from increments inside I and any increment inside $J(I)$ can be written as a sum of an increment in I and increments outside $J(I)$.

For $x \in \mathcal{X}$, $\xi \in \mathbb{R}^d$, $a \geq 0$ define

$$Q_T(x, \xi, a) := \int_{-T}^{0} dt \int_0^T ds\, W(\xi + x_{st}, a + s - t).$$

On W we will make the following assumptions:

H1) $\sup_{x\in\mathcal{X}} |U_I(x)| \leq C|I|^2$ and $\sup_{x\in\mathcal{X}} \int_{-\infty}^{\infty} dt\, |W(x_{0t}, |t|)| < \infty$ (extensiveness of the energy).

H2) For $\xi \in \mathbb{R}^d$, $\sup_{T,a\geq 0} \sup_{x\in\mathcal{X}} |Q_T(x, \xi, a) - Q_T(x, 0, a)| \leq C(1 + |\xi|)$

Then

Theorem 1. *Assume* $(H1)$ *and* $(H2)$*, then there exists at least one Gibbs measure for the specification* Π*.*

Proof. By a theorem of Preston [13] it will be enough to prove that for a path $y \in X$ and for any interval I there exists a measure ω_I on $\mathcal{X}_I$ such that the family $\{\pi_K(\cdot|y)|_I : I \subset K\}$ is uniformly absolutely continuous w.r.t ω_I.

Fix $y \in \mathcal{X}$. For $f \in \mathcal{F}_I$, $f \geq 0$ let

$$\begin{aligned} \pi_K(f|y) &= \frac{\int_{\mathcal{X}_K} f(x \otimes_K y) e^{-\lambda U_K(x) - \lambda V_K(x,y)} \mu_K(dx)}{\int_{\mathcal{X}_K} e^{-\lambda U_K(x) - \lambda V_K(x,y)} \mu_K(dx)} \\ &= \frac{\int_{\mathcal{X}_H} \mu_H(dz) \int_{\mathcal{X}_I} \mu_I(dx) f(x) e^{-\lambda U_I(x) - \lambda V_I(x, z\otimes_H y) - \lambda U_K(z)}}{\int_{\mathcal{X}_H} \mu_H(dz) \int_{\mathcal{X}_I} \mu_I(dx) e^{-\lambda U_I(x) - \lambda V_I(x, z\otimes_H y) - \lambda U_K(z)}} \end{aligned} \tag{8}$$

where $H = K \backslash I$. For V_I and setting $I = [a, b]$, we have the following uniform bound

$$|V_I(x, z \otimes_H y) - V_I(0, z \otimes_H y)| \leq C(1 + |I| + |x_{ab}|) \tag{9}$$

so $\pi_K(f|y) \leq \omega_I(f)$ with

$$\omega_I(dx) := \frac{e^{-\lambda U_I(x) + C\lambda(1+|I|+|x_{ab}|)} \mu_I(dx)}{\int_{\mathcal{X}_I} e^{-\lambda U_I(x) - C\lambda(1+|I|+|x_{ab}|)} \mu_I(dx)}.$$

□

Remark 2. *The hypothesis above is more general than to require the potential to be* absolutely summable *(see* [11]*) , which would be equivalent to have*

$$\int_{-\infty}^{0} ds \int_0^{\infty} dt\, |W(x_{st}, t - s)| < \infty$$

uniformly for $x \in \mathcal{X}$*.*

Remark 3. *The regularized Nelson model:* $W(\xi,t) = -(1+|\xi|^2+|t|^2)^{-1}$ *satisfies assumptions* $(H1)$ *and* $(H2)$*. In particular, for* $(H2)$ *we have*

$$\nabla_\xi Q_T(x,\xi,a) = 2\int_{-T}^0 dt \int_0^T ds\, \frac{1}{(1+|\xi+x_{st}|^2+|a+s-t|^2)^2}(\xi+x_{st}) \tag{10}$$

so that

$$\begin{aligned} |\nabla_\xi Q_T(x,\xi,a)| &\le 2\int_{-T}^0 dt \int_0^T ds\, \frac{1}{(1+|\xi+x_{st}|^2+|a+s-t|^2)^{3/2}} \\ &\le 2\int_{-\infty}^0 dt \int_0^\infty ds\, \frac{1}{(1+|a+s-t|^2)^{3/2}} \\ &\le C(1+|a|)^{-1} \end{aligned} \tag{11}$$

Define the measure

$$\mu_{\lambda,T}(dx) := Z_{\lambda,T}^{-1} e^{-\lambda W_T(x)}\mu(dx) \tag{12}$$

with

$$W_T(x) := \int_{-T}^T dt \int_{-T}^T ds\, W(x_{st}, s-t)$$

As a consequence of the computations of the previous section we can easily derive the following proposition:

Corollary 4. *Under* $(H1)$ *and* $(H2)$ *the family of measures* $\{\mu_{\lambda,T}\}_{T\ge 1}$ *is tight (in the topology of local convergence). Any cluster point is a Gibbs measure for the specification* Π*.*

3.1. **Periodic Boundary Conditions.** Let $\widehat{\mu}_T$ the regular conditional probability $\widehat{\mu}_T = \mu(\cdot|X_{-T,T}=0)$. Under $\widehat{\mu}_T$ the process X is still Gaussian but increments in $[-T,T]$ are no more independent and their covariance structure is

$$\mathbb{E}^{\widehat{\mu}_T}[X_{ab}X_{cd}] = |[b,a]\cap[c,d]| - \frac{|b-a||d-c|}{2T}$$

where $|[b,a]\cap[c,d]|$ is the length of the common subinterval of $[a,b]$ and $[c,d]$. We note that $\widehat{\mu}_T$ converges weakly to μ as $T\to\infty$.

Let

$$\widehat{\mu}_{\lambda,T}(dx) := Z_{\lambda,T}^{-1} e^{\lambda W_T(x)}\widehat{\mu}_T(dx) \tag{13}$$

The advantage of using $\widehat{\mu}_T$ as reference measure is that, due to the choice of the boundary conditions for $-T<s<a<b<t<T$ we have $X_{st} = X_{-Ts}+X_{tT}$, $\widehat{\mu}^T$-a.s so that the r.v. X_{st} is measurable with respect $\mathcal{F}^{T,c}_{[a,b]}$.

This observation leads to:

Proposition 5. *Under* $(H1)$ *the family of measures* $\{\widehat{\mu}_{\lambda,T}\}_{T\ge 1}$ *is tight.*

Proof. Fix $a<b$ and take any positive bounded local function $F\in\mathcal{F}_{ab}$. Then

$$\mathbb{E}^{\mu_{\lambda,T}}[F] = \frac{\mathbb{E}^\mu\left[Fe^{-\lambda W_T(X)}\right]}{\mathbb{E}^\mu\left[e^{-\lambda W_T(X)}\right]}$$

Write $W_T(X)$ as $W_T(X) = W^{(1)}_{T,a,b} + W^{(2)}_{T,a,b} + W^{(3)}_{T,a,b}$ with

$$W^{(1)}_{T,a,b}(x) := \int_{-T}^{a} dt \int_{-T}^{a} ds\, W(x_{st}, s-t) + \int_{b}^{T} dt \int_{b}^{T} ds\, W(x_{st}, s-t) \tag{14}$$

$$\begin{aligned} W^{(2)}_{T,a,b}(x) &:= \int_{-T}^{a} dt \int_{b}^{T} ds\, W(x_{ta} + x_{ab} + x_{bs}, s-t) \\ &\quad + \int_{-T}^{a} ds \int_{b}^{T} dt\, W(x_{sa} + x_{ab} + x_{bt}, s-t) \\ &:= \int_{-T}^{a} dt \int_{b}^{T} ds\, W(x_{-Tt} + x_{sT}, s-t) \\ &\quad + \int_{-T}^{a} ds \int_{b}^{T} dt\, W(x_{-Ts} + x_{tT}, s-t) \end{aligned} \tag{15}$$

where the last equality is true a.e. under $\widehat{\mu}_T$;

$$W^{(3)}_{T,a,b}(x) := \int_{a}^{b} dt \int_{-T}^{T} ds\, W(x_{st}, s-t) + \int_{a}^{b} ds \int_{-T}^{T} dt\, W(x_{st}, s-t) \tag{16}$$

Using $(H1)$ it is easy to prove that, uniformly in T, $|W^{(3)}_{T,a,b}(x)| \le C(b-a)$. Then compute

$$\begin{aligned} \mathbb{E}^{\widehat{\mu}_{\lambda,T}}[F] &= \frac{\mathbb{E}^{\widehat{\mu}_T}\left[F e^{-\lambda W^{(1)} - \lambda W^{(2)} - \lambda W^{(3)}}\right]}{\mathbb{E}^{\widehat{\mu}_T}\left[e^{-\lambda W^{(1)} - \lambda W^{(2)} - \lambda W^{(3)}}\right]} \\ &= \frac{\mathbb{E}^{\widehat{\mu}_T}\left[\mathbb{E}^{\mu}\left(F e^{-\lambda W^{(2)} - \lambda W^{(3)}} | \mathcal{F}^{T,c}_{[a,b]}\right) e^{-\lambda W^{(1)}}\right]}{\mathbb{E}^{\mu}\left[\mathbb{E}^{\widehat{\mu}_T}\left(e^{-\lambda W^{(2)} - \lambda W^{(3)}} | \mathcal{F}^{T,c}_{[a,b]}\right) e^{-\lambda W^{(1)}}\right]} \\ &= \frac{\mathbb{E}^{\widehat{\mu}_T}\left[\mathbb{E}^{\widehat{\mu}_T}\left(F e^{-\lambda W^{(3)}_T(X)} | \mathcal{F}^{T,c}_{[a,b]}\right) e^{-\lambda W^{(1)}_T(X) - \lambda W^{(2)}_T(X)}\right]}{\mathbb{E}^{\widehat{\mu}_T}\left[\mathbb{E}^{\widehat{\mu}_T}\left(e^{-\lambda W^{(3)}_T(X)} | \mathcal{F}^{T,c}_{[a,b]}\right) e^{-\lambda W^{(1)}_T(X) - \lambda W^{(2)}_T(X)}\right]} \\ &\le e^{2C(b-a)} \frac{\mathbb{E}^{\widehat{\mu}_T}\left[\mathbb{E}^{\widehat{\mu}_T}\left(F | \mathcal{F}^{T,c}_{[a,b]}\right) e^{-\lambda W^{(1)}_T(X) - \lambda W^{(2)}_T(X)}\right]}{\mathbb{E}^{\widehat{\mu}_T}\left[e^{-\lambda W^{(1)}_T(X) - \lambda W^{(2)}_T(X)}\right]} \end{aligned} \tag{17}$$

where we used the fact that $W^{(2)}_T$ is measurable w.r.t $\mathcal{F}^{T,c}_{[a,b]}$.

The conditional expectation $\mathbb{E}^{\widehat{\mu}_T}(F|\mathcal{F}^{T,c}_{[a,b]})$ converges as $T \to \infty$ to $\mathbb{E}^{\mu}(F|\mathcal{F}^{T,c}_{[a,b]}) = \mathbb{E}^{\mu}(F)$. Then for any $\varepsilon > 0$, eventually as $T \to \infty$, $|\,\mathbb{E}^{\widehat{\mu}_T}(F|\mathcal{F}^{T,c}_{[a,b]}) - \mathbb{E}^{\mu}(F)| \le \varepsilon$ and $\mathbb{E}^{\widehat{\mu}_{\lambda,T}}[F] \le e^{2C(b-a)}\left[\mathbb{E}^{\mu}[F] + \varepsilon\right]$ which is uniform in T proving the tightness of the family of measures $\{\widehat{\mu}_{\lambda,T}\}_T$. □

4. Uniqueness

In the rest of this paper we will assume always that conditions $(H1)$ and $(H2)$ holds. Then a straighforward condition for uniqueness of the Gibbs measure is given by

H3)

$$\int_0^\infty dt \int_{-\infty}^0 ds\, |W(x_{st}, t-s) - W(0, t-s)| \le C$$

uniformly in $x \in \mathcal{X}$.

Note that $(H3)$ implies $(H2)$.

Proposition 6. *Assuming $(H3)$ there exists a unique Gibbs measure for the specification Π.*

Proof. Just consider the proof of existence and note that if $(H3)$ holds then we can obtain the same upper bound with ω_I replaced by $\widetilde{\omega}_I$:

$$\widetilde{\omega}_I(dx) := e^{2C\lambda} \frac{e^{-\lambda U_I(x)} \mu_I(dx)}{\int_{\mathcal{X}_I} e^{-\lambda U_I(x)} \mu_I(dx)}$$

i.e. $\pi_K(f|y) \le e^{2C\lambda} \widetilde{\omega}_I(f)$ for any positive $f \in \mathcal{F}_I$, $I \subset K$. Since the bound on the interaction energy is now uniform, using a similar argument we obtain also the lower bound $\pi_K(f|y) \ge e^{-2C\lambda} \widetilde{\omega}_I(f)$. Together they imply:

$$\pi_K(f|y) \ge e^{-4C\lambda} \pi_K(f|z) \tag{18}$$

for any couple $z, y \in \mathcal{X}$. This, according to a theorem of Georgii (Proposition 8.38) implies uniqueness of the Gibbs measure. □

Remark 7. *Condition $(H3)$ is satisfied whenever*

$$\sup_\xi |W(\xi, t)| \le C(1 + |t|)^{-(2+\delta)}$$

for some $\delta > 0$ or when

$$W(\xi, t) = C(1 + |\xi|^2 + |t|^2)^{-(3/2+\delta)}$$

always for some $\delta > 0$.

4.1. **Mapping to a discrete model.** For any size $L > 0$ the space $\mathcal{X}$ splits into a product space of countably many copies of $\mathcal{X}_L = C([0, L]^2, \mathbb{R}^d)$ as follows. Introduce the sequence on intervals $\tau_i = [iL, (i+1)L]$, $i \in \mathbb{Z}$ and for $x \in \mathcal{X}$ let $x_i = x|_{\tau_i \times \tau_i}$ considered as an element of $\mathcal{X}_L$. Let $F : \mathcal{X} \to \mathcal{X}_L^{\mathbb{Z}}$ be the map $(Fx)(i) = x_i$ for any $i \in \mathbb{Z}$ and $x \in \mathcal{X}$. Note that it is well defined the inverse map $F^{-1} : \mathcal{X}_L^{\mathbb{Z}} \to \mathcal{X}$ and that through F we can identify $\mathcal{X} \simeq \mathcal{X}_L^{\mathbb{Z}}$. Moreover using F^{-1} the measure μ factorizes accordingly : $\mu = F_*^{-1}(\otimes_{n \in \mathbb{Z}} \mu_L^{(n)})$ where each $\mu_L^{(n)}$ is the Wiener measure over increments in the interval τ_n. Using this mapping the path measure can be seen as a one-dimensional (unbounded) lattice spin system with single spin space $\mathcal{X}_L$.

In particular we can see the total energy $W_I(x|y) = U_I(x) + V_I(x,y)$ in a bounded interval as originating from potentials U_{ij} such that

$$U_{ij}(x) = \int_{iL}^{(i+1)L} dt \int_{jL}^{(j+1)L} ds\, W(x_{ts}, t-s)$$

in the following way: $W_I(x|y) = \sum_{i,j:I\cap I_{ij}\neq\emptyset} U_{ij}(x \otimes_I y)$ where I_{ij} is the smallest closed interval containing the set $\{iL, (i+1)L, jL, (j+1)L\}$.

Introduce condition $(H3b)$:

H3b) There exists constants $C > 0$ and $\gamma > 2$ such that

$$\sup_{\xi\in\mathbb{R}^d} |W(\xi,t)| \leq C(1+|t|)^{-\gamma}.$$

Let $\pi_{I,[-n,n]}(\cdot|\cdot)$ be the following probability kernels $\pi_{I,[-n,n]}(\cdot|y) = \pi_{[-n,n]}(\cdot|y)|_{\mathcal{F}_I}$. Note that the map $y \mapsto \pi_{I,[-n,n]}(\cdot|y)$ is $\mathcal{F}_{[-n,n]^c}$ measurable. Then we can state the following theorem.

Theorem 8. *Under condition $(H3b)$ there exist a unique Gibbs measure μ_λ for the specification Π moreover for any interval $I = [a,b]$, the marginal distribution $\mu_{\lambda,I}$ on $\mathcal{F}_I$, satisfy*

$$\mathbb{E}^\mu \left| \frac{d\mu_{\lambda,I}}{d\mu_I} - \frac{d\pi_{I,[-n,n]}(\cdot|y)}{d\mu_I} \right| \leq \chi(n - \max(|a|,|b|)) \tag{19}$$

uniformly in $y \in \mathcal{X}$ where $\chi(n)$ is a decreasing function which can be chosen to be $\chi(n) = A'|\log n|^{3-\gamma}|n|^{2-\gamma}$ for some constant $A' > 0$.

Proof. The theorem follows directly from Thm. 2 in [9]. Indeed it is easy to check that under condition $(H3b)$ the discretized model (e.g. with $L = 1$) fulfills all the required hypothesis. In particular, setting $M_U(k) = \sup_{|i-j|\geq k} \sup_x |U_{ij}(x)|$, it holds that $\sum_{k\geq 0} kM_U(k) < \infty$ and there exists a non-increasing function $\psi(n) = A|n|^{2-\gamma}$ such that $\sum_{k\geq n} kM_U(k) \leq \psi(n)$. So, according to this result we can choose the function $\chi(n)$ to be $\chi(n) = A'|\log n|^{3-\gamma}|n|^{2-\gamma}$. □

4.2. **The contraction technique.** In a more general setup than conditions $(H3)$ or $(H3b)$ we are able to prove the uniqueness of the Gibbs measure in the small coupling regime (i.e. when λ is small) using the contraction technique introduced by Dobrushin [7, 8].

Let $\|\cdot\|$ denote the sup norm on $\mathcal{X}_L$ and $\|\cdot\|_1$ the Lipschitz semi-norm on $C(\mathcal{X}_L, \mathbb{R})$:

$$\|f\|_1 := \sup_{x\neq y\in\mathcal{X}_L} \frac{|f(x) - f(y)|}{\|x-y\|} \tag{20}$$

Given two probability measures μ, ν on $\mathcal{X}_L$ define the Kantorovich-Rubinstein-Vasserstein (KRV) distance

$$d(\mu,\nu) := \sup_f \frac{|\mu(f) - \nu(f)|}{\|f\|_1}$$

where the sup is taken for $f \in C(\mathcal{X}_L, \mathbb{R})$.

Let $C(\mathcal{X})$ the space of continuous functions on $\mathcal{X}$ which are uniform limit of bounded local functions (i.e. depending only on finitely many $\mathcal{X}_L$ factors). For $f \in C(\mathcal{X})$ let

$$\|f\|_{1,i} := \sup \left\{ \frac{|f(x) - f(y)|}{\|x_i - y_i\|} : x, y \in \mathcal{X} \quad x_j = y_j \text{ for } j \neq i \right\} \tag{21}$$

and let $\mathrm{Lip}(\mathcal{X})$ the class of functions $f \in C(\mathcal{X})$ which satisfy

$$|f(x) - f(y)| \leq \sum_i \|x_i - y_i\| \|f\|_{1,i}, \qquad \sum_i \|f\|_{1,i} < \infty$$

Let $\pi_i := \pi_{\tau_i}$. Define the Dobrushin interaction matrix C as

$$C_{ik} := \sup \left\{ \frac{d(\pi_i(\cdot|y), \pi_i(\cdot|z))}{\|y_k - z_k\|} \,\middle|\, y, z \in \mathcal{X} : y_j = z_j \text{ for } j \neq k \right\} \tag{22}$$

A *tempered measure* ν is a measure on $\mathcal{X}$ for which there exists $z \in \mathcal{X}$ such that

$$\sup_i \int_{\mathcal{X}} \|x_i - z_i\| \nu(dx) < \infty$$

Note that, as a by product of the proof of Thm. 1 we have that, under conditions $(H1)$ and $(H2)$ there exists tempered Gibbs measures for the specification Π.

The following two results are originally due to Dobrushin [7, 8]. The formulation in terms of the KRV metric is taken from Föllmer [10] (see also [12]) where interested readers can find the relative proofs.

Theorem 9 (Dobrushin's uniqueness theorem)**.** *Whenever*

$$\sup_k \lim_{n \to \infty} \sum_i (C^n)_{ik} = 0 \tag{23}$$

there exists a unique tempered Gibbs measure μ_λ for the specification Π.

Remark 10. *Note that a sufficient condition for eq. (23) is given by $\sup_k \sum_i C_{ik} < 1$.*

Next we will apply these general results to our model. Let us introduce another class of potentials which is more general than $(H3)$:

H4) Assume that, for some $K(W) > 0$ and some $\alpha > 3$ we have

$$\sup_{\xi \in \mathbb{R}^d} |W_{\xi\xi}(\xi, t)| \leq \frac{K(W)}{(1 + |t|)^\alpha}$$

Theorem 11. *Under hypothesis $(H4)$ we have*

$$C_{ij} \leq C\lambda K(W)\sigma^2 (1 + |i - j|)^{2-\alpha}. \tag{24}$$

for some constant $C > 0$ and some $\sigma^2 > 0$. Therefore, for small λ there exists a unique tempered Gibbs measure μ_λ for the specification Π.

Proof. We want to check that the specification Π satisfy the requirements for the application of Dobrushin's uniqueness criterion (23).

Take $y, z \in \mathcal{X}$ and

$$\begin{aligned}\pi_i(f|z) - \pi_i(f|y) &= \int_0^1 dr \partial_r \left[Z_i(y^r)^{-1} \int_{\mathcal{X}_L} f(x \otimes_i y) e^{-\lambda U_i(x) - \lambda V_i(x, y^r)} \mu_i(dx) \right] \\ &+ \left[Z_i(z)^{-1} \int_{\mathcal{X}_L} [f(x \otimes_i z) - f(x \otimes_i y)] e^{-\lambda U_i(x) - \lambda V_i(x,z)} \mu_i(dx) \right] \\ &= \int_0^1 dr A(r) + B\end{aligned}$$

where we let

$$Z_i(y) := \int_{\mathcal{X}_L} e^{-\lambda U_i(x) - \lambda V_i(x,y)} \mu_i(dx).$$

The B term is estimated by

$$|B| \le \sum_{k \neq i} \|f\|_{1,k} \|z_k - y_k\| \tag{25}$$

which is finite if $f \in \mathrm{Lip}(\mathcal{X})$.

Let $J_i = J(\tau_i)$ and $J_{ij} = J_i \cap J_j$. For simplicity take first $y, z \in \mathcal{X}$ such that $y = z$ outside τ_j we let $y^r = y + r(z - y)$ with $r \in [0, 1]$ and compute

$$|\partial_r (V_i(x, y^r) - V_i(w, y^r))| \le \int_{J_{ij}} dt ds \sup_\xi |\nabla \nabla W(\xi, t - s)| \, \|x_i - w_i\| \|y_j - z_j\|$$

If $i \neq j$ this integral contains only contributions with time span greater than $L|i - j - 1|$ so if we assume that

$$\sup_\xi |\nabla \nabla W(\xi, t)| \le K(W)(1 + |t|)^{-\alpha}$$

we get

$$|\partial_r (V_i(x, y^r) - V_i(w, y^r))| \le C K(W) (1 + |i - j|)^{2-\alpha} \|x_i - w_i\| \|y_j - z_j\|.$$

For general $y, z \in \mathcal{X}$ we have

$$|\partial_r (V_i(x, y^r) - V_i(w, y^r))| \le C K(W) \sum_{j \neq i} (1 + |i - j|)^{2-\alpha} \|x_i - w_i\| \|y_j - z_j\|.$$

And with $\pi_i(\cdot | y^r) = \nu^r$

$$\begin{aligned} A(r) &= \partial_r \int f(\cdot \otimes_i y) d\nu^r \\ &= \partial_r \left[Z_i(y^r)^{-1} \int_{\mathcal{X}_L} f(x \otimes_i y) e^{-\lambda U_i(x) - \lambda V_i(x, y^r)} \mu_i(dx) \right] \\ &= \int_{\mathcal{X}} (f(x \otimes_i y) - f(w \otimes_i y)) \left[\lambda \partial_r V_i(x \otimes_i y^r) - \partial_r \log Z_i(y^r) \right] \nu^r(dx) \end{aligned} \tag{26}$$

for any $w \in \mathcal{X}_L$.

This implies

$$\begin{aligned}|A(r)|^2 &\leq \int_{\mathcal{X}} (f(x\otimes_i y)-f(w\otimes_i y))^2\nu^r(dx)\mathrm{Var}_{\nu^r}\left[\lambda\partial_r V_i(\cdot\otimes_i y^r)\right]\\ &\leq \|f\|_{1,i}^2\int_{\mathcal{X}}\|x_i-w\|^2\nu^r(dx)\mathrm{Var}_{\nu^r}\left[\lambda\partial_r V_i(\cdot\otimes_i y^r)\right]\\ &\leq \|f\|_{1,i}^2\sigma_i^2\mathrm{Var}_{\nu^r}\left[\lambda\partial_r V_i(\cdot\otimes_i y^r)\right]\end{aligned}$$

where

$$\sigma_i^2 := \sup_{y\in\mathcal{X}}\inf_{w\in\mathcal{X}_L}\int_{\mathcal{X}}\|x_i-w\|^2\pi_i(dx|y).$$

Then we have

$$|A(r)|\leq CK(W)\|f\|_{1,i}\sigma_i^2\sum_j\lambda(1+|i-j|)^{2-\alpha}\|y_j-z_j\|$$

which imply

$$\begin{aligned}|\pi_i(f|y)-\pi_i(f|z)| \leq C\lambda K(W)\|f\|_{1,i}\sigma_i^2\sum_{j\neq i}(1+|i-j|)^{2-\alpha}\|y_j-z_j\|\\ +\sum_{j\neq i}\|f\|_{1,j}\|z_j-y_j\|\end{aligned}$$

then if $f\in\mathrm{Lip}(\mathcal{X})$ we have also $\pi_i(f|\cdot)\in\mathrm{Lip}(\mathcal{X})$ provided $\alpha>3$.

If $f(x)=f(x_i)$

$$|\pi_i(f|y)-\pi_i(f|z)|\leq C\lambda K(W)\|f\|_{1,i}\sigma_i^2\sum_{j\neq i}(1+|i-j|)^{2-\alpha}\|y_j-z_j\|$$

which means that

$$d(\pi_i(\cdot|y),\pi_i(\cdot|z))\leq C\lambda K(W)\sigma_i^2\sum_{j\neq i}(1+|i-j|)^{2-\alpha}\|y_j-z_j\| \tag{27}$$

Moreover if $y=z$ outside τ_j (i.e. if $y|_{\tau_k}=z|_{\tau_k}$ for any $k\neq j$) then

$$d(\pi_i(\cdot|y),\pi_i(\cdot|z))\leq C\lambda K(W)\sigma_i^2(1+|i-j|)^{2-\alpha}\|y_j-z_j\| \tag{28}$$

which in turn implies $C_{ij}\leq C\lambda K(W)\sigma_i^2(1+|i-j|)^{2-\alpha}$. So provided $\sigma=\sup_i\sigma_i<\infty$ and $\alpha>3$ we have that $\sum_i C_{ij}\leq C'\sigma^2\lambda K(W)L^{-1}$ and for λ sufficiently small we obtain $\sup_j\sum_i C_{ij}<1$ and by applying Dobrushin criterion we can conclude that there exists a unique tempered Gibbs measure μ_λ associated to the specification Π. The condition $\sigma^2<\infty$ follows easily from $(H1)$ and $(H2)$ and the computations of Thm. 1. □

Remark 12. *In the case of the Nelson model we have*

$$|\nabla\nabla W(\xi,t)|\leq\frac{C}{(1+|t|^2)^2}$$

uniformly in ξ, so $\alpha=4$ and we can apply the above result.

5. Lower bound on the diffusion constant

Here we provide a lower bound for the second moment of the increments under the Gibbs measure. This will be useful below in the proof of the CLT.

Theorem 13. *Under condition (H4) there exists a positive constant $\sigma_-^2 > 0$ such that*

$$\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \geq \sigma_-^2 |a-b|$$

uniformly in T.

Proof. Integration by parts on μ is given by the formula

$$\mathbb{E}^{\mu}[X_{ab}F] = \int_a^b dt\ \mathbb{E}^{\mu}[D_t F]$$

where D_t is the Malliavin derivative. This, of course, when both sides make sense. By integration by parts we have for disjoint intervals $[a,b]$, $[c,d]$ with $b<c$:

$$\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}X_{cd}] = -\,\mathbb{E}^{\mu_{\lambda,T}}[A_{ab}X_{cd}] = \mathbb{E}^{\mu_{\lambda,T}}[-B_{ab,cd} + A_{ab}A_{cd}] \tag{29}$$

with

$$A_{ab} = \int_a^b ds\, \lambda D_s(W_T(X)) \quad \text{and} \quad B_{ab,cd} = \int_a^b ds \int_c^d dt\, \lambda D_t D_s(W_T(X)).$$

While

$$\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] = |a-b| - \mathbb{E}^{\mu_{\lambda,T}}[A_{ab}X_{ab}] = |a-b| - \mathbb{E}^{\mu_{\lambda,T}}[B_{ab,ab}] + \mathbb{E}^{\mu_{\lambda,T}}[A_{ab}^2] \tag{30}$$

Apply Cauchy-Schwartz inequality to $\mathbb{E}^{\mu_{\lambda,T}}[A_{ab}X_{ab}]$ to get the lower bound

$$\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \geq |a-b| - \left(\mathbb{E}^{\mu_{\lambda,T}}[A_{ab}^2]\right)^{1/2} \left(\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2]\right)^{1/2}$$

Next we have also

$$\mathbb{E}^{\mu_{\lambda,T}}[A_{ab}^2] \leq \mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] + \mathbb{E}^{\mu_{\lambda,T}}[|B_{ab,ab}|]$$

and so

$$\left(\mathbb{E}^{\mu_{\lambda,T}}[A_{ab}^2]\right)^{1/2} \leq \left(\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2]\right)^{1/2} + \left(\mathbb{E}^{\mu_{\lambda,T}}[|B_{ab,ab}|]\right)^{1/2}$$

Putting all together we obtain

$$2\,\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \geq |a-b| - \left(\mathbb{E}^{\mu_{\lambda,T}}[|B_{ab,ab}|]\right)^{1/2} \left(\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2]\right)^{1/2}$$

If we compute the Malliavin derivatives, we get:

$$\begin{aligned} D_t(W_T(X)) &= 2\int_{-T}^{T} du \int_u^T dv\, D_t W(X_{uv}, u-v) \\ &= 2\int_{-T}^{t} du \int_t^T dv\, W_x(X_{uv}, u-v) \end{aligned} \tag{31}$$

and

$$\begin{aligned} D_t D_s(W_T(X)) &= 2\int_{-T}^{T} du \int_u^T dv\, D_t D_s W(X_{uv}, u-v) \\ &= 2\int_{-T}^{s} du \int_t^T dv\, W_{xx}(X_{uv}, u-v) \end{aligned} \tag{32}$$

It is straightforward to see that these equations pass to the limit when $T \to \infty$. Using condition $(H4)$ we get

$$|D_t D_s(W_T(X))| \leq 2 \int_{-T}^{s} du \int_{t}^{T} dv \, \frac{C}{(1+|u-v|)^{\alpha}} \leq C(1+|s-t|)^{2-\alpha}. \tag{33}$$

And since $\alpha > 3$ we obtain the bound $\mathbb{E}^{\mu_{\lambda,T}}[|B_{ab,ab}|] \leq C|a-b|$ uniformly in T.

Using this estimate in the inequality above we get

$$2\,\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \geq |a-b| - C^{1/2}|a-b|^{1/2} \left(\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2]\right)^{1/2}$$

Calling $y = \left(\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2]\right)^{1/2} |a-b|^{-1/2}$ we have $2y^2 - 1 + C^{1/2}y \geq 0$ with $y \geq 0$. This implies that $y \geq \sigma_-^2$ for some $\sigma_-^2 > 0$ and we have obtained the lower bound $\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \geq \sigma_-^2 |a-b|$. □

Remark 14. *Using* $2|ab| \leq |a|^2 + |b|^2$ *we have*

$$\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \leq |a-b| + \frac{1}{2}\,\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] + \frac{1}{2}\,\mathbb{E}^{\mu_{\lambda,T}}\left[A_{ab}^2\right]$$

and then $\mathbb{E}^{\mu_{\lambda,T}}[X_{ab}^2] \leq 2|a-b| + \mathbb{E}^{\mu_{\lambda,T}}[A_{ab}^2]$. *This equation and the decay of correlations can provide upper bounds for the diffusion constant.*

6. Diffusive behavior

According to Thms. 8 and 11 under condition $(H3b)$ or $(H4)$ (the latter in the small coupling regime) we have a unique Gibbs measure which we denote μ_λ as above and for which we have polynomial decay of correlations. Then we address the problem of the long-time behavior of the increment process under diffusive rescaling. Thm. 13 ensure that when the interaction decays sufficiently fast and for any value of the coupling constant the path cannot show a sub-diffusive behavior (this irrespective of the sign of the potential). The next theorem establish that under some more restrictive assumption than needed to prove uniqueness of the Gibbs measure we actually have diffusive behavior of the increment process.

Theorem 15. *Assume* $(H4)$ *holds. Then in either of the following two situations:*

(a) *condition* $(H3b)$ *with* $\gamma > 3$, *or*
(b) *small* λ *and* $\alpha > 4$;

the r.v. $X_{t,s}^{\varepsilon} = \varepsilon^{1/2} X_{\varepsilon^{-1}t,\varepsilon^{-1}s}$ *weakly converges to an isotropic Gaussian vector as* $\varepsilon \to 0$.

Proof. Consider the random variables $Y_i = \langle v, X_{Li,L(i+1)} \rangle \in \mathcal{F}_{\tau_i}$ for some fixed vector $v \in \mathbb{R}^d$ and for $\Lambda \subset \mathbb{Z}$ let $\mathcal{A}_\Lambda = \sigma(X_i, i \in \Lambda)$. If $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$ let $d(\Lambda_1, \Lambda_2) = \inf(|n-k|, n \in \Lambda_1, k \in \Lambda_2)$. Define the following mixing coefficients for the measure μ_λ:

$$\alpha_{l,k}(n) = \sup\{|\mu_\lambda(A_1 \cup A_2) - \mu_\lambda(A_1)\mu_\lambda(A_2)| : A_i \in \mathcal{A}_{\Lambda_i}, |\Lambda_1| \leq k, |\Lambda_2| \leq l, d(\Lambda_1, \Lambda_2) \geq n\}$$

and

$$\rho(n) = \sup\{|\mathrm{Cov}_{\mu_\lambda}(Z_1, Z_2)| : Z_i \in L^2(\mu_\lambda, \mathcal{A}_{\{k_i\}}), \|Y_i\|_2 \leq 1, |k_1 - k_2| \geq n\}$$

where $\|\cdot\|_p$ denote the L^p norm with respect to the measure μ_λ.

Then, according to a theorem of Bolthausen [6], if the following two conditions holds

$$\|Y_i\|_{2+\delta} < \infty, \qquad \sum_{m=1}^{\infty} \alpha_{2,\infty}^{\delta/(2+\delta)}(m) < \infty \tag{34}$$

for some $\delta > 0$ and if

$$\sigma^2 = \sum_n \mathrm{Cov}_{\mu_\lambda}(Y_0, Y_n) > 0 \tag{35}$$

then the r.v. $S_n = (\sigma^2 n)^{-1/2} \sum_{0\le k\le n} Y_k$ converges to a standard Gaussian r.v.

So it will be enough to check the two conditions (34) and (35). For eq. (35) note that $\sigma^2 = \lim_{n\to\infty} n^{-1}\, \mathbb{E}_{\mu_\lambda}(X_{Ln,0})^2$ and by the bound proved in Thm. 13 we have that this quantity is bounded below by $\sigma_-^2 Ln$ with $\sigma_-^2 > 0$, so that we can conclude $\sigma^2 > 0$.

As for condition (34) note that each Y_i has moments of any order due to the fact that the expectation on the measure μ_λ can be bounded above by expectation on the measure $\omega_{[0,L]}$ defined in the proof of Thm. 1 which can the be directly estimated and shown to be finite.

In case (a) the inequality (19) is enough to prove that $\alpha_{2,\infty}(n) \le C\chi(n) = C'|n|^{2-\gamma}$, so that the required condition is fulfilled whenever $\gamma > 3$.

Then it remains to check that the mixing coefficient $\alpha(n)$ is sufficiently summable in case (b). A technical difficulty is that the contraction coefficients for the KRV metric are not suitable to estimate the strong mixing coefficients $\alpha_{l,k}(m)$. This because we cannot uniformly approximate the indicator functions (needed to estimate probabilities) with functions in $\mathrm{Lip}(\mathcal{X})$. However this technical difficulty can be easily overcome by using a slightly different norm in the Dobrushin contraction technique. Let $\|f\|_{*,i}$ the following local "quasi"-Lipschitz semi-norm:

$$\|f\|_{*,i} := \sup\left\{ \frac{|f(x) - f(y)|}{\theta_{x_i,y_i} + \|x_i - y_i\|} : x, y \in \mathcal{X} \quad x_j = y_j \text{ for } j \neq i \right\}$$

where $\theta_{x,y} = 1$ if $x \neq y$ and $\theta_{x,y} = 0$ otherwise, for any two elements $x, y \in \mathcal{X}_L$. This new semi-norm would replace the semi-norm $\|f\|_{1,i}$ defined in eq. (21). Then it is easy to see that all the arguments carry over also with this new semi-norm and that under this semi-norm we can approximate uniformly the indicator function of a set $A \in \mathcal{A}_\Lambda$ by the Lipschitz functions $\Gamma_{A,\rho}(x) = \exp(-\rho^{-1} \inf_{y\in A} \|x - y\|)$. Then, adapting Prop. 2.5 of [12] we are able to prove that, under the condition $\sup_i \sum_j C_{ij}|i-j|^{1+\delta} < \infty$ for some $\delta > 0$ (i.e. $\alpha > 4$) we have $\alpha_{2,\infty}(n) \le C|n|^{-1-\delta}$ and this is enough to conclude the proof. □

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