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The FBSDE approach to sine–Gordon up to 6π

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We develop a stochastic analysis of the sine-Gordon Euclidean quantum field $(\cos(\beta\varphi))_2$ on the full space up to the second threshold, i.e. for $\beta^2 < 6\pi$. The basis of our method is a forward-backward stochastic differential equation (FBSDE) for a decomposition $(X_t)_{t\geqslant 0}$ of the interacting Euclidean field X_∞ along a scale parameter $t \geqslant 0$. This FBSDE describes the optimiser of the stochastic control representation of the Euclidean QFT introduced by Barashkov and one of the authors. We show that the FBSDE provides a description of the interacting field without cut-offs and that it can be used effectively to study the sine-Gordon measure to obtain results about large deviations, sub-gaussian tails, decay of correlations for local observables, singularity with respect to the free field, Osterwalder–Schrader axioms and other properties.

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1. Introduction

The aim of this paper is to provide a rigorous *description* of the two-dimensional sine-Gordon Euclidean quantum field theory (EQFT) on the full space in the regime $\beta^2 < 6\pi$. The sine-Gordon EQFT is formally given by the Gibbs measure

$$\text{“}\nu_{\mathrm{SG}}(\mathrm{d}\varphi) = \Xi^{-1}\exp(-V_{\mathrm{SG}}(\varphi))\mu(\mathrm{d}\varphi)\text{”}, \quad \varphi \in \mathcal{S}'(\mathbb{R}^2), \tag{1.1}$$

where μ is a massive Gaussian free field on the space of Schwartz distributions $\mathcal{S}'(\mathbb{R}^2)$, the constant Ξ is a normalisation to make ν_{SG} a probability measure, and V_{SG} corresponds to the cosine interaction, formally defined as

$$V_{\mathrm{SG}}(\varphi) := \lambda \int_{\mathbb{R}^2} \cos(\beta\varphi(x))\mathrm{d}x.$$

The sine-Gordon model is a prototypical example of a non-Gaussian EQFT and of particular interest as both a theory with infinitely many phase transitions as β^2 varies between 0 and 8π and more generally as a test-bed for non-polynomial interactions.

The approach we take here is based on a scale dependent interpolation $(G_t)_{t\in[0,\infty]}$ of the covariance $G_\infty = (\Delta - m^2)^{-1}$ of the Gaussian free field. This allows us to interpret the Gaussian free field as the terminal value W_∞ of a Brownian martingale $(W_t)_{t\in[0,\infty)}$ defined by

$$W_t := \int_0^t \dot{G}_s^{1/2}\mathrm{d}B_s, \qquad t \geqslant 0, \tag{1.2}$$

where $B = (B_t)_{t\geqslant 0}$ is a cylindrical Brownian motion on $L^2(\mathbb{R}^2)$. From this point of view, we can produce a scale dependent stochastic dynamics $(X_t)_{t\in[0,\infty]}$ for the target measure

(1.1) which provides a pathwise scale-by-scale coupling $(X_t, W_t)_{t\in[0,\infty]}$.

$$\mathrm{d}X_t = -\dot{G}_t \mathbb{E}_t[\mathrm{D}V_{\mathrm{SG}}(X_\infty)]\mathrm{d}t + \dot{G}_t^{1/2}\mathrm{d}B_t, \qquad t \geqslant 0. \tag{1.3}$$

Here, $\mathrm{D}V_{\mathrm{SG}}(\varphi) = -\lambda\beta\sin(\beta\varphi)$ is formally the functional derivative of the interaction potential V_{SG}, and we write $\dot{G}_t := \partial_t G_t$ for the scale derivative of G. The conditional expectation $\mathbb{E}_t$ is taken with respect to the augmentation of the filtration generated by the Brownian motion W and can be interpreted as averaging out the small scales $s^{-1} \leqslant t^{-1}$.

In this paper, we show that, once properly renormalised, the FBSDE (1.3) provides an effective stochastic quantisation equation for (1.1) for any size of the coupling constant $\bar{\lambda} := |\lambda| \in \mathbb{R}$. This allows to construct the measure (1.1) without cut-offs from a straightforward analysis of the equation and only basic estimates of the convolution $\dot{G}$ (see Theorem 1.1). Moreover, we can efficiently transport properties from the Gaussian free field to the sine-Gordon EQFT via (1.3). In particular, we obtain

a) an explicit description of the infinite volume measure via a variational principle (Theorem 7.7);

b) a proof of the mutual singularity of the Gaussian free field and the finite volume sine-Gordon measure for $\beta^2 \geqslant 4\pi$ (Theorem 1.4);

c) a simple proof for the exponential decay of correlations of general local observables (Theorem 1.3);

d) an analysis of the semi-classical limit $\hbar \to 0$ (Theorem 1.5);

e) a full verification of the Osterwalder–Schrader axioms and a proof of non-Gaussianity (Theorem 8.1);

In order to give a rigorous meaning to (1.3) we start, as usual, from a well-defined approximation of the sine-Gordon measure given by

$$\nu^{\rho,T}(\mathrm{d}\varphi) = \Xi_{\rho,T}^{-1}\exp(-V^{\rho,T}(\varphi))\mu^T(\mathrm{d}\varphi), \tag{1.4}$$

where ρ is an infrared cut-off and μ^T denotes the law of an approximation W_T to the massive Gaussian free field W_∞ as in (1.2). In Section 2 we will show that in this regularised setting, the FBSDE (1.3) produces the correct measure, that is the solution $(X_t)_t$ to the FBSDE

$$\mathrm{d}X_t^{\rho,T} = -\dot{G}_t \mathbb{E}_t[\mathrm{D}V^{\rho,T}(X_T^{\rho,T})]\mathrm{d}t + \mathrm{d}W_t, \qquad t \in [0,T] \tag{1.5}$$

has terminal law $\mathrm{Law}(X_T^{\rho,T}) = \nu^{\rho,T}$. As a byproduct, we show that it is associated with the solution to the stochastic optimal control problem

$$-\log\mathbb{E}[\mathrm{e}^{-V^{\rho,T}(W_T)}] = \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V^{\rho,T}(I_T(u)+W_T) + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2\mathrm{d}t\right], \tag{1.6}$$

where $I_t(u) := \int_0^t \dot{G}_s^{1/2} u_s \mathrm{d}s$ and $\mathbb{H}_a$ is an appropriate space of predictable processes. As expected, the representations (1.5) and (1.6) are not stable in the small-scale limit $T \to \infty$, and they require a renormalisation of the potential $V^{\rho,T}$ involving diverging constants. To overcome this problem, suppose that F is a sufficiently nice scale dependent function $F = (F_t)_{t\in[0,T]}$ such that $F_T = \mathrm{D}V^{\rho,T}$. By Ito's formula, solving the FBSDE (1.5) is equivalent to solving the FBSDE

$$\begin{cases} Z_t = -\int_0^t \dot{G}_s(F_s(Z_s + W_s) + R_s)\mathrm{d}s, \\ R_t = \mathbb{E}_t \int_t^T H_s(Z_s + W_s)\mathrm{d}s - \mathbb{E}_t \int_t^T \mathrm{D}F_s \dot{G}_s R_s \mathrm{d}s, \end{cases} \qquad t \in [0,T], \tag{1.7}$$

where the functional $(H_t)_{t\in[0,T]}$ is given by

$$H_t := \partial_t F_t + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t) - \mathrm{D}F_t \dot{G}_t F_t, \qquad t \in [0,T].$$

The solution X to (1.5) can then be obtained from (1.7) with the identification $X_t = Z_t + W_t$. In this representation, the limit $T \to \infty$ is associated to the convergence of the integral over scales in the equation for the *remainder* R. Constructing the measure (1.1) reduces to two tasks:

1. Find an approximation F for the effective force $\mathbb{E}_t[\mathrm{D}V_{\mathrm{SG}}(X_\infty)]$ that makes the source term $H_s(W_s + Z_s)$ of the backward equation in (1.7) integrable as $s \to \infty$, while preserving good continuity and growth properties.

2. Control the associated FBSDE (1.7) uniformly in the regularisations T and ρ and establish global existence for the solutions to (1.7).

The first task involves a good understanding of approximate solutions to the well-known infinite dimensional and non-linear (backward) Polchinski renormalisation flow equation (see e.g. [36] or the recent review [13]),

$$\begin{cases} \partial_t v_t + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 v_t) - \frac{1}{2}\mathrm{D}v_t \dot{G}_t \mathrm{D}v_t = 0, \\ v_\infty(\varphi) = V_{\mathrm{SG}}(\varphi). \end{cases} \tag{1.8}$$

Indeed, given a solution v to (1.8) and taking $F_t = \mathrm{D}v_t$ we would have $H_t = 0$ and therefore $R_t = 0$. The remainder R allows for additional freedom in the choice for the scale interpolation of the force F_t and avoids a precise technical analysis of (1.8).

The second task requires good a priori estimates for the non-standard FBSDE (1.7), which are uniform in the regularisation T. The equation (1.7) is in general nonlinear and thus solutions need not be global so that this step is non-trivial and indeed the reason why the present work is limited to the regime $\beta^2 < 6\pi$. It would be very interesting to better understand the solution theory for FBSDEs of the form (1.7) also in a more general setting for different models, that is different choices of F_t.

Our main result is the following.

Theorem 1.1. *Let $\beta^2 < 6\pi$. For $\rho \in C_c^\infty(\mathbb{R}^2)$ or $\rho \equiv 1$ and $T \in [0,\infty]$, there is scale dependent function $F^{\rho,T} = (F_s^{\rho,T})_{s\in[0,T]}$ such that up to first order (in $\bar\lambda$), $F_T^{\rho,T}$ corresponds perturbatively to the Wick-renormalised sine*

$$F_t^{\rho,T}(W_{t\wedge T})(x) = -\rho(x)\beta\lambda[\![\sin(\beta W_{t\wedge T}(x))]\!] + O(\bar\lambda^2),$$

and the associated FBSDE (1.7) *has a solution $(Z^{\rho,T}, R^{\rho,T}) \in \mathbb{H}^\infty(L^\infty) \times \mathbb{H}^\infty(L^\infty)$.*

If the volume is finite, that is $\rho \in C_c^\infty(\mathbb{R}^2)$, or if the coupling constant $\bar\lambda$ is sufficiently small, $\mathrm{Law}(X_\infty^\rho) = \nu_{\mathrm{SG}}^\rho$ *is unique. For $\rho = 1, T = \infty$ and any $\varepsilon > 0$, there is a version of the drift $Z = Z^{1,\infty}$ with terminal value $Z_\infty \in L^\infty(\mathrm{d}\mathbb{P}; B_{p,p}^{2-\beta^2/4\pi-\varepsilon,-n})$, and the sine-Gordon measure is given as a random shift of the Gaussian free field W_∞,*

$$\nu_{\mathrm{SG}} = \mathrm{Law}(W_\infty + Z_\infty).$$

It should be emphasised that while our analysis provides uniqueness only if the coupling constant $\bar\lambda$ is small or the volume is finite, its existence is guaranteed for any $\lambda \in \mathbb{R}$ also in the full space: we obtain uniform bounds on the FBSDE for any $\lambda \in \mathbb{R}$ which imply tightness for the family $\nu_{\mathrm{SG}}^{\rho,T} = \mathrm{Law}(W_T + Z_T^{\rho,T})$. Indeed, at the level of the approximate measures (1.4), Theorem 1.1 implies following result.

Corollary 1.2. *For any $\lambda \in \mathbb{R}$, the family of measures $(\nu_{\mathrm{SG}}^{\rho,T})_{T,\rho}$ defined in* (1.4) *is tight on $H^{-\varepsilon}(\langle x\rangle^{-2})$. Moreover, there is a unique accumulation point if either of the conditions below is satisfied*

- *$\bar\lambda$ is sufficiently small. In this case $(\nu_{\mathrm{SG}}^{\rho,T})_{T,\rho}$ converges $H^{-\varepsilon}(\langle x\rangle^{-2})$-weakly to a measure ν_{SG}.*
- *$\rho \in C_c^\infty(\mathbb{R}^2)$ remains fixed. Then, as $T \to \infty$, the sequence $(\nu_{\mathrm{SG}}^{\rho,T})_{T\geqslant 0}$ converges $H^{-\varepsilon}(\langle x\rangle^{-2})$-weakly to a limiting measure ν_{SG}^ρ.*

To demonstrate the advantages of the representation, we transport some properties of the free field W_∞ to the sine-Gordon shift $W_\infty + Z_\infty$. A neat application is the exponential decay of correlation via a simple coupling argument as in [39, 28]. In this setting, we can show that for the unique solution Z_t to (1.7) at $T = \infty$, $\rho = 1$, the process $(X_t)_{t\in[0,\infty]} = (Z_t + W_t)_{t\in[0,\infty]}$ inherits the following decay of correlations from W. Note that the theorem below includes $t = \infty$ and thus $\nu_{\mathrm{SG}} = \mathrm{Law}(X_\infty)$.

Theorem 1.3. *Let χ be a smooth function supported on $B_1(0)$ and $x_1, x_2 \in \mathbb{R}^2$. If $\bar\lambda$ is sufficiently small, then there is a constant $\gamma \in (0,1)$ depending only on the mass m such that for any two bounded and Lipschitz observables $\mathcal{O}_1, \mathcal{O}_2 : H^{-\varepsilon,-n} \to \mathbb{R}$, it holds that*

$$|\mathrm{Cov}[\mathcal{O}_1(\chi \cdot X_t(\cdot + x_1)); \mathcal{O}_2(\chi \cdot X_t(\cdot + x_2))]| \lesssim \mathrm{e}^{-m\gamma|x_1 - x_2|}.$$

Here, the implicit constant depends only on the bounds and Lipschitz constants of the observables $\mathcal{O}_1$ and $\mathcal{O}_2$.

In the first region $\beta^2 < 4\pi$, it is not difficult to see that the finite volume sine-Gordon measure is absolutely continuous with respect to the Gaussian free field. Using the FBSDE, we can show that this is no longer the case beyond this threshold. To the best of our knowledge, this is the first proof of this fact.

Theorem 1.4. *For $\beta^2 \in [4\pi, 6\pi)$, the finite volume sine-Gordon measure and the Gaussian free field are mutually singular.*

As a result of this singularity, the control problem (1.6) cannot be transferred to the UV-limit verbatim, in contrast to the simpler setting $\beta^2 < 4\pi$ (treated in [6]). Building on the same ideas used for the change of variables in the FBSDE from (1.3) to (1.7), we reformulate the variational problem (1.6) in terms of an (absolutely continuous) remainder. This reformulation, combined with a localisation property of the limiting measure, allows us to recover a variational problem for the Laplace transform of ν_{SG} in the infinite volume (see 7.7) as long as $\bar{\lambda}$ is sufficiently small.

With this variational formulation at hand, we can show that the limiting measure ν_{SG} defines a non-Gaussian EQFT and derive a Laplace principle for the semi-classical limit $\hbar \to 0$. To make this slightly more precise, let $(\mu^\hbar)_{\hbar\in(0,1)}$ be the family of rescaled Gaussian free fields with covariance $\hbar(m^2-\Delta)^{-1}$. For now, formally define the measures

$$\text{“}\nu^\hbar_{\mathrm{SG}}(\mathrm{d}\varphi) := \Xi_\hbar^{-1} \exp(-\hbar^{-1}V(\varphi))\mu^\hbar(\mathrm{d}\varphi).\text{”}$$

We show the following theorem.

Theorem 1.5. *As $\hbar \to 0$, the family $\nu^\hbar_{\mathrm{SG}}$ satisfies a Laplace principle with rate $\hbar^{-1}$ and rate function*

$$I(\varphi) := \begin{cases} \lambda \int (\cos(\beta\varphi) - 1) + \frac{1}{2}\int \varphi(m^2-\Delta)\varphi, & \varphi \in H^1(\mathbb{R}^2), \\ \infty, & \textit{otherwise}. \end{cases} \tag{1.9}$$

More precisely, for any continuous and bounded $f : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$,

$$\lim_{\hbar\to 0} -\hbar \log \int_{\mathcal{S}'(\mathbb{R}^2)} \exp(-\hbar^{-1} f(\varphi))\nu^\hbar_{\mathrm{SG}}(\mathrm{d}\varphi) = \inf_{\varphi\in H^1}\{f(\varphi) + I(\varphi)\}. \tag{1.10}$$

Finally, we verify the Osterwalder Schrader axioms for small couplings $\bar{\lambda}$.

Remark 1.6. Our approach relies only on some general estimates for the heat kernel of the Laplacian (see Appendix A) and can be easily extended with respect to the dimension of the underlying Euclidean space. In the general d-dimensional setting, the sine-Gordon theory can be considered with respect to the d-dimensional log-correlated Gaussian field and is subcritical for $\beta^2/2\pi \in [0, 2d)$ and the argument presented here allows to construct the sine-Gordon measure in $\beta^2/2\pi \in [0, d+1)$. In one dimension, this means we can cover the full subcritical regime, (see also [31], where a one-dimensional version of the model is considered). We can moreover generalise our results to the (compact) Riemannian manifold context, in analogy to the recent work [2] on Φ^4_3 on a compact Riemannian manifold. For the sake of clarity, we refrain from including these modifications. The required changes are minimal and we do not believe that the associated results would justify extending this contribution.

1.1. Related work

The sine–Gordon model has been subject to many studies in the constructive literature, covering finite or infinite volume interactions and allowing various ranges for $\beta^2 \in [0, 8\pi)$ and the coupling constant $\lambda \in \mathbb{R}$. However, the full mathematical understanding of this model is still lacking and none of these works cover the theory on the full space $\mathbb{R}^2$ for all $\beta^2 \in (0, 8\pi)$ and all $\lambda \in \mathbb{R}$. A comprehensive review of the vast literature on the model can be found in the paper [12] where in the reader will also find a description of the correspondence with certain fermionic Euclidean models.

We single out the pioneering work of Benfatto et al. [14] and Nicoló et al. [32] who establish existence of the model for a finite volume interaction and small coupling constants in the full subcritical range $\beta^2 < 8\pi$ via a probabilistic method initiated by the Roman school of Gallavotti and co-authors. A more modern account is the martingale method of [31] which covers the full subcritical regime in the case $d = 1$, in a bounded domain but without restrictions on the coupling constant $\lambda \in \mathbb{R}$.

The more classical stochastic quantisation using the Langevin dynamics for the sine-Gordon model has only been partially resolved after the preprint of the present article appeared. In contrast to the Φ^4_3 theory, there is no obvious coercive term to ensure global in time existence and even though the local solution theory was established already in [29, 19], the first global in time existence results for $4\pi + \varepsilon$ were obtained by Chandra, Feltes and Weber [20] as a corollary of global existence for the generalised parabolic Anderson model in a finite volume (see also the related paper [37]). Even more recently, Bringmann and Cao [17] managed to show global well-posedness for the Langevin dynamics on the torus for $\beta^2 < 6\pi$.

Due to the analytic treatability of the sine-Gordon interaction, there have been several accounts based on renormalisation group ideas and a direct analysis of the Polchinski flow equation (1.8). In this regard, we want to mention the analysis of Brydges and Kennedy [18], where they lay the foundations for this approach relying on a majorant method to establish convergence of the Mayer expansion up to $\beta^2 < \frac{2}{3}8\pi$. More recently, Bauerschmidt and Bodineau [10] showed convergence for the Mayer expansion up to 6π which allows them to establish a uniform log-Sobolev inequality for a lattice approximation of the model. In a related work, Bauerschmidt and Hofstetter [11] use the solution obtained from the Mayer expansion to construct a multiscale coupling between the Gaussian free field and the sine-Gordon model and analyse the maximum of the sine-Gordon measure. Similar ideas were applied by Barashkov, Gunaratnam and Hofstetter [9] to analyse the maximum of the $P(\varphi)_2$ models in a bounded domain. These last two papers are similar in spirit and complementary to ours, but rely on a direct analysis of the Polchinski equation (1.8) and focus on the extremal analysis in a finite volume instead of a general analysis and properties of the resulting EQFT.

Focusing now on the connection between the FBSDE and stochastic optimal control, a direct precursor of the results presented here is the work of Barashkov [6] (and the related PhD thesis [5]), where the model is studied in the first region $\beta^2 < 4\pi$ on the full space $\mathbb{R}^2$ using a variational approach. This approach is based on the stochastic control problem (1.6) and was first applied to the Φ^4_3 model in bounded volume in [4]. The more

recent extension in [8] to the infinite volume limit for the polynomial and exponential interaction in the 2 dimensional setting relies on a weak formulation of the FBSDE we use here. In the case of a Grassmannian field, the FBSDE approach has been successfully applied in [39] to cover the full subcritical regime. This also includes the complete inductive analysis of the corresponding approximate flow equation. After a first preprint of this article got published, Duch constructed a marginal asymptotically free Fermionic theory [22] by solving the continuous renormalisation group exactly.

Finally, we want to point out a general (tentative) axiomatic framework [3] proposed by Bailleul, Chevyrev and the first author. This framework provides a generalisation of the coupling with the free field given by (1.3) to the construction of random fields endowed with a Wilsonian scale-decomposition and a stochastic dynamics associated to a Gaussian field. These so called Wilson–Itô fields generate interesting questions ranging from the characterisation of measures of the form (1.1) via FBSDEs, to locality properties, the structure of the pre-factorisation algebras generated by the observables, or generalisations of the domain Markov properties, some of which we hope to address in a future study.

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1.2. Notation and assumptions

Let us fix some general notation we will use throughout.

- Let $\langle x\rangle := (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^2$. We will often rely on the following inequality to commute polynomial weights,

$$\langle x\rangle^k \langle y\rangle^{-k} \lesssim \langle x-y\rangle^k, \quad k \in \mathbb{N}. \tag{1.11}$$

- For $\gamma \in (-1,1)$, we define the exponential weights

$$w_\gamma(x) := \mathrm{e}^{\gamma m|x|}.$$

- For a non-negative function $w : \mathbb{R}^2 \to \mathbb{R}_+$, we define the standard weighted Lebesgue, Sobolev and Besov spaces $L^p(w)$, $W^{s,p}(w), H^s(w) = W^{s,2}(w)$ and $B^s_{p,q}(w)$, $p,q \in [1,\infty], s \in \mathbb{R}$ based on the measures $w(x)\mathrm{d}x$ on $\mathbb{R}^2$, e.g. $L^p(w)$ is equipped with the norm

$$\|f\|^p_{L^p(w)} = \|w\cdot f\|^p_{L^p} = \int_{\mathbb{R}^2} |fw|^p = \int_{\mathbb{R}^2} |w(x)f(x)|^p \mathrm{d}x.$$

In the case of $w(x) = \langle x\rangle^k$ for some $k \in \mathbb{R}$, we also write $L^{p,k} := L^p(\langle x\rangle^k)$ and analogously for the Besov and Sobolev spaces. Throughout this paper, we will fix a polynomial weight with n sufficiently large so that $x \mapsto \langle x\rangle^{-n} \in L^1(\mathbb{R}^2)$.

- We denote by $\Delta_i = \varphi_i(\mathrm{D})$ the Littlewood-Paley blocks on $\mathbb{R}^d$ and by $K_i = \mathcal{F}^{-1}(\varphi_i)$ their associated L^p-kernels. We recall that then, for any $i \geqslant -1$ and $p \in [1,\infty)$,

$$\|K_i\|_{L^1} \lesssim 1, \quad \|K_i\|_{L^p} \lesssim 2^{2i\frac{p-1}{p}}. \tag{1.12}$$

For any $\alpha \in \mathbb{R}$, $p,q \in [1,\infty]$ and n we define the usual Besov norms (with the usual modification for $q = \infty$)

$$\|u\|^q_{B^\alpha_{p,q}(\langle x\rangle^{-n})} := \sum_{i\geqslant -1} 2^{\alpha i q}\|\Delta_i u\|^q_{L^{p,-n}},$$

with the corresponding Besov spaces

$$B^\alpha_{p,q}(\langle x\rangle^{-n}) := \left\{u \in \mathcal{S}'(\mathbb{R}^d) : \ \|u\|_{B^\alpha_{p,q}(\langle x\rangle^{-n})} < \infty\right\}.$$

For a more detailed exposition, we refer to [1, Chapter 1].

- For a collection of points $x_I = (x_i)_{i\in I}$ we denote its *Steiner diameter*, that is the shortest tree connecting all points in x_I, by $\mathrm{St}(x_I)$. More precisely, we define

$$\mathrm{St}(x_I) := \min_{x_J \supset x_I} \min_{\tau(x_J)} L(\tau), \tag{1.13}$$

where the second minimum runs over all trees $\tau(x_J)$ connecting the points x_J and $L(\tau)$ measures the length of the tree τ on $\mathbb{R}^d$. We refer to [25] for further details.

- We denote by $B = (B_t)_{t\geqslant 0}$ a cylindrical Brownian motion on $L^2(\mathbb{R}^2)$ and by $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ the augmentation of the filtration generated by B. All considerations are with respect to this filtration and we will not explicitly mention it elsewhere (i.e. adapted always means adapted to the filtration $\mathbb{F}$). The conditional expectation with respect to $\mathcal{F}_t$ is denoted by $\mathbb{E}_t$. Given a measure ν, we write $\nu(f) := \int f \,\mathrm{d}\nu$ for the expectation under this measure and if ν is a probability measure, we write

$$\mathrm{Cov}_\nu(f,g) := \nu(fg) - \nu(f)\nu(g).$$

- For a Banach space $\mathcal{X}$, let $\mathbb{H}_a(\mathcal{X})$ be the space of predictable processes taking values in $\mathcal{X}$ (no integrability restrictions assumed). We also define the spaces, for any $p \in [1,\infty]$

$$\mathbb{H}^p_T(\mathcal{X}) := \left\{u \in \mathbb{H}_a(\mathcal{X}) | \mathbb{E}\int_0^T \|u_s\|^p_{\mathcal{X}} \mathrm{d}s < \infty\right\},$$
$$\mathbb{H}^\infty_T(L^\infty(\mathbb{R}^2)) := \{u \in \mathbb{H}_a(L^\infty(\mathbb{R}^2)) | \mathbb{1}_{\{t\leqslant T\}} u_t \in L^\infty(\mathrm{d}t \otimes \mathrm{d}\mathbb{P} \otimes \mathrm{d}x)\}.$$

If $T = \infty$, we may omit the subscript T in the spaces above.

- We write $\rho \prec 1$ if ρ is a smooth and compactly supported function $\mathbb{R}^2 \to [0,1]$ and analogously, we write $\rho \preceq 1$ if $\rho \prec 1$ or $\rho \equiv 1$. For a family of spatial cut-offs $(\rho_k)_k$ will write $\rho_k \to 1$ if $\rho \preceq 1$ and $\lim_{k\to\infty}(1-\rho_k(x)) = 0$.
- We reserve $\delta := 1 - \beta^2/8\pi > 0$ to denote the distance to criticality of the sine-Gordon model in our normalisation. The relevant thresholds for us, $\beta^2 < 4\pi$, $\beta^2 < 6\pi$ and $\beta^2 < 8\pi$, correspond to $\delta > \frac{1}{2}$, $\delta > \frac{1}{4}$ and $\delta > 0$ respectively.

To study the Laplace transform of ν_{SG}, we will have to consider localised perturbations $g+V$ of the potential V for functionals $g : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$. This localisation will be quantified in terms of the semi-norms

$$|g|_{1,p,k} := \sup_{\varphi \in L^{p,k}} \|\nabla g(\varphi)\|_{L^{p,k}},$$
$$|g|_{2,p,k} := \sup_{\phi_1,\phi_2 \in L^{p,k}} \frac{\|\nabla g(\phi_1) - \nabla g(\phi_2)\|_{L^{p,k}}}{\|\phi_1 - \phi_2\|_{L^{p,k}}},$$

where we drop the parameter k if $k = 0$. We always assume that $g, \nabla g$ is uniformly bounded, that is

$$\sup_{\varphi \in \mathcal{S}'(\mathbb{R}^2)} \|\nabla g(\varphi)\|_{L^\infty} + |g(\varphi)| \leqslant L < \infty,$$

and that $g \in C_b^2(L^{2,-n}) \cap C_b^2(H^{-\varepsilon,-n})$, the space of functions $L^{2,-n} \to \mathbb{R}$ with two continuous and bounded derivatives with a continuous extension in $C_b^2(H^{-\varepsilon,-n})$. The class of functions g satisfying the assumptions above is large enough to be rate function determining (for a proof see e.g. [6, Lemma 9]).

2. Stochastic control set-up for Gibbs measures

In this section, we set up the general variational framework required to study Gibbsian perturbations of the form (1.4) of a Gaussian measure μ^T from a stochastic control perspective. More precisely, for a functional $g : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ satisfying the assumptions laid out in Section 1.2 and suitable functions $U \in C_b^\infty(\mathbb{R})$ and a spatial cut-off $\rho \prec 1$, we consider a generic perturbed potential

$$V^g(\varphi) := (g+V)(\varphi) := \lambda g(\varphi) + \lambda \int_{\mathbb{R}^2} \rho(x) U(\varphi(x)) \mathrm{d}x, \tag{2.1}$$

and study the generic Gibbs measures,

$$\nu(\mathrm{d}\varphi) = \nu^V(\mathrm{d}\varphi) = \Xi_V^{-1} \exp(-V(\varphi)) \mu^T(\mathrm{d}\varphi). \tag{2.2}$$

We agree to drop the superscript g whenever $g = 0$. Note that the measures $\nu_{\mathrm{SG}}^{\rho,T}$ as defined in (1.4) are precisely of this form whenever $\rho \prec 1, T < \infty$.

Before we can begin the analysis of the control problem, we have to construct a suitable probability space. This requires a Brownian martingale W with the Gaussian free field as its terminal value.

2.1. Scale decomposition

Mainly for technical convenience and concreteness, we use a heat kernel decomposition to interpolate the covariance of the free field as

$$(m^2-\Delta)^{-1} = \int_0^\infty Q_t^2 \mathrm{d}t \quad \text{with} \quad Q_t := \left(\frac{1}{t^2}\mathrm{e}^{-(m^2-\Delta)/t}\right)^{1/2}.$$

For a cylindrical Brownian motion B on $L^2(\mathbb{R}^2)$, we then define the Brownian martingale $(W_t)_{t\geqslant 0}$ as the corresponding scale interpolation of the Gaussian free field, that is

$$W_t := \int_0^t Q_s \mathrm{d}B_s.$$

By construction, the measure $\mu^t := \mathrm{Law}(W_t)$ has covariance,

$$G_t(x,y) := G_t(x-y) := \int_0^t Q_s^2(x-y)\mathrm{d}s, \tag{2.3}$$

where we abuse the notation to use the same symbol for the operator and its associated kernel on $L^2(\mathbb{R}^2)$. A standard computation shows that the kernels are explicitly given by,

$$\dot{G}_s^{1/2}(x) = Q_s(x) = \frac{1}{2\pi}\mathrm{e}^{-m^2/2s}\mathrm{e}^{-\frac{s}{2}|x|^2}, \quad \dot{G}_s(x) = \frac{1}{4\pi s}\mathrm{e}^{-m^2/s}\mathrm{e}^{-\frac{s}{4}|x|^2}, \qquad x\in\mathbb{R}^2. \tag{2.4}$$

In addition to the smoothing property of the heat kernel, we rely on the following properties of the scale interpolation

- the kernel of $\dot{G}$ has a positive convolutional square root Q, that is $\dot{G}_t = Q_t * Q_t$.
- the kernels decay exponentially fast on $\mathbb{R}^2$, which is required to show the decay of correlations in Section 5,
- the kernels $\dot{G}_s$ are invariant under Euclidean transformations, which is used in our proof of the Euclidean invariance in Section 8.

Apart from the properties above, the precise choice of the scale interpolation is not important for us and we will only require elementary bounds on the kernels, all of which we collect in Appendix A.

A simple computation shows that the martingale W serves as a smooth approximation to the free field. Before we proceed, let us note this fact for future reference. We postpone the proof to Appendix A.2.

Lemma 2.1. *For any $\varepsilon > 0, p \in [1,\infty)$ and $n > 2$, the sequence $(W_t)_{t\geqslant 0}$ converges in $L^p(\mathrm{d}\mathbb{P}; B^{-\varepsilon,-n}_{p,p})$ and almost surely to a random variable $W_\infty \sim \mu$, where μ is the Gaussian free field, that is the centred Gaussian measure on $\mathcal{S}'(\mathbb{R}^2)$ with covariance $(m^2-\Delta)^{-1}$. Moreover, for any $T < \infty$, the stopped process $(W_{t\wedge T})_{t\geqslant 0}$ is a Gaussian process taking values in the function space $L^{\infty,-n}$.*

2.2. The control problem

With the scale interpolation $(W_t)_t$ of the free field, and thus the probability space, constructed, we can return to the measures (2.2). The goal of this section is to establish the connection between Gibbsian perturbations of a Gaussian and the stochastic control problem which is the basis for the FBSDE formulation.

Theorem 2.2.

a) For any $T \in [0,\infty)$ and $\varphi \in \mathcal{S}'(\mathbb{R}^2)$, the FBSDE

$$Z_t^g(\varphi) = \varphi - \int_0^t \dot{G}_s \mathbb{E}_s[\nabla V^g(Z_T^g(\varphi) + W_T)]\mathrm{d}s, \tag{2.5}$$

has a unique solution in $\mathbb{H}_T^\infty(L^\infty)$.

b) The process $X_t^g = Z_t^g + W_t$, where Z_t^g is the solution to (2.5), satisfies $\mathrm{Law}(X_T^0) = \nu$ and the pair

$$(\bar{u}_t^g, X_t^g)(\varphi) := (-Q_t \mathbb{E}_t[\nabla V^g(X_T^g(\varphi))], X_t^g(\varphi)), \tag{2.6}$$

is the unique optimiser for the stochastic control problem,

$$X_t(u;\varphi) = Z_t(u;\varphi) + W_t, \quad \textit{where} \quad Z_t(u;\varphi) := \varphi + \int_0^t Q_s u_s \mathrm{d}s \tag{2.7}$$

subject to the cost functional

$$\mathcal{V}^{V+g}(\varphi) := \inf_{u\in\mathbb{H}_a} J^{V^g}(u;\varphi) := \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[V^g(Z_T(u;\varphi) + W_T) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s \right]. \tag{2.8}$$

In particular, the Laplace transform of ν satisfies the variational problem,

$$\mathcal{W}^V(g;\varphi) := -\log \nu_\varphi(\mathrm{e}^{-g}) = \inf_{u\in\mathbb{H}_a} J^{V+g}(u;\varphi) - \inf_{u\in\mathbb{H}_a} J^V(u;\varphi). \tag{2.9}$$

Let us agree to drop the dependence on the initial value φ as long as no ambiguities arise. This dependence on the initial value φ will only become relevant in Section 8.3 and can safely be ignored for the rest of the paper. We will arrive at Theorem 2.2 in several steps. We start with the variational description for exponential functionals of Brownian motion by Boué and Dupuis (Lemma 2.3). We then show that any optimally controlled process has the correct law (Lemma 2.4. Finally, we obtain necessary conditions on the optimal control (Lemma 2.5) and use a verification theorem to show existence and uniqueness of an optimal control (Lemma 2.6) which will imply that the optimal dynamics is indeed given by (2.5).

We say a real valued random variable Y is *tame* (with respect to the probability measure $\mathbb{P}$) if there are Hölder conjugates $p, q > 1$ (that is $1/p + 1/q = 1$) such that

$$\mathbb{E}[\exp(-qY)] + \mathbb{E}|Y|^p < \infty.$$

The assumptions on g and the boundedness of V defined in (2.1) imply that this condition is always satisfied for $Y = V^g(W_t)$ and $t \in [0, \infty)$. Recall the the variational formula from [16] in the more general version of [38].

Theorem 2.3 (Boué–Dupuis)**.** *Let B be a cylindrical Brownian motion on a Hilbert space H and let $W = \int_0^\cdot Q_t \mathrm{d}B_t$ be a Brownian motion on $\tilde{H}$ with covariance $G_t = \int_0^t Q_s^2 \mathrm{d}s : H \to \tilde{H}$ and define for $u \in \mathbb{H}_a$,*

$$X_t(u) = Z_t(u) + W_t, \quad \textit{where} \quad Z_t(u) = \int_0^t \mathrm{d}s Q_s u_s. \tag{2.10}$$

For any Borel-measurable functional $F : \tilde{H} \to \mathbb{R}$ such that $F(W)$ is tame, it holds that

$$-\log \mathbb{E}[\mathrm{e}^{-F(W)}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[F(X(u)) + \frac{1}{2} \int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s \right] =: \inf_{u \in \mathbb{H}_a} J^F(u). \tag{2.11}$$

Our interest in this formula is justified by the following observation. If $g : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ satisfies the assumptions laid out in Section 1.2, then V^g is tame and the formula (2.11) provides a variational representation for the Laplace transform of (2.2) via

$$\mathcal{W}^V(g) = -\log \nu(\mathrm{e}^{-g}) = -\log\left(\frac{\mathbb{E}[\mathrm{e}^{-(g+V)(W_T)}]}{\mathbb{E}[\mathrm{e}^{-V(W_T)}]} \right) = \inf_{u \in \mathbb{H}_a} J^{V+g}(u) - \inf_{u \in \mathbb{H}_a} J^V(u). \tag{2.12}$$

If the infimum is a minimum, it turns out that the control problem actually provides a more direct description of the measure ν via the dynamics $X_t(u)$ given by (2.10). We recall Lemma 11 from [8], which is the key to establish this relationship.

Lemma 2.4. *Let $g : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ be bounded and continuous. If for some $\alpha \in \mathbb{R}$ the variational problem $\inf_{u \in \mathbb{H}_a} J_T^{\alpha g}(u)$ has a minimiser $\bar{u}^{\alpha g}$, then $\alpha \mapsto \mathcal{W}^V(\alpha g)$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{W}^V(\alpha g) = \mathbb{E}[g(X_T(\bar{u}^{\alpha g}))].$$

In particular, $\mathrm{Law}(X_T(\bar{u})) = \nu$.

Next, we show a necessary condition for the optimal control, which will also provide a candidate for the minimiser of (2.11) as feedback control.

Lemma 2.5. *If $\bar{u}^g \in \mathbb{H}_a$ is optimal for the control problem* (2.11)*, then $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-almost surely,*

$$\bar{u}_t^g = -Q_t \mathbb{E}_t[\nabla V^g(X_T(\bar{u}^g))]. \tag{2.13}$$

Proof. Standard stability results for SDEs imply that the solution $X(u)$ to (2.7) is differentiable in u. Similarly, the regularity assumed on V and g imply that also $J^{V+g}(u)$ is differentiable along all directions $\delta u \in \mathbb{H}_T^2(L^2)$. We compute

$$\nabla_\varepsilon X_t^{u,\delta u} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} X_t(u + \varepsilon \delta u) = \int_0^t Q_s \delta u_s \mathrm{d}s,$$

$$\nabla_\varepsilon J^{u,\delta u} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} J^{V+g}(u + \varepsilon \delta u) = \mathbb{E}\left[\nabla V^g(X_T(u)) \nabla_\varepsilon X_T^{u,\delta u} + \int_0^T u_s \delta u_s \mathrm{d}s \right].$$

Since the control u has to be adapted, we may insert a conditional expectation to find

$$\nabla_\varepsilon J^{u,\delta u} = \mathbb{E}\int_0^T (Q_s \mathbb{E}_s[\nabla V^g(X_T(u))] + u_s)\delta u_s \mathrm{d}s. \tag{2.14}$$

For an optimal control $u = \bar{u}^g$, it must hold for any direction $\delta u \in \mathbb{H}^2(L^2)$ and $\varepsilon > 0$,

$$J^g(\bar{u}^g + \varepsilon\delta u_s) - J^g(\bar{u}^g) \geqslant 0.$$

Moreover, since $\nabla Y_t^g(u) := \mathbb{E}_t[\nabla V^g(X_T(u))]$ does not depend on the direction δu, we arrive at the claimed first order condition for optimality

$$\bar{u}_t^g + Q_t \nabla Y_t^g(\bar{u}^g) = 0 \Leftrightarrow \bar{u}_t^g = -Q_t \nabla Y_t^g(\bar{u}^g).$$

□

Up until this point, we cannot guarantee existence of a minimiser. For the potentials V as defined in (2.1), we can close this gap with a verification theorem for feedback controls. Given a feedback control $u_t = \hat{u}_t(X_t(u))$, we say that the pair $(u, X(u))$ is *admissible* if $X(u)$ is a strong solution to the SDE (2.10) controlled by u, that is $X(u)$ is a strong solution to the SDE

$$X_t = \int_0^t Q_s \hat{u}_s(X_s)\mathrm{d}s + W_t, \quad t \in [0, T].$$

Lemma 2.6. *The feedback control* (2.13) *is optimal for the control problem* (2.8)*. Moreover, the Hamilton–Jacobi–Bellman equation*

$$\begin{cases} \partial_t v_t + \frac{1}{2}\operatorname{Tr}(\dot{G}_t \mathrm{D}^2 v_t) = \frac{1}{2}\mathrm{D}v_t \dot{G}_t \mathrm{D}v_t, \quad t \in [0, T], \\ v_T = V + g, \end{cases} \tag{2.15}$$

has a unique bounded solution v_t and $\bar{u}^g$ defined in (2.13) *satisfies*

$$\bar{u}_t^g = -Q_t \nabla v_t^g(X_t(\bar{u}^g)). \tag{2.16}$$

Proof. Let us first show the statement for a bounded function g and write $v = v^g$, $V = V^g$. The Hamilton–Jacobi–Bellman equation (HJB-equation; for short) associated to the control problem (2.8) is given by

$$\begin{cases} \partial_t v_t + \inf_{a \in L^2}\left\{\frac{1}{2}\operatorname{Tr}(\dot{G}_t \mathrm{D}^2 v_t) + \langle \mathrm{D}v_t, Q_t a\rangle_{L^2} + \frac{1}{2}\|a\|_{L^2}^2\right\} = 0, \\ v_T = V. \end{cases} \tag{2.17}$$

see e.g. [23, Section 2.5.1]. Solving the quadratic optimisation problem in (2.17) we find that the optimum is attained at $a = -Q_t \mathrm{D}v_t$ so that the PDE (2.17) reduces to (2.15). Define the function

$$v_t(\varphi) := -\log \mathbb{E}[\exp(-V(\varphi + W_T - W_t))]. \tag{2.18}$$

Since V is bounded and smooth by assumption, the representation in (2.18) implies that also v is smooth and bounded, say $v \in C_b^1([0,T], C_b^2(\mathbb{R}^2))$. We readily verify by a direct computation that v is a solution to (2.15).

Having found a solution to the HJB equation (2.17), we have access to the verification theorem (see e.g. [23, Theorem 2.36]): if the feedback control $\bar{u}$ as defined in (2.16) is admissible and satisfies for almost every $s \in [0,T]$ $\mathbb{P}$-almost surely,

$$\bar{u}_s \in \operatorname{argmin}_{a \in L^2} \left\{ \frac{1}{2} \operatorname{Tr}(\dot{G}_s \mathrm{D}^2 v_s(X_s(\bar{u}))) + \langle \mathrm{D} v_s(X_s(\bar{u})), Q_s a \rangle + \frac{1}{2} \|a\|_{L^2}^2 \right\}, \tag{2.19}$$

it follows that $\bar{u}$ is optimal for the control problem. By the same reasoning as before for the HJB-equation, the unique L^2-optimiser of (2.19) is given by $\bar{u}_s = -Q_s \nabla v_s(X_s(\bar{u}))$. Since V is bounded, we see from (2.18) that the solution $\exp(-v_t(\varphi))$ is bounded away from 0 and the gradient is given by

$$\nabla v_t(\varphi) = \frac{-\mathbb{E}[\nabla V(\varphi + W_T - W_t) \exp(-V(\varphi + W_T - W_t))]}{\exp(-v_t(\varphi))}. \tag{2.20}$$

Hence, the gradient ∇v_t inherits the Lipschitz continuity from V and ∇V. As a result, the standard fixed point argument for SDEs with bounded Lipschitz coefficients shows that the pair $(\bar{u}, X(\bar{u}))$ is admissible for the control problem. Finally, expanding the function $f_s := \nabla v_s$ along the flow of the optimally controlled process $X = X(\bar{u})$ using Ito's formula and the fact that v_s solves (2.15), yields

$$\begin{aligned} f_t(X_t) &= \mathbb{E}_t \left[\nabla V(X_T) - \int_t^T \left(\partial_s f_s + \frac{1}{2} \operatorname{Tr} \dot{G}_s \mathrm{D}^2 f_s - \frac{1}{2} \mathrm{D}(f_s \dot{G}_s f_s) \right)(X_s) \mathrm{d}s \right] \\ &= \mathbb{E}_t[\nabla V(X_T)]. \end{aligned} \tag{2.21}$$

which is the missing equality

$$\bar{u}_t = -Q_t \nabla v_t(X_t(\bar{u})) = -Q_t \mathbb{E}_t[\nabla V(X_T(\bar{u}))]. \tag{2.22}$$

□

Proof of Theorem 2.2. To see that (2.5) has a unique solution, note that by (2.22), the SDE (2.5) is equivalent to (2.10) with the feedback control $\bar{u}_t = -Q_t \nabla v_t(X_t(\bar{u}))$. By Lemma 2.6 this control is admissible, i.e. there is a unique strong solution. By (2.12), the variational problem for the Laplace transform is a direct consequence of Lemma 2.3. Lemma 2.5 and 2.6 imply combined that the pair defined in (2.6) is optimal for the control problem. Moreover, the condition (2.13) is necessary and since the solution to the SDE (2.5) is unique, the pair $(\bar{u}^g, X(\bar{u}^g))$ defined by (2.6) is the unique optimiser for (2.8). Finally, Lemma 2.4 shows that the solution X to (2.5) for $g = 0$ has the desired law,

$$\operatorname{Law}(X_T) = \nu.$$

□

Remark 2.7.

a) Compared to the more general setting considered in [8], the fact that the potential is Lipschitz and bounded allows us to directly use the solution to the HJB-equation (2.15) and enables the verification theorem. This means that we do not need to relax the variational problem to ensure existence of a minimiser. The difference is only a technical one and not crucial to our analysis: the subsequent analysis could be carried out verbatim for a relaxed version of the control problem, by possibly enlarging the underlying filtration.

b) We should emphasise the difference between the two formulas

$$\bar{u}_t^g = -Q_t \nabla v_t^g(X_t(\bar{u}^g)), \tag{2.23}$$

via the solution v^g to (2.15) and

$$\bar{u}_t^g = -Q_t \mathbb{E}_t[\nabla V^g(X_T(\bar{u}^g))], \tag{2.24}$$

via the stochastic maximum principle. The PDE (2.15) is not only non-linear but also infinite dimensional. The only reason we were able to easily show well-posedness here are the explicit formulas (2.18) and (2.20) for v and its gradient. Both rely on the boundedness and Lipschitz continuity of V^g and its gradient. In our main application of interest, where $V = V^{\rho,T}$, both of these properties disappear as the regularisations ρ and T are removed. As a result, this strategy does not readily transfer to the unregularised setting.

In contrast, the formula (2.24) yields the entirely self-contained forward-backward dynamics (2.5). This FBSDE is an appealing candidate for a stochastic quantisation equation for the measures $\nu^{\rho,T}$ that we can also transfer to the limit $\rho \to 1, T \to \infty$. Controlling (2.5) uniformly in both regularisations is the objective of the next section.

2.3. The effective FBSDE

Motivated by the issues highlighted in Remark 2.7-b, we move to a reformulation of the FBSDE (2.5), which is stable in the $\rho \to 1$, $T \to \infty$ limit and which can be studied without relying on a direct analysis of the PDE (2.15). This means we do not have access to the exact solution of (2.15). In place of the exact solution, we look for a scale dependent function $(F_t)_t$ such that the error, or remainder, R defined by

$$R_t := \mathbb{E}_t[\mathrm{D}V(X_T)] - F_t(X_t), \tag{2.25}$$

is small in a suitable sense. For $V = V^{\rho,T}$, we would like the bounds to also be uniform in $\rho \prec 1$ and $T < \infty$. While we should keep this goal in mind, the ideas are more general and we therefore first develop them for a generic function $\mathrm{D}V$. Similarly to the computation in (2.21), we develop the function F along the flow of the SDE (2.5) and

obtain a BSDE for the remainder R,

$$\begin{aligned} R_t &= \mathbb{E}_t[F_T(X_T) - F_t(X_t)] \\ &= \mathbb{E}_t \int_t^T H_s(X_s)\mathrm{d}s - \mathbb{E}_t \int_t^T \mathrm{D}F_s(X_s)\dot{G}_s R_s \mathrm{d}s + \mathbb{E}_t \int_t^T \mathrm{D}F_s(X_s)\mathrm{d}W_s, \end{aligned}$$

where

$$H_t(\varphi) = \left(\partial_t F_t + \frac{1}{2}\operatorname{Tr}(\dot{G}_t \mathrm{D}^2 F_t) - \frac{1}{2}\mathrm{D}(F_t \dot{G}_t F_t)\right)(\varphi). \tag{2.26}$$

Since the stochastic integral is a martingale, it vanishes under the conditional expectation. Allowing again a small perturbation g in the potential, the optimal dynamics in (2.5) can equivalently be described by the FBSDE

$$\begin{cases} X_t = \varphi + W_t - \int_0^t \dot{G}_s(F_s(X_s) + R_s)\mathrm{d}s, \\ R_t = \mathbb{E}_t\left[\nabla g(X_T) + \int_t^T H_s(X_s)\mathrm{d}s - \int_t^T \mathrm{D}F_s(X_s)\dot{G}_s R_s \mathrm{d}s\right]. \end{cases} \tag{2.27}$$

If there is no additional perturbation added to the potential V, choosing $F = \nabla v$ for v the exact solution to (2.15), we recover $R = 0$. Even in this unperturbed case, the remainder proves to be useful as it allows us the freedom to choose the function F, and let the remainder R compute the error resulting from this approximation. We, therefore, set out to find a systematic way to construct functions F for which the error term H is small in the next section.

Remark 2.8. Observe that we really treat the function g in (2.27) as a perturbation: we only develop the unperturbed gradient ∇V along the flow. The error due to g is collected entirely in the terminal condition for the remainder R. This means that we only have to analyse the flow equation for the unperturbed periodic potential ∇V. This distinction is of technical importance to us – in the subsequent analysis, we will use the periodicity of ∇V and V to our advantage and including a generic perturbation g would break this symmetry.

3. Analysis of the flow equation

In this section, we inductively derive the bounds on the coefficients of the FBSDE (2.27) using a truncated version of the renormalisation flow equation

$$\partial_t F_t + \frac{1}{2}\operatorname{Tr}(\dot{G}_t \mathrm{D}^2 F_t) - \frac{1}{2}\mathrm{D}(F_t \dot{G}_t F_t) = 0, \quad \text{subject to} \quad F_T = \mathrm{D}V^T. \tag{3.1}$$

3.1. Truncating the flow

Heuristically, we expect that successive Picard iterations of the flow equation (3.1) improve the the approximation. Accordingly, we define an iterative scheme starting from $F^{[0]} := 0$ and define $F^{[\ell]}$ for $\ell > 0$ as the solution to the equation

$$\partial_t F_t^{[\ell]} + \frac{1}{2}\operatorname{Tr}(\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}) = \sum_{\ell'+\ell''=\ell} \frac{1}{2}\mathrm{D}(F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']}), \tag{3.2}$$

subject to the terminal conditions

$$F_T^{[\ell]}(\varphi) = \begin{cases} \nabla V^T(\varphi), & \text{for } \ell = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

for a suitable potential V^T to be determined later. The initial condition $F^{[0]} \equiv 0$ ensures that (3.2) is triangular in ℓ and we can solve (3.2) as a linear PDE with a source term. Proceeding in this way, we define the ℓ^*-th order approximation $F_s^{[\leqslant \ell^*]} := \sum_{\ell \leqslant \ell^*} F_s^{[\ell]}$. With this choice for F in the FBSDE (2.27), the generator of the backward equation as defined in (2.26) reduces to

$$\begin{aligned} H_s^{[\leqslant \ell^*]} &:= \partial_s F_s^{[\leqslant \ell^*]} + \frac{1}{2}\operatorname{Tr}(\dot{G}_s \mathrm{D}^2 F_s^{[\leqslant \ell^*]}) - \frac{1}{2}\mathrm{D}(F_s^{[\leqslant \ell^*]} \dot{G}_s F_s^{[\leqslant \ell^*]}) \\ &= -\frac{1}{2} \sum_{\substack{\ell' + \ell'' > \ell^* \\ \ell', \ell'' \leqslant \ell^*}} \mathrm{D}(F_s^{[\ell']} \dot{G}_s F_s^{[\ell'']}). \end{aligned}$$

The estimates on the flow equation will rely on the following simple Lemma.

Lemma 3.1. *Let $\lambda_t = \lambda \mathrm{e}^{\frac{\beta^2}{2} G_t(0)}$, $\bar{\lambda}_t := |\lambda_t|$ and $\delta = 1 - \frac{\beta^2}{8\pi} > 0$. Then, for any $n \in \mathbb{N}$ and $\alpha > 1 - n\delta$,*

$$\int_t^\infty \bar{\lambda}_s^n \langle s \rangle^{-n} \langle s \rangle^{-\alpha} \mathrm{d}s \lesssim_n \bar{\lambda}_t^n \langle t \rangle^{-(n-1)-\alpha}. \tag{3.4}$$

In particular, for $n\delta > 1$ we can choose $\alpha = 0$ and

$$\int_t^\infty \bar{\lambda}_s^n \langle s \rangle^{-n} \mathrm{d}s \lesssim_n \bar{\lambda}_t^n \langle t \rangle^{-(n-1)}. \tag{3.5}$$

Proof. With the heat kernel estimate (A.1) from Lemma A.1, we see that $\bar{\lambda}_t \asymp C\bar{\lambda}(t \vee 1)^{1-\delta}$ for some $C > 0$. Now the claim follows from $\delta > 0 \Leftrightarrow \beta^2 < 8\pi$. □

Let us take a moment to heuristically explain how successive iterations of (3.2) should improve in ℓ. Starting from the first order approximation $\ell = 1$, the bilinear term does not give any contributions, and the linear equation (3.2) computes the usual Wick-ordering. In the specific case of the cosine interaction, this means more concretely that

$$F_t^{[1]}(\varphi) = -\lambda_t \beta \sin(\beta\varphi), \qquad \text{where} \quad \bar{\lambda}_t := \bar{\lambda} \mathrm{e}^{\frac{\beta^2}{2} G_t(0)} \lesssim \bar{\lambda} C \langle t \rangle^{\beta^2/8\pi} = \bar{\lambda} C \langle t \rangle^{1-\delta}. \tag{3.6}$$

Here, we absorbed the coupling constant $\lambda = \lambda_0$ into the renormalisation constant λ_t. The estimates on λ_t are a direct consequence of basic heat kernel estimates (Lemma A.1). We directly read off the bounds,

$$\|\mathrm{D}F_t^{[1]}(\varphi)\|_{L^\infty} + \|F_t^{[1]}(\varphi)\|_{L^\infty} \lesssim \bar{\lambda}_t \lesssim \bar{\lambda} \langle t \rangle^{1-\delta}. \tag{3.7}$$

Due to the form of the non-linearity of the flow equation (3.2) and Lemma 3.1, we can expect the bound

$$\|\mathrm{D}F_t^{[\ell]}(\varphi)\|_{L^\infty} + \|F_t^{[\ell]}(\varphi)\|_{L^\infty} \lesssim \bar{\lambda}_t^\ell \langle t\rangle^{-(\ell-1)} \lesssim \bar{\lambda}^\ell \langle t\rangle^{1-\ell\delta}, \tag{3.8}$$

to propagate inductively. Indeed, assuming that the bound (3.8) holds for all $\ell', \ell'' < \ell$, we obtain from Young's inequality and the estimate $\|\dot{G}_s\|_{L^1} \lesssim \langle s\rangle^{-2}$ that,

$$\|\mathrm{D}(F_t^{[\ell']}\dot{G}_t F_t^{[\ell'']})\|_{L^\infty} \leqslant \|\mathrm{D}F_t^{[\ell']}\|_{L^\infty}\|\dot{G}_t\|_{L^1}\|F_t^{[\ell'']}\|_{L^\infty} \lesssim \bar{\lambda}_t^{\ell'+\ell''}\langle t\rangle^{-(\ell'+\ell'')}. \tag{3.9}$$

Since $\dot{G}$ is positive, formally integrating out the linear part in (3.2) and passing to the mild formulation (see the next section for details), this suggests

$$\|F_t^{[\ell]}(\varphi)\|_{L^\infty} \lesssim \int_t^T \bar{\lambda}_s^\ell \langle s\rangle^{-(\ell-2)}\langle s\rangle^{-2}\mathrm{d}s \lesssim \int_t^T \bar{\lambda}_s^\ell \langle s\rangle^{-\ell}\mathrm{d}s.$$

Hence, Lemma 3.1 propagates the bound (3.8) only if $\ell\delta > 1$. Otherwise, we will have to improve our analysis and introduce additional regularisations to propagate the bounds from one level to the next. We therefore refer to the terms with $\ell > 1/\delta$ as *irrelevant* and $\ell \leqslant 1/\delta$ as *relevant*. To obtain uniform bounds on the remainder R in (2.27), the source term in (2.27) H should contain only irrelevant terms. The estimates (3.9) suggest that

$$\|H_t^{[\leqslant \ell^*]}(\varphi)\|_{L^\infty} \lesssim \sum_{\substack{\ell'+\ell''>\ell^* \\ \ell',\ell''\leqslant\ell^*}} \bar{\lambda}_t^{\ell'+\ell''}\langle t\rangle^{-(\ell'+\ell'')} \lesssim \bar{\lambda}_t^{\ell^*}\langle t\rangle^{-\ell^*}, \tag{3.10}$$

which is integrable in t from ∞ for $\ell^* > 1/\delta$ by Lemma 3.1. The number of relevant terms depends on the parameter β^2. If

$$\beta^2 < \beta_{\ell^*}^2 := \left(\frac{\ell^*}{\ell^*+1}\right)8\pi, \tag{3.11}$$

then only terms at the levels $\ell \leqslant \ell^*$ are relevant. At $\beta^2 = 8\pi$, the number of relevant terms is infinite and the model reaches criticality. In the subcritical regime, $\beta^2 < 8\pi$, we see that the number of relevant terms is finite, but grows arbitrary large as we approach the critical value $\beta^2 = 8\pi$.

Indeed, for the first region, $\beta^2 < \beta_1^2 = 4\pi$, only the first level $\ell = 1$ is relevant and we can gather all higher order terms in the remainder. Outside the first region, we have to deal with two related issues:

a) due to (3.10), the terms $\ell < \ell^*$ cannot be included in the equation for R, so that we have to iterate (3.2) at least up to ℓ^*;

b) the heuristic considerations suggest that the bound (3.8) cannot naively propagate through the flow equation on its own and these terms require renormalisation.

The goal of our subsequent analysis is to deal with both difficulties and recover estimates to replace (3.8) and (3.10) beyond this first threshold $\beta^2 < 4\pi$.

Since our analysis of the FBSDE is limited to the regime $\beta^2 < \beta_3^2 = 6\pi$, we develop the ideas for the flow equation only up to this threshold, where $\ell^* = 3$ is sufficient. We still emphasise that the inductive reasoning produces (possibly field dependent) bounds on the truncated flow in the entire subcritical regime $\beta^2 < 8\pi$.

3.2. The Fourier representation

To proceed with the iteration defined in (3.2) and finally obtain estimates on $F_s^{[\ell]}$, we restrict our attention to a suitable parametrised space of functions $\mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^d)$. Here, we use the periodicity of the potential to our advantage and pass to a Fourier representation following [18]. For a $\frac{2\pi}{\beta}$-periodic functional $V : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ we introduce the formal power series

$$V_t(\varphi) = \sum_{\ell=0}^{\infty} V_t^{[\ell]}(\varphi), \tag{3.12}$$

where with $\xi = (\sigma, x) \in \{-1, 1\} \times \mathbb{R}^2$ and $\xi_{1:\ell} = (\xi_1, \dots, \xi_\ell)$, we define

$$V_t^{[\ell]}(\varphi) := \sum_{\sigma_i \in \{-1,1\}^\ell} \int_{(\mathbb{R}^2)^\ell} \mathrm{d}x_{1:\ell} f_t^{[\ell]}(\xi_{1:\ell}) \mathrm{e}^{i\beta\sigma_1\varphi(x_1)} \dots \mathrm{e}^{i\beta\sigma_\ell\varphi(x_\ell)}. \tag{3.13}$$

Since the level ℓ is determined uniquely by the number of arguments $\xi_{1:\ell}$, we may drop the superscript ℓ in $f^{[\ell]}$ without introducing ambiguities. For brevity of the subsequent notation, we write $[\ell] := \{1, \dots, \ell\}$ and introduce the following shorthand for the integrals and the exponential fields,

$$\int \mathrm{d}\xi f(\xi) := \sum_{\sigma=\pm 1} \int_{\mathbb{R}^2} \mathrm{d}x f(\sigma, x), \quad \psi_x^\sigma := \mathrm{e}^{i\beta\sigma\varphi(x)}, \quad \psi(\xi_{1:\ell}) := \prod_{i=1}^{\ell} \psi_{x_i}^{\sigma_i},$$

and for a set $I \subset [\ell]$ we write $\xi_I = (\xi_i)_{i \in I}$. Finally, define the covariance matrix

$$\Gamma_{t,s}(\xi_{1:\ell}) := -\frac{\beta^2}{2} \sum_{i,j} \sigma_i \sigma_j (G_s - G_t)(x_i - x_j), \quad t \leqslant s. \tag{3.14}$$

With this notation and basic set-up, we can rewrite the flow equation (3.2) in terms of the coefficients f. Since any additive shift of the potential V_T by a constant does not affect the force, the terminal condition (3.3) translates to

$$f_T^{[\ell],T}(\xi_{1:\ell}) = \begin{cases} \frac{\lambda_T}{2}, & \ell = 1, \\ 0, & \ell > 1. \end{cases} \tag{3.15}$$

The functional $V^{[\ell]}$ satisfies the truncated flow equation (3.17) below at level ℓ if and only if, modulo positive combinatorial coefficients which we gather in $C_{|I_1|,|I_2|}$,

$$f_t^{[\ell],T}(\xi_{1:\ell}) = -\beta^2 \sum_{I_1 \dot\cup I_2 = [\ell]} C_{|I_1|,|I_2|} \int_t^T \mathrm{d}s \mathrm{e}^{\Gamma_{t,s}(\xi_{1:\ell})} f_s(\xi_{I_1}) \Big[\sum_{\substack{i \in I_1 \\ j \in I_2}} \sigma_i \sigma_j \dot G_s(x_i - x_j) \Big] f_s(\xi_{I_2}). \tag{3.16}$$

We include a proof of the relevant implication in Lemma 3.3 at the end of this section. Instead of controlling the functions F and V directly, we now want to inductively derive estimates on these kernels $f^{[\ell]}$. Of course, eventually we will be able to transfer these estimates back to F and V in a straightforward manner (see Section 3.4).

Before we proceed and derive bounds on the kernels f, some remarks about the setup seem appropriate.

Remark 3.2.

a) We are primarily interested in the flow equation for the force. However, for the variational description in Section 7, we will have to work at the level of the potential as well. Since the equations for the force F are readily obtained by differentiating the equations for V, we prefer to use it as a starting point. Up to an additive constant, both descriptions are equivalent on the finite volume and $F^{[\ell]} = \mathrm{D}V^{[\ell]}$ satisfies (3.2) if and only if $V^{[\ell]}$ satisfies the Picard scheme for (2.15), that is

$$\partial_s V_s^{[\ell]} + \frac{1}{2} \mathrm{Tr}(\dot G_s \mathrm{D}^2 V_s^{[\ell]}) = \frac{1}{2} \sum_{\ell' + \ell'' = \ell} (\mathrm{D}V_s^{[\ell']} \dot G_s \mathrm{D}V_s^{[\ell'']}). \tag{3.17}$$

We nonetheless emphasise that we never rely on the fact that F is the gradient of a potential in our analysis.

b) The coefficients $f^{[\ell]}$ are symmetric in their arguments $\xi_{1:\ell}$, i.e. for any permutation π of $[\ell]$,

$$f_t^{[\ell]}(\xi_1, \dots, \xi_\ell) = f_t^{[\ell]}(\xi_{\pi(1)}, \dots, \xi_{\pi(\ell)}). \tag{3.18}$$

c) If $f^{[1]}$ is translation (respectively rotation) invariant, we inductively see from (3.16) and the Euclidean invariance of the heat kernel $\dot G$ that also the kernels $f^{[\ell]}$ at the higher levels $\ell > 1$ are translation (respectively rotation) invariant. Correspondingly, if $f^{[1]}$ is invariant under complex conjugation (that is with $\bar\xi = (-\sigma, x)$ we have $f^{[1]}(\xi) = f^{[1]}(\bar\xi)$) then also $f^{[\ell]}(\xi_{1:\ell}) = f^{[\ell]}(\bar\xi_{1:\ell})$ is true for any $\ell > 1$.

d) We always consider truncations

$$V_t^{[\leqslant \ell^*]} = \sum_{\ell=0}^{\ell^*} V_t^{[\ell]}, \quad \text{and} \quad F_t^{[\leqslant \ell^*]} = \sum_{\ell=0}^{\ell^*} F_t^{[\ell]},$$

of (3.12) for some $\ell^* < \infty$. Therefore, we are not concerned with questions of convergence as $\ell^* \to \infty$. We will refer to the truncated series $F_s^{[\leqslant \ell^*]}$ as the ℓ^*-th order

approximation, even though we do not provide quantitative estimates on the convergence of the series $\sum_\ell F_s^{[\ell]}(\varphi)$. This can at least be motivated by the observation that (3.12) is a formal power series in the coupling constant λ, which formally solves the PDE (3.17). The fact that the representation is not unique (both with respect to the summands in (3.12) and the coefficients in (3.13)) does not cause any inconvenience for us.

e) The representation (3.12) is also known as Mayer expansion in the literature and its convergence was already studied in [18] and more recently in a series of papers [10, 11, 12, 30] for the sine-Gordon model. In contrast to our analysis, these results construct the exact solution to the flow equation (3.17) in different parameter regimes β, λ by showing that the formal series (3.12) converges.

Lemma 3.3. *If* $(f^{[\ell]})_{t\in[0,T]}$ *for* $\ell \leqslant \ell^*$ *satisfy* (3.16)*, then* $V_t^{[\ell]}$ *as defined in* (3.13) *satisfies* (3.17) *for* $\ell \leqslant \ell^*$.

Proof. Suppose that $f_t^{[\ell]}$ satisfies (3.16) and define $V_t^{[\ell]}$ according to (3.13). We compute,

$$\frac{1}{2}\operatorname{Tr}(\dot G_s \mathrm{D}^2 V_s^{[\ell]}(\varphi)) = -\frac{\beta^2}{2}\sum_{i,j}\int \mathrm{d}x_i \int \mathrm{d}x_j \sigma_i\sigma_j \dot G_s(x_i-x_j) f_t^{[\ell]}(\xi_{1:\ell})\psi(\xi_{1:\ell}),$$

and with $|I_1| = \ell'$, $|I_1| = \ell''$,

$$\begin{aligned}&\frac{1}{2}\mathrm{D}V_s^{[\ell']}(\varphi)\dot G_s \mathrm{D}V_s^{[\ell'']}(\varphi)\\ &\quad= \frac{1}{2}\sum_{i\in I_1}\int \mathrm{d}x_{I_1} f_t^{[\ell']}(\xi_{I'})\sum_{j\in I_2}\int \mathrm{d}x_{I_2}\sigma_i\sigma_j\dot G_s(x_i-x_j) f_t^{[\ell'']}(\xi_{I_2})\psi(\xi_{I_1})\psi(\xi_{I_2}).\end{aligned}$$

Differentiating the equation for $f^{[\ell]}$ (3.16) we see that

$$\begin{aligned}&\partial_t f_t^{[\ell]}(\xi_{1:\ell}) - \frac{\beta^2}{2}\sum_{i,j\in[\ell]}\sigma_i\sigma_j\dot G_s(x_i-x_j) f_t^{[\ell]}(\xi_{1:\ell})\\ &\quad= \sum_{I_1\dot\cup I_2=[\ell]} C(|I_1|,|I_2|) f_t^{[\ell']}(\xi_{I_1})\left[\sum_{i\in I_1}\sum_{j\in I_2}\sigma_i\sigma_j\dot G_s(x_i-x_j)\right] f_t^{[\ell'']}(\xi_{I_2}).\end{aligned}$$

Inserting this into

$$\partial_t V_t^{[\ell]}(\varphi) = \sum_{\sigma_i\in\{-1,1\}^\ell}\int_{(\mathbb{R}^2)^\ell}\mathrm{d}x_{1:\ell}\,\partial_t f_t^{[\ell]}(\xi_{1:\ell})\mathrm{e}^{i\beta\sigma_1\varphi(x_1)}\dots\mathrm{e}^{i\beta\sigma_\ell\varphi(x_\ell)},$$

yields the claim. □

3.3. Estimates on the Fourier coefficients

In this section, we derive our main estimates on the kernels f defined in (3.16) to control the flow under the conditional expectation in (2.5). For $\varsigma \in (0,1)$ and some kernel κ to be chosen later (see (3.25) below), we will be using the norms

$$|||f|||_t := \sup_{\xi_1} \int \mathrm{d}\xi_{2:\ell} |f(\xi_{1:\ell})\kappa_t(\xi_{1:\ell})\omega_\varsigma(x_{1:\ell})| \quad \text{where } \omega_\varsigma(x_{1:\ell}) := \mathrm{e}^{\varsigma m(\mathrm{St}(x_{1:\ell}))}, \quad (3.19)$$

for the Fourier kernels (see (1.13) for the definition of the Steiner diameter $\mathrm{St}(x_I)$). If $\kappa_t \equiv 1$ does not depend on t, we may drop the subscript t. Since the coefficients f are symmetric in their arguments (see also (3.18)), the point ξ_1 is not special in any way and the supremum could have been taken over any other ξ_k instead. The exponential tree weights ω_ς allow us to quantify the decay of the coefficients at large separation between the points $x_1, \dots, x_\ell$, which we require to show decay of correlations in Section 5. As

$$\omega_\varsigma(x_{I_1 \cup I_2}) \leqslant \omega_\varsigma(x_{I_1})\omega_\varsigma(x_{I_2})\mathrm{e}^{\varsigma m d(x_{I_1}, x_{I_2})}, \quad \text{where} \quad d(x_{I_1}, x_{I_2}) := \min_{x_i \in x_{I_i}} |x_1 - x_2|,$$

these norms work nicely with the flow equation for the coefficients (3.16) provided we choose $\varsigma \in (0,1)$. Indeed, since the convolution $\dot{G}_s$ in (3.16) always contracts along $(x_i - x_j)$ for $i \in I_1$ and $j \in I_2$, Young's convolution inequality implies that for $\kappa_t \equiv 1$,

$$\begin{aligned} &\sup_{\xi_1} \int \mathrm{d}\xi_{2:\ell}\, \omega_\varsigma(x_{I_1 \cup I_2}) \left| f_t^{[\ell']}(\xi_{I_1}) \left[\sum_{i \in I_1} \sum_{j \in I_2} \sigma_i \sigma_j \dot{G}_t(x_i - x_j) \right] f_t^{[\ell'']}(\xi_{I_2}) \right| \\ &\lesssim |||f_t^{[\ell]}||| \, |||f_t^{[\ell'']}||| \, \|\dot{G}_t\|_{L^1(w_\varsigma)}. \end{aligned} \quad (3.20)$$

We now want to make the heuristic bounds (3.8) precise and for an appropriate choice for κ_t, we will show that for any $\ell \leqslant \ell^*$,

$$|||f_t^{[\ell]}|||_t \lesssim \bar{\lambda}_t^\ell \langle t \rangle^{-(\ell-1)}. \quad (3.21)$$

To motivate our set-up going forward, consider again (3.16). Because $\dot{G}_s$ is a positive definite kernel, it follows immediately from the definition (3.14) of Γ,

$$\Gamma_{t,s}(\xi_{1:\ell}) \leqslant 0, \quad \text{for } t \leqslant s, \quad (3.22)$$

and consequently $\mathrm{e}^{\Gamma_{t,s}(\xi_{1:\ell})} \leqslant 1$. Applying this estimate in (3.16) for $\ell = 2$ yields, with the convolution inequality (3.20), the estimates on the first order term in (3.6) and the heat kernel estimates from Lemma A.4 using the assumption $\varsigma < 1$,

$$|||f_t^{[2]}||| \lesssim \int_t^T \mathrm{d}s \, |||f_s^{[1]}||| \, |||f_s^{[1]}||| \, \|\dot{G}_s\|_{L^1(\omega_\varsigma)} \lesssim \int_t^T \mathrm{d}s \bar{\lambda}_s^2 \langle s \rangle^{-2},$$

which is not integrable from ∞ unless $2\delta > 1$ ($\Leftrightarrow \beta^2 < 4\pi$). Therefore, we need additional help to propagate uniform bounds along the flow. This help will partially come

from the structure of the covariance matrix $\Gamma_{t,s}$, and partially from the choice of κ_t in the definition of the norm (3.19). To this end, define

$$q(\xi_{1:\ell}) := \sum_{k=1}^{\ell} \sigma_k,$$

the *charge of* $\xi_{1:\ell}$. We will call a contribution $\xi_{1:\ell}$ *neutral* if $q(\xi_{1:\ell}) = 0$ and *charged* otherwise. The relevance of the charge is best illustrated by the improved estimates on the covariance matrix $\Gamma_{t,s}$. If $\xi_{1:\ell}$ is charged, the exponential factor in (3.16) can help bring down the scale. As a pleasant side effect, these estimates will also imply that including an additional odd level, that is going from $\ell = 2k$ to $\ell + 1$, introduces no new difficulties to the analysis. To not interrupt the flow of ideas, we postpone the mostly technical proof to Appendix A.3.

Lemma 3.4. *Suppose that $\xi_{1:\ell}$ is charged. Then there is a constant $C > 0$ such that for all $s \geqslant t$,*

$$\Gamma_{t,s}(\xi_{1:\ell}) \leqslant \frac{\beta^2}{2}(G_t(0) - G_s(0)) + C, \tag{3.23}$$

and in particular

$$\mathrm{e}^{\Gamma_{t,s}(\xi_{1:\ell})} \lesssim \bar{\lambda}_t \bar{\lambda}_s^{-1}.$$

Remark 3.5. For neutral contributions, $q(\xi_{1:\ell}) = 0$, the point-wise bound $\mathrm{e}^{\Gamma_{t,s}(\xi_{1:\ell})} \leqslant 1$ is sharp: If $x_i = 0$ for all $i = 1, \dots, \ell$, then we have $\Gamma_{t,s}(\xi_{1:\ell}) = 0$. As a result, point-wise estimates on the linear propagator $\mathrm{e}^{\Gamma_{t,s}(\xi_{1:\ell})}$ cannot help to transport estimates for the kernels f along the flow of (3.16). Conversely, if $|q(\xi_{1:\ell})| > 1$, then it follows from the proof of Lemma 3.4 (see (A.10)) that we could iterate the same procedure until only the neutral part remains and extract more terms from the diagonal. In other words, the tighter bound

$$\Gamma_{t,s}(\xi_{1:\ell}) \leqslant \frac{\beta^2}{8\pi}|q(\xi_{1:\ell})|(\log(t \vee 1) - \log(s \vee 1)) + C,$$

is also true. For our purposes, the bound (3.23) will always be sufficient.

With Lemma 3.4, the integrability estimates from Lemma 3.1 for $\alpha = 1$ show that the charged contributions no longer pose a problem for us, allowing to set $\kappa_t \equiv 1$ in this case. However, for the neutral contributions, this norm is too strong and we will have to rely on the kernel κ_t.

Recall that $\dot{G}$ is exponentially concentrated on $|x| \lesssim t^{-1/2}$, so that (see Lemma A.4),

$$\int_{\mathbb{R}^2} \mathrm{d}x |x|^{2\alpha} \dot{G}_t(x) w_\varsigma(x) \lesssim \langle t \rangle^{-2-\alpha}. \tag{3.24}$$

Combined with Lemma 3.1, we expect that introducing an additional zero of order 2α in $x_i - x_j$ whenever $\sigma_i = -\sigma_j$ should help to propagate a bound on a regularised version of the kernel $f^{[\ell]}$. Of course, this regularisation comes at a price we have to pay later. For now, let us ignore this issue and discuss how we can define a regularised version

of the kernels which allow to propagate the bounds for the neutral contributions. With $\delta_{ij}x = x_1 - x_2$, $c \in \left(0, \frac{1}{4}\right)$ and $\alpha \in [0,1)$ to be chosen later, we introduce the (rotation and translation invariant) kernels

$$\kappa_t(\xi_{1:\ell}) := \begin{cases} t^\alpha |\delta_{12}x|^{2\alpha} \mathrm{e}^{ct|\delta_{12}x|^2}, & \ell = 2 \text{ and } q(\xi_1, \xi_2) = 0, \\ 1, & \text{otherwise.} \end{cases} \tag{3.25}$$

The increment $|\delta_{12}x|^{2\alpha}$ ensures the integrability from ∞ thanks to (3.24), the additional exponential weight in the kernel is included for technical reasons that will become clear later and the factor t^α is included for convenience. Given a charge $q \in \mathbb{Z}$, we will also use the notation

$$f_t^{[\ell](q)}(\xi_{1:\ell}) := \mathbb{1}_{\{q(\xi_{1:\ell})=q\}} f_t^{[\ell]}(\xi_{1:\ell}), \tag{3.26}$$

with analogous notations for the potential

$$V^{[\ell](q)}(\varphi) := \sum_{\sigma_i \in \{-1,1\}^\ell} \int_{(\mathbb{R}^2)^\ell} \mathrm{d}x_{1:\ell} f_t^{[\ell](q)}(\xi_{1:\ell}) \mathrm{e}^{i\beta\sigma_1\varphi(x_1)} \dots \mathrm{e}^{i\beta\sigma_\ell\varphi(x_\ell)}, \tag{3.27}$$

and the force $F^{[\ell],(q)} = \mathrm{D}V^{[\ell],(q)}$ to consider the bounds for charged and neutral contributions separately.

We can now proceed with the estimates on the regularised kernels for the 2-point contributions, whose analysis already contains all additional difficulties resulting from neutral contributions.

Lemma 3.6. *For any $\delta > 0$ and $\alpha > (1-2\delta) \vee 0$,*

$$|||f_t^{[2]}|||_t \lesssim \bar{\lambda}_t^2 \langle t \rangle^{-1}. \tag{3.28}$$

Moreover, the kernels $f_t^{[2]}$ inherit the concentration to $|x_1 - x_2| \leqslant \langle t \rangle^{-1/2}$ from $\dot{G}$. More precisely, letting

$$\tilde{f}_t(\xi_1, \xi_2) = f_t(\xi_1, \xi_2) |\delta_{12}x|^{2\kappa},$$

for some $\kappa \geqslant 0$, it holds that

$$|||\tilde{f}_t|||_t \lesssim \langle t \rangle^{-\kappa} \, |||\, f_t \,|||_t \,. \tag{3.29}$$

Proof. By definition (3.16),

$$f_t^{[2]}(\xi_1, \xi_2) = C \int_t^T \mathrm{d}s \mathrm{e}^{\Gamma_{t,s}(\xi_1,\xi_2)} f_s^{[1]}(\xi_1) \sigma_1\sigma_2 \dot{G}_s(x_1 - x_2) f_s^{[1]}(\xi_2),$$

where for $\ell = 1$,

$$|f_s^{[1]}(\xi)| \lesssim \bar{\lambda} \mathrm{e}^{\frac{1}{2}\beta^2 G_s(0)} = \bar{\lambda}_s = \bar{\lambda}_s^\ell \langle s \rangle^{-(\ell-1)}. \tag{3.30}$$

We only show the bound (3.29), as (3.28) follows directly by letting $\kappa = 0$. We deal with the two cases, charged and neutral, separately. If (ξ_1, ξ_2) is charged, we use Young's

inequality, Lemma 3.4 and the basic estimate (3.24) for $\alpha = 0$, to conclude for any $\delta > 0$,

$$\begin{aligned}|||\tilde{f}_t^{[2](\pm 2)}|||_t &\lesssim \sup_{\xi_1} \bar{\lambda}_t \int_t^T \mathrm{d}s \bar{\lambda}_s^{-1} \, |||\, f_s^{[1]}\, |||_t^2 \, |||x|^\kappa \dot{G}_s\|_{L^1(w_\varsigma)} \lesssim \bar{\lambda}_t \int_t^T \mathrm{d}s \bar{\lambda}_s \langle s\rangle^{-2-\kappa} \\ &\lesssim \bar{\lambda}_t^\ell \langle t\rangle^{-(\ell-1)-\kappa}.\end{aligned}$$

If (ξ_1, ξ_2) is neutral, we have to be more careful. By the definition (3.14) of $\Gamma_{t,s}$, we can absorb the renormalisation constants $\bar{\lambda}_s = \bar{\lambda} \mathrm{e}^{\frac{\beta^2}{2} G_s(0)}$ coming from $f_s^{[1]}$ through,

$$\Gamma_{t,s}(\xi_1, \xi_2) + \beta^2 G_s(0) = \beta^2 G_t(0) - \beta^2 G_t(x_1 - x_2) + \beta^2 G_s(x_1 - x_2).$$

Instead of the worst-case scaling $\beta^2 G_s(0)$, for which only point-wise estimates are possible, this means we only have to deal with $\beta^2 G_s(x_1 - x_2)$. Here, combining the averaging in space with the regularisation from the kernels κ defined in (3.25) allows us to estimate the integral uniformly. Indeed, using the above we obtain

$$\begin{aligned}&|||\, \tilde{f}_t^{[2](0)}|||_t \\ &= C \sup_{\xi_1} \left| \int \mathrm{d}\xi_2 \kappa_t(\xi_1, \xi_2) |x_1 - x_2|^\kappa \omega_\varsigma(x_1, x_2) \int_t^T \mathrm{d}s \mathrm{e}^{\Gamma_{t,s}(\xi_1,\xi_2)} f_s^{[1]}(\xi_1) f_s^{[1]}(\xi_2) \dot{G}_s(x_1 - x_2) \right| \\ &\lesssim \sup_{x_1} \left| \int_t^T \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}x_2 \omega_\varsigma(x_1, x_2) \dot{G}_s(x_1 - x_2) |x_1 - x_2|^{2\alpha+\kappa} t^\alpha \mathrm{e}^{ct|x_1 - x_2|^2} \mathrm{e}^{\Gamma_{t,s}(\xi_1,\xi_2) + \beta^2 G_s(0)} \right| \\ &\lesssim \mathrm{e}^{\beta^2 G_t(0)} t^\alpha \int_{\mathbb{R}^2} \mathrm{d}x |x|^{2\alpha+\kappa} \mathrm{e}^{ct|x|^2 + \varsigma m|x|} \int_t^T \mathrm{d}s \dot{G}_s(x) \mathrm{e}^{\beta^2 G_s(x) - \beta^2 G_t(x)} \\ &\lesssim \mathrm{e}^{\beta^2 G_t(0)} t^\alpha \int_{\mathbb{R}^2} \mathrm{d}x |x|^{2\alpha+\kappa} \mathrm{e}^{ct|x|^2 + \varsigma m|x|} (\mathrm{e}^{\beta^2 (G_\infty - G_t)(x)} - 1),\end{aligned}$$

where we used that $\dot{G}_s$ has a positive kernel in the last inequality to replace G_T by G_∞. Choosing $\alpha > (1 - 2\delta) \vee 0$ we have access to (A.6) to compute the integral over x above and obtain

$$|||\tilde{f}_t^{[2](0)}|||_t \lesssim \bar{\lambda}_t^2 t^\alpha \int_t^T \mathrm{d}s \langle s\rangle^{-2-\alpha-\kappa} \lesssim \bar{\lambda}_t^2 \langle t\rangle^{-1-\kappa} = \bar{\lambda}_t^\ell \langle t\rangle^{-(\ell-1)-\kappa}.$$

□

With the dipole under control, we can propagate bounds on the (charged) subsequent contributions essentially for free using Lemma 3.4. With some additional work, we will show improved estimates that do not rely on the kernel κ in the following lemma.

Lemma 3.7. *For $\alpha \leqslant 1/2$ and $\delta > 1/4$, it holds that*

$$|||f_t^{[3]}||| \lesssim \bar{\lambda}_t^3 \langle t\rangle^{-2}.$$

Proof. By the definition (3.16) of the coefficients the kernel $f^{[3]}$ is given by a linear combination of functions of the form

$$\bar{f}_t(\xi_1, \xi_2, \xi_3) = C \int_t^T \mathrm{d}s \mathrm{e}^{\Gamma_{t,s}(\xi_{1:3})} f_s^{[1]}(\xi_1) f_s^{[2]}(\xi_2, \xi_3) \sigma_1 [\sigma_2 \dot{G}_s(x_1 - x_2) + \sigma_3 \dot{G}_s(x_1 - x_3)],$$

obtained by considering all the permutations of the arguments $(\xi_i)_{i=1,2,3}$. If the 2-point contribution (ξ_2, ξ_3) is charged, applying (3.20) immediately implies the bound on $f^{[3]}$,

$$|||\bar{f}_t||| \lesssim \bar{\lambda}_t \int_t^T \mathrm{d}s \bar{\lambda}_s^{-1} \langle s\rangle^{-2} \,|||\, f_s^{[1]} \,|||\, |||f_s^{[2](\pm)}||| \lesssim \bar{\lambda}_t^3 \langle t\rangle^{-2} = \bar{\lambda}_t^{\ell} \langle t\rangle^{-(\ell-1)}.$$

Otherwise, if (ξ_2, ξ_3) is neutral, we only have uniform bounds on $\kappa_t f_t^{[2]}$ but not on $f_t^{[2]}$. Therefore, we insert $1 = \kappa_t \kappa_t^{-1}$ and absorb κ_t^{-1} with the convolution $\dot{G}_t$ to obtain the bounds on $\bar{f}_t$. Here, we compute with Lemma A.3, using the assumption $\alpha = \frac{1}{2}$, and

$$\omega_\varsigma(x_{1:3}) \leqslant \omega_\varsigma(x_2, x_3)\omega_\varsigma(x_1, x_2).$$

Thus, writing $\delta_{ij}x := x_i - x_j$,

$$\begin{aligned}
&\sup_{\xi_1} \int \mathrm{d}\xi_{2:3} \omega_\varsigma(x_{1:3}) \frac{|\dot{G}_s(x_1 - x_2) - \dot{G}_s(x_1 - x_3)|}{s^\alpha |\delta_{23}x|^{2\alpha}} \mathrm{e}^{-ct|\delta_{23}x|^2} f_s^{[1]}(\xi_1)(\kappa_s f_s^{[2]})(\xi_{2,}\xi_3) \\
\lesssim &\sup_{\xi_1} \int \mathrm{d}\xi_{2:3} (|\delta_{12}x| + |\delta_{23}x|) s^{-1/2} \mathrm{e}^{-ct|\delta_{12}x|^2 + \varsigma m|\delta_{12}x| - m^2/s} f_s^{[1]}(\xi_1)(\omega_\varsigma \kappa_s f_s^{[2]})(\xi_{2,}\xi_3) \\
\lesssim &\,|||\, f_s^{[1]} \,||| \int \mathrm{d}\xi_{2:3} \mathrm{e}^{-cs|\delta_{12}x|^2 + \varsigma m|\delta_{12}x| - m^2/s} [|\delta_{12}x| + |\delta_{23}x|] s^{-1/2} (\omega_\varsigma \kappa_s f_s^{[2]})(\xi_2, \xi_3).
\end{aligned} \tag{3.31}$$

By Young's convolution inequality, the last integral can be estimated as

$$\begin{aligned}
&\sup_{\xi_1} \left| \int \mathrm{d}(\xi_2, \xi_3)(\omega_\varsigma \kappa_s f_s^{[2]})(\xi_2, \xi_3) \mathrm{e}^{-cs|\delta_{12}x|^2 + \varsigma m|\delta_{12}x| - m^2/s} [|\delta_{12}x| + |\delta_{23}x|] \right| \\
\lesssim &\sup_{x_1 \in \mathbb{R}^2} \int \mathrm{d}x_2 \mathrm{e}^{-cs|\delta_{12}x|^2 + \varsigma m|\delta_{12}x| - m^2/s} |\delta_{12}x| \,|||\, f_s^{[2]} |||_s \\
&+ \sup_{x_1 \in \mathbb{R}^2} \int \mathrm{d}x_2 \mathrm{e}^{-cs|\delta_{12}x|^2 + \varsigma m|\delta_{12}x| - m^2/s} \,|||\, f_s^{[2]} |\delta_{23}x| \,|||_s \,.
\end{aligned}$$

Using the scaling properties of $\kappa_s f_s^{[2]}$ from Lemma 3.6 and evaluating the Gaussian integral with Lemma A.4 using $\varsigma < 1$ and choosing $c \in \left(0, \frac{1}{4}\right)$ in (3.25) sufficiently close to $1/4$, we arrive at the required claim

$$|||\bar{f}_t||| \lesssim \bar{\lambda}_t \int_t^T \mathrm{d}s \bar{\lambda}_s^2 \langle s\rangle^{-3} \lesssim \bar{\lambda}_t^3 \langle t\rangle^{-2}.$$

□

Remark 3.8. The proof above more generally shows that with $\alpha \leqslant 1/2$, $\delta > 1/4$ and $|I_1| + |I_2| = \ell > 2$, the following bounds hold,

$$\sup_{\xi_1} \int \mathrm{d}\xi_{2:\ell} \omega_\varsigma(x_{1:\ell}) |f_t(\xi_{I_1})| \left| \sum_{i \in I_1} \sum_{j \in I_2} \sigma_i \sigma_j \dot{G}_t(x_i - x_j) \right| |f_t(\xi_{I_2})| \lesssim (\bar{\lambda}_t \langle t\rangle^{-1})^\ell. \tag{3.32}$$

We will rely on this later to obtain estimates on H in section 3.5.

3.4. The renormalised problem

Given a spatial cut-off $\rho \prec 1$ and a UV cut-off $T < \infty$, let

$$V_t^{[\leqslant \ell^*],\rho,T}(\varphi) = \sum_{\ell \leqslant \ell^*} [V_t^{[\ell],\rho,T}(\varphi) - c^{[\ell],\rho,T}], \tag{3.33}$$

for some suitable renormalisation constants $c^{[\ell],\rho,T}$ to be chosen later. Here, we denote by $V^{[\ell],\rho,T}$ the ℓ-th order contribution as defined via its Fourier expansion as in (3.13) subject to the condition

$$V_T^{[1],\rho,T}(\varphi) = \int_{\mathbb{R}^2} \mathrm{d}x \rho(x) \lambda_T \cos(\beta\varphi).$$

We use the analogous definition and notation for the force $F_t^{[\leqslant \ell^*],\rho,T} = \mathrm{D} V_t^{[\leqslant \ell^*],\rho,T}$ and the remainder $H^{[\leqslant \ell^*],\rho,T}$ defined in (2.26). For $\beta^2 < 6\pi$ and $\ell^* = 3$, we transfer the bounds we obtained for the Fourier coefficients $f^{[\ell]}$ in the previous section to the truncated potential $V^{[\leqslant \ell^*]}$ and the truncated force $F^{[\leqslant \ell^*]}$. In this step, we have to pay the price for the regularisation with the kernels κ defined in (3.25). For $\beta^2 < 6\pi$, these kernels only appear at level $\ell = 2$, and by definition,

$$V_t^{[2],\rho,T}(\varphi) = \int \mathrm{d}\xi_{1:2} f_t^{[2],\rho,T}(\xi_1, \xi_2) \psi(\xi_1) \psi(\xi_2).$$

If (ξ_1, ξ_2) is charged, then $\kappa_t(\xi_1, \xi_2) = 1$ and it follows from $|\psi(\xi)| = 1$ combined with (3.28) from Lemma 3.6,

$$|V_t^{[\ell],\rho,T}(\varphi)| \lesssim |||f_t^{[2]}|||_t \lesssim \bar{\lambda}_t^2 \langle t \rangle^{-1}.$$

If (ξ_1, ξ_2) is neutral, we have by a Taylor expansion

$$\psi(\xi_1)\psi(\xi_2) = 1 + \psi(\xi_1)(x_2 - x_1) \int_0^1 \mathrm{d}\vartheta \nabla_x \psi(x_1 + \vartheta(x_2 - x_1)). \tag{3.34}$$

Therefore, choosing

$$c^{[2],\rho,T} := \int \mathrm{d}\xi_{1:2} \int_0^T \mathrm{d}s \partial_s f_s^{[2](0),\rho,T}(\xi_1, \xi_2), \tag{3.35}$$

it follows that

$$\begin{aligned} V_T^{[2](0),\rho,T}(\varphi) = \int_{(\mathbb{R}^2)^2} \mathrm{d}x_{1:2} (\kappa_T f_T^{[2](0),\rho,T})(\xi_1, \xi_2) \kappa_T(\xi_1, \xi_2)^{-1} (x_2 - x_1) \psi(\xi_1) \times \\ \times \int_0^1 \mathrm{d}\vartheta \nabla_x \psi(x_1 + \vartheta(x_2 - x_1)) + c^{[2],\rho,T}, \end{aligned}$$

and thus,

$$
\begin{aligned}
|V_T^{[2](0),\rho,T}(\varphi) - c^{[2],\rho.T}| &\lesssim \sup_{\xi_1,\xi_2} |\kappa_T^{-1}(\xi_1,\xi_2)(x_1-x_2)| \,|||\, f_T^{[2],\rho,T} \,|||_T \, \|\nabla\varphi\|_{L^\infty} \\
&\lesssim_\rho \bar{\lambda}_T^2 \langle T\rangle^{-1} \left(\langle T\rangle^{-1/2} \|\nabla\varphi\|_{L^\infty}\right).
\end{aligned}
\tag{3.36}
$$

We summarise the bounds in the following Lemma for later reference.

Lemma 3.9. *For $\ell^* = 3$, there is a choice for $c^{[\ell],\rho,T}$ such that for any $\rho \preceq 1$, and $T < \infty$,*

$$
\begin{aligned}
|V_T^{\rho,T}(\varphi)| &\lesssim \|\rho\|_{L^1} \sum_{\substack{\ell \leqslant \ell^* \\ n \leqslant \lfloor \ell/2 \rfloor}} \bar{\lambda}_t^\ell \langle T\rangle^{-(\ell-1)} \left(1 + \langle T\rangle^{-1/2} \|\nabla\varphi\|_{L^\infty}\right)^n, \\
\|F_t^{\rho,T}(\varphi)\|_{L^\infty} &\lesssim \sum_{\substack{\ell \leqslant \ell^* \\ n \leqslant \lfloor \ell/2 \rfloor}} \bar{\lambda}_t^\ell \langle t\rangle^{-(\ell-1)} \left(1 + \langle t\rangle^{-1/2} \|\nabla\varphi\|_{L^\infty}\right)^n.
\end{aligned}
\tag{3.37}
$$

Proof. The estimate on V follows directly by inserting the bound (3.36) and the estimates (3.21) on the coefficients in (3.33). In the same way, we can start from

$$
\|F_t^{[\ell]}(\varphi)\| \lesssim \sup_x \Big| \int \mathrm{d}\xi_{1:\ell} f_t^{[\ell]}(\xi_{1:\ell}) \Big(\sum_{j=1}^{\ell} \sigma_j \delta_x(x_j) \Big) \psi(\xi_{1:\ell}) \Big|,
$$

which with (3.21) yields the desired estimate for all terms except for $\ell = 2, q = 0$. For $F^{[2](0)}$, we follow the same steps as for $V^{[2](0)}$, with the important distinction that the constant term absorbed by (3.35) vanishes since $f(\xi_1,\xi_2) = f(\xi_2,\xi_1)$ which implies

$$
\int \mathrm{d}\xi_{1:\ell} \sigma_1 f_t^{[2](0)}(\xi_{1:2}) \Big(\sum_{j=1}^{\ell} \sigma_j \delta_x(x_j) \Big) = 0.
$$

□

Remark 3.10. If the order of the approximation ℓ^* and the smoothing $\alpha \in (0,1)$ is chosen appropriately ($\alpha > 1 - 2\delta$, according to (3.11) and Lemma 3.1), then by modifying (3.34), one can show that these estimates generalise in the full subcritical regime $\beta^2 < 8\pi$ and $0 < t < T \leqslant \infty$ to bounds of the form

$$
\begin{aligned}
|V_T^{\rho,T}(\varphi)| &\lesssim_\rho \sum_{\substack{\ell \leqslant \ell^* \\ n \leqslant \lfloor \ell/2 \rfloor}} \bar{\lambda}_t^\ell \langle T\rangle^{-(\ell-1)} \left(1 + \langle T\rangle^{-\alpha} \|\varphi\|_{B^{2\alpha}_{\infty,\infty}}\right)^n, \qquad \rho \prec 1, \\
\|F_t^{\rho,T}(\varphi)\|_{L^\infty} &\lesssim \sum_{\substack{\ell \leqslant \ell^* \\ n \leqslant \lfloor \ell/2 \rfloor}} \bar{\lambda}_t^\ell \langle t\rangle^{-(\ell-1)} \left(1 + \langle t\rangle^{-\alpha} \|\varphi\|_{B^{2\alpha}_{\infty,\infty}}\right)^n, \qquad \rho \leqslant 1.
\end{aligned}
\tag{3.38}
$$

However, the field dependency in the estimates (3.37) and (3.38) means that we are currently not able to control the FBSDE uniformly in the UV-cut-off which restricts our

analysis to the regime $\beta^2 < 6\pi$. What saves our analysis in this case is the observation that for the FBSDEs (2.5) and (2.27) the force F_s only appears in combination with the heat kernel. Indeed, it turns out that the smoothing properties of the heat kernel are enough to recover field independent bounds for $Q_s F_s$ and $\mathrm{D}F_s Q_s$.

Remark 3.11. The truncated solutions still satisfy for any function φ and any $T < \infty$,

$$V_T^{\rho,T}(\varphi) = \int_{\mathbb{R}^2} (\lambda_T \cos(\beta\varphi) - c^{\rho,T})\rho(x)\mathrm{d}x, \quad F_t^{\rho,T}(\varphi)(x) = -\beta\lambda_T \rho(x)\sin(\beta\varphi(x)). \tag{3.39}$$

Here, the renormalisation constant $\lambda_t = \lambda \mathrm{e}^{\frac{\beta^2}{2}G_t(0)}$ is the usual Wick-ordering and $c^{\rho,T} := \sum_\ell c^{[\ell],\rho,T}$ is the additive renormalisation resulting from higher order corrections.

3.5. Estimates on the force

From now on we will always assume that $\beta^2 < 6\pi$ and that in the definition (3.25) of the kernel κ_t we fix $c \in (0, 1/4)$ sufficiently close to $1/4$ and $\alpha = 1/2$. Since we only deal with the case $\ell^* = 3$, let us also agree to suppress the dependence on ℓ^* for V, F and H writing e.g. $F := F^{[\leqslant \ell^*]} = F^{[\leqslant 3]}$. Our goal in this section is to recover field independent bounds on all coefficients of (4.1), that is on $Q_s F_s$, $\mathrm{D}F_s Q_s$ and H_s.

Lemma 3.12. *For any $\varphi \in \mathcal{S}'(\mathbb{R}^2)$,*

$$\|Q_t F_t^{[2](0)}(\varphi)\|_{L^\infty} \lesssim (\bar{\lambda}_t \langle t\rangle^{-1})^2.$$

Proof. We follow exactly the same strategy as in the proof of Lemma 3.7, where we now require bounds on

$$\sup_{x\in\mathbb{R}^2} \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 |\kappa_t f_t^{[2](0)}(\xi_1,\xi_2)| \frac{|Q_t(x_1 - x) - Q_t(x_2 - x)|}{|x_1 - x_2|} \mathrm{e}^{-ct|x_1 - x_2|^2}. \tag{3.40}$$

Thanks to the translation invariance, we can apply Lemma A.3 for Q_t and absorb the increment $|x_1 - x_2|^{-1}$,

$$\frac{|Q_t(x_1 - x) - Q_t(x_2 - x)|}{|x_1 - x_2|} \mathrm{e}^{-ct|x_1 - x_2|^2} \lesssim t(|x_1| + |x_1 - x_2|)\mathrm{e}^{-\frac{c}{2}t|x_1|^2} \mathrm{e}^{-m^2/2s}. \tag{3.41}$$

Using this in (3.40) we get from Young's convolution inequality and the scaling properties of the kernels $f^{[2]}$ (see Lemma 3.6),

$$\begin{aligned}
\|Q_t F_t^{[2](0)}(\varphi)\|_{L^\infty} \lesssim & t \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 (|x_1| + |x_1 - x_2|) t^{-1/2} \mathrm{e}^{-\frac{c}{2}t|x_1|^2 - m^2/2t} |\kappa_t f_t^{[2](0)}(\xi_1,\xi_2)| \\
\lesssim & t^{1/2} \int \mathrm{d}x_1 \mathrm{e}^{-\frac{c}{2}t|x_1|^2 - m^2/2t} \sup_{\xi_1} \int \mathrm{d}\xi_2 |x_1 - x_2| |\kappa_t f_t^{[2](0)}(\xi_1,\xi_2)| \\
& + t^{1/2} \int \mathrm{d}x_1 |x_1| \mathrm{e}^{-\frac{c}{2}t|x_1|^2 - m^2/2t} \sup_{\xi_1} \int \mathrm{d}\xi_2 |\kappa_t f_t^{[2](0)}(\xi_1,\xi_2)| \\
\lesssim & \bar{\lambda}_t^2 \langle t\rangle^{-2}.
\end{aligned}$$

□

For the remaining levels, the estimates on the coefficients transfer directly to the force. To remove the cut-offs later, we will also have to control the dependence of the approximate solution F on these parameters. Therefore, let us again keep track of this dependence by writing F^T for the solution to the flow equation on $[0,T]$ with terminal conditions (3.3) at T and in the same way F^ρ for $\rho \preceq 1$. In the estimates it is assumed that the suppressed parameters coincide.

Proposition 3.13. *$E = L^\infty$ or $E = L^{2,k}$ for any $k \in \mathbb{Z}$. For any $\varphi \in \mathcal{S}'(\mathbb{R}^2)$, $R \in L^\infty(\mathbb{R}^2)$, $\rho, \rho_1, \rho_2 \leqslant 1$ and $T, T_1, T_2 < \infty$, the following estimates apply.*

a) ***(Uniform boundedness)*** *For any $p \in [1,\infty]$,*

$$\begin{aligned} \|Q_t F_t(\varphi)\|_{L^\infty} &\lesssim \bar\lambda_t \langle t\rangle^{-1}, \\ \|\mathrm{D}F_t(\varphi) Q_t R\|_{L^\infty} &\lesssim \bar\lambda_t \langle t\rangle^{-1} \|R\|_{L^\infty}, \\ \|H_t(\varphi)\|_{L^\infty} &\lesssim (\bar\lambda_t \langle t\rangle^{-1})^4. \end{aligned} \tag{3.42}$$

b) ***(Uniform Lipschitz condition)***

$$\begin{aligned} \|Q_t F_t(\varphi) - Q_t F_t(\tilde\varphi)\|_E &\lesssim \bar\lambda_t \langle t\rangle^{-1} (\|\varphi - \tilde\varphi\|_E), \\ \|(\mathrm{D}F_t(\varphi) Q_t - \mathrm{D}F_t(\tilde\varphi) Q_t) R\|_E &\lesssim \bar\lambda_t \langle t\rangle^{-1} (\|\varphi - \tilde\varphi\|_E \|R\|_{L^\infty}), \\ \|H_t(\varphi) - H_t(\tilde\varphi)\|_E &\lesssim (\bar\lambda_t \langle t\rangle^{-1})^4 \|\varphi - \tilde\varphi\|_E \end{aligned} \tag{3.43}$$

c) ***(Dependence on T)*** *There is an $\varepsilon > 0$ depending only on β^2 such that,*

$$\begin{aligned} \sup_{t\geq 0} \bar\lambda_t^{-1} \langle t\rangle^{1-2\varepsilon} \|Q_t (F_t^{T_1} - F_t^{T_2})(\varphi)\|_{L^\infty} &\lesssim \bar\lambda \langle T_1 \wedge T_2\rangle^{-\varepsilon}, \\ \sup_{t\geq 0} \bar\lambda_t^{-1} \langle t\rangle^{1-2\varepsilon} \|(\mathrm{D}F_t^{T_1} - \mathrm{D}F_t^{T_2})(\varphi) Q_t R\|_{L^\infty} &\lesssim \bar\lambda \langle T_1 \wedge T_2\rangle^{-\varepsilon} \|R\|_{L^\infty}, \\ \sup_{t\geq 0} \bar\lambda_t^{-1} \langle t\rangle^{1-2\varepsilon} \|H_t^{T_1} - H_t^{T_2}(\varphi)\|_{L^\infty} &\lesssim \bar\lambda^4 \langle T_1 \wedge T_2\rangle^{-\varepsilon}. \end{aligned} \tag{3.44}$$

d) ***(Dependence on ρ)*** *For any $n > 2$, it holds that*

$$\begin{aligned} \|Q_t (F_t^{\rho_1} - F_t^{\rho_2})(\varphi)\|_{L^{2,-n}} &\lesssim \bar\lambda_t \langle t\rangle^{-1} \|\rho_1 - \rho_2\|_{L^{2,-n}}, \\ \|(\mathrm{D}F_t^{\rho_1} - \mathrm{D}F_t^{\rho_2})(\varphi) Q_t R\|_{L^{2,-n}} &\lesssim \bar\lambda_t \langle t\rangle^{-1} \|\rho_1 - \rho_2\|_{L^{2,-n}} \|R\|_{L^\infty}, \\ \|(H_t^{\rho_1} - H_t^{\rho_2})(\varphi)\|_{L^{2,-n}} &\lesssim (\bar\lambda_t \langle t\rangle^{-1})^4 \|\rho_1 - \rho_2\|_{L^{2,-n}}. \end{aligned} \tag{3.45}$$

In particular, for $E = L^\infty$ or $E = L^{2,-n}$ we obtain estimates that are uniform in the volume cut-off ρ.

Proof. a) For all contributions other than for $F^{[2](0)}$ this follows directly from the bounds on the kernels f and $|\psi(\xi_{1:\ell})| \leqslant 1$. In fact, in these cases, we obtain the better bound,

$$\|F_t^{[\ell]}(\varphi)\|_{L^\infty} \lesssim \bar\lambda_t^\ell \langle t\rangle^{-(\ell-1)},$$

so that

$$\|Q_t F_t^{[\ell]}(\varphi)R\|_{L^\infty} \leqslant \|Q_t\|_{L^1}\|F_t^{[\ell]}(\varphi)\|_{L^\infty}\|R\|_{L^\infty} \lesssim \bar{\lambda}_t^\ell \langle t\rangle^{-(\ell-1)}\|R\|_{L^\infty} \lesssim \bar{\lambda}_t.$$

For $F_t^{[2](0)}$, this bound was shown in Lemma 3.12. For the derivative, note that for any test function $R \in L^\infty(\mathbb{R}^2)$

$$\begin{aligned}\|\mathrm{D}F_t^{[\ell]}(\varphi)Q_t R\|_{L^\infty} &\lesssim \sup_{\xi_1}\left|\int \mathrm{d}\xi_{2:\ell}\int \mathrm{d}y\, f_t^{[\ell]}(\xi_{1:\ell})\sum_{k\leqslant\ell} i\beta\sigma_k Q_t(x_k-y)R(y)\psi(\xi_{1:\ell})\right| \\ &\lesssim \|R\|_{L^\infty}\sup_{\xi_1}\int \mathrm{d}\xi_{2:\ell}\int \mathrm{d}y\left|f_t^{[\ell]}(\xi_{1:\ell})\sum_{k\leqslant\ell} i\beta\sigma_k Q_t(x_k-y)\right|,\end{aligned}$$

and now the same reasoning as for F_t applies for the integral on the right-hand side.

Finally, the estimates on H follow from the estimates above and

$$\begin{aligned}\|H_t(\varphi)\|_{L^\infty} &\leqslant \frac{1}{2}\sum_{\ell'+\ell''>3}\|\mathrm{D}(F_t^{[\ell]}\dot{G}_t F_t^{[\ell'']})(\varphi)\|_{L^\infty} \\ &\lesssim \sum_{\ell'+\ell''>3}\|\mathrm{D}F_t^{[\ell']}Q_t(Q_t F_t^{[\ell'']})\|_{L^\infty} \lesssim \sum_{\ell'+\ell''>3}(\bar{\lambda}_t\langle t\rangle^{-1})^{\ell'+\ell''}.\end{aligned}$$

b) The Lipschitz bounds follow as above in part a, combined with the observation that thanks to the boundedness of the complex exponential fields it holds that, writing $\tilde{\psi}(\xi) = \tilde{\psi}(\sigma, x) = \mathrm{e}^{i\sigma\beta\tilde{\varphi}(x)}$,

$$|\psi(\xi_{1:\ell}) - \tilde{\psi}(\xi_{1:\ell})| \leqslant \sum_k |\psi(\xi_k) - \tilde{\psi}(\xi_k)| \leqslant \sum_k |\varphi(\xi_k) - \tilde{\varphi}(\xi_k)|.$$

c) This follows from Lemma B.1 in the same way as Proposition 4.5-a followed from Lemma 3.6 and 3.7.

d) We only show the estimates on QF as the others are a direct consequence as illustrated in the proof of part a. Again, except for $F_t^{[2](0)}$, these estimates follow immediately from the convolution inequalities (see Lemma A.4),

$$\begin{aligned}\|Q_t(F_t^{[\ell],\rho_1} - F_t^{[\ell],\rho_2})(\varphi)\|_{L^{2,-n}} &\leqslant \|Q_t\|_{L^1}\|(F_t^{[\ell],\rho_1} - F_t^{[\ell],\rho_2})(\varphi)\|_{L^{2,-n}} \\ &\lesssim \langle t\rangle^{-1}\|(F_t^{[\ell],\rho_1} - F_t^{[\ell],\rho_2})(\varphi)\|_{L^{2,-n}},\end{aligned}$$

and

$$\begin{aligned}\|(F_t^{[\ell],\rho_1} - F_t^{[\ell],\rho_2})(\varphi)\|_{L^{2,-n}}^2 &= \left\|\int \mathrm{d}\xi_{2:\ell}(\rho_1-\rho_2)(\xi_{1:\ell})f_t^{[\ell]}(\xi_{1:\ell})\right\|^2_{L^2_{x_1}(\langle x\rangle^{-n})} \\ &\lesssim \|\rho_1-\rho_2\|^2_{L^{2,-n}}\,|||\, f_t^{[\ell]}\,|||^2.\end{aligned}$$

The remaining estimate on $F_t^{[2],(0)}$ follows in the same way as before, using Q_t to absorb the increment via Lemma A.3 as in (3.40) and (3.41) and then following the same steps as above.

□

Remark 3.14. Proposition 3.13-c implies that we can define $Q_s F_s^\infty(\varphi)$, $\mathrm{D}F_s^\infty(\varphi)Q_s$ and $H_s^\infty(\varphi)$ as the L^∞-limit of $Q_s F_s^T(\varphi)$, $\mathrm{D}F_s^T(\varphi)Q_s$ and $H_s^T(\varphi)$, respectively as $T \to \infty$.

3.6. Estimates on the potential

The same arguments we used previously for the gradient show the following estimates on the remainder

$$\mathcal{H}_t^{\rho,T}(\varphi) := \left(\partial_t V_t^{\rho,T} + \frac{1}{2}\operatorname{Tr}\dot{G}_t \mathrm{D}^2 V_t^{\rho,T} - \frac{1}{2}\mathrm{D}V_t^{\rho,T}\dot{G}_t \mathrm{D}V_t^{\rho,T}\right)(\varphi), \tag{3.46}$$

at the level of the potential. We should emphasise that in contrast to the estimates on F, they of course only apply in the finite volume, that is for $\rho \prec 1$. The results of this section will be used only later on to recover the variational description for the unregularised measures in Section 7 and we invite the reader skip them at first reading.

Proposition 3.15. *Given a set $A \subset \mathbb{R}^2$, denote its Lebesgue measure by $|A|$. For any spatial cut-off $\rho \prec 1$, UV cut-off $T, T_1, T_2 \leqslant \infty$, and $\varphi, \tilde{\varphi} \in \mathcal{S}'(\mathbb{R}^2)$, it holds that*

a) ***(Lipschitz estimates)***

$$|\mathcal{H}_t^\rho(\varphi) - \mathcal{H}_t^\rho(\tilde{\varphi})| \lesssim |\operatorname{supp}(\rho)|(\bar{\lambda}_t\langle t\rangle^{-1})^4 \|\varphi - \tilde{\varphi}\|_{L^\infty}, \quad t \in [0,T].$$

b) ***(Dependence on the regularisation)*** *There is an $\varepsilon > 0$ such that*

$$|(\mathcal{H}_t^{\rho,T_1} - \mathcal{H}_t^{\rho,T_2})(\varphi)| \lesssim |\operatorname{supp}(\rho)|\langle T_1 \wedge T_2\rangle^{-\varepsilon}, \quad t \in [0,T],$$

and

$$|\mathcal{H}_t^{\rho,T}(\varphi) - \mathcal{H}_t^{\rho,T}(\tilde{\varphi})| \lesssim \|\rho\|_{L^{2,-n}} \|\varphi - \tilde{\varphi}\|_{L^{2,n}}. \quad t \in [0,T]. \tag{3.47}$$

Proof. By the definition (3.46) of $\mathcal{H}^{\rho,T}$ above, we see that $\mathcal{H}^{\rho,T}$ has the Fourier series representation

$$\mathcal{H}_t^{\rho,T}(\varphi) := \sum_{\ell=4}^{6} \int \mathrm{d}\xi_{1:\ell}\rho(\xi_{1:\ell})h_t^T(\xi_{1:\ell})\psi(\xi_{1:\ell}), \tag{3.48}$$

where we used the notation $\rho(\xi_{1:\ell}) = \prod_{k\leqslant\ell}\rho(x_k)$ and for $\ell \in \{4,5,6\}$, we defined

$$h_t^T(\xi_{1:\ell}) := \frac{1}{2}\sum_{I_1 \cup I_2 = [\ell]} C(|I_1|,|I_2|) f_t^T(\xi_{I_1}) \left[\sum_{i\in I_1}\sum_{j\in I_2}\sigma_i\sigma_j \dot{G}_t(x_i - x_j)\right] f_t^T(\xi_{I_2}), \tag{3.49}$$

for a positive combinatorial constant $C(|I_1|,|I_2|)$. Moreover, by Remark 3.8, we have the estimate

$$|||h_t^{[\ell],T}||| \lesssim (\bar{\lambda}_t\langle t\rangle^{-1})^\ell. \tag{3.50}$$

a) This follows directly from the definition of $\mathcal{H}^\rho$, the bounds on the kernels (3.50) and simple rearrangements, using again the boundedness of $|\psi(\xi)| = 1$,

$$\begin{aligned}
&\left|\sum_{\ell=4}^{6}\int \mathrm{d}\xi_{1:\ell}\rho(\xi_{1:\ell})h_t^T(\xi_{1:\ell})(\psi(\xi_{1:\ell})-\tilde{\psi}(\xi_{1:\ell}))\right| \\
\leqslant&\sum_{\ell=4}^{6}\int \mathrm{d}x_1\rho(x_1)\int \mathrm{d}\xi_{2:\ell}h_t^T(\xi_{1:\ell})|\psi(\xi_{1:\ell})-\tilde{\psi}(\xi_{1:\ell})| \\
\leqslant&|\operatorname{supp}(\rho)|\sum_{\ell=4}^{6}\sup_{\xi_1}\int \mathrm{d}\xi_{2:\ell}|h_t^T(\xi_{1:\ell})|\|\varphi-\tilde{\varphi}\|_{L^\infty}, \\
\lesssim&|\operatorname{supp}(\rho)|\|\varphi-\tilde{\varphi}\|_{L^\infty}\sum_{\ell=4}^{6}\bar{\lambda}_t^\ell\langle t\rangle^{-\ell}.
\end{aligned}$$

b) This proof is in complete analogy to Proposition 3.13-c and d using the same ideas as in Proposition 3.15-a above.

□

4. Analysis of the FBSDE

With good approximate solutions to the flow under the conditional expectation and the renormalisation sorted out, we can return to the FBSDE

$$\begin{cases} X_t^{\rho,T,g} = \varphi + W_t - \int_0^t \dot{G}_s(F_s^{\rho,T}(X_s^{\rho,T,g}) + R_s^{\rho,T,g})\mathrm{d}s, \\ R_t^{\rho,T,g} = \mathbb{E}_t\Big[\nabla g(X_T^g) + \int_t^T H_s^{\rho,T}(X_s^{\rho,T,g})\mathrm{d}s - \int_t^T \mathrm{D}F_s^{\rho,T}(X_s^{\rho,T,g})\dot{G}_s R_s^{\rho,T}\mathrm{d}s\Big], \end{cases} \tag{4.1}$$

with F and H as defined in Section 3.4.

We will often work with $Z^{\rho,T,g} := X_t^{\rho,T,g} - (W_t + \varphi)$ directly to obtain deterministic bounds on the drift Z. To lighten the notation, we leave the dependence of the solution on T and ρ implicit whenever possible and fix the perturbation g. Unless explicitly stated otherwise, all estimates are uniform in the parameters ρ and T.

We will furthermore always implicitly assume that the solution to (4.1) is extended to the positive half line $[0,\infty)$ in the standard way, that is

$$(X_t^T, R_t^T) := (X_{T\wedge t}^T, R_{T\wedge t}^T), \quad t \in [0,\infty).$$

4.1. Well-posedness for the FBSDE

As a first step, we show well-posedness for the FBSDE (4.1) with the regularisations in place. We follow a standard Picard-iteration for the solution map $\Gamma(z) = Z^z$ defined by

$$\begin{cases} Z_t^z = -\int_0^t \mathrm{d}s\dot{G}_s(F_s(z_s + W_s) + R_s^z), \\ R_t^z = \mathbb{E}_t\Big[\nabla g(z_s + W_T) + \int_t^T \mathrm{d}sH_s(z_s + W_s) - \int_t^T \mathrm{d}s\mathrm{D}F_s(z_s + W_s)\dot{G}_s R_s^z\Big]. \end{cases} \tag{4.2}$$

Standard well-posedness for decoupled Lipschitz FBSDEs ensures the existence of a unique solution $(Z^z, R^z) \in \mathbb{H}_T^\infty(L^\infty) \times \mathbb{H}_T^\infty(L^\infty)$ to (4.2) for any $z \in \mathbb{H}_T^\infty(L^\infty)$. The only term in (4.2) that cannot be estimated in a linear fashion immediately from Proposition 3.13 is the term $\mathrm{D}F_s(z_s + W_s)\dot{G}_s R_s^z$ in the backward equation. The next Lemma ensures that also this term stays bounded and does not cause any issues.

Lemma 4.1. *For all in $z \in \mathbb{H}_T^\infty(L^\infty)$ and $\lambda \in \mathbb{R}$ (not necessarily small), there are constants $C_{\bar\lambda}, C_{g,\bar\lambda}$ non-decreasing in $\bar\lambda$*

$$\sup_t \|R_t^z\|_{L^\infty} \lesssim C_{\bar\lambda}(|g|_{1,\infty} + \bar\lambda_t^4 \langle t\rangle^{-3}) < \infty, \quad \textit{and} \quad \sup_t \|Z_t^z\|_{L^\infty} \leqslant C_{g,\bar\lambda}$$

Proof. From the definition of R^z, the regularity of $W_t \in L^{2,-n}$ for any $t < \infty$ and the bounds on the flow from Proposition 3.13-a,

$$\begin{aligned}\|R_t^z\|_{L^\infty} &\leqslant \bar\lambda |g|_{1,\infty} + \int_t^T \mathrm{d}s \|H_s(z_s + W_s)\|_{L^\infty} + \bar\lambda \int_t^T \mathrm{d}s \|\mathrm{D}F_s(z_s)\dot{G}_s R_s^z\|_{L^\infty},\\ &\lesssim \bar\lambda |g|_{1,\infty} + \int_t^T \mathrm{d}s \bar\lambda_s^4 \langle s\rangle^{-4} + \int_t^T \mathrm{d}s \bar\lambda_s \langle s\rangle^{-2} \|R_s^z\|_{L^\infty}.\end{aligned}$$

By a backward version of Gronwall's inequality, this implies with $4\delta > 1$ (see Lemma 3.1),

$$\|R_t^z\|_{L^\infty} \lesssim \left(\bar\lambda |g|_{1,\infty} + \int_t^T \mathrm{d}s \bar\lambda_s^4 \langle s\rangle^{-4}\right) \mathrm{e}^{\bar\lambda \int_t^T \mathrm{d}s \langle s\rangle^{-1-\delta}} \lesssim \bar\lambda |g|_{1,\infty} + \bar\lambda_t^4 \langle t\rangle^{-3}. \tag{4.3}$$

Thus, using the bound just derived for R^z in the equation for Z^z,

$$\|Z_t^z\| \leqslant \int_0^t \mathrm{d}s \|\dot{G}_s(F_s(z_s + W_s) + R_s^z)\|_{L^\infty} \lesssim \int_0^\infty \mathrm{d}s \langle s\rangle^{-2}(\bar\lambda_s + \|R_s^z\|_{L^\infty}) \lesssim \bar\lambda C_g.$$

□

With this issue resolved, we are in a position to show that (4.2) defines a contraction on $H_T^\infty(L^\infty)$.

Proposition 4.2. *For $\bar\lambda$ sufficiently small, the map $\Gamma : \mathbb{H}_T^\infty(L^\infty) \to \mathbb{H}_T^\infty(L^\infty); z \mapsto Z^z$ is a contraction.*

Since Z uniquely determines the solution (X, R) to (4.1) via

$$Z \mapsto (\varphi + Z_t + W_t, R_t^Z)_{t\geqslant 0},$$

this immediately implies the existence of a unique solution to (4.1) when combined with the regularity of the stopped Brownian motion $(W_t)_{t\in[0,T]} \in \mathbb{H}_T^2(L^{2,-n})$.

Corollary 4.3. *For any $\rho \preceq 1$, $T < \infty$, $\bar\lambda$ sufficiently small and $p \in [1, \infty)$ the FBSDE (4.1) has a unique solution*

$$(X_t, R_t)_{t\geqslant 0} = (\varphi + Z_t + W_t, R_t)_{t\geqslant 0} \in \mathbb{H}_T^p(L^{p,-n}) \times \mathbb{H}_T^\infty(L^\infty).$$

Proof of Proposition 4.2. Let $z_1, z_2 \in \mathbb{H}_T^\infty(L^\infty)$ and consider the FBSDE for the difference $(\delta Z, \delta R) = (Z^{z_1}, R^{z_1}) - (Z^{z_2}, R^{z_2})$ given by

$$\begin{cases} \delta Z_t = -\int_0^t \mathrm{d}s \dot{G}_s (F_s(X_s^{z_1}) - F_s(X_s^{z_2}) + \delta R_s) \\ \delta R_t = \mathbb{E}_t \Big[\delta_z \nabla g + \int_t^T \mathrm{d}s \delta_z H_s - \int_t^T \mathrm{d}s (\mathrm{D}F_s(X_s^{z_1}) \dot{G}_s R_s^{z_1} - \mathrm{D}F_s(X_s^{z_2}) \dot{G}_s R_s^{z_2}) \Big], \end{cases} \tag{4.4}$$

where we use the shorthand $X_s^z = \varphi + z_s + W_s$ so that $\delta X_s = \delta z_s$ and

$$\delta_z \nabla g := \nabla g(X_s^{z_1}) - \nabla g(X_s^{z_2}), \quad \delta_z H_s := H_s(X_s^{z_1}) - H_s(X_s^{z_2}).$$

To deal with the bilinear term in the backward equation, we combine the estimates from Proposition 3.13-a with the boundedness of R^z provided by Lemma 4.1 to conclude

$$\begin{aligned} \|(\mathrm{D}F_s(X_s^{z_1}) - \mathrm{D}F_s(X_s^{z_2})) \dot{G}_s R_s^{z_2}\|_{L^\infty} &\lesssim \|R_s^{z_2}\|_{L^\infty} \|[\mathrm{D}F_s(X_s^{z_1}) - \mathrm{D}F_s(X_s^{z_2})] \dot{G}_s\|_{L^\infty} \\ &\lesssim \bar{\lambda}_s \langle s \rangle^{-2} \|\delta z_s\|_{L^\infty}. \end{aligned}$$

The remaining terms in the backward equation can all be estimated directly using Proposition 3.13-a and the Lipschitz continuity of ∇g,

$$\|\delta R_t\|_{L^\infty} \lesssim \int_t^T \mathrm{d}s \bar{\lambda}_s^4 \langle s \rangle^{-4} \|\delta z_s\|_{L^\infty} + \int_t^T \mathrm{d}s \bar{\lambda}_s \langle s \rangle^{-2} \|\delta R\|_{L^\infty} + \int_t^T \mathrm{d}s \bar{\lambda}_s \langle s \rangle^{-2} \|\delta z_s\|_{L^\infty},$$

which implies by Gronwall's inequality,

$$\sup_t \|\delta R_t\|_{L^\infty} \leqslant C \bar{\lambda} \sup_t \|\delta z_t\|_{L^\infty}.$$

Using this estimate on R in the equation for the forward component with the Lipschitz estimates from Proposition 3.13-a, we obtain

$$\begin{aligned} \sup_t \|\delta Z_t\|_{L^\infty} &\leqslant C \int_0^T \mathrm{d}s \|\dot{G}_s (F_s(X_s^{z_1}) - F_s(X_s^{z_2}) + \delta R_s)\|_{L^\infty} \\ &\leqslant C \bar{\lambda} \int_0^T \mathrm{d}s \langle s \rangle^{-2} \bar{\lambda}_s \|\delta z_s\|_{L^\infty} + \int_0^T \mathrm{d}s \langle s \rangle^{-2} \|\delta R\|_{L^\infty} \\ &\leqslant C \bar{\lambda} \sup_s \|\delta z_s\|_{L^\infty} + C \sup_s \|\delta R_s\|_{L^\infty} \\ &\leqslant C \bar{\lambda} \sup_s \|\delta z_s\|_{L^\infty}, \end{aligned}$$

which yields the required contraction for $\bar{\lambda}$ small enough. □

Remark 4.4. We crucially rely on the uniform, field independent estimates on the approximate force obtained in Proposition 3.13 which hold only up to 6π. Indeed, assuming only a weaker estimate like (3.38), it is unclear how to obtain suitable replacements for the a priori estimates of Lemma 4.1 and rule out explosion in finite time. Even linear growth in $\mathrm{D}F_t(\varphi)$ would require an additional argument as any trivial estimate for the backward equation results in an exponential dependence on $\|Z\|_{L^\infty}$ in the equation for the remainder through (4.3).

4.2. Stability properties

In this section, we show that the FBSDE (4.1) is stable in both regularisations ρ and T, provided that the coupling constant $\bar{\lambda}$ is chosen sufficiently small. We summarise these properties below.

Proposition 4.5. *For $\bar{\lambda}$ sufficiently small and $n > 2$ (so that $x \mapsto \langle x\rangle^{-n} \in L^1(\mathbb{R}^2)$), the following stability estimates hold.*

a) ***Dependence on the spatial cut-off:*** *Let $\rho_1, \rho_2 \leqslant 1$ and denote the associated solutions to (4.1) by (Z^{ρ_1}, R^{ρ_1}) and (Z^{ρ_2}, R^{ρ_2})respectively. Then, the difference between the solution $(\delta_\rho Z, \delta_\rho R) := (Z^{\rho_1}, R^{\rho_1}) - (Z^{\rho_2}, R^{\rho_2})$ satisfies*

$$\sup_t \|\delta_\rho Z_t\|_{L^{2,-n}} + \sup_t \|\delta_\rho R_t\|_{L^{2,-n}} \lesssim \bar{\lambda}\|\rho_1 - \rho_2\|_{L^{2,-n}}.$$

b) ***Dependence on the UV cut-off:*** *Let $T_1, T_2 < \infty$ and denote the associated solutions to (4.1) by (Z^{T_1}, R^{T_1}) and (Z^{T_2}, R^{T_2}) respectively. Then, the difference between the solution $(\delta_T Z, \delta_T R) := (Z^{T_1}, R^{T_1}) - (Z^{T_2}, R^{T_2})$ satisfies for some $\varepsilon > 0$,*

$$\sup_t \|\delta_T Z_t\|_{L^\infty} + \sup_t \|\delta_T R_t\|_{L^\infty} \lesssim \langle T\rangle^{-\varepsilon}.$$

c) ***Dependence on local perturbations:*** *Let $(Z^{\rho,T,g}, R^{\rho,T,g})$ be the unique solution to (4.1). It holds that*

$$\sup_t \|Z_t^{\rho,T,g} - Z_t^{\rho,T,0}\|_{L^{2,n}} + \sup_t \|R_t^{\rho,T,g} - R_t^{\rho,T,0}\|_{L^{2,n}} \lesssim \bar{\lambda}|g|_{1,2,n}.$$

Proof. a) We follow essentially the same argument as before for the proof of Proposition 4.2, writing the FBSDE for the difference as

$$\begin{cases} \delta_\rho Z_t = -\int_0^t \mathrm{d}s \dot{G}_s (F_s^{\rho_1}(X_s^{\rho_1}) - F_s^{\rho_2}(X_s^{\rho_2}) + \delta_\rho R_s) \\ \delta_\rho R_t = \mathbb{E}_t\Big[\delta_\rho \nabla g + \int_t^T \mathrm{d}s \big(\delta_\rho H_s + \mathrm{D}F_s^{\rho_1}\big(X_s^{\rho_1}\big)\dot{G}_s R_s^{\rho_1} - \mathrm{D}F_s^{\rho_2}(X_s^{\rho_2})\dot{G}_s R_s^{\rho_2}\big)\Big] \end{cases} \tag{4.5}$$

where $\delta_\rho \nabla g = \nabla g(X_T^{\rho_1}) - \nabla g(X_T^{\rho_1})$ and $\delta_\rho H_s = H_s^{\rho_1}(X_s^{\rho_1}) - H_s^{\rho_2}(X_s^{\rho_2})$. Using the estimates from Proposition 3.13-d in the FBSDE for the difference (4.5), we obtain

$$\begin{aligned} \sup_t \|\delta_\rho Z_t\|_{L^{2,-n}} &\lesssim \int_0^T \mathrm{d}s \bar{\lambda}_s \langle s\rangle^{-2} (\|\delta_\rho Z_s\|_{L^{2,-n}} + \|\rho_1 - \rho_2\|_{L^{2,-n}} + \|\delta_\rho R_s\|_{L^{2,-n}}) \\ &\lesssim \bar{\lambda}(\sup_s \|\delta_\rho Z_s\|_{L^{2,-n}} + \|\rho_1 - \rho_2\|_{L^{2,-n}}) + \sup_s \|\delta_\rho R_s\|_{L^{2,-n}}, \end{aligned}$$

and

$$\begin{aligned} \sup_t \|\delta_\rho R_t\|_{L^{2,-n}} \lesssim& \int_0^T \mathrm{d}s \bar{\lambda}_s \langle s\rangle^{-2} (\|\delta_\rho Z_s\|_{L^{2,-n}} + \|\rho_1 - \rho_2\|_{L^{2,-n}}) \\ &+ \int_0^T \mathrm{d}s \bar{\lambda}_s \langle s\rangle^{-2} [\|\rho_1 - \rho_2\|_{L^{2,-n}} + \|\delta_\rho Z_s\|_{L^{2,-n}} + \|\delta_\rho R_s\|_{L^{2,-n}}] \\ \lesssim& \bar{\lambda}(\sup_s \|\delta_\rho Z_s\|_{L^{2,-n}} + \sup_s \|\delta_\rho R_s\|_{L^{2,-n}} + \|\rho_1 - \rho_2\|_{L^{2,-n}}), \end{aligned}$$

which yields the claim after choosing $\bar{\lambda}$ sufficiently small and rearranging.

b) For concreteness, let $T_2 < T_1$. The difference between the two solutions solves the FBSDE,

$$\begin{cases} \delta_T Z_t = -\int_0^t \mathrm{d}s \dot{G}_s (F_s^{T_1}(X_s^{T_1}) - F_s^{T_2}(X_s^{T_2}) + \delta_T R_s) \\ \delta_T R_t = \mathbb{E}_t \left[\delta_T \nabla g + \mathbb{E}_t \int_t^T \delta_T H_s \mathrm{d}s \right] \\ \qquad + \int_t^T \left(-(\mathrm{D}F_s^{T_1}\left(X_s^{T_1}\right) \dot{G}_s R_s^{T_1} - \mathrm{D}F_s^{T_2}(X_s^{T_2}) \dot{G}_s R_s^{T_2}) \right) \mathrm{d}s, \end{cases}$$

where $\delta_T \nabla g = \nabla g(X_{T_1}^{T_1}) - \nabla g(X_{T_2}^{T_2})$ and $\delta_T H_s = H_s^{T_1}(X_s^{T_1}) - H_s^{T_2}(X_s^{T_2})$. The only difference to the estimates before is the additional tail in the backward equation. Other than that we proceed as before. For the forward equation, splitting up the differences as in the proof of part a, we have from Proposition 3.13-a and 3.13-c for ε sufficiently small,

$$\|\delta_T Z_t\|_{L^\infty} \lesssim \bar{\lambda} \sup_s \|\delta_T Z_s\|_{L^\infty} + \|\delta_T R_s\|_{L^\infty} + \langle T_2 \rangle^{-\varepsilon}.$$

For the backward equation, we split the terms as

$$\begin{aligned} \delta_T R_t =& \nabla g(X_{T_1}^{T_1}) - \nabla g(X_{T_2}^{T_2}) \\ &+ \mathbb{E}_t \int_t^{T_2} \mathrm{d}s (H_s^{T_1}(X_s^{T_1}) - H_s^{T_2}(X_s^{T_2})) \\ &- \mathbb{E}_t \int_t^{T_2} \mathrm{d}s (\mathrm{D}F_s^{T_1}(X_s^{T_1}) \dot{G}_s R_s^{T_1} - \mathrm{D}F_s^{T_2}(X_s^{T_2}) \dot{G}_s R_s^{T_2}) \\ &+ \mathbb{E}_t \int_{T_2}^{T_1} \mathrm{d}s H_s^{T_1}(X_s^{T_1}) - \mathbb{E}_t \int_{T_2}^{T_1} \mathrm{d}s \mathrm{D}F_s^{T_1}(X_s^{T_1} \dot{G}_s R_s^{T_1}), \end{aligned}$$

we proceed similarly on $[0, T_2]$ to obtain from Proposition 3.13-c and 3.13-a,

$$\int_t^{T_2} \mathrm{d}s \|H_s^{T_1}(X_s^{T_1}) - H_s^{T_2}(X_s^{T_2})\|_{L^\infty} \lesssim \bar{\lambda} \sup_s \|\delta_T Z_s\|_{L^\infty} + \bar{\lambda} \langle T_2 \rangle^{-\varepsilon},$$

and

$$\begin{aligned} &\int_t^{T_2} \mathrm{d}s \|\mathrm{D}F_s^{T_1}(X_s^{T_1}) \dot{G}_s R_s^{T_1} - \mathrm{D}F_s^{T_2}(X_s^{T_2}) \dot{G}_s R_s^{T_2}\|_{L^\infty} \\ &\qquad \lesssim \bar{\lambda} (\sup_s \|\delta_T Z_s\|_{L^\infty} + \sup_s \|\delta_T R_s\|_{L^\infty} + \langle T_2 \rangle^{-\varepsilon}). \end{aligned}$$

The integrals on $[T_2, T_1]$ can be estimated using just the boundedness of the coefficients provided by Proposition 3.13-a,

$$\begin{aligned} \int_{T_2}^{T_1} \mathrm{d}s \|H_s^{T_1}(X_s^{T_1})\|_{L^\infty} + \int_{T_2}^{T_1} \mathrm{d}s \|\mathrm{D}F_s^{T_1}(X_s^{T_1}) \dot{G}_s R_s^{T_1}\|_{L^\infty} &\lesssim \bar{\lambda} \int_{T_2}^{T_1} \mathrm{d}s \langle s \rangle^{-4\delta} + \langle s \rangle^{-1-\delta} \\ &\lesssim \bar{\lambda} \langle T_2 \rangle^{-\varepsilon}. \end{aligned}$$

Finally, by the regularity assumed on g,

$$\|\nabla g(X_{T_1}^{T_1}) - \nabla g(X_{T_2}^{T_2})\|_{L^\infty} \lesssim \bar{\lambda}|g|_{2,\infty}\|X_{T_2}^{T_1} - X_{T_2}^{T_2}\|_{L^\infty} \lesssim \bar{\lambda}\sup_s \|\delta X_s\|_{L^\infty}.$$

Combining all of the above yields the claim for $\bar{\lambda}$ small enough after rearranging.

c) This proof is straightforward and does not require any new arguments compared to e.g. the proof of part a. Indeed, now the FBSDE for the difference is

$$\begin{cases} \delta_g Z_t = -\int_0^t \mathrm{d}s \dot{G}_s(F_s(X_s^g) - F_s(X_s^0) + \delta_g R_s) \\ \delta_g R_t = \nabla g(X_T^g) + \mathbb{E}_t \int_t^T \mathrm{d}s \delta_g H_s - \mathbb{E}_t \int_t^T \mathrm{d}s \left(\mathrm{D}F_s(X_s^g)\dot{G}_s R_s^g - \mathrm{D}F_s(X_s^0)\dot{G}_s R_s^0 \right), \end{cases}$$

where again $\delta_g H_s = H_s(X_s^g) - H_s(X_s^0)$. The same steps as in part a and b imply the claim with the Lipschitz estimates from Proposition 3.13-b.

□

4.3. Recovering the EQFT

Throughout this section, we assume that $\bar{\lambda}$ is chosen small enough for Corollary 4.3 and Proposition 4.5 to apply. Then, for any $\rho \prec 1$ and $T < \infty$, we denote by $(X^{\rho,T}, R^{\rho,T})$ the unique solution to (4.1). We show the following refined version of Theorem 1.1.

Theorem 4.6. *As the cut-offs are removed, the family $\{(Z^{\rho,T}, R^{\rho,T})\}_{\rho \preceq 1, T<\infty}$ converges in $\mathbb{H}^2(L^{2,-n}) \times \mathbb{H}^2(L^{2,-n})$ to a unique limit $(Z,R) \in \mathbb{H}^\infty(L^\infty) \times \mathbb{H}^\infty(L^\infty)$, that is*

$$\lim_{\substack{\rho \to 1 \\ T \to \infty}} \sup_t \{\|Z_t^{\rho,T} - Z_t\|_{L^{2,-n}} + \|R_t^{\rho,T} - R_t\|_{L^{2,-n}}\} = 0.$$

The limiting process is the unique solution to (4.1) *with $T = \infty, \rho = 1$. Moreover, for any $\varepsilon > 0$, $p \in [0,\infty)$ and $\varphi \in B_{p,p}^{0-\varepsilon,-n}$, there is a version of the drift process Z with terminal value $Z_\infty \in L^\infty\left(\mathrm{d}\mathbb{P};\, B_{p,p}^{2-\beta^2/4\pi-\varepsilon,-n}\right)$, so that*

$$X_\infty = Z_\infty + (\varphi + W_\infty) \in L^\infty\left(\mathrm{d}\mathbb{P};\, B_{p,p}^{2-\beta^2/4\pi-\varepsilon,-n}\right) + L^p(\mathrm{d}\mathbb{P}; B_{p,p}^{0-\varepsilon,-n}). \tag{4.6}$$

In particular, for any $\varepsilon > 0$, the family $(\nu_{\mathrm{SG}}^{\rho,T})_{\rho,T}$ has a unique weak limit in $H^{-\varepsilon,-n}$ as $\rho \to 1$ and $T \to \infty$ which we denote by ν_{SG}. It is given as a random shift of the Gaussian free field,

$$\mathrm{Law}(Z_T^{\rho,T} + W_T) = \nu_{\mathrm{SG}}^{\rho,T} \longrightarrow \nu_{\mathrm{SG}} = \mathrm{Law}(Z_\infty + W_\infty).$$

For $\beta^2 < 4\pi$, we obtain $Z_\infty \in H^{1+,-n}$ and in the finite volume, that is for $\rho \prec 1$, the same argument in the unweighted spaces implies $Z_\infty^\rho \in H^1$. In this case, the Wick ordered cosine $[\![\cos(\beta W_t)]\!]$ converges in H^α for any $\alpha < -\beta^2/4\pi \in (-1,0)$, and we can define all products on the right-hand side of

$$[\![\cos(\beta(Z_\infty + W_\infty))]\!] := \cos(\beta Z_\infty)[\![\cos(\beta W_\infty)]\!] - \sin(\beta Z_\infty)[\![\sin(\beta W_\infty)]\!], \tag{4.7}$$

so that the partition function $\Xi^\rho = \mathbb{E}[\exp(-\bar\lambda V^{\rho,\infty}(Z_\infty + W_\infty))]$ stays bounded. Consequently, we recover that the law of the shift $\nu^\rho_{\mathrm{SG}} = \mathrm{Law}(Z^\rho_\infty + W_\infty)$ is absolutely continuous with respect to the Gaussian free field $\mu = \mathrm{Law}(W_\infty)$. For $\beta^2 \geqslant 4\pi$, this is no longer the case (see Theorem 1.4 and Theorem 6.1) and indeed Theorem 4.6 only ensures the regularity $Z^\rho_\infty \in H^{2-\beta^2/4\pi-}(\mathbb{R}^2) = H^{1/2+}(\mathbb{R}^2)$, which we conjecture to be optimal for $\beta^2 < 6\pi$. This regularity no longer allows to define the products on the right-hand side of (4.7).

Proof of Theorem 4.6. The fact that the limit exists follows from Proposition 4.5-a and b. Note that all constants are uniform in $\rho \leqslant 1$ and $T \leqslant \infty$, so the order in which we take the limits is irrelevant. Denote the limiting processes of $(Z^{\rho,T}, R^{\rho,T})_{\rho,T}$ by (Z,R) and let $X := \varphi + Z + W$. Then, the aforementioned convergence results transfer the bound from Lemma 4.1 to the limit so that,

$$\|Z_\infty\|_{L^\infty} + \|R_\infty\|_{L^\infty} \leqslant \sup_t (\|Z_t\|_{L^\infty} + \|R_t\|_{L^\infty}) \lesssim 1.$$

The convergence of W_t to the Gaussian free field in $H^{0-,-n}$ is the content of Lemma 2.1 and

$$\mathrm{Law}(Z^{\rho,T}_T + W_T) = \nu^{\rho,T}_{\mathrm{SG}},$$

was already shown in Theorem 2.2-b. By Gaussian hypercontractivity and the Besov embeddings, for any $p \in [0,\infty]$ there is a version of the free field such that $W_\infty \in B^{0-,-n}_{p,p}$ holds and it remains to show only that the drift Z_t has a terminal value Z_∞ with the required regularity. Thanks to Lemma A.5, for any $\alpha \in (0,1)$ and $\varepsilon > 0$,

$$\|Z_\infty\|_{B^\alpha_{p,p}} = \left\| \int_0^\infty \mathrm{d}s \dot G_s(F_s(X_s) + R_s) \right\|_{B^{\alpha,-n}_{p,p}} \lesssim \sup_s \langle s\rangle^{\alpha/2+\varepsilon} \|Q_s(F_s(X_s) + R_s)\|_{L^\infty}. \tag{4.8}$$

Using Proposition 3.13 and Lemma 4.1 we know

$$\|Q_s(F_s + R_s)\|_{L^\infty} \lesssim \langle s\rangle^{-1}\bar\lambda_s + \langle s\rangle^{-1} \lesssim s^{-\delta} = \langle s\rangle^{\beta^2/8\pi - 1}.$$

Therefore, we can choose $\varepsilon > 0$ small enough for (4.8) to be finite provided

$$\alpha < 2 - \beta^2/4\pi = 2\delta.$$

□

As a direct consequence of Theorem 4.6, we also get exponential moments for the limiting measure.

Corollary 4.7. *For any $\varepsilon > 0$, there is a constant $\gamma > 0$ such that*

$$\int_{\mathcal{S}'(\mathbb{R}^2)} \mathrm{e}^{\gamma\|\varphi\|^2_{H^{-\varepsilon,-n}}} \nu_{\mathrm{SG}}(\mathrm{d}\varphi) < \infty.$$

Proof. By Fernique's theorem (see e.g. [15, Theorem 2.8.5.]) the Gaussian free field W_∞ has squared exponential moments in $H^{-\varepsilon,-n}$ for some $\gamma > 0$. Combined with the bounds on Z_∞ from Theorem 4.6,

$$\int_{S'(\mathbb{R}^2)} e^{\gamma\|\varphi\|^2_{H^{-\varepsilon,-n}}} \nu_{\mathrm{SG}}(d\varphi) \lesssim \mathbb{E}[\exp(\gamma\|W_\infty\|^2_{H^{-\varepsilon,-n}} + \gamma\|Z_\infty\|^2_{L^{2,-n}})] < \infty.$$

□

For future reference, let us also note the following regularity property of the solution.

Lemma 4.8. *For any $\alpha \in (0,1)$, $\varepsilon > 0$ and $p \in [1,\infty]$, the solution to the FBSDE (4.1) satisfies*

$$\mathbb{E}\left[\sup_t \langle t\rangle^{-\alpha/2}\|W_t\|_{B^{\alpha-\varepsilon,-n}_{p,p}}\right] \lesssim 1, \quad \sup_t \langle t\rangle^{-\alpha/2+\delta}\|Z_t\|_{B^{\alpha-\varepsilon,-n}_{p,p}} \lesssim 1.$$

In particular, $\sup_t \langle t\rangle^{-\alpha/2}\|X_t\|_{B^{\alpha-\varepsilon,-n}_{p,p}} < \infty$ *almost surely.*

Proof. For the estimate on the field $(W_t)_{t\geqslant 0}$, we refer to Appendix A.4. The estimate on Z_t follows in the same way as (4.8). Combining the estimates for Z and W yields the estimate for $(X_t)_{t\geqslant 0}$. □

4.4. Uniqueness for the finite volume measure

The convergence to a *unique* measure in Theorem 4.6 requires the coupling constant $\bar\lambda$ to be sufficiently small to close the argument for the coupled forward backward system. In this section, we show that the limiting measure is still unique as long as the volume remains finite. We do not expect this to be true for large values of the coupling constant in the infinite volume. Let us fix $g = 0$ and a volume cut-off $\rho \prec 1$ left implicit throughout this section.

Proposition 4.9. *Let $\bar\lambda \in \mathbb{R}$, $T \in [0,\infty]$, $\rho \prec 1$ and let $(Z,R) = (Z^\rho, R^\rho)$ be any accumulation point of $(Z^{\rho,T}, R^{\rho,T})_{T\in[0,\infty)}$. Then,* $\mathrm{Law}(\mathrm{Z}_\infty + \mathrm{W}_\infty)$ *is unique.*

Proof. Let O be bounded and continuous $H^{-\varepsilon,-n}(\mathbb{R}^2) \to \mathbb{R}$ for some $\varepsilon > 0$. The claim follows once we show that $X^{\rho,T}_T = Z^{\rho,T}_T + W_T$ satisfies

$$\lim_{T\to\infty} \mathbb{E}[\mathcal{O}(X^T_T)] = \frac{\mathbb{E}\left[\mathcal{O}(\bar X_\infty) e^{-\int_0^\infty \mathcal{H}_s(\bar X_s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^\infty \mathcal{H}_s(\bar X_s)ds}\right]}, \tag{4.9}$$

where $\bar X$ is the unique strong solution to

$$\bar X_t = -\int_0^t \dot G_s F_s(\bar X_s)ds + W_t. \tag{4.10}$$

First, let us note that the estimates on $\dot{G}F$ (bounded and Lipschitz) from Proposition 3.13 guarantee existence of a unique strong solution to (4.10). The same is true for $\bar{X}^T$ the solution to

$$\bar{X}_t^T = -\int_0^t \dot{G}_s F_s^T(\bar{X}_s^T)\mathrm{d}s + W_t \qquad t \in [0,T]. \tag{4.11}$$

Moreover, the estimates on F^T, F in Proposition 3.13 and the arguments used in the proof of Theorem 4.6 show that $\bar{X}^T$ converges to $\bar{X}$ and that $\bar{X}$ has a terminal value $\bar{X}_\infty$. More precisely, with $\bar{Z} = \bar{X} - W$,

$$\lim_{T\to\infty} \sup_{s\in[0,T]} \|\bar{X}_s^T - \bar{X}_s\|_{L^\infty(\mathbb{P};L^\infty)} = 0. \tag{4.12}$$

By definition of X, we know that

$$\mathbb{E}[\mathcal{O}(X_T^T)] = \frac{\mathbb{E}[\mathcal{O}(W_T)\mathrm{e}^{-V_T^T(W_T)}]}{\mathbb{E}[\mathrm{e}^{-V_T^T(W_T)}]} = \frac{\mathbb{E}\left[\mathcal{O}(W_T)\mathrm{e}^{-V_0^T(W_0)-\int_0^T \mathcal{H}_s^T(W_s)\mathrm{d}s}\mathcal{E}_{0,T}\right]}{\mathbb{E}\left[\mathrm{e}^{-V_0^T(W_0)-\int_0^T \mathcal{H}_s^T(W_s)\mathrm{d}s}\mathcal{E}_{0,T}\right]},$$

where

$$\mathcal{E}_{t,T} = \exp\left\{-\int_{t\wedge T}^T F_s^T(W_s)\mathrm{d}W_s - \frac{1}{2}\int_{t\wedge T}^T F_s^T(W_s)\dot{G}_s F_s^T(W_s)\mathrm{d}s\right\}.$$

The estimates on F from Proposition 3.13 show that $(\mathcal{E}_{t,T})_{t\geq 0}$ is a martingale. By Girsanov's theorem,

$$W_t^F := W_t + \int_0^t \dot{G}_s F_s^T(W_s)\mathrm{d}s,$$

is a Brownian motion under $\mathrm{d}\mathbb{P}^{F^T} = \mathcal{E}_{0,T}\mathrm{d}\mathbb{P}$. Therefore, using that $V_0^T(W_0) = V_0^T(0)$ is deterministic,

$$\mathbb{E}[\mathcal{O}(X_T^T)] = \frac{\mathbb{E}\left[\mathcal{O}(\bar{X}_T^T)\mathrm{e}^{-\int_0^T \mathcal{H}_s^T(\bar{X}_s^T)\mathrm{d}s}\right]}{\mathbb{E}\left[\mathrm{e}^{-\int_0^T \mathcal{H}_s^T(\bar{X}_s^T)\mathrm{d}s}\right]},$$

where $\bar{X}^T$ is the unique strong solution to (4.11). Combining the convergence of $\bar{X}_T^T$ from (4.12) with the convergence of $\mathcal{H}_s^T$ to $\mathcal{H}_s$ from Proposition 3.15, we conclude (4.9) and thus the claim. □

Remark 4.10. Let again $\bar{X}$ be the unique strong solution to (4.10) and denote the set of all $\bar{X}$ adapted processes by $\mathcal{A}(\bar{X})$. Consider the $\mathbb{R}$-valued BSDE

$$\mathcal{R}_t = \int_t^\infty \mathcal{H}_s(\bar{X}_s)\mathrm{d}s - \frac{1}{2}\int_t^\infty \|\dot{G}_s^{1/2}\bar{R}_s\|_{L^2}^2\mathrm{d}s - \int_t^\infty \bar{R}_s\mathrm{d}W_s. \tag{4.13}$$

The estimates on $\mathcal{H}$ and $\bar{X}$, imply that (4.13) has a unique bounded solution $(\mathcal{R},\bar{R})$ which is also adapted to $\bar{X}$. Moreover, standard comparison arguments for $\mathbb{R}$-valued BSDEs imply

that the control $\bar{R}$ is the unique optimiser for the following relaxed version of the finite volume control problem discussed in section 7,

$$\mathcal{V}^w = \inf_{R\in\mathcal{A}(\bar{X})} \mathbb{E}^{\mathbb{P}^R}\left[V_0(\bar{X}_0) + \int_0^\infty \mathcal{H}_s(\bar{X}_s)\mathrm{d}s + \frac{1}{2}\int_0^\infty \|\dot{G}_s^{1/2}R_s\|_{L^2}^2\mathrm{d}s\right],$$

where

$$\mathrm{d}\mathbb{P}^R = \exp\left\{\int_0^\infty R_s\mathrm{d}W_s - \frac{1}{2}\int_0^\infty \|\dot{G}_s^{1/2}R_s\|_{L^2}^2\mathrm{d}s\right\}\mathrm{d}\mathbb{P},$$

and $\mathrm{Law}^{\mathbb{P}^{\bar{R}}}(\bar{X}_\infty) = \nu_{\mathrm{SG}}$.

5. Decay of correlations

Using the scale-by-scale coupling via (4.1), a coupling method allows us to transfer the decay of correlations from the massive free field to the sine-Gordon measure and establish Theorem 1.3. We follow mostly [28] but similar arguments can be found e.g. in [39] and the idea appears to originate in [24].

For simplicity and to not distract from the main ideas, let $\mathcal{O}_1, \mathcal{O}_2 : H^{-\varepsilon,-n} \to \mathbb{R}$ be two Lipschitz and bounded observables. Given a smooth bump function χ supported on $B_1(0)$ we want to show that for some $c > 0$, with $X = X_\infty$ the solution to (4.1)

$$\mathrm{Cov}(\mathcal{O}_1(\chi\cdot X(\cdot + x_1)), \mathcal{O}_2(\chi\cdot X(\cdot + x_2))) \lesssim \mathrm{e}^{-c|x_1-x_2|}.$$

Let us agree on some notation to use throughout this proof. We denote by $\mathfrak{l} := |x_1 - x_2|$ the distance between the two points of interest. For $i = 1, 2$, let $D_i(r)$ be the open ball of radius $r > 0$ centred at x_i, where we drop the argument in case $r = \mathfrak{l}/2$. Given a smooth bump function $\vartheta^{(i)}$ supported on $D_i(\mathfrak{l}/4)$ such that $\vartheta(x) \equiv 1$ on $D_i(\mathfrak{l}/8)$, we define the exponential weights

$$q^{(i)}(x) = \mathrm{e}^{-\gamma m|x-x_i|} \quad \text{and} \quad \bar{q}^{(i)}(x) = \vartheta^{(i)}(x)q^{(i)}(x),$$

In order for the heat kernels Q and G to work nicely with the weights $q^{(i)}$, we always assume that $\gamma \in (0, 1)$. To set up the coupling argument, let $W^{(0)} := W$ and $D_0 := \mathbb{R}^2$ and define the identically distributed Brownian motions $W^{(1)}, W^{(2)}$ with covariance

$$\mathbb{E}[W_t^{(i)}(x)W_t^{(j)}(y)] = \int_0^t \mathrm{d}s \int \mathrm{d}z Q_s(z-x)\mathbb{1}_{D_i\cap D_j}(z)Q_s(z-y),$$

so that $W^{(1)}$ and $W^{(2)}$ are independent and $W^{(0)} \approx W^{(i)}$ near x_i. Denote by $X^{(i)}$ the solution to the FBSDE (4.1) with $T = \infty, \rho = 1$ and $g = 0$ driven by $W^{(i)}$. Then, the solutions $X^{(1)}$ and $X^{(2)}$ have the same law as X and inherit the independence from their

driving noise. Inserting the $X^{(i)}$ for X, we find that

$$\begin{aligned}
&\operatorname{Cov}(\mathcal{O}_1(\chi\cdot X(\cdot+x_1)),\mathcal{O}_2(\chi\cdot X(\cdot+x_2)))\\
=&\mathbb{E}[\mathcal{O}_1(\chi\cdot X(\cdot+x_1))\mathcal{O}_2(\chi\cdot X(\cdot+x_2))]\\
&-\mathbb{E}[\mathcal{O}_1(\chi\cdot X^{(1)}(\cdot+x_1))]\mathbb{E}[\mathcal{O}_2(\chi\cdot X^{(2)}(\cdot+x_2))]\\
=&\ \mathbb{E}[\mathcal{O}_2(\chi\cdot X^{(2)}(\cdot+x_2))\,(\mathcal{O}_1(\chi\cdot X(\cdot+x_1))-\mathcal{O}_1(\chi\cdot X^{(1)}(\cdot+x_1)))]\\
&+\mathbb{E}[\mathcal{O}_1(\chi\cdot X(\cdot+x_1))\,(\mathcal{O}_2(\chi\cdot X(\cdot+x_2))-\mathcal{O}_2(\chi\cdot X^{(2)}(\cdot+x_2)))],
\end{aligned}$$

where we denote $X^{(i)}=X^{(i)}_\infty$. Thus, for any $\alpha<0$ and $p\in[1,\infty]$,

$$\begin{aligned}
&|\operatorname{Cov}(\mathcal{O}_1(\chi\cdot X(\cdot+x_1)),\mathcal{O}_2(\chi\cdot X(\cdot+x_2)))|\\
\lesssim_{\mathcal{O}_1,\mathcal{O}_2}&\mathbb{E}\Big[\|\chi\cdot(X(\cdot+x_1)-X^{(1)}(\cdot+x_1))\|_{B^{\alpha,-n}_{p,p}}\Big]\\
&\times\mathbb{E}\Big[\|\chi\cdot(X(\cdot+x_1)-X^{(2)}(\cdot+x_1))\|_{B^{\alpha,-n}_{p,p}}\Big].
\end{aligned}$$

It remains to estimate $\|\chi(X(\cdot+x_i)-X^{(i)}(\cdot+x_i))\|_{B^{\alpha,-n}_{p,p}}$, for $i=1,2$. If x_1,x_2 are close, say $\mathfrak{l}\leqslant 8$, then we use the boundedness of the observables to conclude

$$|\operatorname{Cov}(\mathcal{O}_1(\chi\cdot X(\cdot+x_1)),\mathcal{O}_2(\chi\cdot X(\cdot+x_2)))|\lesssim 1.$$

If on the other hand $\mathfrak{l}>8$, then $D_i(1)\subset D_i(\mathfrak{l}/8)$ and thus $\vartheta^{(i)}\equiv 1$ on $\operatorname{supp}(\chi(\cdot-x_i))\subset D_i(1)$ so that $(\bar q^{(i)})^{-1}\leqslant \mathrm{e}^{\gamma m}$ on D_i. Consequently,

$$\|\chi\cdot X(\cdot+x_i)-\chi\cdot X^{(i)}(\cdot+x_i)\|_{B^{\alpha}_{p,p,\ell}}\leqslant \mathrm{e}^{m\gamma}\|\bar q^{(i)}(X-X^i)\|_{L^p(D_i(1))}\lesssim \mathrm{e}^{m\gamma(1-\mathfrak{l}/8)},$$

where the last inequality follows from Lemma 5.1 below. The remainder of this section will be devoted to its proof.

Lemma 5.1. *Let $\mathfrak{l}>8$ and $|\bar\lambda|$ sufficiently small. The solutions $X^{(i)}$ to the FBSDE* (4.1) *driven by $W^{(i)}$ satisfy for some $\gamma<1$,*

$$\mathbb{E}\Big[\sup_t\|\bar q^{(i)}(X_t-X^{(i)}_t)\|_{L^\infty}\Big]\lesssim \mathrm{e}^{-\gamma m\mathfrak{l}/8}.$$

Proof. The FBSDE for the difference is given by

$$\begin{aligned}
X_t-X^{(i)}_t&=-\int_0^t \mathrm{d}s\dot G_s(F_s(X_s)-F_s(X^{(i)}_s))-\int_0^t\mathrm{d}s\dot G_s(R_s-R^{(i)}_s)+W_t-W^{(i)}_t,\\
R_t-R^{(i)}_t&=\int_t^\infty \mathrm{d}s[H_s(X_s)-H_s(X^{(i)}_s)]-\Big(\int_t^\infty \mathrm{d}s\mathrm{D}F_s(X_s)\dot G_sR_s-\mathrm{D}F_s(X^{(i)}_s)\dot G_sR^{(i)}_s\Big).
\end{aligned}$$

The estimate on the Gaussian part $W_t-W^{(i)}_t$ is proven in Lemma 5.4 below. For the drift, $Z_t-Z^{(i)}_t:=(X_t-X^{(i)}_t)-(W_t-W^{(i)}_t)$, we apply Lemma 5.2 and 5.3 to find

$$\begin{aligned}
&\sup_t\|\bar q^{(i)}(Z_t-Z^{(i)}_t)\|_{L^\infty}\\
&\lesssim\int_0^\infty \mathrm{d}s\bar\lambda_s\langle s\rangle^{-2}(\mathrm{e}^{-\gamma m\mathfrak{l}/8}\|X_s-X^{(i)}_s\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))}+\|\bar q^{(i)}(X_s-X^{(i)}_s)\|_{L^\infty})\\
&\quad+\int_0^\infty \mathrm{d}s\langle s\rangle^{-2}(\mathrm{e}^{-\gamma m\mathfrak{l}/8}\|R_s-R^{(i)}_s\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))}+\|\bar q^{(i)}(R_s-R^{(i)}_s)\|_{L^\infty}).
\end{aligned}\tag{5.1}$$

Similarly for the remainder,

$$\begin{aligned}
&\|\bar{q}^{(i)}(R_t - R_t^{(i)})\|_{L^\infty} \\
&\lesssim \int_t^\infty \mathrm{d}s(\langle s\rangle^{-4}\bar{\lambda}_s^4 + \langle s\rangle^{-2}\bar{\lambda}_s)(\mathrm{e}^{-\gamma m \mathfrak{l}/8}\|X_s - X_s^{(i)}\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} + \|\bar{q}^{(i)}(X_s - X_s^{(i)})\|_{L^\infty}) \\
&\quad + \int_t^\infty \mathrm{d}s\bar{\lambda}_s\langle s\rangle^{-2}(\mathrm{e}^{-\gamma m \mathfrak{l}/8}\|R_s - R_s^{(i)}\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} + \|\bar{q}^{(i)}(R_s - R_s^{(i)})\|_{L^\infty}).
\end{aligned} \tag{5.2}$$

In the region $D_i^c(\mathfrak{l}/8)$, we cannot expect the difference $W - W^{(i)}$ to behave any better than the Brownian motion W_t itself. We therefore control the solution in this region using the uniform bounds on the drift from Lemma 4.1 combined with the bounds on the Brownian motion $(W_t)_t$ from Lemma 4.8. This implies for any $\varepsilon > 0$,

$$\begin{aligned}
\|R_s - R_s^{(i)}\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} &\leqslant 2\|R_s\|_{L^\infty} \lesssim 1, \\
\mathbb{E}[\sup_s \langle s\rangle^{-\varepsilon}\|X_s - X_s^{(i)}\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))}] &\leqslant 2\mathbb{E}[\sup_s \|Z_s\|_{L^\infty}] + 2\mathbb{E}[\sup_s \langle s\rangle^{-\varepsilon}\|W_s\|_{L^{\infty,-n}}] \lesssim 1,
\end{aligned}$$

where the estimate on $(W_t)_t$ follows from Lemma 4.8 and

$$\mathbb{E}\left[\sup_{t\geqslant 1}\|t^{-\varepsilon/2}W_t\|_{L^\infty}\right] \lesssim \mathbb{E}\left[\sup_{t\geqslant 1}\|t^{-\varepsilon/2}W_t\|_{B^{\alpha,-n}_{\infty,\infty}}\right], \quad \alpha \in (0,\varepsilon).$$

Moreover, with Lemma 5.4 below,

$$\mathbb{E}\left[\sup_{s\geqslant 0}\|\bar{q}^{(i)}(X_s - X_s^{(i)})\|_{L^\infty}\right] \lesssim \mathbb{E}\left[\sup_{s\geqslant 0}\|\bar{q}^{(i)}(Z_s - Z_s^{(i)})\|_{L^\infty}\right] + \mathrm{e}^{-\gamma m \mathfrak{l}/8}.$$

With these estimates, the equation (5.2) reduces to

$$\begin{aligned}
\mathbb{E}\|\bar{q}^{(i)}(R_t - R_t^{(i)})\|_{L^\infty} \lesssim &\mathrm{e}^{-\gamma m \mathfrak{l}/8} + \mathbb{E}\left[\sup_{s\geqslant 0}\|\bar{q}^{(i)}(Z_s - Z_s^{(i)})\|_{L^\infty}\right] \\
&+ \int_t^\infty \mathrm{d}s\bar{\lambda}_s\langle s\rangle^{-2}\mathbb{E}\|\bar{q}^{(i)}(R_s - R_s^{(i)})\|_{L^\infty},
\end{aligned}$$

which implies by Gronwall's lemma

$$\mathbb{E}\left[\|\bar{q}^{(i)}(R_t - R_t^{(i)})\|_{L^\infty}\right] \lesssim \mathrm{e}^{-\gamma m \mathfrak{l}/8} + \mathbb{E}\left[\sup_{s\geqslant 0}\|\bar{q}^{(i)}(Z_s - Z_s^{(i)})\|_{L^\infty}\right].$$

Taking expectation (5.1) these bounds imply

$$\mathbb{E}\left[\sup_t \|\bar{q}^{(i)}(Z_t - Z_t^{(i)})\|_{L^\infty}\right] \lesssim \int_0^\infty \mathrm{d}s\bar{\lambda}_s\langle s\rangle^{-2+\varepsilon}\mathrm{e}^{-\gamma m \mathfrak{l}/8} + \bar{\lambda}\mathbb{E}\left[\sup_t \|\bar{q}^{(i)}(Z_t - Z_t^{(i)})\|_{L^\infty}\right],$$

so that for $\bar{\lambda}$ sufficiently small after rearranging

$$\mathbb{E}\left[\sup_t \|\bar{q}^{(i)}(Z_t - Z_t^{(i)})\|_{L^\infty}\right] \lesssim \mathrm{e}^{-\gamma m \mathfrak{l}/8}.$$

□

Lemma 5.2. *For any function v, it holds that*

$$\|q^{(i)}\dot{G}_s v\|_{L^\infty} \lesssim \langle s\rangle^{-2}(\mathrm{e}^{-\gamma m \mathfrak{l}/8}\|v\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} + \|\bar{q}^{(i)}v\|_{L^\infty}). \tag{5.3}$$

Proof. First observe that since $\vartheta^{(i)}$ is compactly supported, the weight $\langle x\rangle^{-n}$ is uniformly bounded away from zero and thus $\vartheta^{(i)}(x) \lesssim \vartheta^{(i)}(x)\langle x\rangle^{-n}$. Combined with the triangle inequality and the estimate (1.11) on the polynomial weights, we have

$$\bar{q}^{(i)}(x) = \vartheta^{(i)}(x)\mathrm{e}^{-\gamma m|x-x_i|} \lesssim \vartheta^{(i)}(x)\langle x-y\rangle^n \mathrm{e}^{\gamma m|x-y|}\mathrm{e}^{-\gamma m|y-x_i|}\langle y\rangle^{-n}. \tag{5.4}$$

Thus, for any function v,

$$\begin{aligned}
\|q^{(i)}\dot{G}_s v\|_{L^\infty} &= \sup_x \left| \int \mathrm{d}y \bar{q}^{(i)}(x)\dot{G}_s(x-y)v(y)\right| \\
&\lesssim \sup_x \int \mathrm{d}y \langle x-y\rangle^n \mathrm{e}^{\gamma m|x-y|}\dot{G}_s(x-y)|\mathrm{e}^{-\gamma m|y-x_i|}\langle y\rangle^{-n}v(y)| \\
&\lesssim \sup_x |\mathrm{e}^{-\gamma m|x-x_i|}\langle x\rangle^{-n}v(x)| \sup_x \int \mathrm{d}y \langle y-x\rangle^n \mathrm{e}^{\gamma m|y-x|}\dot{G}_s(y-x).
\end{aligned}$$

If $x \in D_i^c(\mathfrak{l}/8)$, then $|x-x_i| > \mathfrak{l}/8$ so that

$$\mathbb{1}_{D_i^c(\mathfrak{l}/8)}(x)|\mathrm{e}^{-\gamma m|x-x_i|}\langle x\rangle^{-n}v(x)| \leqslant \mathrm{e}^{-\gamma m\mathfrak{l}/8}\|v(x)\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))}.$$

On the other hand, since $\mathbb{1}_{D_i(\mathfrak{l}/8)} \leqslant \vartheta^{(i)}$, for $x \in D_i(\mathfrak{l}/8)$ we can insert $\vartheta^{(i)}$ to find

$$\mathbb{1}_{D_i(\mathfrak{l}/8)}(x)|\mathrm{e}^{-\gamma m|x-x_i|}\langle x\rangle^{-n}v(x)| \lesssim |\mathrm{e}^{-\gamma m|x-x_i|}\vartheta^{(i)}(x)v(x)| = \|\bar{q}^{(i)}v\|_{L^\infty}.$$

Thus, the estimates on $\dot{G}$ from Lemma A.4 conclude with the assumption that $\gamma < 1$. □

Lemma 5.3. *In the same notation as before, the following Lipschitz estimates apply*

$$\|\bar{q}^{(i)}\dot{G}_s(F_s(\varphi)-F_s(\varphi^{(i)}))\|_{L^\infty} \lesssim \bar{\lambda}_s\langle s\rangle^{-2}(\mathrm{e}^{-\gamma m\mathfrak{l}/8}\|\varphi-\varphi^{(i)}\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} + \|\bar{q}(\varphi-\varphi^{(i)})\|_{L^\infty}) \tag{5.5}$$

$$\|\bar{q}^{(i)}(H_s(\varphi)-H_s(\varphi^{(i)}))\|_{L^\infty} \lesssim \bar{\lambda}_s^4\langle s\rangle^{-4}(\mathrm{e}^{-\gamma m\mathfrak{l}/8}\|\varphi-\varphi^{(i)}\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} + \|\bar{q}(\varphi-\varphi^{(i)})\|_{L^\infty}) \tag{5.6}$$

$$\|\bar{q}^{(i)}\mathrm{D}F_s(\varphi)\dot{G}_s R\|_{L^\infty} \lesssim \bar{\lambda}_s\langle s\rangle^{-2}(\mathrm{e}^{-\gamma m\mathfrak{l}/8}\|R\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))} + \|\bar{q}^{(i)}R\|_{L^\infty}). \tag{5.7}$$

Proof. Regarding (5.5) and (5.6), we only show how to commute the weight through a generic term of the Ansatz for the force (3.13). The optimal, field independent bounds on F_s are then obtained in the same way as in the proof of Proposition 3.13. In this case, we need to estimate expressions of the form

$$\sup_{x_1}\left|\int \mathrm{d}x_{2:\ell} f(x_{1:\ell})[\mathrm{e}^{i\beta\varphi(x_{2:\ell})} - \mathrm{e}^{i\beta\varphi^{(i)}(x_{1:\ell})}]\bar{q}^{(i)}(x_1)\right|,$$

where f is one of the (potentially regularised) force coefficients and $x_{1:\ell} \in (\mathbb{R}^2)^\ell$ for some ℓ. To this end, thanks to the boundedness of the complex exponential function, it is sufficient to estimate the terms

$$\sup_{x_1}\left|\int \mathrm{d}x_{2:\ell} f(x_{1:\ell})|\varphi(x_k)-\varphi^{(i)}(x_k)|\bar{q}^{(i)}(x)\right|, \quad k = 1,\dots,\ell.$$

Using (5.4) we obtain,

$$\begin{aligned}
&f(x_{1:\ell})[\mathrm{e}^{i\beta\varphi(x_{2:\ell})}-\mathrm{e}^{i\beta\varphi^{(i)}(x_{1:\ell})}]\bar{q}^{(i)}(x_1)\\
\lesssim& f(x_{1:\ell})\vartheta^{(i)}(x)\langle x-x_k\rangle^n\mathrm{e}^{\gamma m|x-x_k|}(\varphi_s-\varphi_s^i)(x_k)[\mathrm{e}^{-\gamma m\mathfrak{l}/8}\langle x_k^{-m}\rangle\mathbb{1}_{D_i^c<(\mathfrak{l}/8)}(x_k)\\
&+\bar{q}(x_k)\mathbb{1}_{D_i(\mathfrak{l}/8)}(x_k)].\\
\lesssim&[\mathrm{e}^{-\gamma m\mathfrak{l}/8}\|\varphi_s-\varphi_s^i\|_{L^{\infty,-n}(D_i^c(\mathfrak{l}/8))}+\|(\varphi_s-\varphi_s^i)\bar{q}\|_{L^\infty}]f(x_{1:\ell})\mathrm{e}^{\gamma m|x-x_k|}\langle x-x_k\rangle^n.
\end{aligned}\tag{5.8}$$

From here, thanks to the exponential decay of the force in the separation of the points, the estimates on $f_s^{[\ell]}$ obtained in Section 3.3 conclude since by definition of the Steiner weights ω_ς we have for $\varsigma\in(\gamma,1)$,

$$\sup_{x_1}\left|\int \mathrm{d}x_{2:\ell}f(x_{1:\ell})\mathrm{e}^{\gamma m|x-x_k|}\langle x-x_k\rangle^n\right|\lesssim |||f|||\,.$$

Applying exactly the same reasoning as in the proof of Lemma 5.2 to (5.7), we obtain

$$\begin{aligned}
&|\bar{q}(x)(\mathrm{D}F_s(X)\dot{G}_sR)(x)|\\
&\quad\lesssim\int \mathrm{d}y\mathrm{D}F_s(X)(x,y)\langle x-z\rangle^n\mathrm{e}^{\gamma m|x-y|}\int \mathrm{d}z\dot{G}_s(y-z)q^{(i)}(z)\langle z\rangle^{-n}R(z),
\end{aligned}$$

which implies the claim after splitting up the integral in between $D_i^c(\mathfrak{l}/8)$ and $D_i(\mathfrak{l}/8)$. □

Lemma 5.4. *For i=1,2 and $\mathfrak{l}>8$, it holds that*

$$\mathbb{E}[\sup_t\|\bar{q}^{(i)}(W_t-W_t^{(i)})\|_{L^\infty}]\lesssim\mathrm{e}^{-\gamma m\mathfrak{l}/8}.$$

Proof. First note that since D_i is compact, the restricted weight $q|_{D_i}$ is of order 1 in D_i, that is $1\lesssim q|_{D_i}\lesssim 1$. Therefore, by Besov embeddings,

$$\begin{aligned}
\|\bar{q}(W_t-W_t^{(i)})\|_{L^\infty}&\lesssim\|\vartheta^{(i)}(W_t-W_t^{(i)})\|_{L^\infty}\\
&\lesssim\|\vartheta^{(i)}(W_t-W_t^{(i)})\|_{B^{\alpha-\delta}_{\infty,\infty}}\lesssim\|\vartheta^{(i)}(W_t-W_t^{(i)})\|_{B^{\alpha+\delta}_{p,p}},
\end{aligned}$$

provided $0<\delta<\alpha$ and $p>d/\delta$. Following the same logic as in the proof of Lemma 4.8, we have for any $\alpha\in\mathbb{R}$, $p\in[1,\infty)$ thanks to Gaussian hypercontractivity,

$$\begin{aligned}
\mathbb{E}[\sup_t\|\vartheta^{(i)}(W_t-W_t^{(i)})\|_{B^\alpha_{p,p}}]&\lesssim\mathbb{E}[\sup_t\|\vartheta^{(i)}(W_t-W_t^{(i)})\|^p_{B^\alpha_{p,p}}]^{1/p}\\
&\lesssim\mathbb{E}[\sup_t\|\vartheta^{(i)}(W_t-W_t^{(i)})\|^2_{H^\alpha}]^{1/2}.
\end{aligned}$$

Interpolating between L^2 and H^1, it is therefore sufficient to show that

$$\mathbb{E}\left[\sup_t\|\vartheta^{(i)}(W_t-W_t^{(i)})\|^2_{H^1}\right]\lesssim\mathrm{e}^{-m\gamma\mathfrak{l}/8}.$$

Here, we compute similarly to the argument in Lemma A.4 using now the separation $d(D_i^c, D_i(\mathfrak{l}/4)) > \mathfrak{l}/4$,

$$\begin{aligned}
&\mathbb{E}\Big[\|\vartheta^{(i)}(W_t - W_t^{(i)})\|_{H^1}^2\Big] \\
&\leqslant \int_0^t \mathrm{d}s \int_{D_i(\mathfrak{l}/4)} \mathrm{d}x \int_{D_i^c} \mathrm{d}z (|\nabla Q_s(x-z)|^2 + |Q_s(x-z)|^2) \\
&\leqslant \sup_{\substack{x \in D_i(\mathfrak{l}/4) \\ z \in D_i^c}} \mathrm{e}^{-\frac{m}{2}\gamma|x-z|} \int_0^t \mathrm{d}s \int_{D_i(\mathfrak{l}/4)} \mathrm{d}x \int_{D_i^c} \mathrm{d}z \mathrm{e}^{\frac{m}{2}\gamma|x-z|} (|\nabla Q_s(x-z)|^2 + |Q_s(x-z)|^2) \\
&\lesssim \mathrm{e}^{-m\gamma\mathfrak{l}/8} \int_0^t \mathrm{d}s \int_{D_i(\mathfrak{l}/4)} \mathrm{d}x \int_{D_i^c} \mathrm{d}z (s|x-z|+1) \mathrm{e}^{-2s|x-z|^2 + \frac{m}{2}\gamma|x-z| - m^2/s} \\
&\lesssim \mathrm{e}^{-m\gamma\mathfrak{l}/8}.
\end{aligned}$$

Finally, the maximal martingale inequalities allow to take the supremum inside the expectation and we arrive at the claim. □

6. Singularity for $\beta^2 \geqslant 4\pi$

We use the FBSDE (4.1) to show Theorem 1.4, that is that the finite volume sine-Gordon measure and the Gaussian free field are mutually singular for $\beta^2 \geqslant 4\pi$. Our proof relies on the asymptotics for the regularised cosine potential. It is similar in spirit to the method used in [7], but does not rely on a change of measure. We also refer to [33], where the authors show singularity of the Φ_3^3 measure using a variational problem.

Before we go into more detail, let us discuss the heuristics behind our argument. To show that absolute continuity is violated it is enough to find an event S such that $\mu(S) = 0$ but $\nu_{\mathrm{SG}}^{\rho}(S) > 0$. Good candidates for events S are limits of approximations of W and X, chosen such that the almost sure limits are different. If such an event exists, then we also get the reverse singularity result for free: since $\mu(S^c) = 1$ and $\nu_{\mathrm{SG}}^{\rho}(S^c) = 0$, both measures are singular with respect to each other.

A natural choice is an appropriate rescaling of the cosine itself. For W, Gaussian hypercontractivity suggests that as $t \to \infty$ the martingale behaves like

$$[\![\cos(\beta W_t(0))]\!] - 1 = \int_0^t -\beta [\![\sin(\beta W_s(0))]\!] \mathrm{d}W_s(0) \asymp t^{\frac{1}{2}-\delta} \vee 1.$$

For the interacting field $X_t = W_t + Z_t$, the Wick ordered cosine is no longer a martingale as we pick up an additional drift term,

$$[\![\cos(\beta X_t(0))]\!] - 1 = \int_0^t -\beta [\![\sin(\beta X_s(0))]\!] \dot{Z}_s(0) \mathrm{d}s + \int_0^t -\beta [\![\sin(\beta X_s(0))]\!] \mathrm{d}W_s(0).$$

For the martingale part, we expect the same asymptotics as in the Gaussian case. However, the drift should behave like

$$\int_0^t -\beta[\![\sin(\beta X_s(0))]\!]\dot{Z}_s(0)\mathrm{d}s \asymp \int_0^t \langle s\rangle^{2(1-\delta)}\langle s\rangle^{-2}\mathrm{d}s \asymp t^{1-2\delta}\vee 1.$$

Note that there is no blow up for $\delta > \frac{1}{2} \iff \beta^2 < 4\pi$. However, for $\delta \leqslant \frac{1}{2}$, the drift and martingale blow up in the ultraviolet limit at different rates and so we are able to find a rescaling of the Wick ordered cosine which extracts only the divergence coming from the drift. In other words, we want to find $c(t)$ such that as $t \to \infty$,

$$c(t)[\![\cos(\beta W_t(0))]\!] - 1 \to 0, \qquad c(t)[\![\cos(\beta X_t(0))]\!] - 1 \to \infty.$$

To make the above heuristics precise, we have to make some adjustments. First, instead of the pointwise cosine, we will consider an average in space. Moreover, we will have to formulate all of the above only in terms of the terminal values X_∞ and W_∞ correspond to the physically meaningful EQFT measures. The bulk of this section then consists in showing that the divergence heuristics above are precise and that the drift and the martingale can be separated by tuning $c(t)$. The result is the following theorem.

Theorem 6.1. *For*

$$\gamma \in \begin{cases} (1/2,1), & \delta = 1/2 \\ 2(1/2-\delta\vee 1-3\delta, 1-2\delta), & \delta \in (\frac{1}{4},\frac{1}{2}) \end{cases},$$

define

$$r(\varepsilon) = \begin{cases} \log(\varepsilon^{-2}\vee 1)^{-\gamma}, & \delta = 1/2 \\ \varepsilon^{\gamma}, & \delta \in (\frac{1}{4},\frac{1}{2}) \end{cases},$$

and let $\chi^\varepsilon = \varepsilon^{-2}\chi(\cdot/\varepsilon)$ be a standard mollifier with radially symmetric and compactly supported Fourier transform $\hat{\chi}^\varepsilon = \chi(\varepsilon\cdot)$. Define the observable

$$U^\varepsilon(\varphi) = \int r(\varepsilon)\left(\mathrm{e}^{\frac{\beta^2}{2}G^\varepsilon(0)}\cos(\beta(\chi^\varepsilon * \varphi)(x)) - 1\right)\rho(x)\mathrm{d}x, \tag{6.1}$$

where $G^\varepsilon = \mathrm{Cov}(\chi^\varepsilon * W_\infty)$ is the covariance of the mollified Gaussian free field. Then, there is a subsequence $\varepsilon_n \to 0$ such that

$$|U^{\varepsilon_n}(Z_\infty + W_\infty)| \overset{n\to\infty}{\longrightarrow} \infty \quad \textit{while} \quad U^{\varepsilon_n}(W_\infty) \overset{n\to\infty}{\longrightarrow} 0, \tag{6.2}$$

where $Z_\infty = Z_\infty^\rho$ solves (4.1).

Before we prove this statement, let us note the following consequence.

Corollary 6.2. *For $\delta \leqslant 1/2$, that is $\beta^2 \geqslant 4\pi$, the finite volume sine-Gordon measure and the Gaussian free field are mutually singular.*

Proof. For some $\alpha > 0$ arbitrarily small, define the event

$$S = \left\{ \varphi \in H^{-\alpha}(\mathbb{R}^2) : \lim_{n\to\infty} U^{\varepsilon_n}(\varphi) = 0 \right\},$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a suitable but fixed subsequence and U^ε is defined as in (6.1). It follows from Theorem 6.1 that $\nu^\rho_{\mathrm{SG}}(S) = 0 = \mu(S^c)$ while $\mu(S) = 1 = \nu^\rho_{\mathrm{SG}}(S^c)$, which implies the claim. □

Proof of Theorem 6.1. Let $(U^\varepsilon_t)_{t\geqslant 0}$ be the scale interpolation of the functional U^ε such that $(U^\varepsilon_t(W_t))_t$ is a martingale, that is

$$U^\varepsilon_t(\varphi) = \int r(\varepsilon)\left(\mathrm{e}^{\frac{\beta^2}{2} G^\varepsilon_t(0)} \cos(\beta(\chi^\varepsilon * \varphi)(x)) - 1 \right)\rho(x)\mathrm{d}x, \tag{6.3}$$

where $G^\varepsilon_t = \mathrm{Cov}(\chi^\varepsilon * W_t)$. For convenience, we always assume $\varepsilon < 1$ and we also write $\lambda^\varepsilon_t := \mathrm{e}^{\frac{\beta^2}{2} G^\varepsilon_t(0)}$, $\bar\lambda^\varepsilon_t := |\lambda^\varepsilon_t|$, $\varphi^\varepsilon = \chi^\varepsilon * \varphi$, and $W^\varepsilon = \chi^\varepsilon * W$. It follows from Ito's formula that

$$\begin{aligned} U^\varepsilon(Z_\infty + W_\infty) = &\int_0^\infty \mathrm{d}s \left(\partial_s U^\varepsilon_s + \frac{1}{2}\mathrm{Tr}(\dot G^\varepsilon_s \mathrm{D}^2 U^\varepsilon_s) \right)(Z_s + W_s) && (\mathrm{I}^\varepsilon) \\ &+ \int_0^\infty \mathrm{d}s \mathrm{D}U^\varepsilon_s(Z_s + W_s)\dot Z_s && (\mathrm{II}^\varepsilon) \\ &+ \int_0^\infty \mathrm{D}U^\varepsilon_s(Z_s + W_s)\mathrm{d}W_s. && (\mathrm{III}^\varepsilon) \end{aligned}$$

Thanks to the choice of the interpolation (6.3), the term (I^ε) vanishes for all ε.

$(\mathrm{III}^\varepsilon)$ Let for $\varphi \in \{X, W\}$,

$$M^\varepsilon_\infty = \int_0^\infty \mathrm{D}U^\varepsilon_s(\varphi_s)\mathrm{d}W_s = -\beta r(\varepsilon)\int_0^\infty \bar\lambda^\varepsilon_s \int \mathrm{d}x \rho(x)\sin(\beta\varphi^\varepsilon_s(x))\mathrm{d}W^\varepsilon_s(x),$$

so that

$$\begin{aligned} &\mathbb{E}|M^\varepsilon_\infty|^2 \\ &= \beta^2 r(\varepsilon)^2 \mathbb{E}\int_0^\infty (\bar\lambda^\varepsilon_s)^2 \int \mathrm{d}x_1 \rho(x_1)\int \mathrm{d}x_2 \rho(x_2)\sin(\beta\varphi^\varepsilon(x_1))\sin(\beta\varphi^\varepsilon(x_2))\mathrm{d}\langle W^\varepsilon(x_1)W^\varepsilon(x_2)\rangle_s, \end{aligned}$$

where we compute

$$\mathrm{d}\langle W^\varepsilon(x_1)W^\varepsilon(x_2)\rangle_s = \int \mathrm{d}y_1 \int \mathrm{d}y_2 \chi_\varepsilon(x_1 - y_1)\dot G_s(y_1 - y_2)\chi_\varepsilon(x_2 - y_2)\mathrm{d}s.$$

Using $\chi^\varepsilon = \varepsilon^{-2}\chi(\cdot/\varepsilon)$ and $\|\chi^\varepsilon\|_{L^p} \lesssim \varepsilon^{-2(1-1/p)}$ we obtain with Young's convolution in-

equalities,

$$\begin{aligned}\mathbb{E}|M_\infty^\varepsilon|^2 &\leqslant \beta^2 r(\varepsilon)^2 \int_0^\infty \mathrm{d}s(\bar\lambda_s^\varepsilon)^2 \int \mathrm{d}x_1\rho(x_1)\int \mathrm{d}x_2\rho(x_2)\times\\ &\quad\times \int \mathrm{d}y_1 \int \mathrm{d}y_2 \chi_\varepsilon(x_1-y_1)\dot G_s(y_1-y_2)\chi_\varepsilon(x_2-y_2)\\ &\leqslant \beta^2 r(\varepsilon)^2 \int_0^\infty \mathrm{d}s(\bar\lambda_s^\varepsilon)^2 \|\chi^\varepsilon\|_{L^1}\|\chi^\varepsilon\|_{L^1}\|\dot G_s\|_{L^1}\\ &\lesssim r(\varepsilon)^2 \int_0^\infty \mathrm{d}s(\bar\lambda_s^\varepsilon)^2 s^{-2}\mathrm{e}^{-m^2/s}.\end{aligned}$$

Using Lemma 6.3 below,

$$\begin{aligned}\mathbb{E}|M_\infty^\varepsilon|^2 &\lesssim r(\varepsilon)^2 \int_0^{\varepsilon^{-2}} \mathrm{d}s s^{-2}\mathrm{e}^{-m^2/s}s^{2(1-\delta)} + r(\varepsilon)^2\varepsilon^{-4(1-\delta)}\int_{\varepsilon^{-2}}^\infty s^{-2}\mathrm{d}s\\ &\lesssim r(\varepsilon)^2 \begin{cases}\log(\varepsilon^{-2}\vee 1)+1, & \delta = 1/2,\\ \varepsilon^{-2+4\delta}+1 & \delta<1/2.\end{cases}\end{aligned}$$

Combined, choosing $\gamma > 1/2$ implies for $\delta = 1/2$,

$$\mathbb{E}|M_\infty^\varepsilon|^2 \lesssim r(\varepsilon)^2(\log(\varepsilon^{-2})+1) = \log(\varepsilon^{-2})^{-2\gamma}(\log(\varepsilon^{-2})+1)\xrightarrow{\varepsilon\to 0} 0.$$

Otherwise, if $\delta < 1/2$,

$$\mathbb{E}|M_\infty^\varepsilon|^2 \lesssim \varepsilon^{2\gamma-2+4\delta}\xrightarrow{\varepsilon\to 0} 0,$$

provided $\gamma > 2(1/2-\delta)$. Passing to a subsequence, this implies $M_\infty^{\varepsilon_n}\to 0$ almost surely and consequently also $\sup_n |M^{\varepsilon_n}| < \infty$ almost surely. For the Gaussian, we can set $Z\equiv 0$ to find $U^{\varepsilon_n}(W_\infty) = M_\infty^{\varepsilon_n}$, which gives the second claim in (6.2).

(II$^\varepsilon$) To get started, we split this term into the two parts,

$$\begin{aligned}(\mathrm{II}^\varepsilon) &= -\beta r(\varepsilon)\int_0^\infty \mathrm{d}s\int \mathrm{d}x\rho(x)\sin(\beta\varphi^\varepsilon(x))\chi_\varepsilon * \dot G_s F_s^{[1]}(\varphi_s)\\ &\quad -\beta r(\varepsilon)\int_0^\infty \mathrm{d}s\lambda_s^\varepsilon\int \mathrm{d}x\rho(x)\sin(\beta\varphi^\varepsilon(x))\chi_\varepsilon * \dot G_s[(F_s - F_s^{[1]})(\varphi_s)+R_s]\\ &=: (\mathrm{II}_1^\varepsilon)+(\mathrm{II}_{\geqslant 2}^\varepsilon).\end{aligned}$$

(II$_{\geqslant 2}^\varepsilon$) We claim that under the assumptions of Theorem 6.1, this term is uniformly bounded in $\varepsilon > 0$ and Ω. Indeed, using again the estimate on λ_s^ε from Lemma 6.3, we compute

with Proposition 3.13 and the a priori estimates on the remainder (Proposition 4.1),

$$\begin{aligned}
(\mathrm{II}^{\varepsilon}_{\geqslant 2}) &\leqslant r(\varepsilon)\int_0^{\infty} \mathrm{d}s\bar{\lambda}^{\varepsilon}_s \left|\beta\int \mathrm{d}x\rho(x)\sin(\beta\varphi^{\varepsilon}(x))(\chi_{\varepsilon}*\dot{G}_s)[(F_s-F^{[1]}_s)(\varphi_s)+R_s]\right| \\
&\lesssim r(\varepsilon)\|\chi_{\varepsilon}\|_{L^1}\int_0^{\infty}\mathrm{d}s\bar{\lambda}^{\varepsilon}_s\|\dot{G}_s[(F_s-F^{[1]}_s)(\varphi_s)+R_s]\|_{L^{\infty}} \\
&\lesssim r(\varepsilon)\left[1+\int_1^{\varepsilon^{-2}\vee 1}\mathrm{d}s\langle s\rangle^{(1-\delta)}\langle s\rangle^{-2}\langle s\rangle^{1-2\delta}\right]+r(\varepsilon)\varepsilon^{-2(1-\delta)}\int_{\varepsilon^{-2}}^{\infty}\mathrm{d}s\langle s\rangle^{-2}\langle s\rangle^{1-2\delta}. \\
&\lesssim r(\varepsilon)((\langle s\rangle^{1-3\delta})|_{s=1}^{s\equiv\varepsilon^{-2}}+1)+r(\varepsilon)\varepsilon^{-2+6\delta} \\
&\lesssim r(\varepsilon)(\varepsilon^{-2+6\delta}+1)+r(\varepsilon)\varepsilon^{-2+6\delta}.
\end{aligned}$$

For $\delta = 1/2$, we see that $-2+6\delta = 1 > 0$ so that $\sup_{\varepsilon>0}(\mathrm{II}^{\varepsilon}_{\geqslant 2}) < \infty$. For $\delta < 1/2$, the assumption $\gamma > 2(1-3\delta)$ implies the analogous bound.

($\mathrm{II}^{\varepsilon}_1$) We show that for γ small enough according to the assumptions, this term can be split into a divergent term, and uniformly bounded almost surely finite term. To get started note that

$$U^{\varepsilon}_t(\varphi) = \frac{\lambda^{\varepsilon}_t}{2}\int\rho(x)\left(\sum_{\sigma=\pm 1}(\mathrm{e}^{i\sigma\beta\varphi^{\varepsilon}(x)}-1)\right)\mathrm{d}x,$$

so that in the same way as in Section 3, we find

$$\begin{aligned}
&\mathrm{D}U_t(\varphi_t)\dot{G}_tF^{[1]}_t(\varphi_t) \\
=&-\sum_{\sigma_1,\sigma_2\in\{\pm 1\}}\int\mathrm{d}x_1\rho(x_1)\int\mathrm{d}x_2\lambda^{\varepsilon}_t\lambda_t\mathrm{e}^{i\beta(\sigma_1\varphi^{\varepsilon}_t(x_1)+\sigma_2\varphi_t(x_2))}\sigma_1\sigma_2\beta^2(\chi^{\varepsilon}*\dot{G}_t)(x_1-x_2).
\end{aligned}$$

Motivated by the renormalisation constant produced by the neutral contribution (c.f. Section 3.4), we treat the summands for the charged case,

$$\mathcal{C}^{\varepsilon}_t = -\sum_{\sigma_1=\sigma_2\in\{\pm 1\}}\int\mathrm{d}x_1\rho(x_1)\int\mathrm{d}x_2\lambda^{\varepsilon}_t\lambda_t\mathrm{e}^{i\beta(\sigma_1\varphi^{\varepsilon}_t(x_1)+\sigma_2\varphi_t(x_2))}\sigma_1\sigma_2\beta^2(\chi^{\varepsilon}*\dot{G}_t)(x_1-x_2),$$

and the neutral case

$$\mathcal{N}^{\varepsilon}_t = -\sum_{\sigma_1=-\sigma_2\in\{\pm 1\}}\int\mathrm{d}x_1\rho(x_1)\int\mathrm{d}x_2\lambda^{\varepsilon}_t\lambda_t\mathrm{e}^{i\beta(\sigma_1\varphi^{\varepsilon}_t(x_1)+\sigma_2\varphi_t(x_2))}\sigma_1\sigma_2\beta^2(\chi^{\varepsilon}*\dot{G}_t)(x_1-x_2),$$

separately.

($\mathcal{C}^{\varepsilon}_t$). We start by rewriting this sum again as a trigonometric function,

$$\mathcal{C}^{\varepsilon}_t = r(\varepsilon)\int\mathrm{d}z\beta^2(\chi^{\varepsilon}*\dot{G}_t)(z)\lambda_t\lambda^{\varepsilon}_t\int\mathrm{d}x\rho(x)\cos(\beta(\varphi_t(z-x)+\varphi^{\varepsilon}_t(x))),$$

where we can add and subtract $\varphi_t(x)$ to obtain using the trigonometric identities,

$$\begin{aligned}\cos(\beta(\varphi_t(z-x)+\varphi_t^\varepsilon(x))) &= \cos(\beta(\varphi_t(z-x)+\varphi_t(x)))\cos(\beta(\varphi_t^\varepsilon(x)-\varphi_t(x)))\\ &\quad -\sin(\beta(\varphi_t(z-x)+\varphi_t(x)))\sin(\beta(\varphi_t^\varepsilon(x)-\varphi_t(x))).\end{aligned}$$

Since both of these terms are estimated in the exact same way, let us only consider the contribution coming from the cosine. Here, we apply the trigonometric identities again, now for $\varphi_t = Z_t + W_t$, to rewrite

$$\begin{aligned}\cos(\beta(\varphi_t^\varepsilon(x)-\varphi_t(x))) &= \cos(\beta(Z_t^\varepsilon(x)-Z_t(x)))\cos(\beta(W_t^\varepsilon(x)-W_t(x)))\\ &\quad -\sin(\beta(Z_t^\varepsilon(x)-Z_t(x)))\sin(\beta(W_t^\varepsilon(x)-W_t(x))).\end{aligned}$$

Use the trivial estimate $|\cos(\beta(W_t^\varepsilon(x)-W_t(x)))|\leqslant 1$ for the contribution from the GFF while we use the additional regularity $\sup_t \|Z_t\|_{B^{2\delta-}_{\infty,\infty}} < \infty$ (see Theorem 4.6) in the drift Z_t to get the improved bound

$$\sup_x |\cos(\beta(Z_t^\varepsilon(x)-Z_t(x)))| \lesssim \varepsilon^{\gamma_1}\|Z_t\|_{B^{\gamma_2}_{\infty,\infty}}, \quad \text{provided } \gamma_1 < 2\delta.$$

It remains to deal with $\int \mathrm{d}x\rho(x)\cos(\beta(\varphi_t(z-x)+\varphi_t(x)))$, for which we follow the same procedure,

$$\begin{aligned}\cos(\beta(\varphi_t(z-x)+\varphi_t(x))) &= \cos(\beta(Z_t(z-x)+Z_t(x)))\cos(\beta(W_t(z-x)+W_t(x)))\\ &\quad -\sin(\beta(Z_t(z-x)+Z_t(x)))\sin(\beta(W_t(z-x)+W_t(x))),\end{aligned}$$

so that for any $s>0$,

$$\begin{aligned}&\int \mathrm{d}x\rho(x)[\![\cos(\beta(\varphi_t(z-x)+\varphi_t(x)))]\!]\\ \lesssim &\|\cos(\beta(Z_t(z-\cdot)+Z_t(\cdot)))\|_{B^s_{p,p}}\|[\![\cos(\beta(W_t(z-\cdot)+W_t(\cdot)))]\!]\rho(\cdot)\|_{B^{-s}_{q,q}}\\ &+\|\sin(\beta(Z_t(z-\cdot)+Z_t(\cdot)))\|_{B^s_{p,p}}\|[\![\sin(\beta(W_t(z-\cdot)+W_t(\cdot)))]\!]\rho(\cdot)\|_{B^{-s}_{q,q}}\end{aligned}$$

Here, we defined the Wick ordered cosine with respect to the Gaussian $W_t(z-x)\pm W_t(x)$ in the usual way,

$$[\![\cos(\beta(\varphi_t(z-x)\pm\varphi_t(x)))]\!] := \mathrm{e}^{\frac{\beta^2}{2}\mathbb{E}[|W_t(z-x)\pm W_t(x)|^2]}\cos(\beta(\varphi_t(z-x)\pm\varphi_t(x))), \quad (6.4)$$

and similarly for the sine. In what follows we will focus on the cosine term. The sine can be dealt with in the same way. It follows from Lemma 6.5 below that for any $\gamma_1>0$, $\gamma_3 > 2-3\delta$ and $\alpha<2\delta$ sufficiently close to 2δ,

$$\sup_{z,\varepsilon}\mathrm{e}^{-\beta^2 G_t(z)}t^{-\gamma_3}|z|^{\gamma_1/2}\|[\![\cos(\beta(W_t(z-\cdot)+W_t(\cdot)))]\!]\rho(\cdot)\|_{B^{-\alpha}_{q,q}} < \infty \quad a.s.$$

Moreover, from Theorem 4.6, $\sup_t \|\cos(\beta(Z_t(z-\cdot)+Z_t(\cdot)))\|_{B^\alpha_{p,p}} \lesssim 1$ for any $\alpha<2\delta$ so that by a Kolmogorov argument,

$$\mathbb{M}^c := \sup_{z,\varepsilon}|z|^{\gamma_1}\mathrm{e}^{-\beta^2 G_t(z)}t^{-\gamma_3}\int \mathrm{d}x\rho(x)[\![\cos(\beta(\varphi_t(z-x)+\varphi_t(x)))]\!] < \infty \quad a.s.$$

Combined, this implies

$$\begin{aligned}|\mathcal{C}_t^\varepsilon| &\leqslant \left| r(\varepsilon) \int \mathrm{d}z \beta^2 (\chi^\varepsilon * \dot{G}_t)(z) \bar{\lambda}_t \bar{\lambda}_t^\varepsilon \int \mathrm{d}x \rho(x) \cos(\beta(\varphi_t(z-x) + \varphi_t^\varepsilon(x))) \right| \\ &\lesssim r(\varepsilon) \varepsilon^{\gamma_2} \|\dot{G}_t(z)/|z|^{\gamma_1}\|_{L^1(\mathrm{d}z)} \|\chi^\varepsilon\|_{L^1} \bar{\lambda}_t \bar{\lambda}_t^\varepsilon \bar{\lambda}_t^{-2} t^{\gamma_3} \mathbb{M}^c \\ &\lesssim r(\varepsilon) \varepsilon^{\gamma_2} \langle t \rangle^{-2+\gamma_1/2+\gamma_3} \bar{\lambda}_t^\varepsilon \bar{\lambda}_t^{-1} \mathbb{M}^c.\end{aligned}$$

Integrating over the scales and using the usual estimate for λ_t^ε for $c \in (0,1)$ we find

$$\int_0^\infty |\mathcal{C}_t^\varepsilon| \mathrm{d}t \lesssim_{\mathbb{M}^c} \quad r(\varepsilon) \varepsilon^{\gamma_2} \int_0^\infty \mathrm{d}t \langle t \rangle^{-3+\gamma_1/2+\gamma_3+\delta} \bar{\lambda}_s^\varepsilon.$$

For $\delta = 1/2$, we find for $\gamma_3 > 2 - 3\delta$ and $\gamma_1 > 0$ sufficiently small for some $\bar{\gamma} \in (0, 1/2)$,

$$\int_0^\infty \mathrm{d}t \langle t \rangle^{-3+\gamma_1/2+\gamma_3+\delta} \langle t \rangle^{1-\delta} \leqslant \int_0^\infty \mathrm{d}t \langle t \rangle^{-3/2+\bar{\gamma}} < \infty,$$

so that for any $\gamma > 0$,

$$\sup_{\varepsilon \in (0,1)} \int_0^\infty |\mathcal{C}_t^\varepsilon| \mathrm{d}t \lesssim C(\mathbb{M}^c) \sup_{\varepsilon \in (0,1)} \log(\varepsilon^{-2})^{-\gamma} \varepsilon^{2\delta} < \infty, \quad a.s.$$

For $\delta < 1/2$, choosing $\gamma_1 > 0$, $\gamma_3 > 2 - 3\delta$ sufficiently small, we find for some $\bar{\gamma}$

$$\begin{aligned}&\int_0^\infty |\mathcal{C}_t^\varepsilon| \mathrm{d}t \\ &\lesssim r(\varepsilon) \varepsilon^{\gamma_2} \int_0^\infty \mathrm{d}t \langle t \rangle^{-3+\gamma_1/2+\gamma_3+\delta} \bar{\lambda}_s^\varepsilon \\ &= r(\varepsilon) \varepsilon^{\gamma_2} \int_0^{\varepsilon^{-2}} \langle t \rangle^{-2+\gamma_1/2+\gamma_3} + r(\varepsilon) \varepsilon^{\gamma_2} \varepsilon^{-2(1-\delta)} \int_{\varepsilon^{-2}}^\infty \langle t \rangle^{-3+\gamma_1/2+\gamma_3+\delta} \\ &\lesssim r(\varepsilon) \varepsilon^{2\delta} (1 + \varepsilon^{-2+6\delta-\bar{\gamma}}),\end{aligned}$$

so that the right-hand side is uniformly bounded provided $\gamma > 2(1-4\delta)$.

($\mathcal{N}_t^\varepsilon$). In the same way as for $\mathcal{C}_t^\varepsilon$, we rewrite $\mathcal{N}_t^\varepsilon$ in terms of the trigonometric functions

$$\mathcal{N}_t^\varepsilon = r(\varepsilon) \int \mathrm{d}z \beta^2 (\chi^\varepsilon * \dot{G}_t)(z) \lambda_t \lambda_t^\varepsilon \int \mathrm{d}x \rho(x) \cos(\beta(\varphi_t(z-x) - \varphi_t^\varepsilon(x))),$$

In contrast to the charged contribution, the Wick ordering (6.4) now introduces a divergent contribution instead. Therefore, we split $\mathcal{N}_t^\varepsilon$ once more as

$$\begin{aligned}\mathcal{N}_t^\varepsilon =& r(\varepsilon) \int \mathrm{d}z \beta^2 (\chi^\varepsilon * \dot{G}_t)(z) \lambda_t \lambda_t^\varepsilon \int \mathrm{d}x \rho(x) \\ &+ r(\varepsilon) \int \mathrm{d}z \beta^2 (\chi^\varepsilon * \dot{G}_t)(z) \lambda_t \lambda_t^\varepsilon \int \mathrm{d}x \rho(x) (\cos(\beta(\varphi_t(z-x) - \varphi_t^\varepsilon(x))) - 1) \\ =:& c_t^\varepsilon + r(\varepsilon) \int \mathrm{d}z \beta^2 (\chi^\varepsilon * \dot{G}_t)(z) \lambda_t \lambda_t^\varepsilon \int \mathrm{d}x \rho(x) (\cos(\beta(\varphi_t(z-x) - \varphi_t^\varepsilon(x))) - 1).\end{aligned}$$

We claim that under the assumptions on $r(\varepsilon)$, the constant c_t^ε diverges, while the difference $\sup_{\varepsilon\in(0,1)}|\mathcal{N}_t^\varepsilon - c_t^\varepsilon|$ is almost surely finite. Indeed, from the asymptotics of G^ε in Lemma 6.3 below, it follows that for any $c>0$, and a constant C allowed to change from line to line,

$$\begin{aligned} c_t^\varepsilon &= r(\varepsilon)\int \mathrm{d}z\beta^2(\chi^\varepsilon * \dot{G}_t)(z)\lambda_t\lambda_t^\varepsilon = Cr(\varepsilon)\|\chi^\varepsilon\|_{L^1}\|\dot{G}_t\|_{L^1}(\lambda_t\lambda_t^\varepsilon + O(1)) \\ &= Cr(\varepsilon)\langle t\rangle^{-2}(\lambda_t\lambda_t^\varepsilon + O(1)), \end{aligned}$$

where we used that $\|\chi^\varepsilon\|_{L^1} = \|\chi\|_{L^1}$. We again split the integral over the scales at ε^{-2} to extract the divergent contribution, using the bounds λ_t^ε from Lemma 6.3,

$$\begin{aligned} \int_0^{\varepsilon^{-2}} c_t^\varepsilon \mathrm{d}t &= \beta^2 r(\varepsilon)\int_0^{\varepsilon^{-2}} \mathrm{d}t\langle t\rangle^{-1-\delta}\langle t\rangle^{1-\delta} + O(1), \\ &= \beta^2 r(\varepsilon)\int_0^{\varepsilon^{-2}} \mathrm{d}t\langle t\rangle^{-2\delta} + O(1) \\ &= \begin{cases} C\log(\varepsilon^{-2})^{-\gamma}(\log(\varepsilon^{-2}) + O(1)), & \delta = 1/2, \\ Cr(\varepsilon)(1+\varepsilon^{-2+4\delta}), & \delta < 1/2, \end{cases} \end{aligned}$$

while

$$\int_{\varepsilon^{-2}}^\infty c_t^\varepsilon \mathrm{d}t = Cr(\varepsilon)\varepsilon^{-2(1-\delta)}\int_{\varepsilon^{-2}}^\infty \langle t\rangle^{-1-\delta}\mathrm{d}t = Cr(\varepsilon)\varepsilon^{-2+2\delta} = \begin{cases} Cr(\varepsilon), & \delta = 1/2, \\ C\varepsilon^{\gamma-2+2\delta}, & \delta < 1/2. \end{cases}$$

Combined this implies with the assumptions $\gamma<1$, in case $\delta = 1/2$ and $\gamma < 2(1-2\delta)$, that

$$\int_0^\infty c_t^\varepsilon \mathrm{d}t = \left.\begin{cases} C\log(\varepsilon^{-2})^{-\gamma}(\log(\varepsilon^{-2}) + O(1)), & \delta = 1/2 \\ Cr(\varepsilon)(1+\varepsilon^{-2+4\delta}), & \delta < 1/2 \end{cases}\right\} \xrightarrow{\varepsilon\to 0} \infty.$$

It remains to show that $\sup_{\varepsilon\in(0,1)}\int_0^\infty |\mathcal{N}_t^\varepsilon - c_t^\varepsilon|\mathrm{d}t < \infty$ almost surely. To this end, we Taylor expand the cosine as

$$(\cos(\beta(\varphi_t(z-x) - \varphi_t^\varepsilon(x))) - 1) = |\varphi_t(z-x) - \varphi_t^\varepsilon(x)|^2\int_0^1 \mathrm{d}\vartheta\cos(\vartheta\beta((\varphi_t(z-x) - \varphi_t^\varepsilon(x)))).$$

Recall that for any $\alpha\in(0,1)$, $\sup_t \left\|\langle t\rangle^{-\alpha/2}\varphi_t\right\|_{B^{\alpha-}_{\infty,\infty}} < \infty$ almost surely by Lemma 4.8, so that for any γ_1

$$|\varphi_t(z-x) - \varphi_t^\varepsilon(x)| \lesssim \left(|z|^2\|\varphi_t\|_{B^2_{\infty,\infty}} + \varepsilon^{\gamma_1}\|\varphi_t\|_{B^{\gamma_1}_{\infty,\infty}}\right) \lesssim \mathbb{X}(|z|^2\langle t\rangle^{1+} + \varepsilon^{\gamma_1}\langle t\rangle^{\gamma_1/2+}),$$

where $\mathbb{X} := \sup_t \|\langle t\rangle^{-\gamma_1/2-}\varphi_t\|_{B^{\gamma_1}_{\infty,\infty}} \vee \sup_t \|\langle t\rangle^{-1-}\varphi_t\|_{B^2_{\infty,\infty}}$ is almost surely finite.

Regarding the cosine term, we proceed similarly to the charged case, repeatedly applying the trigonometric identities

$$\begin{aligned} &\cos(\vartheta\beta((\varphi_t(z-x) - \varphi_t^\varepsilon(x)))) \\ =&\cos(\vartheta\beta((\varphi_t(z-x) - \varphi_t(x))))\cos(\vartheta\beta((\varphi_t(x) - \varphi_t^\varepsilon(x)))) \\ &-\sin(\vartheta\beta((\varphi_t(z-x) - \varphi_t(x))))\sin(\vartheta\beta((\varphi_t(x) - \varphi_t^\varepsilon(x)))). \end{aligned}$$

As before, we restrict our attention to the cosine term, with the analysis for the sines being analogous. The difference due to the mollification can be repeated verbatim to obtain

$$\cos(\vartheta\beta((\varphi_t(x)-\varphi_t^\varepsilon(x)))) \lesssim \varepsilon^{\gamma_2}\|Z_t\|_{B^{\gamma_2}_{\infty,\infty}},$$

knowing that $\sup_t \|Z_t\|_{B^{\gamma_2}_{\infty,\infty}} < \infty$ almost surely provided $\gamma_2 < 2\delta$. For the remaining term, we again insert the Wick ordering and apply the trigonometric identities for $\varphi = Z + W$ to obtain

$$\begin{aligned}&\int \mathrm{d}x\rho(x)[\![\cos(\vartheta\beta((\varphi_t(z-x)-\varphi_t(x))))]\!]\\ \leqslant&\|[\![\cos(\vartheta\beta(W_t(z-\cdot)-W_t(\cdot)))]\!]\rho(x)\|_{L^1}\|\cos(\vartheta\beta(Z_t(z-\cdot)-Z_t(\cdot)))\|_{L^\infty}.\end{aligned}$$

It follows from Lemma A.1 and Lemma 6.4 below that

$$\|\cos(\vartheta\beta(Z_t(z-\cdot)-Z_t(\cdot)))\|_{L^\infty} \lesssim \|Z_t(z-\cdot)-Z_t(\cdot)\|_{L^\infty} \lesssim t^{1/2-\delta}|z|.$$

Combined with Lemma 6.5, for $\delta' = 1-(\beta\vartheta)^2/8\pi$, $\gamma_3 > -1+4(1-\delta')$ and $\gamma_4 > 1/2$

$$\int \mathrm{d}x\rho(x)[\![\cos(\vartheta\beta((\varphi_t(z-x)-\varphi_t(x))))]\!] \leqslant t^{\gamma_4+1/2-\delta}|z|^{-\gamma_3+1}\mathrm{e}^{-\vartheta\beta^2 G_t(z)}\mathbb{M}^n,$$

where

$$\mathbb{M}^n := \sup_{t,z} t^{-\gamma_4}|z|^{\gamma_3}\mathrm{e}^{\vartheta\beta^2 G_t(z)}\|\rho(\cdot)[\![\cos(\vartheta\beta(W_t(z-\cdot)-W_t(\cdot)))]\!]\|_{L^1} < \infty, \quad a.s.$$

Combined, for some implicit random but almost surely finite constant depending on $\mathbb{X}, \mathbb{M}^n$ and $\|Z_t\|_{B^{\gamma_2}_{\infty,\infty}}$,

$$\begin{aligned}&|\mathcal{N}_t^\varepsilon - c_t^\varepsilon|\\ \lesssim& r(\varepsilon)\varepsilon^{\gamma_2}\int \mathrm{d}z(\chi^\varepsilon * \dot{G}_t)(z)\bar{\lambda}_t\bar{\lambda}_t^\varepsilon(|z|^2\langle t\rangle^{1+} + \varepsilon^{\gamma_1}\langle t\rangle^{\gamma_1/2+})t^{-\gamma_4+1/2-\delta}|z|^{-\gamma_3+1}\mathrm{e}^{-(\beta\vartheta)^2 G_t(0)}.\end{aligned} \tag{6.5}$$

Since $2-4(1-\delta) > -d$, we can choose $\tilde{\gamma}_3 = -\gamma_3+1 < 2-4(1-\delta')$ sufficiently large so that $\tilde{\gamma}_3 > -d$ and

$$\varepsilon^2\int(\chi^\varepsilon * \dot{G}_t)(z)|z|^{\tilde{\gamma}_3} \lesssim \varepsilon^{2+\tilde{\gamma}_3}\langle t\rangle^{-2},$$

and in the same way

$$\int(\chi^\varepsilon * \dot{G}_t)(z)|z|^{\tilde{\gamma}_3+2} \lesssim \langle t\rangle^{-2-\tilde{\gamma}_3/2-1}.$$

Inserting these bounds in (6.5), we obtain

$$\begin{aligned}|\mathcal{N}_t^\varepsilon - c_t^\varepsilon| \lesssim& r(\varepsilon)\varepsilon^{\gamma_2}\int_0^1 \mathrm{d}\vartheta\int \mathrm{d}z(\chi^\varepsilon * \dot{G}_t)(z)|z|^{-\tilde{\gamma}_3+2}\bar{\lambda}_t\bar{\lambda}_t^\varepsilon t^{\gamma_4+1/2-\delta+1}\mathrm{e}^{-(\beta\vartheta)^2 G_t(0)}\\ &+ r(\varepsilon)\varepsilon^{\gamma_2}\varepsilon^{\gamma_1}\int_0^1 \mathrm{d}\vartheta\int \mathrm{d}z(\chi^\varepsilon * \dot{G}_t)(z)|z|^{-\tilde{\gamma}_3}\bar{\lambda}_t\bar{\lambda}_t^\varepsilon\langle t\rangle^{\gamma_1/2}t^{\gamma_4+1/2-\delta}\mathrm{e}^{-(\beta\vartheta)^2 G_t(0)}\\ =:&(I_t^a)+(I_t^b).\end{aligned}$$

Integrating over the scales with the estimate on λ_t^ε from Lemma 6.3, we obtain for the first term after using the conditions on γ_i, $i = 1,\dots,4$, we have for some $\bar\gamma > 0$ arbitrarily small,

$$\begin{aligned}
&\int_0^{\varepsilon^{-2}} (I_t^a)\mathrm{d}t \\
&\lesssim \int_0^{\varepsilon^{-2}} \mathrm{d}t\, r(\varepsilon)\varepsilon^{\gamma_2} \int_0^1 \mathrm{d}\vartheta \int \mathrm{d}z (\chi^\varepsilon * \dot G_t)(z)|z|^{-\tilde\gamma_3+2} t^{2(1-\delta)} t^{\gamma_4+1/2-\delta+1} \mathrm{e}^{-(\beta\vartheta)^2 G_t(0)} \\
&\lesssim r(\varepsilon)\varepsilon^{\gamma_2} \int_0^{\varepsilon^{-2}} t^{-3\delta} t^{\bar\gamma/2} \mathrm{d}t \\
&\lesssim r(\varepsilon)\varepsilon^{\gamma_2}(\varepsilon^{-2+6\delta-\bar\gamma}+1),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\varepsilon^{-2}}^{\infty} (I_t^a)\mathrm{d}t \\
&\lesssim r(\varepsilon)\varepsilon^{\gamma_2}\varepsilon^{-2(1-\delta)} \int_{\varepsilon^{-2}}^{\infty} \mathrm{d}t \int_0^1 \mathrm{d}\vartheta \int \mathrm{d}z (\chi^\varepsilon * \dot G_t)(z)|z|^{-\tilde\gamma_3+2} t^{1-\delta} t^{\gamma_4+1/2-\delta+1} \mathrm{e}^{-(\beta\vartheta)^2 G_t(0)} \\
&\lesssim r(\varepsilon)\varepsilon^{\gamma_2}\varepsilon^{-2+6\delta-\bar\gamma}.
\end{aligned}$$

Choosing γ_i, $i = 1,\dots,4$ such that $\bar\gamma < \gamma_2 < 2\delta$, we see that this term is uniformly bounded in ε in the case $\delta = 1/2$. Similarly, choosing $\bar\gamma$ sufficiently small and γ_2 sufficiently large, the condition $\gamma > 2(1-3\delta) > 2(1-4\delta)$ implies the boundedness for $\delta < 1/2$.

For the second term, we argue similarly, again using the conditions on γ_i we have for some arbitrarily small $\bar\gamma > 0$,

$$\sup_{\varepsilon\in(0,1)} \int_0^\infty (I_t^b)\mathrm{d}t \lesssim \sup_{\varepsilon\in(0,1)} r(\varepsilon)(\varepsilon^{\gamma_2}\varepsilon^{-2+6\delta-\bar\gamma}+1) < \infty,$$

due to the assumptions on γ. □

Lemma 6.3. *Using the notation introduced in* (6.3)*, it holds that*

$$G_t^\varepsilon(0) = \frac{1}{4\pi}\log((\varepsilon^{-2}\wedge t)\vee 1) + O(1). \tag{6.6}$$

Proof. By definition of G^ε, it holds that

$$G_t^\varepsilon(0) = \mathbb{E}[|W_t^\varepsilon(0)|^2] = \mathbb{E}|\langle W_t, \chi_\varepsilon\rangle|^2 = \langle \chi_\varepsilon, G_t\chi_\varepsilon\rangle.$$

For $t > \varepsilon^{-2}$, passing to Fourier space this implies with

$$G_t = \int_0^t \frac{\mathrm{d}s}{s^{-2}} \mathrm{e}^{-(m^2-\Delta)/s} = \mathrm{e}^{-(m^2-\Delta)/t}(m^2-\Delta)^{-1},$$

$$\langle \chi^\varepsilon, G_t \chi^\varepsilon \rangle = \int \mathrm{d}\xi \frac{|\hat{\chi}^\varepsilon(\xi)|^2}{m^2+|\xi|^2} - \int \mathrm{d}\xi \frac{|\hat{\chi}^\varepsilon(\xi)|^2}{m^2+|\xi|^2}(1-\mathrm{e}^{-(m^2+|\xi|^2)/t}).$$

The second term is uniformly bounded using $t < \varepsilon^{-2}$, and moreover as $(1-\mathrm{e}^{-\varepsilon^2(m^2+|\xi|^2)}) \to 0$ as $\varepsilon \to 0$, vanishes in the limit by dominated convergence. For the first term, using $\operatorname{supp} \hat{\chi} \subset B_1(0)$, we find

$$\int \mathrm{d}\xi \frac{|\hat{\chi}^\varepsilon(\xi)|^2}{m^2+|\xi|^2} = \frac{1}{2\pi}\int_0^{\varepsilon^{-1}} \frac{r}{m^2+r^2}\mathrm{d}r = O(1) + \frac{1}{2\pi}\int_1^{\varepsilon^{-1}} \frac{\mathrm{d}r}{r} = \frac{1}{4\pi}\log(\varepsilon^{-2}\vee 1) + O(1).$$

In the case $t < \varepsilon^{-2}$, we use

$$\int \mathrm{d}x(\chi^\varepsilon(x)-\delta(x))G_t(x) \lesssim \varepsilon^\alpha \|G_t\|_{B^\alpha_{\infty,\infty}} \lesssim t^{-\alpha/2}t^{\alpha/2} = O(1),$$

to obtain,

$$\begin{aligned} G_t^\varepsilon(0) &= \int \mathrm{d}x \chi^\varepsilon(x) \int \mathrm{d}y \chi^\varepsilon(y) G_t(x-y) \\ &= G_t(0) + \int \mathrm{d}x(\chi^\varepsilon(x)-\delta(x))G_t(x) + \int \mathrm{d}x \chi^\varepsilon(x) \int \mathrm{d}y(\chi^\varepsilon(y)-\delta(y))G_t(x-y) \\ &= \frac{1}{4\pi}\log(t\vee 1) + O(1). \end{aligned}$$

□

Lemma 6.4. *For any $z \in \mathbb{R}^2$, $t \in [0,\infty)$,*

$$\|Z_t(z-\cdot) - Z_t(\cdot)\|_{L^\infty} \lesssim t^{1/2-\delta}|z|.$$

Proof. This follows directly from (A.4) in Lemma A.1, and the FBSDE (4.1) for $Z = X - W$. □

Lemma 6.5. ***(N)*** *For any $\gamma_1 > 1/2$, $\gamma_2 > -1+4(1-\delta)$ it holds that*

$$\sup_{t,z} t^{-\gamma_1}|z|^{\gamma_2}\mathrm{e}^{\beta^2 G_t(z)}\|[\![\cos(\beta(W_t(z-\cdot)-W_t(\cdot)))]\!]\rho(\cdot)\|_{L^1} < \infty, \quad a.s.$$

(C) *For any $\gamma_1 > 0$, $\gamma_2 > 2-3\delta$ and $s < 2\delta$ sufficiently large, it holds that*

$$\sup_{t,z} \|\mathrm{e}^{-\beta^2 G_t(z)}|z|^{\gamma_1/2}t^{-\gamma_2}[\![\cos(\beta(W_t(\cdot-z)+W_t(\cdot)))]\!]\rho(\cdot)\|_{B^{-s}_{p,p}(\mathrm{d}x)} < \infty, \quad a.s.$$

The proof of this lemma is given on page 89 at the end of Appendix C.

7. Variational description and large deviations

7.1. Finite volume

If $\beta^2 < 4\pi$, the variational description in the finite volume is essentially a direct consequence of the convergence of the Wick-ordered cosine and the refinement of the Boué–Dupuis formula (Lemma 2.3) from [38]. Beyond the first threshold, the apparent singularity of the sine-Gordon measure means that both the renormalised potential, and the quadratic part in the cost functional $J^{\rho,g} := J^{V^\rho+g}$ as defined in (2.8) cannot be expected to stay bounded as $T \to \infty$. To overcome this difficulty, we follow the same strategy as for the FBSDE and introduce a change of variables that isolates the singular part of the control from a more regular remainder. In these new variables, we can again recover uniform estimates and pass to the limit for any coupling constant $\bar{\lambda}$. Throughout this section, we assume that $\rho \prec 1$ is fixed and suppress the dependency on ρ and g whenever no ambiguities arise.

Translating the same ideas as before now to the level of the variational problem, we begin by developing the potential along the flow. This yields by Ito's formula

$$V_T(X_T^u) = V_t(X_t^u) + \int_t^T \left[\left(\partial_s V_s + \frac{1}{2}\operatorname{Tr}\dot{G}_s \mathrm{D}^2 V_s\right)(X_s^u) + \mathrm{D}V_s(X_s^u)Q_s u_s\right]\mathrm{d}s + \text{mart.}, \quad (7.1)$$

where we use the shorthand

$$X_t^u := X_t(u) = \varphi + \int_0^t Q_s u_s \mathrm{d}s + W_t.$$

Again, we want to use the fact that V approximately solves the flow equation (3.17). Adding the missing terms we can insert the remainder $\mathcal{H}$ as defined in (3.46) and rewrite (7.1) as

$$V_T(X_T^u) = V_t(X_t^u) + \int_t^T \mathrm{d}s \left\{\mathcal{H}_s(X_s^u) + \mathrm{D}V_s(X_s^u)Q_s u_s + \frac{1}{2}(\mathrm{D}V_s \dot{G}_s \mathrm{D}V_s)(X_s^u)\right\} + \text{mart.}. \quad (7.2)$$

Since $\mathcal{H}$ is integrable in the scale parameter t from ∞, it remains to deal with the quadratic terms. Using the notation

$$z_s^u := -Q_s F_s(X_s^u),$$

for the singular part of the control, the variational problem (2.8) becomes upon inserting (7.2) for V_T and completing the square,

$$\begin{aligned}
&\inf_{u\in\mathcal{A}} J_T^g(u) \\
&= \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[g(X_T^u) + V_0(X_0^u) + \int_0^T \mathrm{d}s \left\{\mathcal{H}_s^T\left(X_s^u\right) - \langle z_s^u, u_s\rangle + \frac{1}{2}\|z_s^u\|_{L^2}^2 + \frac{1}{2}\|u_s\|_{L^2}^2\right\}\right], \quad (7.3)\\
&= \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[g(X_T^u) + V_0(X_0^u) + \int_0^T \mathrm{d}s \left\{\mathcal{H}_s^T(X_s^u) + \frac{1}{2}\|u_s - z_s^u\|_{L^2}^2\right\}\right].
\end{aligned}$$

Importantly, this reformulation no longer imposes square integrability on the control u but only on $u - z^u$, which heuristically corresponds to the (more regular) remainder. We take this as an invitation to introduce the change of variables

$$r_t^u := u_t - z_t^u. \tag{7.4}$$

The following Lemma ensures that this change of variables does not affect the variational problem (7.3).

Lemma 7.1. *For any $r \in \mathbb{H}^2(L^2)$, $T \leqslant \infty$ and $\rho \prec 1$, there is a unique solution $\hat{Z}^r \in \mathbb{H}^\infty(L^\infty)$ to the SDE*

$$\hat{Z}_t^r = \int_0^t Q_s(r_s - Q_s F_s^{\rho,T}(\hat{Z}_s^r + W_s))\mathrm{d}s. \tag{7.5}$$

In particular, with $\hat{X} = \hat{Z} + W$, defining $u_t^r := -Q_t F_t^{\rho,T}(\hat{X}_t^r) + r_t$, the control u^r is admissible for the finite-horizon control problem (7.3) *and we have $\hat{X}^r \equiv X^{u^r}$ and $r_t = u_t^r - z_t^{u^r}$ almost surely.*

Proof. The estimates on the approximate solution to the flow equation imply that $Q_t F_t^{\rho,T}$ is globally Lipschitz and bounded, uniformly in t. The result now follows from standard well-posedness for SDEs with Lipschitz coefficients. □

This allows yet another reformulation of (7.3) in terms of the remainder. To avoid confusion with the infinite volume control problem later, let us make the dependence on ρ explicit again and define the cost functional

$$\hat{J}_T^{\rho,g}(r) := \mathbb{E}\left[g(\hat{X}_T^r) + V_0^\rho(\hat{X}_0^r) + \int_0^T \mathrm{d}s \left\{ \mathcal{H}_s^{\rho,T}(\hat{X}_s^r) + \frac{1}{2}\|r_s\|_{L^2}^2 \right\} \right], \tag{7.6}$$

with $\hat{X}^r$ defined as the unique solution to (7.5). Observe that in contrast to $J_T^{\rho,g}$, the functional $\hat{J}_T^{\rho,g}$ satisfies for some $C = C_g > 0$,

$$\hat{J}_T^{\rho,g}(r) \geqslant -C + \frac{1}{2}\int_0^T \mathrm{d}s \|r_s\|_{L^2}^2, \tag{7.7}$$

which we immediately verify from (7.6) and the estimates on $\mathcal{H}$ (see Proposition 3.13-b). In particular, the cost functional $\hat{J}_T^{\rho,g}$ makes sense also at $T = \infty$. From (7.7) we see that $\hat{J}_T^{\rho,g}(r) = \infty$, whenever $r \notin \mathbb{H}_T^2(L^2)$. We can thus enlarge the set over which we take the infimum and use Lemma 7.1 to see for any $T < \infty$,

$$\inf_{r \in \mathbb{H}_a} \hat{J}_T^{\rho,g}(r) = \inf_{r \in \mathbb{H}^2(L^2)} \hat{J}_T^{\rho,g}(r) = \inf_{u \in \mathbb{H}_a} J_T^{\rho,g}(u).$$

Therefore, the relation between the FBSDE (4.1) and the variational problem (2.8) obtained in Theorem 2.2-b is also valid for the UV-limit, provided we renormalise the form of the cost functional.

Theorem 7.2. *Denote by* $(Z^{\rho,g}, R^{\rho,g})$ *the unique solution to* (4.1) *with* $\rho \prec 1$ *and* $T = \infty$. *Then,*

$$r_t^{\rho,g} := -Q_t R_t^{\rho,g}, \tag{7.8}$$

is admissible and optimal for the control problem (7.6) *at* $T = \infty$.

Proof. Let us fix a cut-off $\rho \prec 1$ and leave the dependence implicit for this proof. Thanks to Lemma 4.1, the candidate for the optimal control for the control problem defined in (7.8) satisfies

$$\mathbb{E}\int_0^\infty \|r_s^g\|_{L^2}^2 \mathrm{d}s < \infty,$$

and thus $r_s^g \in \mathbb{H}^2(L^2(\mathbb{R}^2))$ so that

$$\inf_{r\in\mathbb{H}^2(L^2)} \hat{J}_\infty^g(r) \leqslant \hat{J}_\infty^g(r^g).$$

It remains to show the reverse inequality. To this end, let $\bar{u}^{T,g} = -Q_t(R_t^{T,g} + F_t(X_t^{T,g}))$ be the optimal control for the control problem (2.8). Then, for any finite T, (writing $\bar{z}^{T,g} \equiv z^{\bar{u}^{T,g}}$ and $X^{T,g} = X^{u^{T,g}} = Z^{T,g} + W$), it holds that

$$\bar{r}_t^{T,g} := \bar{u}_t^{T,g} - \bar{z}_t^{T,g} = -Q_t(R_t^{T,g} + F_t(X_t^{T,g})) + Q_t F_t(X_t^{T,g}) = -Q_t R_t^{T,g},$$

is optimal for the control problem (7.3) as shown in Theorem 2.2-b. On the finite volume Proposition 3.13-a, implies for $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ and $\varepsilon, \tilde{\varepsilon} > 0$ sufficiently small,

$$\begin{aligned}\int_0^\infty |(\mathcal{H}_t^\infty - \mathbb{1}_{\{t\leqslant T\}}\mathcal{H}_t^T)(\varphi)|\mathrm{d}t &\lesssim \int_T^\infty |\mathcal{H}_t^\infty(\varphi)|\mathrm{d}t + \int_0^T |(\mathcal{H}_t^\infty - \mathcal{H}_t^T)(\varphi)|\mathrm{d}t \\ &\lesssim_\rho \langle T\rangle^{1-4\delta} + \langle T\rangle^{-\varepsilon} \underbrace{\int_0^T \langle t\rangle^{-4\delta+\tilde{\varepsilon}}\mathrm{d}t}_{<\infty}.\end{aligned}$$

Combined with the convergence of $X_T^g \to X_\infty^g$ from Theorem 4.6 and the continuity of g, this implies for any fixed $r \in \mathbb{H}^2(L^2)$,

$$\hat{J}_\infty^g(r) = \lim_{T\to\infty} \hat{J}_T^g(r).$$

By the optimality of $r^{T,g}$ for $T < \infty$ this implies for any $r \in \mathbb{H}^2(L^2)$,

$$\liminf_{T\to\infty} \hat{J}_T^g(r^{T,g}) = \liminf_{T\to\infty} \inf_{r\in\mathbb{H}^2(L^2)} \hat{J}_T^g(r) \leqslant \lim_{T\to\infty} \hat{J}_T^g(r) = \hat{J}_\infty^g(r). \tag{7.9}$$

From the continuity of ∇g and the convergence of the solution $(Z^{T,g}, R^{T,g}) \to (Z^g, R^g)$ derived in Theorem 4.6, we immediately get in the $T \to \infty$ limit

$$\mathbb{E}\int_0^\infty \|r_t^g - r_t^{T,g}\|_{L^2}^2 \mathrm{d}t = \mathbb{E}\int_0^\infty \|Q_t(R_t^{T,g} - R_t^g)\|_{L^2}^2 \mathrm{d}t \lesssim \mathbb{E}\int_0^\infty \mathrm{d}t \langle t\rangle^{-2}\langle T\rangle^{-\varepsilon} \to 0,$$

Therefore, by Fatou's Lemma and the continuity of g and V_0,

$$\hat{J}^g_\infty(r^g) = \mathbb{E}\left[g(X^g_\infty) + V_0(X^g_0) + \int_0^\infty \mathcal{H}^\infty_s(X^g_s)\mathrm{d}s + \int_0^\infty \|r^g_t\|^2_{L^2}\mathrm{d}t \right] \leqslant \liminf_{T\to\infty} \hat{J}^g_T(r^{T,g}). \tag{7.10}$$

Combining (7.9) and (7.10) we obtain the missing inequality,

$$\hat{J}^g_\infty(r^g) \leqslant \liminf_{T\to\infty} \hat{J}^g_T(r^{T,g}) \leqslant \inf_{r\in\mathbb{H}^2(L^2)} \hat{J}^g_\infty(r).$$

□

Remark 7.3. The boundedness of the cosine interaction is the reason we have good bounds over the optimisers to the control problem (2.8) uniformly in T. This allows us to bypass the technically more involved Γ-convergence for the cost functionals $\hat{J}^T \to J$ to remove the small-scale regularisation T as was instead necessary in [4] in the case of the Φ^4_3 model on a bounded domain.

The variational description for the Laplace transform is now an immediate consequence of the description in Theorem 7.2.

Corollary 7.4. *The variational problem for the Laplace transform* (2.9) *also holds for* $T = \infty$*, that is,*

$$-\log \nu^\rho_{\mathrm{SG}}(\mathrm{e}^{-g}) = \mathcal{W}^\rho(g) := \inf_{r\in\mathbb{H}^2(L^2)} \hat{J}^{\rho,g}_\infty(r) - \inf_{r\in\mathbb{H}^2(L^2)} \hat{J}^{\rho,0}_\infty(r).$$

Proof. From the weak convergence of $\nu^{\rho,T}_{\mathrm{SG}} \to \nu^\rho_{\mathrm{SG}}$ in $H^{-\varepsilon}$,

$$-\log \nu^\rho_{\mathrm{SG}}(\mathrm{e}^{-g}) = \lim_{T\to\infty} -\log \nu^{\rho,T}_{\mathrm{SG}}(\mathrm{e}^{-g}) = \lim_{T\to\infty} \mathcal{W}^\rho_T(g) = \lim_{T\to\infty} \mathcal{V}^{V^\rho+g}_T - \lim_{T\to\infty} \mathcal{V}^{V^\rho}_T = \mathcal{W}^\rho(g),$$

where we used (3.14) and Theorem 7.2 to justify the last two equalities. □

Remark 7.5. It should be emphasised that the change of variables (7.4) makes the extension to $T = \infty$ possible: passing to the remainder term $r^u = u - z^u$ allows us to incorporate the singular part z^u of the control into the flow equation remainder $\mathcal{H}^{\rho,T}_s$ while optimising only over absolutely continuous shifts r^u. Indeed, while we have $z^u \in \mathbb{H}^2_T(L^2)$ for any $u \in \mathbb{H}^2_T(L^2)$, our estimates on z^u only allow

$$\|z^u_t\|^2_{L^2} \lesssim \langle t\rangle^{-2\delta},$$

which is not sufficient to conclude $z^u \in \mathbb{H}(L^2)$ unless $\delta > 1/2$ ($\Leftrightarrow \beta^2 < 4\pi$). In contrast, the estimates on the optimal FBSDE (see e.g. Lemma 4.1) suggests that the remainder $r = z^u - u$ remains square-integrable for the whole subcritical regime $\delta > 0$ (provided of course that an appropriate approximate solution V to the flow equation is used).

Remark 7.6. Differentiating (7.3) with respect to the initial value $X_0 = \varphi$, we obtain a formula for the gradient of the value function in terms of the solution to the optimal FBSDE (4.1)

$$\nabla \mathcal{V}^{V+g}(\varphi) = \nabla J^g(\bar{u}^g;\varphi) = (\nabla g + F_0)(\varphi) + R^g_0(\varphi). \tag{7.11}$$

7.2. Infinite volume

We finally want to remove the restriction to the finite volume. Of course, the potential will not be meaningful without a spatial cut-off. What saves the variational problem for the Laplace transform in the infinite volume are the localisation properties we derived earlier: since the effect of a local perturbation only has a localised effect on the optimal control by Proposition 4.5-c we are able to show that the functional

$$\hat{\mathcal{J}}^{g,\rho}(v) := \hat{J}^{g,\rho}(v + \bar{r}^\rho) - \hat{J}^{0,\rho}(\bar{r}^\rho),$$

stays meaningful in the infinite volume limit, at least if the functional g is sufficiently localised and the coupling constant $\bar{\lambda}$ is small enough. This change of variables follows the same idea we used for the finite volume variational problem: it again allows us to absorb the singular part in a normalisation while we only optimise along the absolutely continuous directions, which in this case corresponds to controls in $\mathbb{D} = \mathbb{H}^2(L^{2,n})$.

The aim is to show the following.

Theorem 7.7. *Let R^0 be the backward component of the solution to the FBSDE* (4.1) *for $g = 0$, $\rho = 1, T = \infty$ and define $\bar{r} := -QR^0$. Then, with*

$$\hat{\mathcal{J}}^g(v) = \mathbb{E}\left[g(\hat{X}_\infty^{\bar{r}+v}) + \int_0^\infty \mathcal{H}_s^1(\bar{r}_s, v_s)\mathrm{d}s + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s + \int_0^\infty \langle \bar{r}_s, v_s\rangle_{L^2}\mathrm{d}s \right],$$

the Laplace transform of the infinite volume sine-Gordon measure ν_{SG} satisfies the variational problem

$$\mathcal{W}(g) := -\log \nu_{\mathrm{SG}}(\mathrm{e}^{-g}) = \inf_{v\in\mathbb{D}} \hat{\mathcal{J}}^g(v).$$

Here, the functional $\mathcal{H}_s^\rho(\bar{r}_s, v_s)$ is defined for any $\rho \preceq 1$ in terms of (3.49)*, in complete analogy to* (3.48)*,*

$$\mathcal{H}_t^1(\bar{r}_t, v_t) := \sum_{\ell=4}^{6} \int \mathrm{d}\xi_{1:\ell} h_t(\xi_{1:\ell})[\psi_t^{\bar{r}+v} - \psi_t^{\bar{r}}](\xi_{1:\ell}),$$

with $\psi_t^{\bar{r}+v}(\xi_{1:\ell}) := \exp\Big(i\beta \sum_{k=1}^\ell \sigma_k \hat{X}_t^{\bar{r}+v}(x_k)\Big)$.

Proof. Restriction to $\mathbb{D}_g$: Motivated by Proposition 4.5-c, we expect that the regular part of the control is captured nicely by the domain,

$$\mathbb{D}_g = \mathbb{D}_g(C) := \left\{ v \in \mathbb{H}_a : \mathbb{E}\int_0^\infty \|v_s\|_{L^{2,n}}^2 \mathrm{d}s \leqslant C|g|_{1,2,n} \right\}, \tag{7.12}$$

provided $C > 0$ is chosen sufficiently large. We first show convergence of the restricted variational problem

$$\widehat{\mathcal{W}}_g^\rho(g) := \inf_{v\in\mathbb{D}_g(C)} \hat{\mathcal{J}}^{g,\rho}(v) \xrightarrow{\rho\to 1} \mathcal{W}(g) := -\log \nu_{\mathrm{SG}}(\mathrm{e}^{-g}) = \inf_{v\in\mathbb{D}_g} \hat{\mathcal{J}}^g(v).$$

We claim that this restriction does not change the finite volume variational problem, that is for any $\rho \prec 1$,

$$\inf_{\nu\in\mathbb{D}_g} \hat{\mathcal{J}}^{g,\rho}(\nu) = \widehat{\mathcal{W}}_g^\rho(g) = \mathcal{W}^\rho(g) = \inf_{\nu\in\mathbb{H}^2(L^2)} \hat{\mathcal{J}}^{g,\rho}(\nu). \tag{7.13}$$

We know from Theorem 7.2 that $\bar{r}_t^{g,\rho} = -Q_t R_t^{g,\rho}$ is optimal for the variational problem (7.6), and from Proposition 3.13-c that $\|R_t^{g,\rho} - R_t^{0,\rho}\|_{L^{2,n}} \lesssim |g|_{1,2,n}$. Thus, for some constant $C_g > 0$,

$$\mathbb{E}\int_0^\infty \|\overline{r_s}^{g,\rho} - \bar{r}_s^\rho\|_{L^{2,n}}^2 \mathrm{d}s \lesssim \mathbb{E}\int_0^\infty \langle s\rangle^{-2}\|R_t^{g,\rho} - R_t^{0,\rho}\|_{L^{2,n}}^2 \leqslant C_g |g|_{1,2,n}^2.$$

But then $\bar{\nu}^{g,\rho} = \bar{r}^{g,\rho} - \bar{r}^\rho \in \mathbb{D}_g$ for C sufficiently large this implies (7.13) for any $C \geqslant C_g$.

Convergence: We show that uniformly on $\mathbb{D}_g$,

$$\begin{aligned}\hat{\mathcal{J}}^{g,\rho}(\nu) &= \mathbb{E}\left[g(\hat{X}_\infty^{\bar{r}^\rho+\nu}) + \int_0^\infty \mathcal{H}_s^\rho(\bar{r}_s^\rho, \nu_s)\mathrm{d}s + \frac{1}{2}\int_0^\infty \|\bar{r}_s^\rho + \nu_s\|_{L^2}^2 \mathrm{d}s - \frac{1}{2}\int_0^\infty \|\bar{r}_s^\rho\|_{L^2}^2 \right] \\ &\xrightarrow{\rho\to 1} \mathbb{E}\left[g(\hat{X}_\infty^{\bar{r}+\nu}) + \int_0^\infty \mathcal{H}_s^1(\bar{r}_s, \nu_s)\mathrm{d}s + \frac{1}{2}\int_0^\infty \|\nu_s\|_{L^2}^2 \mathrm{d}s + \int_0^\infty \langle \bar{r}_s, \nu_s\rangle_{L^2}\mathrm{d}s \right].\end{aligned}$$

We proceed term by term. Going left to right, we start by estimating

$$|g(\phi_1) - g(\phi_2)| = \int_0^1 (\nabla g(\phi_1 + \vartheta(\phi_2 - \phi_1)))(\phi_2 - \phi_1)\mathrm{d}\vartheta \lesssim |g|_{1,2,n}\|\phi_2 - \phi_1\|_{L^{2,n}}.$$

The convergence $g(\hat{X}_\infty^{\rho,\bar{r}^\rho+\nu}) \to g(\hat{X}_\infty^{\bar{r}+\nu})$ then follows from Lemma 7.8 below. For the remainder term, we write,

$$|\mathcal{H}_s^1(\bar{r}_s, \nu_s) - \mathcal{H}_s^\rho(\bar{r}_s^\rho, \nu_s)| \leqslant |\mathcal{H}_s^1(\bar{r}_s, \nu_s) - \mathcal{H}_s^1(\bar{r}_s^\rho, \nu_s)| + |\mathcal{H}_s^1(\bar{r}_s^\rho, \nu_s) - \mathcal{H}_s^\rho(\bar{r}_s^\rho, \nu_s)|.$$

For the first term, the definition of $\mathcal{H}^1$ and the estimates on the coefficients h in (3.50) imply,

$$\begin{aligned}|\mathcal{H}_s^1(\bar{r}_t, \nu_t) - \mathcal{H}_t^1(\bar{r}_t^\rho, \nu_t)| &\leqslant \sum_{\ell=4}^6 \int \mathrm{d}\xi_{1:\ell} h_t(\xi_{1:\ell})[\psi_t^{\bar{r}+\nu} - \psi_t^{\bar{r}} - (\psi_t^{\bar{r}^\rho+\nu} - \psi_t^{\bar{r}^\rho})](\xi_{1:\ell}) \\ &\lesssim |||h_t|||\,(\|\delta_\nu \hat{X}_t^{\bar{r}+\nu} - \delta_\nu \hat{X}_t^{\rho,\bar{r}^\rho+\nu}\|_{L^{2,n}}) \\ &\lesssim \bar{\lambda}_t^4\langle t\rangle^{-4}\|\delta_\nu \hat{X}_t^{\bar{r}+\nu} - \delta_\nu \hat{X}_t^{\rho,\bar{r}^\rho+\nu}\|_{L^{2,n}}.\end{aligned}$$

Since $\bar{\lambda}_t^4\langle t\rangle^{-4} \in L^1(\mathbb{R}_+)$, the desired convergence will follow from Lemma 7.8 below. For the second term,

$$\begin{aligned}|\mathcal{H}_s^1(\bar{r}_t^\rho, \nu_t) - \mathcal{H}_s^\rho(\bar{r}_t^\rho, \nu_t)| &\leqslant \sum_{\ell=4}^6 \int \mathrm{d}\xi_{1:\ell}|1-\rho(\xi_{1:\ell})||h_t(\xi_{1:\ell})[\psi_t^{\bar{r}^\rho+\nu} - \psi_t^{\bar{r}^\rho}](\xi_{1:\ell})| \\ &\leqslant \|1-\rho\|_{L^{2,-n}}\,|||\,h_t\,|||\,\|\hat{X}_t^{\bar{r}^\rho+\nu} - \hat{X}_t^{\bar{r}^\rho}\|_{L^{2,n}} \\ &\lesssim \|1-\rho\|_{L^{2,-n}}\bar{\lambda}_t^4\langle t\rangle^{-4}\|\nu_t\|_{L^{2,n}}.\end{aligned}$$

Hence,

$$\lim_{\rho\to 1}\sup_{v\in\mathbb{D}_g}\mathbb{E}\int_0^\infty |\mathcal{H}^1_s(\bar r_s,v_s)-\mathcal{H}^1_s(\bar r^\rho_s,v_s)|\mathrm{d}s=0.$$

Finally, for the quadratic terms, we expand the square to find,

$$\frac{1}{2}\|\bar r^\rho_s+v_s\|^2_{L^2}-\frac{1}{2}\|\bar r^\rho_s\|^2_{L^2}=\frac{1}{2}\|v_s\|^2_{L^2}+\langle \bar r^\rho_s,v_s\rangle_{L^2}.$$

Consequently, for any $v\in\mathbb{D}_g$,

$$\mathbb{E}\int_0^\infty \mathrm{d}s\langle \bar r^\rho_s-\bar r_s,v_s\rangle_{L^2}\lesssim\left(\mathbb{E}\int_0^\infty\|\bar r^\rho_s-\bar r_s\|^2_{L^{2,-n}}\mathrm{d}s\right)^{1/2}\left(\mathbb{E}\int_0^\infty\|v_s\|^2_{L^{2,n}}\mathrm{d}s\right)^{1/2},$$

which with $\bar r_t-\bar r^\rho_t=-Q_t(R_t-R^\rho_t)$ and the estimates on Q in Lemma A.4 and Proposition 4.5-c converges to 0 uniformly on $\mathbb{D}_g$.

Recovering the full domain: Finally, we show that for any $C\geqslant C_g$,

$$\inf_{v\in\mathbb{D}_g(C)}\hat{\mathcal{J}}^g(v)=\inf_{v\in\mathbb{D}}\hat{\mathcal{J}}^g(v).$$

Since $\mathbb{D}_g(C)\subset\mathbb{D}$, clearly $\inf_{v\in\mathbb{D}_g(C)}\hat{\mathcal{J}}^g(v)\leqslant\inf_{v\in\mathbb{D}}\hat{\mathcal{J}}^g(v)$. For the reverse inequality, let $\bar v\in\mathbb{D}$ and let $\bar C>\|\bar v\|^2_{\mathbb{D}}:=\|\bar v\|^2_{\mathbb{H}^2(L^{2,n})}$ so that $\bar v\in\hat{\mathbb{D}}_g(\bar C)$ and thus by the argument used to show convergence on $\mathbb{D}_f(C)$,

$$\mathcal{J}^g(\bar v)\geqslant\inf_{v\in\mathbb{D}_g(\bar C)}\mathcal{J}^g(v)=\mathcal{J}^g(v^*)=\inf_{v\in\mathbb{D}_g(C)}\mathcal{J}^g(v).$$

Taking the infinitum over $\bar v\in\mathbb{D}$ in the inequality above then yields the claim. □

We still have to supplement the following two convergence results to finish up the proof of Theorem 7.7.

Lemma 7.8. *Using the notation defined in* (7.5)*, it holds uniformly in* $v\in\mathbb{D}_g$*,*

$$\lim_{\rho\to 1}\sup_{s\geqslant 0}\|\hat X^{\bar r+v}_s-\hat X^{\bar r^\rho+v}_s\|_{L^{2,-n}}=0.$$

Proof. This follows immediately from Proposition 3.13 following the arguments in Proposition 4.5-a and Proposition 4.5-d. □

7.3. Large deviations

We apply the variational problem for the Laplace transform just derived in Theorem 7.7 to show the Laplace principle from Theorem 1.5 for the limiting measure ν_{SG}. More precisely, we want to study the family of rescaled measures are formally given by

$$\text{“}\nu^\hbar_{\mathrm{SG}}(\mathrm{d}\varphi)=\Xi^{-1}_\hbar\exp(\hbar^{-1}V(\varphi))\mu^\hbar(\mathrm{d}\varphi),\text{”}\tag{7.14}$$

in the limit $\hbar \to 0$. Here $V(\varphi) = \lambda \int \mathrm{d}x \cos(\beta\varphi(x))$ denotes as before the cosine interaction and $\mu^\hbar$ is the Gaussian measure with covariance $\hbar(m^2 - \Delta)^{-1}$. Taking the Wick-ordering with respect to $\mu^\hbar$ and the obvious modification to the interpolation $G_s^\hbar := \hbar G_s$, the same derivation as before yields a description for $\nu_{\mathrm{SG}}^\hbar$ via the rescaled FBSDE,

$$\begin{cases} Z_t^\hbar = -\int_0^t \mathrm{d}s \dot{G}_s^\hbar(F_s^\hbar(Z_s^\hbar + \hbar^{1/2}W_s)) - \int_0^t \mathrm{d}s \dot{G}_s R_s^\hbar, \\ R_t^\hbar = \mathbb{E}_t \int_t^\infty \mathrm{d}s \hbar H_s^\hbar(Z_s^\hbar + \hbar^{1/2}W_s) - \mathbb{E}_t \int_t^\infty \mathrm{d}s \hbar \mathrm{D}F_s^\hbar(Z_s^\hbar + \hbar^{1/2}W_s)\dot{G}_s^\hbar R_s^\hbar, \end{cases} \tag{7.15}$$

where $F^\hbar = F^{\hbar,\infty}$ and $F^{\hbar,T}$ is the approximate solution to the flow equation (3.2) with covariance $G^\hbar$ and the rescaled initial data,

$$F_T^{[1],T,\hbar}(\varphi) := -\beta\hbar^{-1}\lambda_0 \mathrm{e}^{\frac{\hbar\beta^2}{2}G_T(0)} \sin(\beta\varphi) = -\hbar^{-1}\lambda_t^\hbar \beta \sin(\beta\varphi).$$

At least when $\hbar \in [0,1]$, we have $\bar{\lambda}_t^\hbar \leqslant \bar{\lambda}_t$ and the well-posedness of (7.15) follows in the exact same way as the well-posedness of (4.1) in Proposition 4.2. Moreover, rescaling the analysis of Theorem 4.6, we see that also the drift $Z^\hbar$ has a terminal value with regularity

$$Z_\infty^\hbar \in L^\infty(\mathrm{d}\mathbb{P}; H^{2-\hbar\beta^2/4\pi,-n}) \subset L^\infty(\mathrm{d}\mathbb{P}; H^{2-\beta^2/4\pi,-n}). \tag{7.16}$$

Thus, the same reasoning as in Theorem 4.6 applies and can use the solution to the FBSDE (7.15) to make the formal definition (7.14) precise. Define the measures $\nu_{\mathrm{SG}}^\hbar$ as a random shift of the rescaled Gaussian free field,

$$\nu_{\mathrm{SG}}^\hbar := \mathrm{Law}(Z_\infty^\hbar + \hbar^{1/2}W_\infty).$$

Since we are only interested in the limit $\hbar \to 0$, we can limit our considerations to a small neighbourhood of 0. This has the advantage that the measure $\nu_{\mathrm{SG}}^\hbar$ will essentially behave like the sine-Gordon measure with parameter $\beta\hbar^{1/2}$ (see also [21, Section C]): if $\hbar$ is sufficiently small (say $\hbar < \hbar_0$ where $\beta^2\hbar_0 < 4\pi$), then, by (7.16), the measure $\nu_{\mathrm{SG}}^\hbar$ is a Girsanov shift of the free field. More concretely this means we can carry out the analysis of (7.15) by relying on the convergence of the Wick-ordered cosine illustrated in (4.7), as in the absolutely continuous first region $\beta^2 < 4\pi$ already covered in [6]. It only remains to check that this approach is compatible with our definition of the measures via (7.15). This is in part resolved by the following Lemma.

Lemma 7.9. *Let $\hbar < \hbar_0$. The solution $Z^\hbar$ to* (7.15) *satisfies*

$$Z_t^\hbar = \beta\lambda_0 \int_0^t \mathrm{d}s \dot{G}_s \mathbb{E}_s[[\![\sin(\beta(Z_\infty^\hbar + \hbar^{1/2}W_\infty))]\!]] = \beta\lambda_0 I_t(\bar{u}_t^\hbar), \tag{7.17}$$

where $[\![\sin(\varphi + \hbar^{1/2}W_\infty)]\!]$ is defined for any $\varphi \in H^{1,-n}$ via (7.20) *below.*

Proof. Let us first introduce again the approximate FBSDEs with the cut-off $T < \infty$. Then, we know from the definition of $F^{T,\hbar}$ and $R^{T,\hbar}$ that

$$F_s^{T,\hbar}(X_s^{T,\hbar}) + R_s^{T,\hbar} = \mathbb{E}_s[\nabla V_T^{T,\hbar}(\beta(Z_T^{T,\hbar} + \hbar^{1/2}W_t))]. \tag{7.18}$$

For $\hbar < \hbar_0$, it follows from Lemma A.5 and the convergence of $Z_T^T \to Z_\infty$ in $\mathbb{H}^\infty(L^\infty)$,

$$\lim_{T\to\infty} \|Z_T^{T,\hbar} - Z_\infty^\hbar\|_{H^{1,-n}}^2 = 0.$$

Moreover, we can use the trigonometric identities to rewrite

$$[\![\sin(\beta(Z_T^\hbar + \hbar^{1/2}W_T))]\!] := \cos(\beta Z_T^\hbar)[\![\sin(\beta\hbar^{1/2}W_T)]\!] + \sin(\beta Z_T^\hbar)[\![\cos(\beta\hbar^{1/2}W_T)]\!], \quad (7.19)$$

and similarly for the cosine. By Lemma C.1, the Wick-ordered sine (cosine) $[\![\sin(\beta\hbar^{1/2}W_T)]\!]$ converge in $H^{-1+\varepsilon,-n}$. Thus, the products on the right-hand side of (7.19) stay well-defined in the limit as $T \to \infty$ and consequently $[\![\sin(\beta(Z_T^{T,\hbar}+\hbar^{1/2}W_T))]\!]$ and $[\![\cos(\beta(Z_T^{T,\hbar}+\hbar^{1/2}W_T))]\!]$ converge in $L^2(\mathrm{d}\mathbb{P}; H^{-1+\varepsilon,-n})$ and almost surely to a well-defined limit which we denote by

$$[\![\sin(\beta(Z_\infty^\hbar + \hbar^{1/2}W_\infty))]\!] := \cos(\beta Z_\infty^\hbar)[\![\sin(\beta\hbar^{1/2}W_\infty)]\!] + \sin(\beta Z_\infty^\hbar)[\![\cos(\beta\hbar^{1/2}W_\infty)]\!], \quad (7.20)$$

and respectively

$$[\![\cos(\beta(Z_\infty^\hbar + \hbar^{1/2}W_\infty))]\!] := \cos(\beta Z_\infty^\hbar)[\![\cos(\beta\hbar^{1/2}W_\infty)]\!] + \sin(\beta Z_\infty^\hbar)[\![\sin(\beta\hbar^{1/2}W_\infty)]\!].$$

By uniqueness of the limit, we can pass to the limit $T \to \infty$ in (7.18) to conclude

$$F_t^\hbar(Z_t^\hbar + \hbar^{1/2}W_t) + R_t^\hbar = -\beta\lambda\mathbb{E}_t[\![\sin(\beta(Z_\infty^\hbar + \hbar^{1/2}W_\infty))]\!],$$

which immediately implies (7.17). □

Applying the same argument as in Lemma 7.9 to the cost functional $\hat{\mathcal{J}}^\hbar$ we can undo the change of variables to the remainder in (7.3) and (7.4) provided $\hbar < \hbar_0$. In this case, we obtain the cost functionals

$$\mathcal{J}^{g,\hbar}(w) := \mathbb{E}[g(I_\infty(\bar{u}^\hbar + w) + W_s^\hbar) + \hbar V_\infty^\hbar(u^\hbar, w)\mathrm{d}s + \mathcal{E}(w, u^\hbar)], \quad (7.21)$$

where $W^\hbar = \hbar^{1/2}W$ is the rescaled Brownian motion, $\bar{u}_t^\hbar := Q_t\hbar F_t^\hbar(Z_t^\hbar + \hbar^{1/2}W_t) + Q_t R_t^\hbar$ is the candidate for optimal control,

$$V_\infty^\hbar(u, w) := \lambda\int_{\mathbb{R}^2} \mathrm{d}x([\![\cos(\beta(I_\infty(u+w) + W_\infty^\hbar))]\!] - [\![\cos(\beta(I_\infty(u) + W_\infty^\hbar))]\!])(x),$$

and

$$\mathcal{E}(w, u) := \frac{1}{2}\int_0^\infty \|w_s\|_{L^2}^2\mathrm{d}s + \int_0^\infty \langle w_s, u_s\rangle_{L^2}\mathrm{d}s.$$

Since the functional $\mathcal{J}^{g,\hbar}$ depends on $\hbar$ also through the optimal control $\bar{u}_t^\hbar = -Q_t\hbar F_t^\hbar(Z_t^\hbar + \hbar^{1/2}W_t) - Q_t R_t^\hbar$, we have to first identify the limit of $\bar{u}^\hbar$ before we can find the limiting candidate for $\mathcal{J}^{g,0}$.

Lemma 7.10. *With $\bar{u}_t^\hbar = -Q_t \hbar F_t^\hbar(Z_t^\hbar + \hbar^{1/2} W_t) - Q_t R_t^\hbar$, it holds that,*

$$\lim_{\hbar \to 0} \mathbb{E} \int_0^\infty \|\bar{u}_t^\hbar\|_{L^{2,-n}}^2 \mathrm{d}t = 0. \tag{7.22}$$

Proof. We use the linear flow approximation $\tilde{F} := F^{[1]}$ for the SDE (7.17) to obtain again a FBSDE. With $X_t^\hbar = Z_t^\hbar + \hbar^{1/2} W_t$, this results in the FBSDE,

$$\begin{cases} Z_t^\hbar = \int_0^t \mathrm{d}s \dot{G}_s (\lambda_s^\hbar \beta \sin(\beta X_s^\hbar) - \tilde{R}_s^\hbar), \\ \tilde{R}_t^\hbar = \mathbb{E}_t \int_t^\infty \mathrm{d}s (\lambda_s^\hbar)^2 \beta \cos(\beta X_s^\hbar) \dot{G}_s \sin\big(\beta(X_s^\hbar)\big) - \mathbb{E}_t \int_t^\infty \lambda_s^\hbar \cos(\beta X_s^\hbar) \dot{G}_s \tilde{R}_s^\hbar \mathrm{d}s. \end{cases}$$

Thus, the same arguments as before show using $\hbar \beta^2 < 4\pi$,

$$\|R_t^\hbar\|_{L^{2,-n}} \lesssim (\bar{\lambda}_t^\hbar)^2 \langle t \rangle^{-1} \sup_s \|Z_s^\hbar\|_{L^{2,-n}} \lesssim \bar{\lambda}^2 \sup_s \|Z_s^\hbar\|_{L^{2,-n}}.$$

Using this estimate in the equation for $Z^\hbar$,

$$\begin{aligned} \|Z_t^\hbar\|_{L^{2,-n}} \lesssim \int_0^t \mathrm{d}s \bar{\lambda}_s^\hbar \langle s \rangle^{-2} \|Z_s^\hbar\|_{L^{2,-n}} + \hbar^{1/2} \int_0^t \mathrm{d}s \langle s \rangle^{-2} \|W_s\|_{L^{2,-n}} \\ + \bar{\lambda}^2 \int_0^t \mathrm{d}s \langle s \rangle^{-2} \sup_r \|Z_r^\hbar\|_{L^{2,-n}}. \end{aligned}$$

Keeping in mind that

$$\mathbb{E}\|W_s\|_{L^{2,-n}}^2 = G_s(0) \lesssim \log(s \vee 1) + 1,$$

we rearrange and take expectation to find for $\bar{\lambda}$ sufficiently small,

$$\mathbb{E}\|Z_t^\hbar\|_{L^{2,-n}} \lesssim \hbar^{1/2} \int_0^t \mathrm{d}s \langle s \rangle^{-2} \mathbb{E}[\|W_s\|_{L^{2,-n}}^2]^{1/2} \lesssim \hbar^{1/2} \int_0^t \langle s \rangle^{-2} (\log(s \vee 1) + 1) \lesssim \hbar.$$

Putting everything together,

$$\begin{aligned} \mathbb{E} \int_0^\infty \|\bar{u}_t^\hbar\|_{L^{2,-n}}^2 \mathrm{d}t &= \mathbb{E} \int_0^\infty \lambda_0 \beta \|Q_t \sin((\beta Z_t^\hbar + \hbar^{1/2} W_t))\|_{L^{2,-n}}^2 \mathrm{d}t \\ &\lesssim \bar{\lambda} \int_0^\infty \langle t \rangle^{-2} (\mathbb{E}\|Z_t^\hbar\|_{L^{2,-n}}^2 + \hbar \mathbb{E}\|W_t\|_{L^{2,-n}}^2) \mathrm{d}t \\ &\lesssim \bar{\lambda}\hbar + \hbar \int_0^\infty \mathrm{d}t \langle t \rangle^{-2} \log(t \vee 1) \\ &\lesssim \bar{\lambda}\hbar. \end{aligned}$$

□

With the limiting optimal control sorted out, we can now show convergence as $\hbar \to 0$ in the exact same way as for the case $\beta^2 < 4\pi$ and we refer the reader to [6, Section 5] for details.

8. Osterwalder–Schrader axioms

The Osterwalder–Schrader axioms (OS axioms; for short), as introduced in [34, 35], provide sufficient conditions under which the (Euclidean) Schwinger functions define a relativistic QFT satisfying the Wightman axioms. We only briefly introduce the aspects that are immediately relevant to our discussion, for a more detailed exposition we refer to Chapter 6 in [26] or Section 5 in [27]. For a Radon measure ν on $\mathcal{S}'(\mathbb{R}^2)$ for $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^2)$, we define the associated Schwinger functions $S_n^\nu \in (\mathcal{S}(\mathbb{R}^2))^{\otimes n}$, by

$$S_n^\nu(f_1 \otimes \dots \otimes f_n) := \int_{\mathcal{S}'(\mathbb{R}^2)} \langle \varphi, f_1 \rangle \dots \langle \varphi, f_n \rangle \, \nu(\mathrm{d}\varphi). \tag{8.1}$$

We say that ν satisfies the OS axioms, if its associated Schwinger functions (8.1) satisfy the OS-Axioms. We already reformulate the axioms as conditions on the measures ν instead of the Schwinger functions above. It is easy to verify that the conditions on ν below imply the OS axioms for the Schwinger functions (8.1).

1. (**OS1-Regularity**) There is a Schwartz-norm $\|\cdot\|_S$ and a $\gamma > 0$ such that
$$\int_{\mathcal{S}'(\mathbb{R}^2)} \mathrm{e}^{\gamma \|\varphi\|_S^2} \, \nu(\mathrm{d}\varphi) < \infty.$$

2. (**OS2-Euclidean invariance**) The measure ν is invariant under the action of the Euclidean group. More precisely, for any $\mathcal{G} = (\mathcal{R}, a) \in O(2) \times \mathbb{R}^2$, it holds that $\nu = \mathcal{G}_\# \nu$, where $\mathcal{G}_\# \nu(\cdot) := \nu(\mathcal{G}^{-1}(\cdot))$ denotes the push forward measure of ν under $\mathcal{G}$.

3. (**OS3-Reflection positivity**) Define the reflection $\Theta : \mathbb{R}^2 \to \mathbb{R}^2, (x_0, x_1) \mapsto (-x_0, x_1)$ along the first coordinate axis. Then, for any exponential observable of the form $\mathcal{O}(\varphi) = \prod_{i=1}^n c_i \exp\{\langle \varphi, f^i \rangle\}$ for $f_i \in \mathcal{S}(\mathbb{R}^2)$ with support on $\{(x_0, x_1) \in \mathbb{R}^2;\ x^0 \geqslant 0\}$,
$$\int_{\mathcal{S}'(\mathbb{R}^2)} \overline{(\Theta\mathcal{O})(\varphi)} \mathcal{O}(\varphi) \, \nu(\mathrm{d}\varphi) \geqslant 0.$$
Here, for $z \in \mathbb{C}$, we denote the complex conjugate by $\bar{z}$ and extended the reflection map Θ to functions $f \in \mathcal{S}(\mathbb{R}^2)$ and the observables $\mathcal{O}$ via
$$\Theta f(x_0, x_1) := f(-x_0, x_1), \quad \Theta\mathcal{O}(\varphi) := \prod_{i=1}^n \exp\{\langle \varphi, \Theta f_i \rangle\}.$$

If the measures ν satisfies the conditions above, then the reconstruction theorem [35] (see also [26, Theorem 6.1.3]) ensures the existence of a Wightman theory corresponding to the measure ν.

Theorem 8.1. *Every accumulation point of* $(\mathrm{Law}(X_T^{\rho,T}))_{\rho,T}$ *satisfies* ***OS1*** *and* ***OS3****. If moreover,* $\bar{\lambda}$ *is sufficiently small, then also* ***OS2*** *holds and the measure is not Gaussian.*

The regularity property for ν_{SG} was already shown in Corollary 4.7. The remaining claims are proven in the next three sections.

8.1. Euclidean invariance

The Euclidean invariance in this setting is a straightforward consequence of the uniqueness obtained in Theorem 4.6.

Proposition 8.2. *The joint law of (Z_∞, W_∞) is invariant under the action of the Euclidean group defined by*

$$\mathcal{G}f(x) = f(\mathcal{R}(x - \mathcal{R}^{-1}a)) \quad \textit{for} \quad \mathcal{G} = (\mathcal{R}, a) \in O(2) \times \mathbb{R}^2.$$

Proof. Since the kernels $f_s^{[\ell]}$ are translation and rotation invariant (see Remark 3.2-c),

$$\mathcal{G}F_s(X_s) = F_s(\mathcal{G}X_s).$$

Moreover, immediately from the definition of $\dot{G}_s$, we have $\mathcal{G}(\dot{G}_s f) = \dot{G}_s(\mathcal{G}f)$. Therefore, for any $\rho \preceq 1$ and $T \leqslant \infty$, the transformed solution $\mathcal{G}X^{\rho,T}$ satisfies the equation,

$$\begin{aligned}\mathcal{G}X_t^{\rho,T} &= -\int_0^t \mathrm{d}s \mathcal{G}\dot{G}_s(F_s^{\rho,T}(X_s^{\rho,T}) + R_s^{\rho,T}) + \mathcal{G}W_t \\ &= -\int_0^t \mathrm{d}s \dot{G}_s(F_s^{\mathcal{G}\rho,T}(\mathcal{G}X_s^{\rho,T}) + \mathcal{G}R_s^{\rho,T}) + \mathcal{G}W_t.\end{aligned}$$

With the same reasoning,

$$\mathcal{G}R_t^{\rho,T} = \mathbb{E}_t \int_t^T \mathrm{d}s H_s^{\mathcal{G}\rho,T}(\mathcal{G}X_s^{\rho,T}) - \mathbb{E}_t \int_t^T \mathrm{d}s \mathrm{D}F_s^{\mathcal{G}\rho,T}(\mathcal{G}X_s^{\rho,T})\dot{G}_s \mathcal{G}R_s^{\rho,T}.$$

In other words, $(\tilde{X}^{\rho,T}, \tilde{R}^{\rho,T}) := \mathcal{G}(X^{\rho,T}, R^{\rho,T})$ is a solution to

$$\begin{cases}\tilde{X}_t^{\rho,T} = \tilde{W}_t - \int_0^t \dot{G}_s(F_s^{\mathcal{G}\rho,T}(\tilde{X}_s^{\rho,T}) + \tilde{R}_{s,T}^{\rho})\mathrm{d}s, \\ \tilde{R}_t^{\rho,T} = \mathbb{E}_t \int_t^T H_s^{\mathcal{G}\rho,T}(\tilde{X}_s^{\rho,T})\mathrm{d}s - \mathbb{E}_t \int_t^T \mathrm{D}F_s^{\mathcal{G}\rho,T}(\tilde{X}_s^{\rho,T})\dot{G}_s \tilde{R}_s^{\rho,T}\mathrm{d}s,\end{cases}$$

where $\tilde{W} := \mathcal{G}W = \int_0^\cdot Q_s \mathrm{d}(\mathcal{G}B_s)$ is again a Brownian motion with the same covariance as W. By the uniqueness of the solution to (4.1) (see Corollary 4.3) we then have

$$\mathrm{Law}(\tilde{X}^{\mathcal{G}^{-1}\rho,T}, \tilde{W}) = \mathrm{Law}(X^{\rho,T}, W).$$

In other words, the joint law is invariant under the action of the Euclidean group $\mathcal{G}$ provided that $\rho = \mathcal{G}\rho$, which holds only when the weight ρ is flat, that is $\rho \propto 1$. In this case, we have for any $T \leqslant \infty$,

$$\mathrm{Law}(\tilde{X}_T^T, \tilde{W}_T) = \mathrm{Law}(\mathcal{G}(X_T^T, W_T)) = \mathrm{Law}(X_T^T, W_T).$$

□

8.2. Reflection positivity

To show that ν_{SG} is reflection positive, we show that it is the weak limit of reflection positive measures. We cannot use the approximating sequence $\nu_{\mathrm{SG}}^{\rho,T}$ because the small scale regularisation for $T < \infty$ mollifies the measure in all directions and consequently breaks reflection positivity. Instead, we will construct a new sequence of reflection positive measures $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ such that $\nu_{\mathrm{SG}}^{\varepsilon,\rho} \to \nu_{\mathrm{SG}}^{\rho}$ for any spatial cut-off $\rho \prec 1$. Since weak limits of reflection positive measures are reflection positive and since ν_{SG} is the weak limit of the finite volume measures ν_{SG}^{ρ}, this will prove the claim. Throughout this section, we fix a symmetric cut-off ρ and suppress the dependency whenever it does not lead to ambiguities.

To preserve reflection positivity, we cannot mollify in the direction of physical time, and we instead mollify along only one of the coordinate axes. For $\eta \in C_c^\infty(\mathbb{R})$ supported on $|x| < 1$, define the family of mollifiers with $\tilde{\eta}_\varepsilon = \varepsilon^{-1}\eta(\cdot\varepsilon^{-1})$ on $\mathbb{R}^1$ and introduce the corresponding mollifiers $\eta_\varepsilon = \delta_0 \otimes \widetilde{\eta_\varepsilon}$ on $\mathbb{R}^2$. Then, using a variant of Theorem 2.2-b, we can define the measures,

$$\nu_{\mathrm{SG}}^{\varepsilon,\rho,T} = \mathrm{Law}(X_\infty^{\varepsilon,\rho,T}),$$

where

$$X_t^{\varepsilon,\rho,T} = -\int_0^t \mathrm{d}s \dot{G}_s^\varepsilon (F_s^{\varepsilon,\rho,T}(X_s^{\varepsilon,\rho,T}) + R_s^{\varepsilon,\rho,T}) + W_t^\varepsilon = -\int_0^t \mathrm{d}s \dot{G}_s^\varepsilon \mathbb{E}_s[\nabla V_T^\varepsilon(X_s^{\varepsilon,\rho,T})] + W_t^\varepsilon. \tag{8.2}$$

Here, we defined $G_t^\varepsilon := Q_t^\varepsilon * Q_t^\varepsilon$ with $Q_t^\varepsilon = \eta^\varepsilon * Q_t$, and obtain F^ε and W^ε as before by replacing G by its mollification G^ε. Then, denoting $\mu^{\varepsilon,\rho,T} := \mathrm{Law}(W_T^\varepsilon)$, the same argument as in Theorem 2.2-b shows that

$$\mathrm{Law}(X_T^{\varepsilon,\rho,T}) \propto \exp\left(-\lambda_T^\varepsilon \int_{\mathbb{R}^2} \cos(\beta\varphi)\right) \mu^{\varepsilon,\rho,T}(\mathrm{d}\varphi),$$

where $\lambda_t^\varepsilon = \lambda \mathrm{e}^{-\frac{\beta^2}{2} G_t^\varepsilon(0)}$. The point is now that the additional convolution with η_ε ensures that the measures $\mu^\varepsilon = \mu^{\varepsilon,\infty}$ are supported on a function space. Indeed, we compute for any $\varepsilon > 0$,

$$G_T^\varepsilon(x) = \int_0^T \eta_\varepsilon * \dot{G}_s(x)\mathrm{d}s \leqslant \int_0^\infty \eta_\varepsilon * \dot{G}_s(x)\mathrm{d}s = G_\infty^\varepsilon(x) = \frac{1}{4\pi}\log\left(\frac{1}{|x|^2 \vee \varepsilon}\right) + r_\varepsilon(x),$$

where r_ε is bounded uniformly in $x \in \mathbb{R}^2$ and $\varepsilon > 0$. In particular, the Wick-ordering with respect to the Gaussian measure $\mu^{\varepsilon,\rho,T}$ is given by

$$[\![\sin(\beta W_T)]\!] = \lambda_T^\varepsilon \sin(\beta W_T), \quad \text{where} \quad \lambda_T^\varepsilon = \lambda \mathrm{e}^{\frac{\beta^2}{2} G_T^\varepsilon(0)} \lesssim \lambda\varepsilon^{-1},$$

which is not only bounded uniformly in T but also converges to a limit at $T = \infty$ with $\lambda^\varepsilon := \lambda_\infty^\varepsilon = \mathrm{e}^{\frac{\beta^2}{2} G_\infty(0)}$. Therefore, the same argument as used in Theorem 2.2-b implies

that for any $\varepsilon > 0$, the SDE (8.2) is meaningful also for $T = \infty$, with

$$X_t^{\varepsilon,\rho} = -\int_0^t \mathrm{d}s \dot{G}_s^{\varepsilon} \mathbb{E}_s[\nabla V_\infty^{\varepsilon}(X_\infty^{\varepsilon,\rho})] + W_\infty^{\varepsilon}, \tag{8.3}$$

has a unique solution for $\bar{\lambda}$ sufficiently small. Theorem 2.2-b also implies that the law of $X_\infty^{\varepsilon,\rho}$ is absolutely continuous with respect to μ^ε and we define

$$\nu_{\mathrm{SG}}^{\rho,\varepsilon} := \mathrm{Law}(X_\infty^{\varepsilon,\rho}) \propto \exp\left(-\lambda^\varepsilon \int_{\mathbb{R}^2} \mathrm{d}x \rho(x) \cos(\beta\varphi(x))\right) \mu^\varepsilon(\mathrm{d}\varphi). \tag{8.4}$$

For these measures, reflection positivity will follow directly from the reflection positivity of μ^ε.

Lemma 8.3. *For any $\varepsilon > 0$ and $\rho \prec 1$, the measures $\nu_{\mathrm{SG}}^{\rho,\varepsilon}$ defined in* (8.4) *are reflection positive.*

Proof. Denote the projection on the positive half-plane $\mathbb{R}_+ \times \mathbb{R}$ by π_+ and let again

$$\Theta f(x_0, x_1) := f(-x_0, x_1),$$

be the reflection around the first coordinate axis. We first show that the Gaussian measure $\mu^\varepsilon = \mathrm{Law}(W_\infty^\varepsilon)$ is reflection positive. Since a Gaussian measure is reflection positive if and only if its covariance is reflection positive (see e.g. [26, Theorem 6.2.2.]), it is sufficient to check that for any function $f \in L^2(\mathbb{R}^2)$,

$$G^\varepsilon(\pi_+ f, \Theta\pi_+ f) = \langle \eta_\varepsilon * \pi_+ f, (m^2 - \Delta)^{-1} \Theta \eta_\varepsilon * \pi_+ f \rangle_{L^2} \geqslant 0, \tag{8.5}$$

where $G^\varepsilon = G_\infty^\varepsilon$ is the covariance μ^ε. Because η_ε leaves the first coordinate invariant, the convolution with η_ε commutes with the projection π_+. The reflection positivity of $(m^2 - \Delta)^{-1}$ now implies (8.5).

To see that the measures $(\nu_{\mathrm{SG}}^{\varepsilon,\rho})_{\varepsilon>0}$ defined by (8.4) are also reflection positive, we split the potential between the two half-planes $\{x_0 \geqslant 0\}$ and $\{x_0 < 0\}$, as

$$V_\varepsilon^{\rho,\pm}(\varphi) := \lambda^\varepsilon \int_{\mathbb{R}_\pm \times \mathbb{R}} \rho(x) \cos(\beta\varphi(x)) \mathrm{d}x,$$

so that

$$\nu_{\mathrm{SG}}^{\rho,\varepsilon}(\mathrm{d}\varphi) = \exp(-(V_\varepsilon^{\rho,+}(\varphi) + V_\varepsilon^{\rho,-}(\varphi)))\mu^\varepsilon(\mathrm{d}\varphi).$$

For the symmetric cut-off ρ, the reflection Θ acts on this decomposition as $\Theta V_\varepsilon^{\rho,\pm} = V_\varepsilon^{\rho,\mp}$. Consequently, we have for any exponential observable $\mathcal{O}$ supported on the positive half plane as defined in **(OS3)**,

$$\begin{aligned}\int_{\mathcal{S}'(\mathbb{R}^2)} \mathcal{O}(\varphi)\Theta\overline{\mathcal{O}(\varphi)} \nu_{\mathrm{SG}}^{\varepsilon,\rho}(\mathrm{d}\varphi) &= \int_{\mathcal{S}'(\mathbb{R}^2)} \mathcal{O}(\varphi)\Theta\mathcal{O}(\varphi) \mathrm{e}^{-V_\varepsilon^{\rho,+}(\varphi)} \mathrm{e}^{-V_\varepsilon^{\rho,-}(\varphi)} \mu^\varepsilon(\mathrm{d}\varphi) \\ &= \int_{\mathcal{S}'(\mathbb{R}^2)} \mathcal{O}(\varphi) \mathrm{e}^{-V_\varepsilon^{\rho,+}(\varphi)} \Theta(\mathcal{O}(\varphi) \mathrm{e}^{-V_\varepsilon^{\rho,+}(\varphi)}) \mu^\varepsilon(\mathrm{d}\varphi).\end{aligned}$$

Since $V^{\rho,+}$ is supported on the positive half plane $\mathbb{R}_+ \times \mathbb{R}$, the last integral is non-negative as a result of the reflection positivity of μ^ε. In other words, for any symmetric cut-off $\rho \prec 1$, also $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ is reflection positive. □

Having established reflection positivity for $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ for any $\varepsilon > 0$, we want to extract a subsequence which converges to the desired limiting measure ν_{SG}^{ρ} to conclude this proof. That is, it remains to show that for any $\alpha > 0$, there is a sequence $\varepsilon_N \downarrow 0$ such that

$$\sup_t \mathbb{E}\|X_t^{\varepsilon_N} - X_t\|^2_{H^{-\alpha,-n}} \to 0. \tag{8.6}$$

Adapting the definitions (3.39) to the current situation, we see that with the usual remainder R^ε, the FBSDE for the difference is given by

$$\begin{cases} X_t^\varepsilon - X_t = -\int_0^t \mathrm{d}s[\dot{G}_s^\varepsilon(F_s^\varepsilon(X_s^\varepsilon) + R_s^\varepsilon) - \dot{G}_s(F_s(X_s) + R_s)] + W_t^\varepsilon - W_t, \\ R_t^\varepsilon - R_t = \mathbb{E}_t \int_t^\infty (H_s^\varepsilon(X_s^{\varepsilon,\rho}) - H_s(X_s) - (\mathrm{D}F_s^{\varepsilon,\rho}(X_s^{\varepsilon,\rho})\dot{G}_s^\varepsilon R_s^\varepsilon - \mathrm{D}F_s(X_s)\dot{G}_s R_s))\mathrm{d}s. \end{cases} \tag{8.7}$$

By definition of G^ε, it holds for any $\alpha > 0$,

$$\lim_{\varepsilon \to 0} \sup_t \mathbb{E}\|W_t^\varepsilon - W_t\|^2_{H^{-\alpha,-n}} = 0.$$

For some subsequence $\varepsilon_N \downarrow 0$ and any $\varphi \in \mathcal{S}'(\mathbb{R}^2)$, Lemma B.2 combined with Proposition 3.13-a implies,

$$\|Q_s(F_s^{\varepsilon_N} - F_s)(\varphi)\|^2_{L^\infty} + \|Q_s(\mathrm{D}F_s^{\varepsilon_N} - \mathrm{D}F_s)(\varphi)\|^2_{L^\infty} \lesssim N^{-1}\bar{\lambda}_s\langle s\rangle^{-1},$$

and

$$\|(H_s^\varepsilon - H_s)(\varphi)\|^2_{L^\infty} \lesssim N^{-1}(\bar{\lambda}_s\langle s\rangle^{-1})^4.$$

Following the (by now standard) procedure for the FBSDE (8.7) yields (8.6) and thus concludes the proof.

Remark 8.4. A slight modification of the argument allows to show reflection positivity for any accumulation point of $(\nu_{\mathrm{SG}}^{\rho,T})_{T\geqslant 0}$. Therefore, reflection positivity holds also without the smallness assumption on the coupling constant $\bar{\lambda}$.

8.3. Non-Gaussianity

We want to show now that for $\bar{\lambda}$ small the measure ν_{SG} is non-Gaussian. For any Gaussian measure ν with support on a Hilbert space $\mathcal{H}$ with dual pairing $\langle\cdot,\cdot\rangle$, mean $b \in \mathcal{H}^*$, covariance C, Cameron-Martin space $H_{\mathrm{CM}}(\nu)$ it holds for $\psi \in \mathcal{H}$,

$$\frac{1}{2}\|C\psi\|^2_{H_{\mathrm{CM}}(\nu)} = \frac{1}{2}\|C^{1/2}\psi\|^2_{\mathcal{H}} = \log \int \exp(-\langle\varphi,\psi\rangle)\,\nu(\mathrm{d}\varphi) + \langle b,\psi\rangle. \tag{8.8}$$

Showing that the generating function of ν_{SG} does not satisfy (8.8) for will therefore yield the claim.

Step 1. We first show that for any $\psi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned} -\log \int \exp(-\langle \varphi, \psi \rangle) \nu_{\mathrm{SG}}(\mathrm{d}\varphi) = &-\frac{1}{2} \|(m^2 - \Delta)^{-1/2} \psi\|_{L^2}^2 \\ &+ \lim_{\rho \to 1} [\mathcal{V}^\rho(-(m^2 - \Delta)^{-1} \psi) - \mathcal{V}^\rho(0)], \end{aligned} \tag{8.9}$$

where $\mathcal{V}^\rho$ is the value function as defined in (2.8). By Theorem 4.6 the measure ν_{SG} on $\mathcal{H}$ can be obtained as a weak limit of measures $\nu_{\mathrm{SG}}^{\rho,T}$ that are absolutely continuous with respect to the Gaussian measure $\mu_T = \mathcal{N}(0, G_T)$. Combined with the Gaussian tails from Corollary 4.7 this implies convergence of $-\log \nu_{\mathrm{sG}}^{\rho,T}(\exp(-\langle \psi, \cdot \rangle)) \to -\log \nu_{\mathrm{sG}}(\exp(-\langle \psi, \cdot \rangle))$ by uniform integrability for the functionals $\exp(-\langle \psi, \cdot \rangle)$. Indeed, for every $L > 0$,

$$\begin{aligned} &\left| \int \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}^{\rho,T}(\mathrm{d}\varphi) - \int \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}(\mathrm{d}\varphi) \right| \\ \leqslant &\left| \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \leqslant L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}^{\rho,T}(\mathrm{d}\varphi) - \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \leqslant L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}(\mathrm{d}\varphi) \right| \\ &+ \left| \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \geq L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}^{\rho,T}(\mathrm{d}\varphi) \right| + \left| \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \geq L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}(\mathrm{d}\varphi) \right|. \end{aligned}$$

Regarding the two tail integrals, note that for every $\gamma > 0$, there is a $K_\gamma > 0$ such that

$$|-\langle \psi, \varphi \rangle| \leqslant \|\psi\|_{H^{\varepsilon,n}} \|\varphi\|_{H^{-\varepsilon,-n}} \leqslant \gamma \|\varphi\|_{H^{-\varepsilon,-n}}^2 + K_\gamma \|\psi\|_{H^{\varepsilon,n}}^2 .$$

Hence, using Corollary 4.7, for any $\kappa > 0$, we can choose $L > 0$ sufficiently large so that

$$\sup_{\rho \prec 1, T \leqslant \infty} \left| \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \geq L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}^{\rho,T}(\mathrm{d}\varphi) \right| \leqslant \frac{1}{3}\kappa.$$

The first difference converges by weak convergence of $\nu_{\mathrm{sG}}^{\rho,T}$ to ν_{sG} and the boundedness of $\exp(-\langle \psi, \varphi \rangle)$ on $\{\|\varphi\|_{H^{-\varepsilon,-n}} \leqslant L\}$. Thus, for L fixed as above, we may choose ρ, T sufficiently close to $1, \infty$ so that also

$$\left| \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \leqslant L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}^{\rho,T}(\mathrm{d}\varphi) - \int_{\{\|\varphi\|_{H^{-\varepsilon,-n}} \leqslant L\}} \exp(-\langle \psi, \varphi \rangle) \nu_{\mathrm{sG}}(\mathrm{d}\varphi) \right| \leqslant \frac{\kappa}{3}.$$

Finally, it remains only to show that also the log converges. To this end, note that by Lemma 4.1, there are universal constants $K, K_Z > 0$ such that uniformly in $\rho \preceq 1$, $T \in (0, \infty]$

$$\begin{aligned} \nu_{\mathrm{sG}}^{\rho,T}(\exp(-\langle \psi, \cdot \rangle)) = &\mathbb{E}[\exp(-\langle \psi, Z_T^{\rho,T} \rangle) \exp(-\langle \psi, W_T \rangle)] \\ \geq &\mathrm{e}^{-K_Z \|\psi\|_{H^{\varepsilon,n}}^2} \mathbb{E}[\exp(-\gamma \|W_\infty\|_{H^{-\varepsilon,-n}}^2)] \geq K > 0. \end{aligned}$$

Therefore, the convergence $-\log \nu_{\mathrm{sG}}^{\rho,T}(\exp(-\langle\psi,\cdot\rangle)) \to -\log \nu_{\mathrm{sG}}^{\rho,T}(\exp(-\langle\psi,\cdot\rangle))$ also follows immediately. Combined, this allows us to write

$$\begin{aligned}
&-\log \int \exp(-\langle\varphi,\psi\rangle)\nu_{\mathrm{SG}}(\mathrm{d}\varphi)\\
&= \lim_{\substack{T\to\infty\\ \rho\to 1}} -\log\left\{\Xi_{T,\rho}^{-1}\int \exp(-\langle\varphi,\psi\rangle - V_T^{T,\rho}(\varphi))\mu_T(\mathrm{d}\varphi)\right\}.
\end{aligned}$$

For any $h \in H_{\mathrm{CM}}(\mu) = H^1$, by the Cameron–Martin theorem (see e.g. [15, Corollary 2.4.3]),

$$\mu_T(\mathrm{d}(\varphi - h)) = \exp\left(\langle\varphi, G_T^{-1}h\rangle - \frac{1}{2}\langle h, G_T^{-1}h\rangle\right)\mu_T(\mathrm{d}\varphi).$$

Thus,

$$\begin{aligned}
&-\log\left\{\Xi_{T,\rho}^{-1}\int \exp(-\langle\varphi,\psi\rangle - V_T^{T,\rho}(\varphi))\mu_T(\mathrm{d}\varphi)\right\}\\
&= -\log\left\{\Xi_{T,\rho}^{-1}\int \exp\left(-\langle\varphi,\psi\rangle - V_T^{T,\rho}(\varphi) + \frac{1}{2}\langle h, G_T^{-1}h\rangle - \langle\varphi, G_T^{-1}h\rangle\right)\mu_T(\mathrm{d}(\varphi-h))\right\}\\
&= -\log\left\{\Xi_{T,\rho}^{-1}\int \exp\left(-\langle(\varphi-h),\psi + G_T^{-1}h\rangle - V_T^{T,\rho}(\varphi+h) + \frac{1}{2}\langle h, G_T^{-1}h\rangle\right)\mu_T(\mathrm{d}\varphi)\right\}.
\end{aligned}$$

Since $\psi \in H^{-1}$ and $G_\infty H^{-1} = H^1 = H_{\mathrm{CM}}(\mu)$, we may perform a shift along $h_T = -G_T\psi$ to obtain

$$\begin{aligned}
&-\log \int \exp(-\langle\varphi,\psi\rangle)\nu_{\mathrm{SG}}(\mathrm{d}\varphi)\\
&\quad= \lim_{\substack{T\to\infty\\ \rho\to 1}} -\log\left\{\exp\left(\frac{1}{2}\langle\psi, G_T\psi\rangle\right)\Xi_{T,\rho}^{-1}\int \exp(-V_T^{T,\rho}(\varphi - G_T\psi))\mu_T(\mathrm{d}\varphi)\right\}\\
&\quad= -\frac{1}{2}\|(m^2-\Delta)^{\frac{1}{2}}h\|_{L^2}^2 - \lim_{\substack{T\to\infty\\ \rho\to 1}} \log\left\{\Xi_{T,\rho}^{-1}\int \exp(-V_T^{T,\rho}(\varphi + h_T))\mu_T(\mathrm{d}\varphi)\right\}.
\end{aligned}$$

For the latter, Theorem 7.7 with $h = -G_\infty\psi$ implies,

$$\lim_{T\to\infty} -\log\left\{\Xi_{T,\rho}^{-1}\int \exp(-V_T^{T,\rho}(\varphi + h_T))\mu_T(\mathrm{d}\varphi)\right\} = \mathcal{V}^\rho(h) - \mathcal{V}^\rho(0),$$

which yields (8.9).

Step 2. We now show that (8.8) implies that $C\psi \in H_{\mathrm{CM}}(\nu)$ whenever $\psi \in C_c^\infty$. To this end it is sufficient to show that $\|\psi\|_{H_{\mathrm{CM}}(\nu_{\mathrm{SG}})}^2 \leqslant \infty$ whenever $\psi \in C_c^\infty$. From Theorem

4.6, we know that $\mathrm{supp}(\nu_{\mathrm{SG}}) \subset H^{-1,-n}$. Therefore, applying (8.8) for $\mathcal{H} = H^{-1,-n}$ and inserting the estimate from Step 1,

$$\begin{aligned}
&\|C\psi\|^2_{H_{\mathrm{CM}}(\nu_{\mathrm{SG}})} \\
&\quad \leqslant |\langle \psi, b\rangle| + |\log \int \exp(-\langle \varphi, \psi\rangle)\, \nu_{\mathrm{SG}}(\mathrm{d}\varphi)| \\
&\quad \leqslant \|\psi\|_{H^{-1,-n}} \|b\|_{H^{1,n}} + \frac{1}{2}\|(m^2-\Delta)^{-\frac{1}{2}}\psi\|^2_{L^2} + \sup_{\rho \leqslant 1} |\mathcal{V}^\rho(-G_\infty \psi) - \mathcal{V}^\rho(0)|.
\end{aligned}$$

All the terms except the last one are finite for $\psi \in C_c^\infty$. For the difference of the value functions, we use (7.11) combined with Lemma 3.9 and Lemma 4.1,

$$\sup_{\rho \leqslant 1} \|\nabla \mathcal{V}^\rho(h)\|_{L^\infty} \leqslant \sup_{\rho \leqslant 1} \|F_0^\rho(h)\|_{L^\infty} + \sup_{\rho \leqslant 1} \|R_0^\rho(h)\|_{L^\infty} \lesssim 1, \qquad h \in C_c^\infty(\mathbb{R}^2).$$

Then, a simple Taylor expansion combined with the estimates on F and R

$$\sup_{\rho \leqslant 1} |\mathcal{V}^\rho(-G_\infty \psi) - \mathcal{V}^\rho(0)| \leqslant \sup_{\theta \in [0,1]} \sup_{\rho \leqslant 1} \|\nabla \mathcal{V}^{\rho,T}(\theta(m^2-\Delta)^{-1}\psi)\|_{L^\infty} \|(m^2-\Delta)^{-1}\psi\|_{L^1} < \infty.$$

Step 3. We conclude by showing that (8.9) is not quadratic for $\psi \in C_c^\infty$, which is valid by Step 2. It will be more convenient to show that its gradient is non-linear. We know from Lemma 3.9 that

$$\|F_0^\rho(h) + \rho\lambda \sin(\beta h)\|_{L^\infty} \lesssim \bar{\lambda}^2 \|h\|_{W^{1,\infty}}.$$

Combined with $\|R_0\|_{L^\infty} \lesssim \bar{\lambda}^4$ from Lemma 4.1, we can gather all contributions $\ell > 1$ in a uniformly bounded function c^ρ satisfying

$$\sup_{\rho \preceq 1} \sup_{\|h\|_{W^{1,\infty}} \leqslant C} \|c^\rho(h)\|_{L^\infty} \lesssim 1,$$

so that for any fixed $C > 0$ and $\|h\|_{W^{1,\infty}} \leqslant C$,

$$\nabla \mathcal{V}^\rho(h) = -\rho\beta\lambda \sin(\beta h) + \lambda^2 c^\rho(h).$$

From here, we can see that $\nabla \mathcal{V}^\rho$ is not additive. For example, we may choose $\psi, \tilde{\psi} \in C_c^\infty \subset H_{\mathrm{CM}}(\nu_{\mathrm{SG}})$ such that $\mathbb{1}_{\{|x| \leqslant 1\}}\psi(x) = \frac{\pi}{2\beta}$ and $\mathbb{1}_{\{|x| \leqslant 1\}}\tilde{\psi} = \frac{\pi}{4\beta}$. If $\rho \equiv 1$ on $B_1(0)$, then we verify that on $B_1(0)$ for $K > 0$ and $\lambda > 0$,

$$\nabla \mathcal{V}^\rho(\psi + \tilde{\psi}) + \nabla \mathcal{V}^\rho(\psi - \tilde{\psi}) - 2\nabla \mathcal{V}^\rho(\psi) \tag{8.10}$$

$$= -\beta\lambda\left[\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) - 2\sin\left(\frac{\pi}{2}\right)\right] + O(\lambda^2) \tag{8.11}$$

$$\geqslant \lambda\left(2 - \sqrt{2}\right) - K\lambda^2. \tag{8.12}$$

For $\bar{\lambda}$ sufficiently small, uniformly in $\rho \preceq 1$ it holds that $\lambda\left(2-\sqrt{2}\right) - K\lambda^2 \geqslant \tilde{K} > 0$ and we conclude that $\nabla \mathcal{V}^\rho$ is non-linear for any $\rho \preceq 1$. For $\lambda < 0$ the the claim follows in the same way.

A. Heat kernel estimates

This Appendix contains some basic estimates on the heat kernel which we use throughout as well as some technical proofs which have been postponed.

A.1. General estimates

Lemma A.1. *With G as defined in* (2.3)*, there are uniformly bounded functions g_1, g_2 and constants $C \geqslant 0$ such that for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^2$,*

$$G_t(0) = \frac{1}{4\pi}\log(t \vee 1) + g_1(t), \tag{A.1}$$

$$(G_\infty - G_t)(x) = \frac{1}{4\pi}\log(|x|^{-2}t^{-1} \vee 1) + g_2(t,x) \tag{A.2}$$

$$\nabla \dot{G}_t(x) = C|x|\mathrm{e}^{-m^2/t - \frac{t}{4}|x|^2}, \tag{A.3}$$

and consequently

$$|\dot{G}_t(0) - \dot{G}_t(x)| \lesssim |x|\langle t\rangle^{-1/2}. \tag{A.4}$$

Proof. The estimate (A.1) follows immediately from the L^2-kernel representation (2.4), noting that

$$G_t(0) = \frac{1}{4\pi}\int_0^t \frac{\mathrm{d}s}{s}\mathrm{e}^{-m^2/s} = \int_0^{1\wedge t}\frac{\mathrm{d}s}{4\pi s}\mathrm{e}^{-m^2/s} + \int_{t\wedge 1}^{t\vee 1}\frac{\mathrm{d}s}{4\pi s}\mathrm{e}^{-m^2/s} = g_1(t) + \frac{1}{4\pi}\log(t\vee 1).$$

Regarding (A.2), we obtain after a substitution with $u = s^{-1}|x|^{-2}$,

$$\begin{aligned}
(G_\infty - G_t)(x) &= \int_t^\infty \frac{\mathrm{d}s}{4\pi s}\mathrm{e}^{-m^2/s}\mathrm{e}^{-\frac{s}{4}|x|^2} \\
&= \int_0^{t^{-1}|x|^{-2}} \frac{\mathrm{d}s}{4\pi s}\mathrm{e}^{-m^2|x|^2 s}\mathrm{e}^{-\frac{1}{4s}} \\
&= \int_0^{1\wedge t^{-1}|x|^{-2}} \frac{\mathrm{d}s}{4\pi s}\mathrm{e}^{-m^2|x|^2 s}\mathrm{e}^{-\frac{1}{4s}} + \int_{1\wedge t^{-1}|x|^{-2}}^{t^{-1}|x|^2\vee 1} \frac{\mathrm{d}s}{4\pi s}\mathrm{e}^{-m^2|x|^2 s}\mathrm{e}^{-\frac{1}{4s}} \\
&= g_2(t,x) + \frac{1}{4\pi}\log(t^{-1}|x|^{-2}\vee 1).
\end{aligned}$$

Finally, (A.3) is a direct computation and (A.4) follows from

$$\frac{|\dot{G}_t(0) - \dot{G}_t(x)|}{|x|} \lesssim \sup_y |\nabla \dot{G}_t(y)|.$$

Maximising the right-hand side, we see that the maximum is attained at $y = Cs^{-1/2}$ for some constant C which gives the claim. □

Lemma A.2. *For $\gamma^2 < 4c$, it holds that*

$$\int_{\mathbb{R}^2} \mathrm{d}x \mathrm{e}^{-ct|x|^2+m\gamma|x|-m^2/s}|x|^{2k} \lesssim \langle t\rangle^{-1-k}.$$

Proof. We treat the small and large scales separately. For $t > 1$,

$$\mathrm{e}^{-\frac{t}{4}|x|^2+\gamma m|x|} \lesssim \mathrm{e}^{-\frac{t}{8}|x|^2},$$

so that the estimate follows as in the unweighed case. To deal with the large scales, note that for any $c \geqslant 0$ the polynomial

$$p(x) := ct|x|^2 - m(\gamma-\varepsilon)|x| + \frac{m^2}{t},$$

attains its minimum at $x^{\pm} = \pm\frac{m(\gamma-\varepsilon)}{2ct}$ and $p(x) \geqslant p(x^{\pm}) \geqslant -\varepsilon^2/t$ provided $(\gamma-\varepsilon)^2 < 4c$. Therefore, choosing $\varepsilon > 0$ sufficiently small depending on $\gamma^2 < 4c$, we have

$$\mathrm{e}^{-ct|x|^2+\gamma|x|}\mathrm{e}^{-m^2/t} \leqslant \mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{-\varepsilon^2/t},$$

and thus

$$\sup_{t\in[0,1]}\int_{\mathbb{R}^2}\mathrm{d}y\frac{1}{4\pi s}\mathrm{e}^{-\frac{t}{4}|x|^2+\gamma|x|}\mathrm{e}^{-m^2/t} \lesssim \sup_{t\in[0,1]}\int_{\mathbb{R}^2}\mathrm{d}y\frac{1}{4\pi}t^{-1}\mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{-\varepsilon^2/t} < \infty.$$

□

Lemma A.3. *For any $t \in \mathbb{R}_+, x \in \mathbb{R}^2$ and $c \in (0,1)$, we have*

$$|\dot{G}_t(x) - \dot{G}_t(y)|\mathrm{e}^{-c\frac{t}{4}|x-y|^2} \lesssim |x-y|(|x|+|x-y|)\mathrm{e}^{-\frac{c}{2}\frac{t}{4}|x|^2}\mathrm{e}^{-m^2/t}.$$

The same estimate holds for $\dot{G}_t$ replaced by $t^{-1}\mathrm{e}^{-m^2/2t}Q_t$.

Proof. We start by rewriting the difference as

$$\dot{G}_t(x) - \dot{G}_t(y) = (y-x)\int_0^1 \mathrm{d}\vartheta \nabla\dot{G}_t(x-\vartheta(x-y)).$$

For any $z = x-\vartheta(x-y)$ and $\vartheta \in [0,1]$, we have $\frac{1}{2}|x|^2 \leqslant |x-z|^2+|z|^2 \leqslant \vartheta^2|x-y|^2+|z|^2$. Combined with (A.3) this means

$$|\nabla\dot{G}_t(z)| \leqslant C|z|\mathrm{e}^{-\frac{t}{4}|z|^2}\mathrm{e}^{-m^2/t} \leqslant C|z|\mathrm{e}^{-\frac{c}{2}\frac{t}{4}|x|^2}\mathrm{e}^{c\frac{t}{4}|x-y|^2}\mathrm{e}^{-m^2/t},$$

and consequently,

$$|\nabla\dot{G}_t(z)|\mathrm{e}^{-c\frac{t}{4}|x-y|^2} \leqslant C|z|\mathrm{e}^{-\frac{c}{2}|x|^2}\mathrm{e}^{-m^2/t} \leqslant C(|x|+|x-y|)\mathrm{e}^{-\frac{c}{2}\frac{t}{4}|x|^2}\mathrm{e}^{-m^2/t}.$$

The estimate on Q follows in the exact same way, only replacing the estimate on the gradient by

$$|\nabla Q_t(x)| \lesssim t|x|\mathrm{e}^{-2t|x|^2-m^2/2t}.$$

□

Lemma A.4. *For $k \in \mathbb{R}$, $\alpha > 0$, and $\gamma \in (-1,1)$, consider the weight $w(x) = \langle x\rangle^k$ or $w(x) = \exp(\gamma m|x|)$. Then, for any $t \in \mathbb{R}_+$ and $u \in L^p(w)$,*

$$\begin{aligned} \||x|^{2\alpha/p} Q_t\|_{L^p(w)} &= \langle t\rangle^{-1/p-\alpha/p}, \qquad \|Q_t u\|_{L^p(w)} \lesssim \langle t\rangle^{-1/p}\|u\|_{L^1(w)}, \\ \||x|^{2\alpha/p} \dot{G}_t\|_{L^p(w)} &= \langle t\rangle^{-1-1/p-\alpha/p}, \qquad \|\dot{G}_t u\|_{L^p(w)} \lesssim \langle t\rangle^{-1-1/p}\|u\|_{L^1(w)}. \end{aligned} \tag{A.5}$$

Moreover, with $C_t := (G_\infty - G_t)$ it holds for any $s \geqslant t$ and $c > 0$ sufficiently small, and $\alpha > (1-2\delta)\vee 0 = \left(\frac{\beta^2}{4\pi} - 1\right)\vee 0$,

$$\int_{\mathbb{R}^2} \dot{G}_s(x) \mathrm{e}^{\beta^2 C_s(x)} |x|^{2\alpha} \mathrm{e}^{ct|x|^2} \mathrm{d}x \lesssim \langle s\rangle^{-2-\alpha}. \tag{A.6}$$

Proof. For the first estimate we simply compute from (2.4) and Lemma A.2,

$$\|Q_t\|^p_{L^p(w)} \lesssim \int_{\mathbb{R}^2} \mathrm{e}^{-pm^2/2s} \mathrm{e}^{-2pt|x|^2} |x|^{2\alpha} w(x) \mathrm{d}x \lesssim \langle t\rangle^{-1-\alpha}.$$

In the polynomial case, the second bound is now a simple consequence of Young's convolution inequality and (1.11). For the case of the exponential weights, observe that by the triangle inequality,

$$\|Q_s u_s\|^p_{L^p(w_\gamma)} \leqslant \int_{\mathbb{R}^2} \mathrm{d}x \left(\int_{\mathbb{R}^2} \mathrm{d}y \mathrm{e}^{\gamma m|x-y|} Q_s(x-y) \mathrm{e}^{\gamma m|y|} u_s(y) \right)^p \leqslant \|Q_s\|^p_{L^1(w)} \|u_s\|^p_{L^p(w)},$$

which again implies the claim with Lemma A.2. The estimates on $\dot{G}_t$ follow from the estimates on Q_t the convolution inequalities, since

$$\|\dot{G}_t\|_{L^p(w)} = \|Q_t * Q_t\|_{L^p(w)} \leqslant \|Q_t\|_{L^1} \|Q_t\|_{L^p(w)} \lesssim \langle t\rangle^{-1-1/p}.$$

For the estimate (A.6), we start from Lemma A.1 and the definition of $\dot{G}_s$, to estimate

$$\dot{G}_s(x) \mathrm{e}^{\beta^2 C_s(x)} \lesssim s^{-1} \mathrm{e}^{-m^2/s} \mathrm{e}^{-\frac{s}{4}|x|^2} |x|^{-\beta^2/2\pi} \langle s\rangle^{-\beta^2/4\pi}.$$

Moreover, for $s \geqslant t$ and $c, \tilde{c} > 0$ sufficiently small (more precisely, $c \in \left(0, \frac{1}{4}\right)$ and $\tilde{c} \in \left(0, \frac{1}{4} - c\right)$), we have $\mathrm{e}^{-\frac{s}{4}|x|^2} \mathrm{e}^{ct|x|^2} \leqslant \mathrm{e}^{-\tilde{c}s|x|^2}$. Combining both observations we see that

$$\begin{aligned} \int_{\mathbb{R}^2} \dot{G}_s(x) \mathrm{e}^{\beta^2 C_s(x)} |x|^{2\alpha} \mathrm{e}^{ct|x|^2} \mathrm{d}x &\lesssim \int |x|^{2(\alpha-\beta^2/4\pi)} \mathrm{e}^{-\tilde{c}s|x|^2} \mathrm{e}^{-m^2/s} s^{-1-\beta^2/4\pi} \mathrm{d}x \\ &\lesssim s^{-\beta^2/4\pi} \mathrm{e}^{-m^2/s} \int_0^\infty r^{2(\alpha-\beta^2/4\pi)+1} \mathrm{e}^{-\tilde{c}sr^2} \mathrm{d}r \\ &\lesssim \langle s\rangle^{-2-\alpha}, \end{aligned}$$

provided that $r \mapsto r^{2(\alpha-\beta^2/4\pi)+1} \mathrm{e}^{-\tilde{c}sr^2}$ is integrable over $\mathbb{R}_+$, which is exactly the condition $\alpha > \frac{\beta^2}{4\pi} - 1$. □

Lemma A.5. *For any $\alpha \in [0,1]$, $k \in \mathbb{R}$ and $p \in [1,\infty]$, we have*

$$\left\| \int_0^T Q_s u_s \mathrm{d}s \right\|_{B^{\alpha,k}_{p,p}} \lesssim \sup_{s\in[0,T]} \|\langle s\rangle^{\alpha/2+\varepsilon} u_s\|_{L^{p,k}}. \tag{A.7}$$

Moreover, in L^2 the improved bound

$$\left\| \int_0^T Q_s u_s \mathrm{d}s \right\|^2_{H^{1,k}} \lesssim \int_0^T \|u_s\|^2_{L^{2,k}} \mathrm{d}s. \tag{A.8}$$

holds.

Proof. For any $\bar\varepsilon > 0$ and $\bar p > p$, it holds

$$\begin{aligned}
\left\| \int_0^t Q_s u_s \mathrm{d}s \right\|_{B^{\alpha,k}_{p,p}} &\leqslant \left\| \int_0^t Q_s u_s \mathrm{d}s \right\|_{B^{\alpha,k}_{\bar p,\bar p}} \\
&\leqslant \int_0^t \mathrm{d}s \|Q_s\|_{B^{\alpha,n}_{1,p}} \|u_s\|_{L^{p,k}} \\
&\leqslant \sup_s \|\langle s\rangle^{\alpha/2+\varepsilon} u_s\|_{L^{p,k}} \int_0^\infty \mathrm{d}s \langle s\rangle^{-\alpha/2-\varepsilon} \|Q_s\|_{B^{\alpha,n}_{1,p}}.
\end{aligned}$$

Moreover, by the interpolation of Besov spaces, for any $p > p(\alpha) = (1-\alpha)^{-1}$ and $\tilde\varepsilon > 0$ sufficiently small,

$$\|Q_s\|_{B^{\alpha,n}_{1,p}} \leqslant \|Q_s\|_{B^{\alpha,n}_{1,p(\alpha)}} \lesssim \|Q_s\|^{1-\alpha}_{L^{1,n}} \|Q_s\|^{\alpha}_{B^{1-\tilde\varepsilon,n}_{1,\infty}} \lesssim \langle s\rangle^{-1+\alpha} \|Q_s\|^{\alpha}_{B^{1-\tilde\varepsilon,n}_{1,\infty}},$$

so that the claim will follow once we compute

$$\|Q_s\|_{B^{1-\tilde\varepsilon,n}_{1,\infty}} \lesssim \sup_{|y|\leqslant 1} \int \mathrm{d}x \frac{|Q_s(x-y)-Q_s(x)|}{|y|} \langle x\rangle^n \lesssim \langle s\rangle^{-1/2}.$$

But this follows from a simple Taylor expansion,

$$\begin{aligned}
\int \mathrm{d}x |Q_s(x-y)-Q_s(x)| &= |y| s \mathrm{e}^{-m^2/2s} \int_0^1 \mathrm{d}\vartheta \int \mathrm{d}x \langle x\rangle^n |x-\vartheta y| \mathrm{e}^{-2s|x-\vartheta y|^2} \\
&= |y| s \mathrm{e}^{-m^2/2s} \int \mathrm{d}x |x| \mathrm{e}^{-s|x|^2} \\
&\lesssim |y| s^{-1/2} \mathrm{e}^{-m^2/2s}.
\end{aligned}$$

To remove the ε in the L^2 estimates, we pass to the Fourier transform and use the fact that Q_s is diagonal in Fourier space. Since $w(x) := \langle x\rangle^k$ grows at most polynomially, we have $w \in \mathcal{S}'(\mathbb{R}^2)$. For this computation only, we denote the Fourier transform of a distribution

$f \in \mathcal{S}'(\mathbb{R}^2)$ by $\hat{f} = \mathcal{F}(f)$. Repeatedly applying Hölder's inequality and Parseval's identity yields after some manipulation,

$$
\begin{aligned}
&\left\| \int_0^t \mathrm{d}s Q_s u_s \right\|_{H^\alpha(w)}^2 \\
=& C \left\| \int_0^t \mathrm{d}s \mathcal{F}(w(1-\Delta)^{\alpha/2} Q_s u_s) \right\|_{L^2}^2 \\
=& C \left\| \int_0^t \mathrm{d}s \int \mathrm{d}k \hat{w}(\xi - k)(1+|k|^2)^{\alpha/2} s \mathrm{e}^{-\frac{m^2+|k|^2}{2s}} \hat{u}(k) \right\|_{L^2(\mathrm{d}\xi)}^2 \\
\lesssim& \left\| \int_{\mathbb{R}^2} \mathrm{d}k (1+|k|^2)^{\alpha/2} \hat{w}(\xi - k) \left(\int_0^t \mathrm{d}s \frac{1}{s^{1+\alpha}} \mathrm{e}^{-\frac{m^2+|k|^2}{2s}} \right)^{1/2} \times \right. \\
&\qquad \left. \times \left(\int_0^t \mathrm{d}s \mathrm{e}^{-m^2/2s} s^{-1+\alpha} \hat{u}_s^2(k) \right)^{1/2} \right\|_{L^2(\mathrm{d}\xi)}^2 \\
\lesssim& \left\| \int_{\mathbb{R}^2} \mathrm{d}k \frac{(1+|k|^2)^{\alpha/2}}{(|k|^2+m^2)^{\alpha/2}} \mathrm{e}^{-\frac{m^2+|k|^2}{2t}} \hat{w}(\xi - k) \left(\int_0^t \mathrm{d}s \langle s \rangle^{-1+\alpha} \hat{u}_s^2(k) \right)^{1/2} \right\|_{L^2(\mathrm{d}\xi)}^2 \\
\lesssim& \sup_k \frac{(1+|k|^2)^\alpha}{(|k|^2+m^2)^\alpha} \left\| \int_{\mathbb{R}^2} \mathrm{d}k \hat{w}(\xi - k) \left(\int_0^t \mathrm{d}s \langle s \rangle^{-1+\alpha} \hat{u}_s^2(k) \right)^{1/2} \right\|_{L^2(\mathrm{d}\xi)}^2 \\
=& \int_0^t \mathrm{d}s \langle s \rangle^{-1+\alpha} \| \hat{w} * \hat{u} \|_{L^2}^2 \\
\lesssim& \int_0^t \mathrm{d}s \langle s \rangle^{-1+\alpha} \| u_s \|_{L^2(w)}^2 .
\end{aligned}
$$

□

Remark A.6. Lemma A.5 takes advantage of the concrete choice for the scale interpolation to get the optimal regularity estimates (A.8) in L^2. For a general scale interpolation, not necessarily diagonal in Fourier space, we have to use (A.7) and give up an arbitrarily small $\varepsilon > 0$ in regularity. This is not crucial to the analysis, but would in general lead to slightly worse results, e.g. replacing $\mathbb{D} = L^{2,n}$ by $\mathbb{D} = H^{\varepsilon,n}$ in the infinite volume variational problem in Theorem 7.7.

A.2. Proof of Lemma 2.1

We first restrict ourselves to the case $p \in [1, \infty)$. To this end, we use the translation invariance of the Law of W_t, and hypercontractivity to estimate

$$
\mathbb{E} \| W_t \|_{B_{p,p}^{-\varepsilon,-n}}^p = \sum_{i \geqslant -1} 2^{-i\varepsilon p} \int \mathbb{E}[|\Delta_i W_t(x)|^p] \langle x \rangle^{-pn} \mathrm{d}x \lesssim \sum_{i \geqslant -1} 2^{-i\varepsilon p} \mathbb{E}[|\Delta_i W_t(0)|^2]^{p/2}. \quad \text{(A.9)}
$$

Since $\Delta_i W_t(0) = \langle W_t, K_i\rangle$ and $\mathrm{Cov}(W_t) = G_t = (m^2-\Delta)^{-1}\mathrm{e}^{-(m^2-\Delta)/t}$, we can compute the expectation on the right hand side as

$$\begin{aligned}\mathbb{E}[|\Delta_i W_t(0)|^2] &= \mathbb{E}[\langle W_t, K_i\rangle\langle W_t, K_i\rangle] \\ &= \int \mathrm{d}\xi \frac{|\varphi_i(\xi)|^2}{m^2+|\xi|^2}\mathrm{e}^{-(m^2-\Delta)/t} \\ &\leqslant \int_{R_1 2^i}^{R_2 2^i} \frac{r}{(m^2+r^2)}\mathrm{d}r \\ &\lesssim \log(2^i).\end{aligned}$$

Here we used the fact that φ_i is supported on an annulus with radii $R_1 2^i$, $R_2 2^i$ in the second to last estimate. Inserting this bound in (A.9) yields the claim for $p \in [1,\infty)$. For $p=\infty$, we use the Besov embedding $\|\cdot\|_{B^{-\alpha,-n}_{\infty,\infty}} \lesssim \|\cdot\|_{B^{-\alpha+\varepsilon}_{p,p}}$ for $p > 2/\varepsilon$.

Finally, applying exactly the same reasoning to the increment $W_\infty - W_t$ instead shows the convergence in $L^p(\mathrm{d}\mathbb{P}, B^{-\varepsilon,-n}_{p,p})$ for any $p \in [1,\infty)$.

A.3. Proof of Lemma 3.4

Suppose for concreteness that $q(\xi_{1:\ell}) > 0$ and recall that we want to show

$$\Gamma_{t,s}(\xi_1,\dots,\xi_\ell) \leqslant \frac{\beta^2}{2}(G_t(0) - G_s(0)) + C.$$

We assume that (possibly after relabelling),

$$\sigma_k = \begin{cases} +1, & k \leqslant q, \\ (-1)^k & k > q, \end{cases}$$

and that ξ_1 is the most isolated charge, that is $\min_{k>q}|x_1 - x_k| = \max_{j\leqslant q}\min_{k>q}|x_j - x_k|$. We split the matrix into the 3 components

$$\Gamma_{t,s}(\xi_1,\dots,\xi_n) = \Gamma_{t,s}(\xi_1) + \Gamma_{t,s}(\xi_2,\dots,\xi_n) + \sigma_1\sum_{i>1}\sigma_i(G_t - G_s)(x_1 - x_i). \tag{A.10}$$

By the definition of Γ and the basic heat kernel estimates (A.1), the first summand is

$$\frac{\beta^2}{2}(G_t - G_s)(0) \leqslant \frac{\beta^2}{8\pi}(\log(t\vee 1) - \log(s\vee 1)) + C.$$

The second summand is bounded from above by (3.22). So (3.23) will follow once we establish an upper bound for the last term in (A.10). Towards this goal, we start by extracting the charged and neutral part,

$$\sum_{i>1}\sigma_1\sigma_i(G_t - G_s)(x_1 - x_i) = \sum_{i=2}^{q}\sigma_1\sigma_i(G_t - G_s)(x_1 - x_i) + \sum_{i=q+1}^{\ell}\sigma_1\sigma_i(G_t - G_s)(x_1 - x_i).$$

The first (charged) part satisfies $\sigma_1\sigma_i = 1$ and we can use the same reasoning as in (3.22) to conclude boundedness from the positivity of G. The second (neutral) part also contains contributions with the "bad" signs $\sigma_1\sigma_i = -1$ and requires special attention. Since this part is neutral, we know that the sum contains an even number of points and we can proceed by considering the neutral pairs (x_{q+2i}, x_{q+2i+1}), $i = 0, 1, \dots$ one at a time. In other words, the claim will follow if there is a constant $C > 0$ such that for any $(y, z) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$(G_t - G_s)(x_1 - y) - (G_t - G_s)(x_1 - z) \leqslant C. \tag{A.11}$$

By construction, one of the terms in these pairings comes with the "good" sign, which we are going to use to bound the neutral contribution. We start by rewriting the covariance using the kernel representation,

$$\begin{aligned}(G_t - G_s)(x_1 - y) - (G_t - G_s)(x_1 - z) &= -\int_t^s \mathrm{d}\tau [\dot{G}_\tau(x_1 - y) - \dot{G}_\tau(x_1 - z)] \\ &= -\int_t^s \mathrm{d}\tau \tau^{-1} \mathrm{e}^{-m^2/\tau} \left(\mathrm{e}^{-\frac{\tau}{4}|x_1 - y|^2} - \mathrm{e}^{-\frac{\tau}{4}|x_1 - z|^2} \right).\end{aligned}$$

If the charged edge is the shortest edge in the triangle connecting x_1, y, z, that is $|x_1 - y| \leqslant |x_1 - z|$, then

$$\mathrm{e}^{-\frac{\tau}{4}|x_1 - y|^2} - \mathrm{e}^{-\frac{\tau}{4}|x_1 - z|^2} \geqslant 0,$$

and we can bound (A.11) with $C = 0$. Otherwise, one of the neutral edges $|x_1 - z|$ or $|z - y|$ is the shortest edge. Thanks to the choice of x_1 as the most isolated charge, $|y - z| \leqslant |x_1 - z|$. If $|y - z| = 0$, then $(G_t - G_s)(x_1 - y) - (G_t - G_s)(x_1 - z) = 0$ and (A.11) is trivially true. Thus, we may assume that all edges have positive lengths. On $\tau > |y - z|^{-2}$, we directly compute

$$\int_{|z-y|^{-2}}^s \mathrm{d}\tau \tau^{-1} \mathrm{e}^{-m^2/r} \left| \mathrm{e}^{-\frac{\tau}{4}|x_1 - y|^2} - \mathrm{e}^{-\frac{\tau}{4}|x_1 - z|^2} \right| \leqslant 2 \int_{|z-y|^{-2}}^s \mathrm{d}\tau \tau^{-1} \mathrm{e}^{-m^2/\tau} \mathrm{e}^{-\frac{\tau}{4}|z-y|^2} \lesssim 1.$$

On $\tau \leqslant |y - z|^{-2}$, we use (A.4) combined with the translation invariance of $\dot{G}$ to conclude

$$\begin{aligned}\int_t^{|z-y|^{-2}} \mathrm{d}\tau [\dot{G}_\tau(x_1 - y) - \dot{G}_\tau(x_1 - z)] &\lesssim |y - z| \int_t^{|y-z|^{-2}} \mathrm{d}\tau |\nabla \dot{G}_\tau(x_1 - y + \tau(y - z))| \\ &\lesssim |y - z| \int_t^{|y-z|^{-2}} \mathrm{d}\tau \langle \tau \rangle^{-1/2} \lesssim 1,\end{aligned}$$

which completes the proof of Lemma 3.4.

A.4. Proof of Lemma 4.8

We first show the claim for $p < \infty$. To this end, we start by rewriting $t^{-\alpha/2} W_t$ for $t \geqslant 1$ using Ito's formula

$$t^{-\alpha/2} W_t = W_1 - \frac{\alpha}{2} \int_1^t s^{-1-\alpha/2} W_s \mathrm{d}s + \int_1^t s^{-\alpha/2} \mathrm{d}W_s,$$

so that

$$\sup_t \|t^{-\alpha/2}W_t\|_{B^{\alpha-\varepsilon,-n}_{p,p}} \lesssim \|W_1\|_{B^{\alpha-\varepsilon,-n}_{p,p}} + \int_1^\infty s^{-1-\alpha/2}\|W_s\|_{B^{\alpha-\varepsilon,-n}_{p,p}}\mathrm{d}s + \sup_t \left\| \int_1^t s^{-\alpha/2}\mathrm{d}W_s \right\|_{B^{\alpha-\varepsilon,-n}_{p,p}}. \tag{A.12}$$

For the bounded variation part, a similar computation to the proof of Lemma 2.1 shows that for any $\tilde{\varepsilon} > 0$ we have

$$\mathbb{E}\|W_s\|_{B^{\alpha-\varepsilon}_{p,p}} \lesssim \langle s\rangle^{\alpha/2-(\varepsilon-\tilde{\varepsilon})/2}.$$

Therefore, choosing $\tilde{\varepsilon} \in (0, \varepsilon)$,

$$\begin{aligned}\mathbb{E}\int_1^\infty s^{-1-\alpha/2}\|W_s\|_{B^{\alpha,-n}_{p,p}}\mathrm{d}s &\lesssim \int_1^\infty s^{-1-\tilde{\varepsilon}/2}\mathbb{E}[\langle s\rangle^{\alpha/2-(\varepsilon-\tilde{\varepsilon})/2}\|W_s\|_{B^{\alpha-\varepsilon}_{p,p}}]\mathrm{d}s \\ &\lesssim \int_1^\infty s^{-1-\tilde{\varepsilon}/2} < \infty.\end{aligned}$$

Regarding the martingale $M_t = \int_1^t s^{-\varepsilon/2}\mathrm{d}W_s$, we compute by translation invariance, the maximal inequalities and Gaussian hypercontractivity,

$$\begin{aligned}\mathbb{E}[\sup_t \|M_t\|^p_{B^{\alpha,-n}_{p,p}}] &= \mathbb{E}\left[\sup_t \sum_{i\geqslant -1} 2^{i(\alpha-\varepsilon)p}\int |\Delta_i M_t(x)|^p \langle x\rangle^{-pn}\mathrm{d}x\right] \\ &\lesssim \sum_{i\geqslant -1} 2^{i(\alpha-\varepsilon)p}\mathbb{E}[\sup_t |\Delta_i M_t(0)|^p] \\ &\lesssim \sum_{i\geqslant -1} 2^{i(\alpha-\varepsilon)p}\mathbb{E}[|\Delta_i M_\infty(0)|^p] \\ &\lesssim \sum_{i\geqslant -1} 2^{i(\alpha-\varepsilon)p}\mathbb{E}[|\Delta_i M_\infty(0)|^2]^{p/2}.\end{aligned}$$

The covariance of M_∞ can be computed directly, as for some constant C,

$$\begin{aligned}\mathbb{E}[\langle M_\infty, f\rangle\langle M_\infty, g\rangle] &= \int_0^\infty \mathrm{d}s \int \mathrm{d}x f(x)\int \mathrm{d}y s^{-\alpha}\dot{G}_s(x-y)g(y) \\ &= C\int \mathrm{d}x \int \mathrm{d}y f(x)((m^2-\Delta)^{-1-\alpha}g)(x) \\ &= C\langle f, (m^2-\Delta)^{-1-\alpha}g\rangle.\end{aligned}$$

Since $\Delta_i M_\infty(0) = \langle M_\infty, K_i\rangle$, we have

$$\mathbb{E}[|\Delta_i M_\infty(0)|^2] = \mathbb{E}[|\langle M_\infty, K_i\rangle|^2] = C\langle K_i, (m^2-\Delta)^{-1-\alpha}K_i\rangle = \int \mathrm{d}\xi \frac{|\varphi_i(\xi)|^2}{(m^2+|\xi|^2)^{1+\alpha}},$$

where we used $K_i = \mathcal{F}^{-1}(\varphi_i)$. Since φ is radially symmetric and supported on an annulus with radii R_1, R_2, we see passing to spherical coordinates using $\varphi_i = \varphi(2^{-i}\cdot)$

$$\int \mathrm{d}\xi \frac{|\varphi_i(\xi)|^2}{(m^2+|\xi|^2)^{1+\alpha}} \lesssim \int_{R_1 2^i}^{R_2 2^i} \frac{r^{d-1}}{(m^2+r^2)^{1+\alpha}} \mathrm{d}r \overset{d=2}{\lesssim} (R_1 2^i)^{-2\alpha} \lesssim 2^{-2\alpha i}.$$

Therefore, for $0 < \alpha < \varepsilon$,

$$\mathbb{E}[\sup_t \|M_t\|^p_{B^{(\alpha-\varepsilon),-n}_{p,p}}] \lesssim \sum_{i\geqslant -1} 2^{i(\alpha-\varepsilon)p} \mathbb{E}[|\Delta_i M_\infty(0)|^2]^{p/2} \lesssim \sum_{i\geqslant -1} 2^{-pi\varepsilon} < \infty,$$

and inserting the bounds in the (A.12)

$$\mathbb{E}[\sup_t \|t^{-\alpha/2} W_t\|_{B^{\alpha-\varepsilon,-n}_{p,p}}] \lesssim 1 + \mathbb{E}\int_1^\infty s^{-1-\alpha/2} \|W_s\|_{B^{\alpha-\varepsilon,-n}_{p,p}} \mathrm{d}s + \mathbb{E}[\sup_{t\geqslant 1} \|M_t\|_{B^{\alpha-\varepsilon,-n}_{p,p}}] < \infty.$$

Finally, for the case $p = \infty$, we use the Besov embedding $\|\cdot\|_{B^{\alpha-\gamma,-n}_{\infty,\infty}} \lesssim \|\cdot\|_{B^{\alpha,-n}_{p,p}}$ for $\alpha - \gamma > 0$ and $p > 2/\gamma$. Then choosing $\gamma \in (0,\varepsilon)$ and p sufficiently large it holds that

$$\mathbb{E}[\sup_{t\geqslant 1} \|t^{-\alpha/2} W_t\|_{B^{\alpha-\varepsilon,-n}_{\infty,\infty}}] \lesssim \mathbb{E}[\sup_{t\geqslant 1} \|t^{-\alpha/2} W_t\|^p_{B^{\alpha-(\varepsilon-\gamma),-n}_{p,p}}]^{1/p} \lesssim 1.$$

B. Auxiliary estimates on the Fourier coefficients

We collect some additional estimates on the kernels f defined in (3.16). No new ideas are needed for these estimates and we only want to briefly illustrate how the proofs in the previous section can be modified to obtain the additional results.

B.1. Dependence on the terminal condition

The following Lemma quantifies the dependence on the terminal condition (3.3) and is used to show convergence as the small-scale cut-off is removed in Proposition 3.13.

Lemma B.1. *Let $T_1, T_2 > 0$. For $\varepsilon > 0$ sufficiently small depending only on δ, it holds for all $t < \infty$,*

$$|||f^{[\ell],T_1}_t - f^{[\ell],T_2}_t|||_t \lesssim \bar{\lambda}^\ell_t \langle t\rangle^{-(\ell-1)+2\varepsilon} \langle T_1 \wedge T_2\rangle^{-\varepsilon}. \tag{B.1}$$

Proof. We proceed inductively (mod the procedure required to remove the regularising kernel for $\ell \geqslant 3$) starting from $\ell = 1$. For simplicity, let us assume that $T_2 \leqslant T_1$, then

$$|f^{[1],T_1}_t - f^{[1],T_2}_t| = |\mathbb{1}_{\{t\geqslant T_2\}}(f^{[\ell],T_1}_t - f^{[\ell],T_2}_{T_2})| = \mathbb{1}_{\{t\geqslant T_2\}} \lambda_t \left(1 - \frac{\lambda_{T_2}}{\lambda_t}\right).$$

And using that $G_t - G_{T_2} \geqslant 0$ on $\{t \geqslant T_2\}$,

$$0 \leqslant 1 - \frac{\lambda_{T_2}}{\lambda_t} = 1 - \mathrm{e}^{-\beta^2(G_t - G_{T_2})(0)} \leqslant \beta^2 (G_t - G_{T_2})(0) \leqslant \frac{\beta^2}{4\pi} \log(t \vee 1) + C.$$

Therefore,

$$|||f_t^{[1],T_1} - f_t^{[1],T_2}|||_t = |f_t^{[1],T_1} - f_t^{[1],T_2}| \lesssim \mathbb{1}_{\{t \geqslant T_2\}} \bar{\lambda}_t \log(t \vee 1) \lesssim \bar{\lambda}_t t^{2\varepsilon} \langle T_2 \rangle^{-\varepsilon},$$

as required. Moving on now to $\ell \geqslant 2$, assuming that (B.1) holds for $\ell' < \ell$, we have

$$\begin{aligned}\delta_T f_t^{[\ell]} = &\delta_T f_{T_2}^{[\ell]} - \sum_{\substack{I_1 \dot\cup I_2 = [\ell] \\ k \in \{1,2\}}} \sum_{\substack{i \in I_1 \\ j \in I_2}} C(|I_1|, |I_2|) \sigma_i \sigma_j \frac{\beta^2}{2} \times \\ &\times \int_{t \vee T_2}^{T_2} \delta_T f_s(\xi_{I_1}) f_s^{[T_k]}(\xi_{I_2}) \mathrm{e}^{\Gamma_{t,s}(\xi_I)} \dot{G}_s(x_i - x_j) \mathrm{d}s.\end{aligned}$$

Inserting the bounds on $\delta_T f^{[\ell']}$ for $\ell' < \ell$, and the bounds on $|||f_t^{[\ell']}|||_t$, we find in the same way as in section 3,

$$||| \int_t^{T_2} \delta_T f_s(\xi_{I_1}) f_s(\xi_{I_2}) \mathrm{e}^{\Gamma_{t,s}(\xi_I)} \dot{G}_s(x_i - x_j) \mathrm{d}s |||_t \lesssim \bar{\lambda}_t^\ell t^{-(\ell-1)+2\varepsilon} \langle T_2 \rangle^{-\varepsilon}.$$

Regarding the remaining term, for $\ell > 1$, it holds for $t \leqslant T_2$,

$$\begin{aligned}&|||\delta_T f_{T_2}^{[\ell]}|||_t \\ &= |||f_{T_2}^{[\ell],T_1}|||_t \\ &\lesssim \sum_{I_1 \dot\cup I_2 = [\ell]} \int_{T_2}^{T_1} |||f_s^{T_1}(\xi_{I_1})|||_s \, |||f_s^{T_1}(\xi_{I_2})|||_s \times \\ &\quad \times ||| \sum_{\substack{i \in I_1 \\ j \in I_2}} \mathrm{e}^{\Gamma_{t,s}(\xi_I)} \frac{\kappa_t(\xi_I) \sigma_i \sigma_j}{\kappa_s(\xi_{I_1}) \kappa_s(\xi_{I_2})} \dot{G}_s(x_i - x_j) ||| \, \mathrm{d}s, \\ &\lesssim \left\{ \begin{array}{ll} t^{1/2} \bar{\lambda}_{T_2}^2 \langle T_2 \rangle^{-3/2}, & \ell = 2, q(\xi_I) = 0 \\ \bar{\lambda}_t^1 \bar{\lambda}_{T_2}^{\ell-1} \langle T_2 \rangle^{-(\ell-1)}, & \text{otherwise} \end{array} \right\} \\ &\lesssim \bar{\lambda}_t^\ell \langle t \rangle^{-(\ell-1)+2\varepsilon} \langle T_2 \rangle^{-\varepsilon},\end{aligned}$$

as required. □

B.2. Dependence on the mollification

Suppose that $(\eta^\varepsilon)_{\varepsilon > 0}$ is an approximation to the identity on $\mathbb{R}^2$ such that for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$,

$$\lim_{\varepsilon \to 0} G_t^\varepsilon(x) := \lim_{\varepsilon \to 0} \int \mathrm{d}y \, \eta^\varepsilon(x - y) G_t(y) = G_t(x).$$

Define the truncated solution to the flow equation F^ε, its Fourier coefficients f^ε and the renormalisation constants λ^ε in the same way as before, with G replaced by G^ε. To prove

reflection positivity in Section 8, we have to understand the dependency of the flow equation on this mollification. The required argument is essentially the same as for the dependence on T in the previous section, with the main difference being that we do not need to distinguish cases $t \geq T_1 \wedge T_2$ and $t \leqslant T_1 \wedge T_2$.

Lemma B.2. *If $\bar{\lambda}_t^\varepsilon \leqslant C\bar{\lambda}_t$ for some constant $C > 0$, then there is a subsequence $\varepsilon_N \to 0$ such that*

$$|||f_t^{[\ell]} - f_t^{[\ell],\varepsilon_N}||| \lesssim N^{-1} |||f_t^{[\ell]}|||.$$

Proof. Let us first derive the dependency of the renormalisation constant λ^ε on ε. Since $\bar{\lambda}_t^\varepsilon \lesssim \bar{\lambda}_t$ uniformly in $\varepsilon > 0$,

$$|\lambda_t^\varepsilon - \lambda_t| = \left| e^{\frac{\beta^2}{2} G_t^\varepsilon(0)} - e^{\frac{\beta^2}{2} G_t(0)} \right| \lesssim e^{\frac{\beta^2}{2} G_t(0)} |G_t^\varepsilon(0) - G_t(0)|.$$

Choosing $(\varepsilon_N)_{N\in\mathbb{N}}$ such that $|G_t^{\varepsilon_N}(0) - G_t(0)| \leqslant N^{-1}$, this implies with the definition of λ_t,

$$|\lambda_t^\varepsilon - \lambda_t| \lesssim N^{-1}\bar{\lambda}_t,$$

As a by-product, since $f_t^{[1]} = -\frac{\beta\lambda_t}{2i}$, this shows the claim for $\ell = 1$. For $\ell = 2, 3$ we proceed as in the bounds derived in Lemma 3.6 and 3.7. For example, for the charged case

$$\begin{aligned}
&|||f_t^{[2](\pm 2)} - f_t^{[2],(\pm 2),\varepsilon}||| \\
&= C \sup_{x_1} \left| \int_t^T ds \int dx_2 [(\lambda_s^\varepsilon)^2 \dot{G}_s^\varepsilon(x_1 - x_2) e^{\Gamma_{t,s}^\varepsilon(\xi_1,\xi_2)} - (\lambda_s)^2 \dot{G}_s(x_1 - x_2) e^{\Gamma_{t,s}(\xi_1,\xi_2)}] \right| \\
&= C \left| \int_t^T ds \int dx_2 [\lambda_t^\varepsilon \lambda_s^\varepsilon \dot{G}_s^\varepsilon(x_2) - \lambda_t \lambda_s \dot{G}_s(x_2)] \right| \\
&= C \left| \int_t^T ds \left[\lambda_t^\varepsilon \lambda_s^\varepsilon \int dx_2 \left(\dot{G}_s^\varepsilon(x_2) - \dot{G}_s(x_2) \right) - (\lambda_t \lambda_s - \lambda_t^\varepsilon \lambda_s^\varepsilon) \int dx_2 \dot{G}_s(x_2) \right] \right|.
\end{aligned}$$

But thanks to the translation invariance,

$$\int dx_2 (\dot{G}_s^\varepsilon(x_2) - \dot{G}_s(x_2)) = \int dx_1 \eta^\varepsilon(x_1) \int dx_2 (\dot{G}_s(x_1 - x_2) - \dot{G}_s(x_2)) = 0.$$

Thus, with the estimates on $|\lambda_s^\varepsilon - \lambda_s| \lesssim N^{-1}\bar{\lambda}_s$ and $|\lambda_t\lambda_s - \lambda_t^\varepsilon\lambda_s^\varepsilon| \lesssim \bar{\lambda}_t\bar{\lambda}_s N^{-1}$

$$|||f_t^{[2](\pm 2)} - f_t^{[2](\pm 2),\varepsilon_N}||| \lesssim \int_t^T ds |\lambda_t\lambda_s - \lambda_t^\varepsilon\lambda_s^\varepsilon| \int dx_2 \dot{G}_s(x_2) \lesssim \bar{\lambda}_t^2 \langle t \rangle^{-1} N^{-1}.$$

The remaining bounds on the neutral part $f^{[2](0)}$ and $f^{[3]}$ follow in the same way. □

C. Wick-ordered cosine

For the large deviations principle in Section 7.3, we rely on the convergence of the Wick-ordered sine and cosine in the first region $\beta^2 < 4\pi$.

Lemma C.1. *Let $\beta^2 \in [0, 4\pi)$. For any $p \in [1, \infty)$ and $\alpha > \beta^2/4\pi$, it holds that*

$$\sup_T \mathbb{E} \| \llbracket \cos(\beta W_T) \rrbracket - 1 \|^p_{B^{-\alpha}_{p,p}(\langle x \rangle^{-n})} \lesssim \beta^2.$$

Moreover, as $T \to \infty$, the martingale $(\llbracket \cos(\beta W_T) \rrbracket)$ converges in $L^p(\mathrm{d}\mathbb{P}; B^{-\alpha}_{p,p}(\langle x \rangle^{-n})$ and almost surely. We denote the unique limit by $\llbracket \cos(\beta W_\infty) \rrbracket$. An analogous statement holds for the Wick-ordered sine.

The main ingredient in for the proof of Lemma C.1 is the following point-wise estimate on the quadratic variation.

Lemma C.2. *Let $N_t = \llbracket \cos(\beta W_t) \rrbracket$ and $\beta^2 \in [0, 4\pi)$. Then, its quadratic variation satisfies for any $\varepsilon > 0$,*

$$|\langle \Delta_i N \rangle_t(x)| \lesssim \beta^2 2^{2i(\beta^2/4\pi + \varepsilon)}.$$

The analogous statement holds for the cosine replaced by the sine.

Proof. Expanding the Wick-ordered cosine with Ito's formula we find,

$$\llbracket \cos(\beta W_t) \rrbracket = 1 - \beta \int_0^t \llbracket \sin(\beta W_s) \rrbracket \mathrm{d}W_s = 1 - \beta \int_0^t \mathrm{e}^{\frac{\beta^2}{2} G_s(0)} \sin(\beta W_s) \mathrm{d}W_s.$$

Therefore, using $\mathrm{Cov}(W_s) = G_s$ and applying Young's inequality repeatedly,

$$\begin{aligned}
&|\langle \Delta_i N \rangle_t(x)| \\
\leqslant &\beta^2 \int_0^t \mathrm{d}s \mathrm{e}^{\beta^2 G_s(0)} \int \mathrm{d}y_1 K_i(x - y_1) \int \mathrm{d}y_2 K_i(x - y_2) \times \\
&\quad \times \sin(\beta W_s(y_1)) \sin(\beta W_s(y_2)) \mathrm{d}\langle W(y_1), W(y_2) \rangle_s \\
\leqslant &\beta^2 \int_0^t \mathrm{d}s \mathrm{e}^{\beta^2 G_s(0)} \sup_x \int \mathrm{d}y_1 K_i(x - y_1) \int \mathrm{d}y_2 K_i(x - y_2) \dot{G}_s(y_1 - y_2) \qquad \text{(C.1)} \\
\leqslant &\beta^2 \|K_i\|_{L^1} \|K_i\|_{L^p} \sup_{y_1} \int_0^t \mathrm{d}s \mathrm{e}^{\beta^2 G_s(0)} \| \dot{G}_s(y_1 - y_2) \|_{L^q(\mathrm{d}y_2)} \\
\leqslant &\beta^2 \|K_i\|_{L^1} \|K_i\|_{L^p} \int_0^t \mathrm{d}s \mathrm{e}^{\beta^2 G_s(0)} \| \dot{G}_s \|_{L^q},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ are to be determined later. So using the estimates on the heat kernel

$$\| \dot{G}_s \|_{L^q} \lesssim \langle s \rangle^{-1} \| Q_s \|_{L^q} \lesssim \langle s \rangle^{-1-1/q},$$

combined with the estimates on the Littlewood–Paley kernels (1.12), the previous computation (C.1) and $e^{\beta^2 G_s(0)} \lesssim \langle s\rangle^{\beta^2/4\pi}$ from Lemma A.1 gives

$$|\langle \Delta_i N\rangle_t(x)| \lesssim \beta^2 2^{2i\frac{p-1}{p}} \int_0^t \mathrm{d}s \langle s\rangle^{\beta^2/4\pi}\langle s\rangle^{-1-1/q}.$$

For the integral to be bounded uniformly in t, we need $\frac{1}{q} > \beta^2/4\pi$ and since p,q are Hölder conjugates this means

$$\frac{p-1}{p} = \frac{1}{q} > \beta^2/4\pi.$$

Consequently, we can choose $p,q \in [1,\infty)$ if and only if $\beta^2 < 4\pi$ which gives the claim. □

Proof of Lemma C.1. Recall the definition of the Besov norms,

$$\mathbb{E}[\|N_t\|^p_{B^\alpha_{p,p}(\langle x\rangle^{-n})}] = \sum_{i\geqslant -1} 2^{-ip\alpha}\mathbb{E}\|\Delta_i N_t\|^p_{L^{p,-n}}. \tag{C.2}$$

We compute by Lemma C.2 and the Burkholder–Davis–Gundy's inequalities, for any $\varepsilon > 0$,

$$\mathbb{E}\|\Delta_i N_t\|^p_{L^p(\langle x\rangle^{-n})} \lesssim \int \mathrm{d}x \langle x\rangle^{-pn}\mathbb{E}[|\langle \Delta_i[\![\cos(\beta W)]\!](x)\rangle_t|]^{p/2} \lesssim \beta^2 2^{pi(\beta^2/4\pi+\varepsilon)}.$$

Therefore, (C.2) is finite provided $\beta^2/4\pi < \alpha$ and the convergence now follows from the martingale convergence theorem. □

We also owe the proof of Lemma 6.5.

Proof of Lemma 6.5. **(N)** Let

$$M_t(x) = [\![\cos(\beta(W_t(z-x)-W_t(x)))]\!] := [\![\cos(\beta(\delta_z W_t(x)))]\!],$$

which by Ito's formula can be written as

$$\begin{aligned} M_t(x) &= \int_0^t -\beta[\![\sin(\beta\delta_z W_s(x))]\!]\mathrm{d}(\delta_z W_s(x)) \\ &= \int_0^t -\beta[\![\sin(\beta\delta_z W_s(x))]\!]\int \mathrm{d}y(Q_s(x-z-y)-Q_s(x-y))\mathrm{d}B_s(y), \end{aligned}$$

where we recall that $(B_t)_t$ is a cylindrical Brownian motion on $L^2(\mathbb{R}^2)$. Now

$$Q_s(x-z-y)-Q_s(x-y) = |z|\int_0^1 \mathrm{d}\vartheta \nabla Q_s(x-y-\vartheta z),$$

so that by translation invariance and since ρ has compact support, we conclude $\mathbb{E}\|M_t\|_{L^1} \lesssim \mathbb{E}[|M_t(0)|^2]^{1/2}$. The latter can be estimates as follows

$$\begin{aligned}
&\mathbb{E}|M_t(0)|^2\\
=&\mathbb{E}\langle M(0), M(0)\rangle_t\\
=&\beta^2|z|^2\int_0^t [\![\sin(\beta\delta_z W_s(0))]\!][\![\sin(\beta\delta_z W_s(0))]\!]\times\\
&\qquad\times\int \mathrm{d}y\int_0^1\mathrm{d}\vartheta_1\int_0^1\mathrm{d}\vartheta_2\nabla Q_s(y-\vartheta_1 z)\nabla Q_s(y-\vartheta_2 z)\mathrm{d}s\\
\leqslant&\beta^2|z|^2\int_0^t \mathrm{d}s\mathrm{e}^{2\beta^2 G_s(0)-2\beta^2 G_s(z)}\int_0^1\mathrm{d}\vartheta_1\int_0^1\mathrm{d}\vartheta_2\int \mathrm{d}y\nabla Q_s(y-(\vartheta_1-\vartheta_2)z)\nabla Q_s(y).
\end{aligned}$$

Using the estimates from Lemma A.4, $\|\nabla^\alpha Q_s\|_{L^p}\lesssim\langle s\rangle^{-1/p+\alpha/2}$ so that

$$\begin{aligned}
\mathbb{E}|\mathrm{e}^{\beta^2 G_t(z)}M_t(0)|^2 &\leqslant \beta^2|z|^2\int_0^t \mathrm{d}s\mathrm{e}^{2\beta^2 G_s(0)}\mathrm{e}^{2\beta^2(G_t-G_s)(z)}\times\\
&\qquad\times\int_0^1\mathrm{d}\vartheta_1\int_0^1\mathrm{d}\vartheta_2\int\mathrm{d}y\nabla Q_s(y-(\vartheta_1-\vartheta_2)z)\nabla Q_s(y)\\
&\lesssim |z|^{2-8(1-\delta)}\int_0^t\mathrm{d}s\int_0^1\mathrm{d}\vartheta_1\int_0^1\mathrm{d}\vartheta_2\int\mathrm{d}y\nabla Q_s(y-(\vartheta_1-\vartheta_2)z)\nabla Q_s(y)\\
&\leqslant |z|^{2-8(1-\delta)}\int_0^t\mathrm{d}s\|\nabla Q_s\|_{L^1}\|\nabla Q_s\|_{L^\infty}\\
&\lesssim |z|^{2-8(1-\delta)}\int_0^t\mathrm{d}s\langle s\rangle^{-1/2}\langle s\rangle^{1/2}\\
&\lesssim |z|^{2-8(1-\delta)}t.
\end{aligned}$$

Using the same argument as in the proof of Lemma 4.8 in the scales as well as the usual Kolmogorov argument for the L^∞-norm in z we obtain for any $\gamma_1>1/2$, $\gamma_2>-1+4(1-\delta)$

$$\mathbb{E}[\sup_{t,z} t^{-\gamma_1}|z|^{\gamma_2}\mathrm{e}^{\beta^2 G_t(z)}\|M_t\|_{L^1}]\lesssim 1,$$

and thus

$$\sup_{t,z} t^{-\gamma_1}|z|^{\gamma_2}\mathrm{e}^{\beta^2 G_t(z)}\|M_t\|_{L^1}<\infty,\quad a.s.$$

(C) We start by estimating

$$\begin{aligned}
&\mathbb{E}\|[\![\cos(\beta(W_t(\cdot-z))+W_t(\cdot))]\!]\rho(\cdot)\|_{B^{-s}_{p,p}(\mathrm{d}x)}\\
&=\sum_{i\geqslant -1}2^{-isp}\mathbb{E}\|\Delta_i[\![\cos(\beta(W_t(\cdot-z)+W_t(\cdot)))]\!]\|^p_{L^p(\mathrm{d}x)}\\
&\lesssim_\rho \sum_{i\geqslant -1}2^{-isp}\mathbb{E}[|\Delta_i[\![\cos(\beta(W_t(\cdot-z)+W_t(\cdot)))]\!](0)|^2]^{p/2},
\end{aligned}$$

where we again used that ρ is smooth and compactly supported and that the law of W is translation invariant. Here, the Littlewood-Paley blocks act only in x, that is

$$\Delta_i[\![\cos(\beta(W_t(\cdot-z)+W_t(\cdot)))]\!](x)=\int \mathrm{d}y K_i(x-y)[\![\cos(\beta(W_t(y-z)+W_t(y)))]\!].$$

Developing the martingale $[\![\cos(\beta(W_t(y-z)+W_t(y)))]\!]$ along the scales with Ito's formula, we obtain,

$$\begin{aligned}
&\mathbb{E}[|\Delta_i[\![\cos(\beta(W_t(z)+W_t(0)))]\!]|^2]\\
\leqslant&\beta^2\int_0^t\int \mathrm{d}y_1\int \mathrm{d}y_2 K_i(y_1)K_i(y_2)\mathrm{e}^{\frac{\beta^2}{2}\mathbb{E}[|W_s(y_1-z)+W_s(y_1)|^2]}\mathrm{e}_s^{\frac{\beta^2}{2}\mathbb{E}[|W_s(y_2-z)+W_s(y_2)|^2]}\\
&\times \mathrm{d}\langle W(y_1-z)+W(y_1),W(y_2-z)+W(y_2)\rangle_s\\
=&\int_0^t\int \mathrm{d}y_1\int \mathrm{d}y_2 K_i(y_1)K_i(y_2)\mathrm{e}^{2\beta^2 G_s(z)}\mathrm{e}^{2\beta^2 G_s(0)}[\dot G_s(y_1-y_2)+\dot G_s(y_1-y_2-z)]\mathrm{d}s.
\end{aligned}$$

Thanks to the positivity of G, we have the estimate $\mathrm{e}^{2\beta^2(G_s-G_t)(z)}\leqslant 1$ for $t\geqslant s$ so that for any $1/r+1/q=1$,

$$\begin{aligned}
&\int_0^t\int \mathrm{d}y_1\int \mathrm{d}y_2 K_i(y_1)K_i(y_2)\mathrm{e}^{2\beta^2(G_s-G_t)(z)}\mathrm{e}^{2\beta^2 G_s(0)}[\dot G_s(y_1-y_2)+\dot G_s(y_1-y_2-z)]\mathrm{d}s\\
\leqslant&\int_0^t\int \mathrm{d}y_1\int \mathrm{d}y_2 K_i(y_1)K_i(y_2)\mathrm{e}^{2\beta^2 G_s(0)}[\dot G_s(y_1-y_2)+\dot G_s(y_1-y_2-z)]\mathrm{d}s\\
\leqslant&\|K_i\|_{L^1}\|K_i\|_{L^r}\int_0^t\langle s\rangle^{4(1-\delta)}\|\dot G_s\|_{L^q}\\
\lesssim&2^{2i/q}\int_0^t \mathrm{d}s\langle s\rangle^{-1-1/q+4(1-\delta)}\\
\lesssim&(t^{-1/q+4(1-\delta)}\vee 1)2^{2i/q}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\|\mathrm{e}^{-\beta^2 G_t(z)}[\![\cos(\beta(W_t(\cdot-z)+W_t(\cdot)))]\!]\rho(\cdot)\|^p_{B^{-s}_{p,p}(\mathrm{d}x)}\lesssim(t^{-1/2q+2(1-\delta)}\vee 1)^p\sum_{i\geqslant -1}2^{-ip\left(s-\frac{1}{q}\right)}.$$

Using the same argument as in the proof of Lemma 4.8, we can choose $1/2q$ sufficiently close to $s\in(0,2\delta)$ sufficiently large to conclude for any $\gamma_1>0$, $\gamma_2>2-3\delta$,

$$\sup_{t,z}\|\mathrm{e}^{-\beta^2 G_t(z)}|z|^{\gamma_1/2}t^{-\gamma_2}[\![\cos(\beta(W_t(\cdot-z)+W_t(\cdot)))]\!]\rho(\cdot)\|_{B^{-s}_{p,p}(\mathrm{d}x)}<\infty,\quad a.s.$$

□

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