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Well-posedness of the transport equation by stochastic perturbation

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Abstract

We consider the linear transport equation with a globally Hölder continuous and bounded vector field. While this deterministic PDE may not be well-posed, we prove that a multiplicative stochastic perturbation of Brownian type is enough to render the equation well-posed. This seems to be the first explicit example of partial differential equation that become well-posed under the influece of noise. The key tool is a differentiable stochastic flow constructed and analysed by means of a special transformation of the drift of Tanaka type.

1 Introduction

The transport equation in $[0,T]\times\mathbb{R}^d$

$$\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^d,$$
$$u(0,x) = u_0(x), \quad x\in\mathbb{R}^d.$$

driven by a vector field $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ which is Lipschitz continuous in x, uniformly in t, can be solved in several classical ways and in different function spaces. A remarkable extension has been obtained by R.J. Di Perna and P.L. Lions [5] who proved, under the assumption $b\in L^1(0,T;W^{1,1}_{loc}(\mathbb{R}^d,\mathbb{R}^d))$ with a linear growth condition and $\mathrm{div} b \in L^1(0,T;L^\infty(\mathbb{R}^d))$, that a unique $L^\infty(0,T;L^\infty(\mathbb{R}^d))$ solution exists, weak-$*$ continuous in time, for a given $u_0\in L^\infty(\mathbb{R}^d)$. Moreover, a generalized notion of flow is introduced and its existence and uniqueness is proved under the assumption that the negative part of $\mathrm{div}\, b$ is $L^1(0,T;L^\infty(\mathbb{R}^d))$. L. Ambrosio [1] extended further the theory including in particular the case of vector fields $b\in L^1(0,T;BV_{loc}(\mathbb{R}^d,\mathbb{R}^d))$ with similar conditions on $\mathrm{div}\, b$. The literature following [5] is wide, see a partial review in [2]. See also a generalization to transport-diffusion equations and the associated stochastic differential equations by Figalli [6] (in a direction different from the one of the present paper).

In all these results some regularity assumption on ∇b is imposed, at least L^1 or a signed finite measure. If we assume only that b is Hölder continuous, couterexamples to well-posedness can be constructed.

The aim of this paper is to show that suitable random perturbations of the transport equation improve the theory. We prove existence and uniqueness of L^∞-solutions when b is continuous, bounded, globally α-Hölder continuous in space, $\alpha \in (0,1)$, uniformly in time, and satisfy a suitable integrability condition on $\operatorname{div} b$. This result is obtained at the price of introducing a multiplicative white noise in the PDE. Precisely, we consider the equation

$$\begin{aligned} &\partial_t u(t,x) + (b(t,x)\cdot\nabla u(t,x))dt + \sum_{i=1}^{d} e_i\cdot\nabla u(t,x)\circ dW_t^i = 0, \\ &u(0,x) = u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{1}$$

where $e_1,...,e_d$ is the canonical basis of $\mathbb{R}^d$ and $W_t = (W_t^1,...,W_t^d)$ is a standard Brownian motion in $\mathbb{R}^d$. The stochastic integration is understood in the Stratonovich sense.

This equation seems to be the first concrete example of partial differential equation that is not well-posed without noise, but is well-posed with a suitable noise. This phenomenon is well known for ordinary differential equations but it is one of the more interesting direction of investigations in SPDEs, where people have in mind the hope to prove well-posedness for some relevant equation of fluid dynamics.

The well-posedness of equation (1) relies on two facts. The associated stochastic ordinary differential equation (the stochastic characteristics)

$$dX_t^x = b\left(t, X_t^x\right)dt + dW_t, \quad t\geq 0, \quad X_0^x = x \tag{2}$$

under the assumption $b(t,\cdot)\in C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)$, uniformly in time, generates a stochastic flow of diffeomorphisms. Ultimately, the reason is that integrals of the form

$$\int_0^t f\left(s, X_s^x\right)ds \tag{3}$$

are more regular in x in the case of diffusion processes X_s than in the case of deterministic solutions to ODEs. The existence of a stochastic flow of diffeomorphisms implies the existence of weak solutions to equation (1) in L^∞.

The uniqueness of such weak solutions is more difficult and, even under the existence of a stochastic flow of diffeomorphisms, it must be related to some form of commutator lemma, as in the deterministic case (see [5], [1]), since we need to perform differential computations on regularizations of L^∞-solutions. We use a weak form for the commutator composed with the flow, which does not require weak differentiability of $b(t,\cdot)$ but only requires weak differentiability of the Jacobian of the flow itself. The Jacobian of the flow is given by $\exp(\int_0^t \operatorname{div} b(X_s^x)ds)$. In the deterministic case there is no hope to differentiate in x without further differentiability assumptions on $\operatorname{div} b$, but in the stochastic case we use again the improved regularity of integrals of the form (3). We have only to impose an integrability condition on $\operatorname{div} b$, which turns out to be different from the one required in the deterministic setting [5], [1].

The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem. What looks very difficult is to generalize to an adapted random field $b(t,x,\omega)$. Indeed, it is very easy to produce examples,

both for the linear SPDE (1) and for a stochastic version of Euler equation which show that the particular noise we use does not have any regularizing effect. Thus new ideas are necessary to approach nonlinear problems.

Plan. In Sect. 2 we prove the existence of a global stochastic flow associated to eq. (2) and its differentiability properties. In Sect.3 we use the stochastic flow to prove existence of weak solutions to the SPDE (1). Sect. 4 and 5 are devoted to prove uniqueness for solutions to the SPDE (1). Finally, in Sect. 6 we point out the basic difficulty present when trying to use the same approach for non-linear transport equations arising from fluid dynamics.

Notations. Let $T > 0$ be fixed. For $\alpha \in (0,1)$ define the space $C_b^\alpha([0,T] \times \mathbb{R}^d; \mathbb{R})$ as the set of all bounded continuous functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ for which

$$[f]_{\alpha,T} := \sup_{t\in[0,T]} \sup_{x\neq y\in\mathbb{R}^d} \frac{|f(t,x) - f(t,y)|}{|x-y|^\alpha} < \infty$$

($|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$ for every d, if no confusion may arise). This is a Banach space with respect to the usual norm $\|f\|_{\alpha,T} = \|f\|_0 + [f]_{\alpha,T}$ where $\|f\|_0 = \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |f(t,x)|$.

We write $C_b^\alpha([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ or simply $C_{b,T}^\alpha$ for the space of all vector fields $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ having all components in $C_b^\alpha([0,T] \times \mathbb{R}^d; \mathbb{R})$.

Moreover $f \in C_b^{n+\alpha}([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$, briefly $f \in C_{b,T}^{n+\alpha}$, if for all $k = 0, \dots, n$, the spatial Fréchet derivatives $D^k f$ are all bounded and θ-Hölder continuous. Define the corresponding norm as

$$\|f\|_{n+\alpha,T} = \|f\|_0 + \sum_{k=1}^{n} \|D^k f\|_0 + [D^n f]_{\alpha,T}.$$

The spaces $C_b^{n+\alpha} = C_b^{n+\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ are defined as before but only involve functions $f : \mathbb{R}^d \to \mathbb{R}^d$ which do not depend on time.

We say that $f : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $C^{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, if f is continuous on $\mathbb{R}^d$, k-times differentiable with all continuous derivatives and the derivatives of order k are locally α-Hölder continuous.

Finally, $C_0^0(\mathbb{R}^d)$ denotes the space of all real continuous functions defined on $\mathbb{R}^d$, having compact support and $C_0^\infty(\mathbb{R}^d)$ its subspace consisting of infinitely differentiable functions.

Throughout the paper we will assume a stochastic basis with a d-dimensional Brownian motion $(\Omega, (\mathcal{F}_t), \mathcal{F}, P, (W_t))$ to be given. Denote by $\mathcal{F}_{s,t}$ the completed σ-algebra generated by $W_u - W_r$, $s \le r \le u \le t$, for each $0 \le s < t$, and set $\mathcal{F}_{s,s} = \{\varnothing, \Omega\}$.

We will consider the following assumptions.

Hypothesis 1 *There exists $\alpha \in (0,1)$ such that $b \in C_{b,T}^\alpha$.*

Hypothesis 2 *There exists $p, q \in (2, +\infty)$ such that*

$$\mathrm{div} b \in L^q(0,T; L^p(\mathbb{R}^d)) \quad \textit{with } \frac{d}{p} + \frac{2}{q} < 1 \tag{4}$$

(where $\mathrm{div}\, b(t,\cdot)$ is understood in distributional sense).

Remark 3 *In [10] condition* (4) *is imposed locally on* b *in order to prove existence and uniqueness of strong solution to* (2)*. This hypothesis is satisfied if* $d \geq 2$ *and*

$$\mathrm{div} b \in L^{\infty}(0,T;L^{p}(\mathbb{R}^{d})) \quad \textit{for some } \ p > d.$$

For a given function f depending on $t \in [0,T]$ and $x \in \mathbb{R}^d$, we will also adopt the notation

$$f_t(x) = f(t,x).$$

2 Differentiable stochastic flow for an SDE with $C^{\alpha}_{b,T}$-drift

In this section we assume that b *satisfies Hypothesis 1.*

Given $s \in [0,T]$ and $x \in \mathbb{R}^d$, consider the stochastic differential equation in $\mathbb{R}^d$:

$$dX_t = b\left(t, X_t\right) dt + dW_t, \quad t \in [s,T], \quad X_s = x. \tag{5}$$

A classical fact is the existence and uniqueness of a weak solution, obtained for instance by Girsanov transform. Yu. Veretennikov [18] prove that boundedness of b (independent on t) is enough to have path-wise uniqueness and existence of strong probabilistic solutions. For related work see also the recent paper [10] by N.V. Krylov and M. Röckner. Recently, A.M. Davie has proved that for a.e. Brownian path W it is possible to prove uniqueness for equation (5) between all deterministic continuous functions which solve equation (5) with respect to the choosen W (for any initial condition and any bounded measurable and time dependend drift b). The analysis of the associated flow is however still missing.

Our key result is the existence of a *differentiable* stochastic flow $(x,s,t) \mapsto \varphi_{s,t}(x)$ for equation (5). Recall the relevant definition from H. Kunita [12]:

Definition 4 *A* stochastic flow of diffeomorphisms *(resp.* of class $C^{1,\alpha}$*) on* $(\Omega, (\mathcal{F}_t), \mathcal{F}, P, (W_t))$ *associated to equation (5) is a map* $(s,t,x,\omega) \mapsto \phi_{s,t}(x)(\omega)$*, defined for* $0 \leq s \leq t \leq T$*,* $x \in \mathbb{R}^d$*,* $\omega \in \Omega$ *with values in* $\mathbb{R}^d$*, such that*

(a) given any $s \in [0,T]$*,* $x \in \mathbb{R}^d$*, the process* $X^{s,x} = (X_t^{s,x}(\omega), t \in [s,T], \omega \in \Omega)$ *defined as* $X_t^{s,x} = \phi_{s,t}(x)$ *is a continuous* $\mathcal{F}_{s,t}$*-measurable solution of equation (5)*

(b) P*-a.s.,* $\phi_{s,t}$ *is a diffeomorphism, for all* $0 \leq s \leq t \leq T$*, and the functions* $\phi_{s,t}(x)$*,* $\phi^{-1}_{s,t}(x)$*,* $D\phi_{s,t}(x)$*,* $D\phi^{-1}_{s,t}(x)$ *are continuous in* (s,t,x) *(resp. of class* $C^{1,\alpha}$ *in* x *uniformly in* (s,t)*),*

(c) P*-a.s.,* $\phi_{s,t}(x) = \phi_{s,u}(\phi_{u,t}(x))$ *for all* $0 \leq s \leq u \leq t \leq T$ *and* $x \in \mathbb{R}^d$ *and* $\phi_{s,s}(x) = x$*.*

As already mentioned the main ingredient to obtain the regularity of the flow is the observation that the time integral $\int_0^t b\left(s, X_s^x\right) ds$ has richer regularity properties than expected only on the basis of the regularity of b; such properties, coming from the interaction between b and the local behavior of W_t, and ultimately related to the local time (or occupation measure) of the Brownian motion W_t, may be revealed by the Tanaka approach to the definition of

local time: the integral $\int_0^t b\,(s, X_s^x)\,ds$ may be considered as the corrector of an Itô formula, and the other terms of the formula have better properties than $\int_0^t b\,(s, X_s^x)\,ds$.

Let $\lambda > 0$ be fixed. Let us extend b to the whole $[0,\infty)\times\mathbb{R}^d$ by setting

$$b(t,x) = b(T,x), \quad t \geq T, \;\; x \in \mathbb{R}^d. \tag{6}$$

Given a function $f : [0,+\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ consider the following parabolic system (collecting d independent equations):

$$\partial_t u_\lambda + L^b u_\lambda - \lambda u_\lambda = f, \quad (t,x) \in (0,\infty)\times\mathbb{R}^d, \tag{7}$$

where

$$L^b u = \frac{1}{2}\Delta u + b\cdot\nabla u \tag{8}$$

and $u : [0,+\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ (clearly (8) has to be interpreted componentwise).

The next result deals with Schauder estimates and is known even in a more general form (see [11] and the references therein). Schauder estimates of the type (9) were first proved in [3]. We will only sketch a proof for the sake of completeness.

Lemma 5 *Let $\lambda > 0$, $b \in C^\alpha_{b,T}$, $\alpha \in (0,1)$, and $f : [0,+\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ be a continuous and bounded such that $f(t,\cdot) \in C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)$, $t \geq 0$. Then there exists a unique bounded classical solution $u = u_\lambda$ to equation (7). The functions $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and $\partial_t u$ are continuous. Moreover there exists $C > 0$ (independent on u and f) such that*

$$\sup_{t\geq 0}\|u(t,\cdot)\|_{C_b^{2+\alpha}} \leq C \sup_{t\geq 0}\|f(t,\cdot)\|_{C_b^\alpha} \tag{9}$$

Proof. Uniqueness follows by the maximum principle (see [7]). Let us consider existence.

If $L^b = \frac{1}{2}\triangle$, then the result follows by using the explicit formula

$$u(t,x) = \int_t^{+\infty} e^{-\lambda(r-t)} P_{r-t} f(r,\cdot)(x)dr = \int_0^{+\infty} e^{-\lambda s} P_s f(t+s,\cdot)(x)ds \tag{10}$$

(where (P_t) denotes the heat semigroup) and well known estimates on the spatial derivatives of $P_t g$ when $g \in C_b^\alpha$ and $t > 0$. Let $L^b = \frac{1}{2}\triangle + b\cdot\nabla$. Thanks to the boundedness of b we get easily a-priori estimates for equation (7) (assuming that there exists a classical bounded solution u such that $\sup_{t\geq 0}\|u(t,\cdot)\|_{C_b^{2+\alpha}} < \infty$). Then a standard continuity method (see for instance [7]) allows to get the existence of the solution which verifies equation (7). The proof is complete. ■

We also need the following simple lemma.

Lemma 6 *Under the assumptions of Lemma 5 let u_λ be the solution to (7). Then*

$$\sup_{t\geq 0}\|\nabla u_\lambda(t,\cdot)\|_0 \to 0, \quad \textit{as} \;\; \lambda \to +\infty.$$

Proof. We write $\partial_t u_\lambda + \frac{1}{2}\triangle u_\lambda - \lambda u_\lambda = f - b\cdot\nabla u_\lambda$.

Using the well known estimate for the heat semigroup $\|\nabla P_t g\|_0 \le \frac{C}{t^{\frac{1}{2}}}\|g\|_0$, $g \in C_b(\mathbb{R}^d)$, $t > 0$, and, differentiating in equation (7), we get easily, for any $\lambda > 0$,

$$\|\nabla u_\lambda\|_0 \le \frac{c}{\lambda^{\frac{1}{2}}}(\|f\|_0 + \|b\cdot\nabla u_\lambda\|_0) \le \frac{c}{\lambda^{\frac{1}{2}}}\|f\|_0 + \frac{c}{\lambda^{\frac{1}{2}}}\|b\|_0\|\nabla u_\lambda\|_0,$$

where the suprema are taken on $[0,\infty)\times\mathbb{R}^d$. Considering $\lambda > c^2\|b\|_0^2$, we get

$$(1 - \frac{c}{\lambda^{\frac{1}{2}}}\|b\|_0)\,\|\nabla u_\lambda\|_0 \le \frac{c}{\lambda^{\frac{1}{2}}}\|f\|_0$$

and the assertion follows. The proof is complete. ■

Theorem 7 *Assume $b \in C^\alpha_{b,T}$. Then we have the following facts.*

(i) (pathwise uniqueness) For every $s \in [0,T]$, $x \in \mathbb{R}^d$, the stochastic equation (5) has a unique continuous adapted solution $X^{s,x} = \left(X_t^{s,x}(\omega)\,, t \in [s,T]\,, \omega \in \Omega\right)$.

(ii) (differentiable flow) There exists a stochastic flow $\phi_{s,t}$ of diffeomorphisms for equation (5). The flow is also of class $C^{1+\alpha'}$ for any $\alpha' < \alpha$.

(iii) (stability) Let $(b^n) \subset C^\alpha_{b,T}$ be a sequence of vector fields and ϕ^n the corresponding stochastic flows. If $b^n \to b$ in $C^\alpha_{b,T}$, then, for any $p \ge 1$,

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le T} E[\sup_{u\in[s,T]}|\phi^n_{s,u}(x) - \phi_{s,u}(x)|^p] = 0. \tag{11}$$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le T} E[\sup_{u\in[s,T]}\|D\phi^n_{s,u}(x) - D\phi_{s,u}(x)\|^p] = 0. \tag{12}$$

(here $\|\cdot\|$ denotes the Hilbert-Schmidt norm).

Proof. Step 1 (auxiliary parabolic systems). For $\lambda > 0$ consider the (vector valued) solution ψ to the parabolic system

$$\partial_t\psi_\lambda + L^b\psi_\lambda - \lambda\psi_\lambda = -b, \quad (t,x) \in (0,\infty)\times\mathbb{R}^d \tag{13}$$

provided by an application of Lemma 5 (recall (6)). Define

$$\Psi_\lambda(t,x) = x + \psi_\lambda(t,x).$$

Lemma 8 *For λ large enough, such that $\sup_{t\ge0}\|\nabla\psi_\lambda(t,\cdot)\|_0 < 1$ (see Lemma 6), the following statements hold:*

(i) Uniformly in $t \in [0,+\infty)$, Ψ_λ has bounded first and second spatial derivatives and moreover the second (Fréchet) derivative $D^2_x\Psi_\lambda$ is globally α-Hölder continuous.

(ii) for any $t \ge 0$, $\Psi_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism of class C^2.

(iii) Ψ_λ^{-1} has bounded first and second spatial derivatives, uniformly in $t \in [0,+\infty)$.

Proof. Assertion (i) follows by Lemma 5.

(ii) Recall the classical Hadamard theorem (see, for instance, [17, page 330]): Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be of class C^2. Suppose that $\lim_{|x|\to\infty} |g(x)| = +\infty$ and that the Jacobian matrix $Dg(x)$ is an isomorphism of $\mathbb{R}^d$ for all $x \in \mathbb{R}^d$. Then g is a C^2-diffeomorphism of $\mathbb{R}^d$. Applying this result to Ψ_λ, we get the assertion.

(iii) We know that Ψ_λ^{-1} is of class C^2. Moreover

$$D\Psi_\lambda^{-1}(t,y) = [D\Psi_\lambda(t,\Psi_\lambda^{-1}y)]^{-1} = [I + D\psi_\lambda(t,\Psi_\lambda^{-1}y)]^{-1}$$

$$= \sum_{k\geq 0}(-D\psi_\lambda(t,\Psi_\lambda^{-1}y))^k, \quad y \in \mathbb{R}^d.$$

It follows that

$$\sup_{t\geq 0} \|D\Psi_\lambda^{-1}(t,\cdot)\|_0 \leq \sum_{k\geq 0} \left(\sup_{t\geq 0} \|D\psi_\lambda(t,\cdot)\|_0\right)^k < \infty.$$

This shows the boundedness of the first derivative. Arguing in a similar way we get also the boundedness of the second derivative since

$$D^2\Psi_\lambda^{-1}(t,y) = -[D\Psi_\lambda(t,\Psi_\lambda^{-1}y)]^{-1} D^2\Psi_\lambda(t,\Psi_\lambda^{-1}(t,y)) \left\{[D\Psi_\lambda(t,\Psi_\lambda^{-1}y)]^{-1}\right\}^{\otimes 2}.$$

■

In the sequel we will use a value of λ for which Lemma 8 holds and simply write ψ and Ψ for ψ_λ and Ψ_λ.

Step 2 (conjugated SDE). Define

$$\widetilde{b}(t,y) = -\lambda\psi(t,\Psi^{-1}(t,y)), \quad \widetilde{\sigma}(t,y) = D\Psi(t,\Psi^{-1}(t,y))$$

and consider, for every $s \in [0,T]$ and $y \in \mathbb{R}^d$, the SDE

$$Y_t = y + \int_s^t \tilde{\sigma}(u,Y_u)dW_u + \int_s^t \widetilde{b}(u,Y_u)du, \qquad t \in [s,T]. \tag{14}$$

This equation is equivalent to equation (5), in the following sense. If X_t is a solution to (5), then $Y_t = \Psi(t,X_t)$ verifies equation (14) with $y = \Psi(s,x)$: it is sufficient to apply Itô formula to $\Psi(t,X_t)$ and use equation (13). Viceversa, given a solution Y_t of equation (14), let $X_t = \Psi^{-1}(t,Y_t)$; let us prove that X_t is a solution of (5) with $x = \Psi^{-1}(s,y)$. In the sequel summation over repeated indices is understood. We set $\rho(t,\cdot) = \Psi^{-1}(t,\cdot)$. Using Lemma 8, a direct application of Itô formula gives

$$dX_t^i = \nabla_j\rho_t^i(Y_t)\,\widetilde{b}_t^j(Y_t)dt \,+\, \nabla_j\rho_t^i(Y_t)\widetilde{\sigma}_t^{jk}(Y_t)dW_t^k$$

$$+\frac{1}{2}\nabla_j\nabla_k\rho_t^i(Y_t)\widetilde{\sigma}_t^{j\ell}(Y_t)\widetilde{\sigma}_t^{k\ell}(Y_t)dt + \partial_t\rho_t^i(Y_t)dt$$

and since $(\nabla\rho_t)^{ij}(y) = \nabla_j\rho_t^i(y) = [((\nabla\Psi_t)(\rho_t(y)))^{-1}]^{ij} = [\tilde{\sigma}_t(y)^{-1}]^{ij}$,

$$\nabla_k\nabla_\ell\rho_t^i(y) = -[\tilde{\sigma}_t(y)^{-1}]^{im}(\nabla_a\nabla_b\Psi_t^m)(\rho_t(y))[\tilde{\sigma}_t(y)^{-1}]^{ak}[\tilde{\sigma}_t(y)^{-1}]^{b\ell}$$

and $\partial_t \rho_t(y)^i = [\tilde{\sigma}_t(y)^{-1}]^{ij} \partial_t \Psi_t(\rho_t(y))^j$. The diffusion coefficient in the equation for dX is $\nabla \rho_t(y) \widetilde{\sigma}_t(y) = I$ while the computation of the drift gives:

$$\begin{aligned}
&\nabla_j \rho_t(y)^i \tilde{b}_t(y)^j + \frac{1}{2} \nabla_j \nabla_k \rho_t^i(y) \widetilde{\sigma}_t^{j\ell}(y) \widetilde{\sigma}_t^{k\ell}(y) + \partial_t \rho_t^i(y) \\
&\quad = [\tilde{\sigma}_t(y)^{-1}]^{ij} \left[-\lambda \psi_t(\rho_t(y))^j - \frac{1}{2} (\Delta \psi_t^j)(\rho_t(y)) + \partial_t \psi_t(\rho_t(y))^j \right] \\
&\quad = [\tilde{\sigma}_t(y)^{-1}]^{ij} \left[b_t(\rho_t(y))^k (\nabla_k \Psi_t^j)(\rho_t(y)) \right] \\
&\quad = [\tilde{\sigma}_t(y)^{-1}]^{ij} \tilde{\sigma}_t(\rho_t(y))^{jk} b_t(\rho_t(y))^k = b_t(\rho_t(y))^i
\end{aligned}$$

so we obtain equation (5).

Step 3 (proof of (i) and (ii)). We have clearly $\widetilde{b}$ and $\widetilde{\sigma} \in C_{b,T}^{1+\alpha}$. By classical results (see [12, Chapter 2]) this implies existence and uniqueness of a strong solution Y of equation (14) and even the existence of a $C^{1,\alpha'}$ ($\alpha' < \alpha$) stochastic flow of diffeomorphisms $\varphi_{s,t}$ associated to equation (14). By the results of steps 1 and 2, these facts hold true for equation (5) and $\phi_{s,t} = \Psi_t^{-1} \circ \varphi_{s,t} \circ \Psi_s$ is the corresponding flow.

Step 4 (proof of (iii)). Let ψ^n be the solution of the parabolic problem associated to b_n. Since $b^n \to b$ in $C_{b,T}^{\alpha}$. By Lemma 5 we have $\psi^n \to \psi$ in $C_{b,T}^{2+\alpha}$. Consider the flows $\varphi_{s,t}^n = \Psi_t^n \circ \phi_{s,t}^n \circ (\Psi_s^n)^{-1}$ which satisfy

$$\varphi_{s,t}^n(y) = y + \int_s^t \widetilde{b}_u^n \circ \varphi_{s,u}^n(y) du + \int_s^t \widetilde{\sigma}_u^n \circ \varphi_{s,u}^n(y) \cdot dW_u, \tag{15}$$

We have $\widetilde{\sigma}_u^n \to \widetilde{\sigma}_u$ and $\widetilde{b}_u^n \to \widetilde{b}_u$ in $C_{b,T}^{1+\alpha}$. By standard argument using the Gronwall lemma, the Doob inequality and the Burkholder inequality (compare, for instance, with the proof of [12, Theorem II.2.1])) we obtain (11).

Concerning the derivative of the stochastic flow, we first prove that there exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \sup_{0 \le s \le T} E[\sup_{u \in [s,T]} \|D\phi_{s,u}^n(x)\|^p] \le C, \tag{16}$$

for any $n \in \mathbb{N}$ and $p \ge 1$ (here $\|\cdot\|$ denotes the Hilbert-Schmidt norm).

Arguing as in the proof of [12, Theorem II.3.1], we get the following linear equation for the derivative $D\phi_{s,t}(x)$

$$\begin{aligned}
[D\Psi_t(\phi_{s,t}(x))] D\phi_{s,t}(x) = D\Psi_s(x) &+ \int_s^t [D^2 \Psi_u(\phi_{s,u}(x))] D\phi_{s,u}(x) dW_u \\
&- \lambda \int_s^t [D\psi_u(\phi_{s,u}(x))] D\phi_{s,u}(x) du,
\end{aligned} \tag{17}$$

$0 \le s \le t \le T$, $x \in \mathbb{R}^d$. Thanks to (9), we get that $\psi^n \to \psi$ in $C_{b,T}^{2+\alpha}$. Using this fact together with (16) and (17), we finally obtain, for any $p \ge 1$,

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{0 \le s \le T} E[\sup_{u \in [s,T]} \|D\phi_{s,u}^n(x) - D\phi_{s,u}(x)\|^p] = 0. \tag{18}$$

■

We show now that the inverse flow $\phi_{s,t}^{-1}$ is directly related to the solutions of a simple backward stochastic differential equations, of the same form as the original one (only the drift has opposite sign). This result will be useful to prove uniqueness for our SPDE.

Lemma 9 *Under Hypothesis 1 the process* $\left(\phi_{s,t}^{-1}(y)\right)_{s\in[0,t]}$ *is the unique solution of the backward SDE*

$$\phi_{s,t}^{-1}(y) = y - \int_s^t b_r\left(\phi_{r,t}^{-1}(y)\right) dr - \left[W_t(\omega) - W_s(\omega)\right].$$

and

$$\sup_{x\in\mathbb{R}^d}\sup_{0\le u\le T} E[\sup_{s\in[0,u]} \|D\phi_{s,u}^{-1}(x)\|^p] < \infty \tag{19}$$

for any $p \ge 1$.

Proof. We have

$$\phi_{s,t,\omega}(x) = x + \int_s^t b_r\left(\phi_{s,r,\omega}(x)\right) dr + W_t(\omega) - W_s(\omega)$$

and then

$$\phi_{s,t,\omega}\left(\phi_{s,t,\omega}^{-1}(y)\right) = \phi_{s,t,\omega}^{-1}(y) + \int_s^t b_r\left(\phi_{s,r,\omega}\left(\phi_{s,t,\omega}^{-1}(y)\right)\right) dr + W_t(\omega) - W_s(\omega).$$

But $\phi_{s,r,\omega}\circ\phi_{s,t,\omega}^{-1} = \phi_{r,t,\omega}^{-1}$ and thus

$$y = \phi_{s,t,\omega}^{-1}(y) + \int_s^t b_r\left(\phi_{r,t,\omega}^{-1}(y)\right) dr + W_t(\omega) - W_s(\omega).$$

The proof of the bound (19) is then similar to that of eq. (18) once taken into account the backward character of the equation. ■

Remark 10 *Note that, for any* $M > 0$ *and* $s \in [0,T]$*, using the stability estimates in Th. 7 and the bounds*

$$E\Big[\int_{|y|\le M}\sup_{u\in[s,T]}|\phi_{s,u}^n(y)-\phi_{s,u}(y)|^p dy + \int_{|y|\le M}\sup_{u\in[s,T]}\|D\phi_{s,u}^n(y)-D\phi_{s,u}(y)\|^p dy\Big]$$

$$\le C\Big(\sup_{x\in\mathbb{R}^d} E[\sup_{u\in[s,T]}|\phi_{s,u}^n(x)-\phi_{s,u}(x)|^p] + \sup_{x\in\mathbb{R}^d} E[\sup_{u\in[s,T]}\|D\phi_{s,u}^n(x)-D\phi_{s,u}(x)\|^p]\Big),$$

with $p \ge 1$*, and possibly passing to a subsequence, still denoted by* ϕ^n*, we have that* $\phi_{s,t}^n(x)$ *and* $D\phi_{s,t}^n(x)$ *converge respectively to* $\phi_{s,t}(x)$ *and* $D\phi_{s,t}(x)$*, a.s in* x *with* $|x| \le M$*, P-a.s. in* $\omega\in\Omega$ *and uniformly in* $t\in[s,T]$.

In the sequel, we will often use the standard *mollifiers approximation.* Let $\vartheta : \mathbb{R}^d \to \mathbb{R}$ be a test function such that $0 \le \vartheta(x) \le 1$, $x \in \mathbb{R}^d$, $\vartheta(x) = \vartheta(-x)$, $\int_{\mathbb{R}^d}\theta(x)dx = 1$, $\mathrm{Supp}(\vartheta) \subset B(2)$, $\vartheta(x) = 1$, $x \in B(1)$ ($B(r)$ denotes the euclidean ball centered in 0 and with radius $r > 0$). For any $\varepsilon > 0$, define (ϑ_ε),

$\vartheta_\varepsilon(z) = \varepsilon^{-d}\vartheta(z/\varepsilon)$. Let $g : \mathbb{R}^d \to \mathbb{R}^n$ be a measurable function. The mollifiers approximation of g is

$$g^\varepsilon(x) = g * \vartheta_\varepsilon(x) = \int_{\mathbb{R}^d} g(x-y)\vartheta_\varepsilon(y)dy, \quad x \in \mathbb{R}^d. \tag{20}$$

If g depends also on time t, we consider $g^\varepsilon(t,x) = (g(t,\cdot) * \vartheta_\varepsilon)(x)$, $t \in [0,T]$, $x \in \mathbb{R}^d$.

Let $\phi_t = \phi_{0,t}$ be the stochastic flow associated to equation (5) starting at 0. Let us define the Jacobian determinant of the flow ϕ:

$$J_{s,t}(x) = \det[D\phi_{s,t}(x)], \ \ 0 \le s \le t \le T, \ \text{ and } \ J_t(x) = \det[D\phi_t(x)], \ \ t \in [0,T].$$

We finish the section with a result of independent interest. It concerns a special situation when the given vector field b is of zero distributional divergence and give rise to a measure-preserving flow.

Lemma 11 *Assume* $\operatorname{div} b(t,\cdot) = 0$, $t \in [0,T]$, *in the sense of distributions. Then the stochastic flow* ϕ *is* P*-a.s. measure-preserving, i.e.,* $J_{s,t}(x) = 1$, *for all* $0 \le s \le t \le T$ *and all* $x \in \mathbb{R}^d$, P*-a.s..*

Proof. Let b_n be $C^\infty_{b,T}$ vector fields that converges to b in $C^\alpha_{b,T}$ and such that $\operatorname{div} b_n = 0$. The functions b_n can be constructed as in (20). Let ϕ^n be the associated smooth diffeomorphism. Applying [12, Theorem II.3.1] and the well known Liouville theorem, we get that the diffeomorphism ϕ^n preserves the Lebsegue measure since b_n is a divergenceless vector field. Then $\det[D\phi^n_{s,t}(x)] = 1$ for all $0 \le s < t \le T$ and all x, P-a.s.. Fix $x \in \mathbb{R}^d$ and $s \in [0,T]$. By (18), there exists a subsequence (possibly depending on x, s and still denoted by $D\phi^n_{s,u}$) such that P-a.s.

$$\sup_{s\le u\le T} \|D\phi^n_{s,u}(x) - D\phi_{s,u}(x)\|^2 \to 0, \ \text{ as } \ n \to \infty.$$

We find that $J_{s,t}(x) = 1$, for any $x \in \mathbb{R}^d$, and so $\phi_{s,t}$ is a measure-preserving diffeomorphism P-a.s. for any $s < t \le T$. ■

3 Stochastic transport equation. Existence of weak solutions

Definition 12 *Given* $u_0 \in L^\infty(\mathbb{R}^d)$, *a weak* L^∞*-solution of the Cauchy problem (1) is a measurable function* $(u(t,x,\omega), t \ge 0, x \in \mathbb{R}^d, \omega \in \Omega)$ *such that*

(i) $\sup_{t\in[0,T], x\in\mathbb{R}^d, \omega\in\Omega} |u(t,x,\omega)| < \infty$

(ii) for any test function $\theta \in C^0_0(\mathbb{R}^d)$, *the process* $t \mapsto \int_{\mathbb{R}^d} u(t,x)\theta(x)dx$ *is continuous and* $\mathcal{F}_t$*-adapted;*

(iii) for any test function $\theta \in C^\infty_0(\mathbb{R}^d)$, *the process* $t \mapsto \int_{\mathbb{R}^d} u(t,x)\theta(x)dx$ *is an* $\mathcal{F}_t$*-semimartingale and*

$$\int_{\mathbb{R}^d} u(t,x)\theta(x)dx = \int_{\mathbb{R}^d} u_0(x)\theta(x)dx + \int_0^t ds \int_{\mathbb{R}^d} u(s,x)[b_s(x)\cdot\nabla\theta(x) + \operatorname{div} b_s(x)\theta(x)]dx + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x)D_i\theta(x)dx\right) \circ dW^i_s$$

Theorem 13 *Assume Hypothesis 1 and* $\operatorname{div} b \in L^1\left(0,T;L^1_{loc}(\mathbb{R}^d)\right)$. *Given* $u_0 \in L^\infty(\mathbb{R}^d)$, *the stochastic process* $u(t,x)$ *defined as* $u(t,x) := u_0(\phi_t^{-1}(x))$ *is a solution of (1).*

Proof. The function $u_0(\phi_t^{-1}(x))$ is bounded and measurable in (t,x,ω). If $\theta \in C_0^0(\mathbb{R}^d)$, by changing variable we have:

$$\int_{\mathbb{R}^d} u(t,x)\theta(x)dx = \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t(y)dy.$$

This easily yields property (ii) of Definition 12.

To prove property (iii) consider the flow ϕ_t^ε for the regularized vector field b^ε (see (20)) and let $J_t^\varepsilon(y)$ be its Jacobian determinant. Note that $u_0 \circ (\phi_t^\varepsilon)^{-1} \to u_0 \circ \phi_t^{-1}$ weakly in $L^\infty(\mathbb{R}^d)$, uniformly in $t \in [0,T]$ and P-a.s., indeed for $\theta \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (u_0 \circ (\phi_t^\varepsilon)^{-1})(y)\theta(y)dy = \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dy \to \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t(y)dy$$

using Remark 10. By density we can extend this convergence to any $\theta \in L^1(\mathbb{R}^d)$. Moreover since b^ε is smooth, it is easy to prove that

$$dJ_t^\varepsilon(y) = \operatorname{div} b_t^\varepsilon(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dt.$$

and by the Itô formula we find

$$\begin{aligned}\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dy = &\int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)L^{b^\varepsilon}\theta(\phi_s^\varepsilon(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s^\varepsilon(y))\operatorname{div} b_s^\varepsilon(\phi_s^\varepsilon(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s^\varepsilon(y))J_s^\varepsilon(y)dy.\end{aligned} \tag{21}$$

To pass to the limit as $\varepsilon \to 0$ the only difficulty is given by the term

$$\int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s^\varepsilon(y))\operatorname{div} b_s^\varepsilon(\phi_s^\varepsilon(y))J_s^\varepsilon(y)dy = \int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(y))\theta(y)\operatorname{div} b_s^\varepsilon(y)dy$$

Now $\operatorname{div} b^\varepsilon$ converges in $L^1\left(0,T;L^1_{loc}(\mathbb{R}^d)\right)$ to $\operatorname{div} b$ and, using also the previous weak convergence of $u_0 \circ (\phi_t^\varepsilon)^{-1}$, we can conclude the convergence, for any fixed ω, of this term toward

$$\int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s)^{-1}(y))\theta(y)\operatorname{div} b_s(y)dy$$

as required. For the stochastic integral we have

$$\int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s^\varepsilon(y))J_s^\varepsilon(y)dy \to \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s(y))J_s(y)dy$$

uniformly on $[0,T]$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Then we get

$$\begin{aligned}\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t(y)dy = &\int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)L^b\theta(\phi_s(y))J_s(y)dy \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s(y))\mathrm{div}\, b(\phi_s(y))J_s(y)dy \\ &+ \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s(y))J_s(y)dy.\end{aligned}$$

By writing the above formula with θ replaced by $\nabla_i\eta$, where η is a test function, we discover that $\int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s(y))J_s(y)dy$ is a semimartingale and so it can be integrated in the Stratonovich sense.

Using the Stratonovich version of Itô formula

$$\theta(\phi_t(y)) = \theta(y) + \int_0^t b(\phi_s(y)) \cdot \nabla\theta(\phi_s(y))ds + \int_0^t \nabla\theta(\phi_s(y)) \circ dW_s.$$

and a stochastic Fubini theorem (see [14]) gives finally

$$\begin{aligned}\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t(y)dy = &\int_{\mathbb{R}^d} u_0(y)\theta(y)dy \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u_0(y)b(\phi_s(y)) \cdot \nabla\theta(\phi_s(y))J_s(y)dy \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s(y))\mathrm{div}\, b(\phi_s(y))J_s(y)dy \\ &+ \int_0^t dW_s \circ \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s(y))J_s(y)dy\end{aligned}$$

which is exactly the equation we want to prove. ■

4 Uniqueness of weak solutions

In the deterministic case with non-regular vector fields b, uniqueness of weak L^∞-solutions is proved by means of the concept of renormalized solutions, see [5], [1]. The technical tool is a commutator lemma, where the role of the assumptions on b is immediately visible.

One of the usual forms of the commutator lemma is the following one, taken from [15], Chapter 2. Let ϑ_ε be the mollifiers introduced in (20); the convolution $\vartheta_\varepsilon * f = \int \vartheta_\varepsilon(\cdot - x) f(x)\, dx$ is extended also to distributions f.

Let $\eta : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1-diffeomorphism; in this section, we will use the following notation for the Jacobian determinant

$$J\eta(x) = \det[D\eta(x)], \quad x \in \mathbb{R}^d.$$

Given $g \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R})$, $v \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $\mathrm{div}\, v \in L^1_{loc}(\mathbb{R}^d, \mathbb{R})$, denote by $v \cdot \nabla g$ the distribution defined on smooth compact support test functions f as

$$(v \cdot \nabla g)(f) := -\int gv \cdot \nabla f dx - \int gf \mathrm{div}\, v\, dx.$$

Define also the commutator $\mathcal{R}_\varepsilon[g,v](x)$ as the (smooth) function

$$\mathcal{R}_\varepsilon[g,v](x) = [\vartheta_\varepsilon * (v \cdot \nabla g) - v \cdot \nabla(\vartheta_\varepsilon * g)](x).$$

Lemma 14 *Given*

$$v \in W^{1,1}\left(\mathbb{R}^d, \mathbb{R}^d\right), \quad g \in L^\infty\left(\mathbb{R}^d, \mathbb{R}\right),$$

we have

$$\int \left|\mathcal{R}_\varepsilon\left[g, v\right](x)\right| dx \leq C \left\|v\right\|_{W^{1,1}} \left\|g\right\|_{L^\infty}$$

for some $C > 0$ independent of ε, v and g. Moreover,

$$\lim_{\varepsilon \to 0} \int \left|\mathcal{R}_\varepsilon\left[g, v\right](x)\right| dx = 0.$$

The strong convergence of commutators requires some weak form of differentiability of v or g. We do not impose it, so we have to use a relaxed form of this result. First, essentially with the same proof of the previous lemma one gets the following fact: distributional convergence of commutators holds under very weak conditions.

Distributional convergence is not sufficient to study the deterministic PDE, while we can work with it in the stochastic case, as explained below.

For any $r > 0$ denote by $B(r)$ is the ball centered at the origin of radius r and by $C_r^\infty(\mathbb{R}^d, \mathbb{R})$ the space of smooth functions with compact support in $B(r)$. Moreover, denote by $\left\|\cdot\right\|_{L_r^1}$ and $\left\|\cdot\right\|_{L_r^\infty}$ respectively, the L^1-norm and the L^∞-norm on $B(r+1)$.

Lemma 15 *Given*

$$v \in L_{loc}^\infty\left(\mathbb{R}^d, \mathbb{R}^d\right), \quad \mathrm{div} v \in L_{loc}^1\left(\mathbb{R}^d, \mathbb{R}\right), \qquad g \in L_{loc}^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$$

for any $\rho \in C_r^\infty(\mathbb{R}^d, \mathbb{R})$ and for sufficiently small ε we have

$$\left|\int \mathcal{R}_\varepsilon\left[g, v\right](x)\, \rho(x)\, dx\right| \leq 2 \left\|g\right\|_{L_r^\infty} \left[\left\|\mathrm{div} v\right\|_{L_r^1} \left\|\rho\right\|_{L_r^\infty} + \left\|v\right\|_{L_r^\infty} \left\|\nabla \rho\right\|_{L_r^1}\right]$$

and

$$\lim_{\varepsilon \to 0} \int \mathcal{R}_\varepsilon\left[g, v\right](x)\, \rho(x)\, dx = 0.$$

Proof. The proof proceeds as the one of the previous lemma from [15], Chapter 2: we prove the inequality for regular fields, then one can extend to non-regular ones and prove the convergence again first in the regular case. We omit the details of the second part and only prove the inequality.

Step 1. Let us rewrite the expressions is a suitable way.

$$\begin{aligned}
\int \mathcal{R}_\varepsilon\left[g, v\right](x)\rho(x)dx &= \int \left[\vartheta_\varepsilon * (v \cdot \nabla g) - v \cdot \nabla\left(\vartheta_\varepsilon * g\right)\right](x)\rho(x)dx \\
&= \iint \left[-g(x')v(x')\nabla_{x'}\vartheta_\varepsilon(x-x') - g(x')\vartheta_\varepsilon(x-x')\mathrm{div}\, v(x')\right]\rho(x)\, dxdx' \\
&\quad + \iint \left[g(x')v(x)\vartheta_\varepsilon(x-x')\nabla_x\rho(x) + g(x')\rho(x)\vartheta_\varepsilon(x-x')\mathrm{div}\, v(x)\right] dxdx'
\end{aligned}$$

by using $\nabla_{x'}\vartheta_\varepsilon(x-x') = -\nabla_x\vartheta_\varepsilon(x-x')$ and integrating by parts in x the first term we get

$$\begin{aligned}
&= \iint g(x')\vartheta_\varepsilon(x-x')\nabla_x\rho(x)[v(x) - v(x')]\, dxdx' \\
&\quad + \iint g(x')\rho(x)\vartheta_\varepsilon(x-x')[\mathrm{div}\, v(x) - \mathrm{div}\, v(x')]\, dxdx'
\end{aligned}$$

Step 2. Assume ε so small that the support of ϑ_ε has diameter less than one. We have (using standard estimates on convolutions)

$$\left|\int \mathcal{R}_\varepsilon [g,v](x)\rho(x)dx\right| \leq 2\|g\|_{L^\infty_r}\|v\|_{L^\infty_r}\|\nabla\rho\|_{L^1_r} + 2\|\rho\|_{L^\infty_r}\|g\|_{L^\infty_r}\|\mathrm{div}\, v\|_{L^1_r}$$

The inequality is proved. The second assertion follows using standard properties of mollifiers. The proof is complete. ■

In the proof of the uniqueness theorem the distributional convergence of commutators is sufficient but the commutator must be composed with the flow. This strategy would work in the deterministic case as well. But the main difficulty is that composition with the flow put into play the first derivatives of the Jacobian of the flow. This would impose too much regularity on b in the deterministic case. On the contrary, in the stochastic case, we have a weakly differentiable Jacobian of the flow under mild assumptions on b, as explained in the previous section.

First, let us write the previous weak commutator lemma in the case when commutators are applied to a diffeomorphism.

Corollary 16 *Let ϕ be a C^1 diffeomorphism of $\mathbb{R}^d$. Assume*

$$v \in L^\infty_{loc}\left(\mathbb{R}^d,\mathbb{R}^d\right), \quad \mathrm{div}\, v \in L^1_{loc}\left(\mathbb{R}^d,\mathbb{R}\right), \quad g \in L^\infty_{loc}\left(\mathbb{R}^d,\mathbb{R}\right)$$
$$J\phi^{-1} \in W^{1,1}_{loc}\left(\mathbb{R}^d,\mathbb{R}\right).$$

then for any $\rho \in C^\infty_r(\mathbb{R}^d,\mathbb{R})$

$$\lim_{\varepsilon\to 0}\int \mathcal{R}_\varepsilon [g,v](\phi(x))\rho(x)\,dx = 0.$$

Proof. Set $R_{r,\phi} := |\phi(0)| + (r+1)\|D\phi\|_{L^\infty_r}$. Using

$$|\phi(x)| \leq |\phi(0)| + \|D\phi\|_{L^\infty_\rho}|x| \leq |\phi(0)| + r\|D\phi\|_{L^\infty_\rho}$$

we check that the map ϕ send the support of ρ strictly in the ball of radius $R_{r,\phi}$. Standing this preliminary fact, we simply have

$$\int \mathcal{R}_\varepsilon [g,v](\phi(x))\rho(x)\,dx = \int \mathcal{R}_\varepsilon [g,v](y)\rho_\phi(y)\,dy$$

where the function $\rho_\phi(y) := \rho\left(\phi^{-1}(y)\right)J\phi^{-1}(y)$ has the support strictly contained in the ball of radius $R_{r,\phi}$. Thus by the previous lemma

$$\left|\int \mathcal{R}_\varepsilon [g,v](\phi(x))\rho(x)\,dx\right| \leq 2\|g\|_{L^\infty_{R_{r,\phi}}}\Big[\|\mathrm{div}\, v\|_{L^1_{R_{r,\phi}}}\|\rho_\phi\|_{L^\infty_{R_{r,\phi}}} + \|v\|_{L^\infty_{R_{r,\phi}}}\|\nabla\rho_\phi\|_{L^1_{R_{r,\phi}}}\Big].$$

To conclude, it is sufficient to notice that $\|\rho_\phi\|_{L^\infty_{R_{r,\phi}}} \leq \|\rho\|_{L^\infty_r}\left\|J\phi^{-1}\right\|_{L^\infty_{R_{r,\phi}}}$ and, denoting by $[D\phi^{-1}(y)]^*$ the adjoint matrix of $[D\phi^{-1}(y)]$,

$$\begin{aligned}\|\nabla\rho_\phi\|_{L^1_{R_{r,\phi}}} &\leq \left\|[D\phi^{-1}(\cdot)]^*\left(\nabla\rho\circ\phi^{-1}\right)(\cdot)\,J\phi^{-1}(\cdot)\right\|_{L^1_{R_{r,\phi}}} \\ &\quad + \left\|\left(\rho\circ\phi^{-1}\right)\cdot\nabla J\phi^{-1}\right\|_{L^1_{R_{r,\phi}}} \\ &\leq \left\|D\phi^{-1}\right\|_{L^\infty_{R_{r,\phi}}}\|\nabla\rho\|_{L^1_r} + \|\rho\|_{L^\infty_r}\left\|\nabla J\phi^{-1}\right\|_{L^1_{R_{r,\phi}}}.\end{aligned}$$

The proof is complete. ■

We can now prove the uniqueness of weak L^∞-solutions. We postpone the main estimate on the derivatives of the Jacobian of the flow to the next section.

Theorem 17 *Under Hypothesis 1 and 2, for every $u_0 \in L^\infty$ there exists a unique weak L^∞-solution of the Cauchy problem (1). Thus every solution $u(t,x,\omega)$ has the form $u(t,x,\omega) := u_0(\phi_t^{-1}(\omega)\, x)$.*

Proof. Step 1. By linearity we have to prove that a solution with initial condition $u_0 = 0$ vanishes identically. Let us denote by u such a solution.

For $y \in \mathbb{R}^d$ fixed, $\varepsilon > 0$, let us choose the test function $\theta(x) := \vartheta_\varepsilon(y - x)$ in property (iii) of Definition 12. Let us define $u_\varepsilon(t,\cdot) = \vartheta_\varepsilon * u(t,\cdot)$. We get

$$u_\varepsilon(t,y) = u_{0,\varepsilon}(y) + \int_0^t A_\varepsilon(s,y)\, ds + \sum_{i=1}^d \int_0^t B_\varepsilon^{(i)}(s,y) \circ dW_s^i$$

where

$$A_\varepsilon(t,y) := \int_{\mathbb{R}^d} u(t,x) \left\{ b_t(x) \cdot \nabla_x [\vartheta_\varepsilon(y-x)] + \operatorname{div} b_t(x) \vartheta_\varepsilon(y-x) \right\} dx$$
$$B_\varepsilon^{(i)}(t,y) := \int_{\mathbb{R}^d} u(t,x) D_{x_i} [\vartheta_\varepsilon(y-x)] dx.$$

All these functions of y, namely $u_\varepsilon(t,y)$, $u_{0,\varepsilon}(y)$, ... , $B_\varepsilon^{(i)}(t,y)$, are continuous in t, adapted, smooth (of class C^3 is required for the next computation) in y (from (i) and (ii) of definition 12). By the Stratonovich version of Kunita-Itô-Wentzel formula (see [12, Th. 8.3 page 188])

$$\begin{aligned} du_\varepsilon(t,\phi_t(x)) &= A_\varepsilon(t,\phi_t(x))\, dt + \sum_{i=1}^d B_\varepsilon^{(i)}(t,\phi_t(x)) \circ dW_t^i \\ &\quad + (b \cdot \nabla u_\varepsilon)(t,\phi_t(x)) dt + \sum_{i=1}^d (D_i u_\varepsilon)(t,\phi_t(x)) \circ dW_t^i \end{aligned}$$

But

$$(D_i u_\varepsilon)(t,y) = \int_{\mathbb{R}^d} u(t,x) D_{y_i} [\vartheta_\varepsilon(y-x)]\, dx = -\int_{\mathbb{R}^d} u(t,x) D_{x_i} [\vartheta_\varepsilon(y-x)]\, dx$$

hence

$$du_\varepsilon(t,\phi_t(x)) = [A_\varepsilon(t,\phi_t(x)) + (b \cdot \nabla u_\varepsilon)(t,\phi_t(x))]\, dt$$

namely (recall that the initial condition is zero)

$$u_\varepsilon(t,\phi_t(x)) = -\int_0^t \mathcal{R}_\varepsilon[u_s, b_s](\phi_s(x))\, ds$$

where $\mathcal{R}_\varepsilon[u_s, b_s](y)$ is the commutator defined above.

Step 2. For every $\rho \in C_r^\infty(\mathbb{R}^d)$ (see the definition above) we have

$$\int u_\varepsilon(t,\phi_t(x)) \rho(x)\, dx = \int u_\varepsilon(t,y) \rho\left(\phi_t^{-1}(y)\right) J\phi_t^{-1}(y) dy.$$

Given $t \in [0,T]$, with probability one, $u_\varepsilon(t,\cdot)$ converges weak-$*$ to $u(t,\cdot)$ as $\varepsilon \to 0$. With probability one, the function $y \mapsto \rho(\phi_t^{-1}(y))$ is at least continuous with compact support, and also the function $y \mapsto J\phi_t^{-1}(y)$ is continuous. Hence, P-a.s.,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int u_\varepsilon(t,\phi_t(x))\rho\left(x\right)dx &= \int u(t,y)\rho\left(\phi_t^{-1}(y)\right)J\phi_t^{-1}(y)dy\\ &= \int u(t,\phi_t(x))\rho\left(x\right)dx.\end{aligned}$$

Therefore $\int\left(\int_0^t \mathcal{R}_\varepsilon\left[u_s,b_s\right]\left(\phi_s(x)\right)ds\right)\rho\left(x\right)dx$ converges P-a.s. as $\varepsilon \to 0$ and we have

$$\int u(t,\phi_t(x))\rho\left(x\right)dx = \lim_{\varepsilon\to 0}\int\left(\int_0^t \mathcal{R}_\varepsilon\left[u_s,b_s\right]\left(\phi_s(x)\right)ds\right)\rho\left(x\right)dx.$$

The function $(s,x) \mapsto \mathcal{R}_\varepsilon\left[u_s,b_s\right]\left(\phi_s(x)\right)\rho\left(x\right)$ is continuous with compact support, P-a.s., hence

$$\int u(t,\phi_t(x))\rho\left(x\right)dx = \lim_{\varepsilon\to 0}\int_0^t\left(\int \mathcal{R}_\varepsilon\left[u_s,b_s\right]\left(\phi_s(x)\right)\rho\left(x\right)dx\right)ds.$$

If we prove that, given $\rho \in C_0^\infty(\mathbb{R}^d)$ and $t \in [0,T]$, this P-a.s.-limit (which exists) is zero, then we have u identically zero (because ϕ_t is a bijection).

Step 3. Let us check, by means of Corollary 16, that

$$s \mapsto \int_{\mathbb{R}^d} \mathcal{R}_\varepsilon\left[u_s,b_s\right]\left(\phi_s(x)\right)\rho\left(x\right)dx$$

satisfies the assumptions of Lebesgue dominated convergence theorem (P-a.s.) on $[0,T]$. This will complete the proof. For every $s \in [0,T]$ we have (recall the notation $R_{r,\phi}$ from Corollary 16)

$$\begin{aligned}\left|\int \mathcal{R}_\varepsilon\left[u_s,b_s\right]\left(\phi_s\left(x\right)\right)\rho\left(x\right)dx\right| &\le 2\left\|u_s\right\|_{L^\infty_{R_{r,\phi_s}}}\left\|\operatorname{div} b_s\right\|_{L^1_{R_{r,\phi_s}}}\left\|\rho\right\|_{L^\infty_r}\left\|J\phi_s^{-1}\right\|_{L^\infty_{R_{r,\phi_s}}}\\ &+2\left\|u_s\right\|_{L^\infty_{R_{r,\phi_s}}}\left\|b_s\right\|_{L^\infty_{R_{r,\phi_s}}}\left(\left\|D\phi_s^{-1}\right\|_{L^\infty_{R_{r,\phi_s}}}\left\|\nabla\rho\right\|_{L^1_r}+\left\|\rho\right\|_{L^\infty_r}\left\|\nabla J\phi_s^{-1}\right\|_{L^1_{R_{r,\phi_s}}}\right)\\ &\le C_{u,\rho}\Big(1+\left\|\operatorname{div} b_s\right\|_{L^1_{R_{r,\phi_s}}}\left\|J\phi_s^{-1}\right\|_{L^\infty_{R_{r,\phi_s}}}+\left\|D\phi_s^{-1}\right\|_{L^\infty_{R_{r,\phi_s}}}+\left\|\nabla J\phi_s^{-1}\right\|_{L^1_{R_{r,\phi_s}}}\Big)\end{aligned}$$

because of the global boundedness of u, b, ρ.

We know, from the properties of the stochatic flow ϕ, that, for any $R > 0$,

$$P\left(\sup_{s\in[0,T]}\left(\left\|D\phi_s\right\|_{L^\infty_R}+\left\|\phi_s\right\|_{L^\infty_R}\right)<\infty\right)=1$$

hence (recall the definition of R_{r,ϕ_s}) there exists a positive random variable X, P-a.s. finite, such that

$$\left\|\operatorname{div} b_s\right\|_{L^1_{R_{r,\phi_s}}}\left\|J\phi_s^{-1}\right\|_{L^\infty_{R_{r,\phi_s}}}+\left\|D\phi_s^{-1}\right\|_{L^\infty_{R_{r,\phi_s}}}$$

$$+\left\|\nabla J\phi_s^{-1}\right\|_{L^1_{R_{r,\phi_s}}} \leq \|\operatorname{div} b_s\|_{L^1_X}\left\|J\phi_s^{-1}\right\|_{L^\infty_X} + \left\|D\phi_s^{-1}\right\|_{L^\infty_X} + \left\|\nabla J\phi_s^{-1}\right\|_{L^1_X},$$

uniformly in $s\in[0,T]$, with probability one. The term $\int_0^T \|\operatorname{div} b_s\|_{L^1_X}\,ds$ is finite (P-a.s.). Since $(s,x)\mapsto D\phi_s^{-1}(x)$ is P-a.s. continuous, we also have

$$\sup_{s\in[0,T]}\left(\left\|J\phi_s^{-1}\right\|_{L^\infty_X} + \left\|D\phi_s^{-1}\right\|_{L^\infty_X}\right) < \infty, \ \text{ P-a.s..}$$

Finally, we just need

$$\int_0^T \left\|\nabla J\phi_s^{-1}\right\|_{L^1_X} ds < \infty, \quad P\text{-a.s.} \tag{22}$$

This property is crucial and will be proved in the next section. The proof is complete. ■

5 Estimates on derivatives of the Jacobian

The aim of this section is to prove the estimate (22).

Since by Lemma 9 the equation for ϕ_s^{-1} is equal to the equation for ϕ_s (up to a sign and inversion of time), for simplicity of notations we prove the estimate

$$\int_0^T \|\nabla J\phi_s\|_{L^1_X} ds < \infty, \quad P\text{-a.s.} \tag{23}$$

in place of estimate (22). Here X is a generic P-a.s. finite random variable. Recall that $\|\cdot\|_{L^1_X}$ denotes the L^1-norm on the ball $B(X+1)$.

We know that

$$J\phi_t(x) = e^{\int_0^t \operatorname{div} b_s(\phi_s(x))ds}$$

hence

$$\nabla J\phi_t(x) = J\phi_t(x)\cdot\nabla\int_0^t \operatorname{div} b_s(\phi_s(x))\,ds$$

if the integral, as a function of x, is differentiable (at least weakly). The first factor, $J\phi_t(x)$ is P-a.s. bounded in (t,x) on $[0,T]\times B(X+1)$ (because it is continuous). Hence, to prove (22), it is sufficient to prove

$$\int_0^T \left\|\nabla\int_0^t \operatorname{div} b_s(\phi_s(\cdot))\,ds\right\|_{L^1_X} dt < \infty, \quad P\text{-a.s.} \tag{24}$$

Again the basic tool is a Tanaka type formula, based on a result for parabolic equations proved in [8, Theorem 1.2] (see also [10, Theorem 10.3]).

We introduce a function space from [8] and [10]. For $p,q\in(1,+\infty)$, we consider the space $H_p^{2,q}(T)$ consisting of all functions u defined on $[0,T]$ with values in distributions on $\mathbb{R}^d$ such that $u\in L^q([0,T];W_p^2(\mathbb{R}^d))$ and there exists $D_t u\in L^q([0,T];L^p(\mathbb{R}^d))$.

Theorem 18 *Under Hypothesis 1 and 2, in the space $H_p^{2,q}(T)$ there exists a unique solution to the parabolic equation*

$$\begin{cases} \dfrac{\partial F}{\partial t} + L^b F = \operatorname{div} b, & t\in[0,T[\\ F(T,x) = 0, & x\in\mathbb{R}^d. \end{cases}$$

Writing $\frac{\partial F}{\partial t}+\frac{1}{2}\triangle F=-b\cdot\nabla u+\operatorname{div} b$, this results follows from [8, Theorem 1.2] (using the fact that b is globally bounded). We mention that an extension of [8, Theorem 1.2] is given in [10, Theorem 10.3].

Proposition 19 *Under Hypothesis 1 and 2, estimate (24) holds true.*

Proof. The condition $\frac{d}{p}+\frac{2}{q}<1$ implies that if $u\in H_p^{2,q}(T)$ then u can be represented by a real function $u(t,x)$ continuous on $[0,T]\times\mathbb{R}^d$ such that there exists $\nabla u(t,x)$ and is continuous on $[0,T]\times\mathbb{R}^d$ (see [10, Lemma 10.1]).

The following Ito formula holds (it is a special case of [10, Theorem 3.7])

$$F\left(t,\phi_t\left(x\right)\right)=F\left(0,x\right)+\int_0^t\nabla F\left(s,\phi_s\left(x\right)\right)\cdot dW_s+\int_0^t\left(\frac{\partial F}{\partial s}+L_bF\right)\left(s,\phi_s\left(x\right)\right)ds. \tag{25}$$

Thus

$$\begin{aligned}\nabla\int_0^t\operatorname{div}b_s\left(\phi_s\left(x\right)\right)ds&=\nabla\left[F\left(t,\phi_t\left(x\right)\right)-F\left(0,x\right)-\int_0^t\nabla F\left(s,\phi_s\left(x\right)\right)dW_s\right]\\&=\nabla F\left(t,\phi_t\left(x\right)\right)\cdot D\phi_t\left(x\right)-\nabla F\left(0,x\right)-\int_0^t D^2F\left(s,\phi_s\left(x\right)\right)D\phi_s\left(x\right)dW_s.\end{aligned}$$

From the properties of F we readily have

$$\int_0^T\left\|\nabla F\left(0,\cdot\right)\right\|_{L_X^1}dt<\infty,\quad P\text{-a.s.}$$

Because of the P-a.s. continuity of $D\phi_t\left(x\right)$ in (t,x), we get

$$\int_0^T\left\|\nabla F\left(t,\phi_t\left(\cdot\right)\right)\cdot D\phi_t\left(\cdot\right)\right\|_{L_X^1}dt<\infty,\quad P\text{-a.s.}$$

which is another part of the estimate we need to prove.

Finally, we need to prove

$$\int_0^T\left\|\int_0^t D^2F\left(s,\phi_s\left(\cdot\right)\right)D\phi_s\left(\cdot\right)dW_s\right\|_{L_X^1}dt<\infty,\quad P\text{-a.s.}$$

If we prove that

$$E\left[\int_0^T\int_K\left|\int_0^t D^2F\left(s,\phi_s\left(x\right)\right)D\phi_s\left(x\right)dW_s\right|dxdt\right]<\infty$$

for every (deterministic) compact set $\mathbb{R}^d$, then it is not difficult to deduce the previous property; (using the inequality $|a|\leq a^2+1$) it is sufficient to bound

$$\begin{aligned}&E\left[\int_0^T\int_K\left|\int_0^t D^2F\left(s,\phi_s\left(x\right)\right)D\phi_s\left(x\right)dW_s\right|^2dxdt\right]\\&=\int_0^T\int_K E\left[\left|\int_0^t D^2F\left(s,\phi_s\left(x\right)\right)D\phi_s\left(x\right)dW_s\right|^2\right]dxdt\\&\leq\int_0^T\int_K E\left[\int_0^t\left|D^2F\left(s,\phi_s\left(x\right)\right)D\phi_s\left(x\right)\right|^2ds\right]dxdt\\&\leq T\int_K E\left[\int_0^T\left|D^2F\left(s,\phi_s\left(x\right)\right)D\phi_s\left(x\right)\right|^2ds\right]dx.\end{aligned}$$

Let p' such that $2 < p' < p$ and $2 < p' < q$ where p and q are given in Hypothesis 2. Since

$$\int_K \left(\int_0^T E\left[\left| D\phi_s(x) \right|^r \right] ds \right) dx < \infty,$$

for every $r \geq 1$ (see (16)), by the Hölder inequality on $\Omega \times K \times [0,T]$, it is sufficient to prove that

$$\int_K E\left[\left(\int_0^T \left| D^2 F(s, \phi_s(x)) \right|^{p'} ds \right) \right] dx < \infty.$$

But we have

$$\begin{aligned} & E\left[\left(\int_0^T \int_K \left| D^2 F(s, \phi_s(x)) \right|^{p'} dx ds \right) \right] \\ = & E\left[\left(\int_0^T \int_{\phi_s(K)} \left| D^2 F(s,y) \right|^{p'} J\phi_s^{-1}(y) dy ds \right) \right] \\ \leq & \int_0^T \int_{\mathbb{R}^d} \left| D^2 F(s,y) \right|^{p'} E\left[J\phi_s^{-1}(y) \right] dy ds. \end{aligned}$$

By the Hölder inequality,

$$\int_0^T \int_{\mathbb{R}^d} \left| D^2 F(s,y) \right|^{p'} E\left[J\phi_s^{-1}(y) \right] dy ds$$

$$\leq \int_0^T \left(\int_{\mathbb{R}^d} \left| D^2 F(s,y) \right|^p dy \right)^{p'/p} \left(\int_{\mathbb{R}^d} (E\left| J\phi_s^{-1}(y) \right|)^{\frac{p'}{p'-p}} dy \right)^{\frac{p'-p}{p'}} ds.$$

Using the bound (19) we have

$$\sup_{x \in \mathbb{R}^d} \sup_{s \in [0,T]} E\left[\left| J\phi_s^{-1}(x) \right|^r \right] < \infty$$

it is sufficient to have

$$\int_0^T \left(\int_{\mathbb{R}^d} \left| D^2 F(s,y) \right|^p dy \right)^{p'/p} ds < \infty.$$

This is true since $p'/p < q/p$ and $D^2 F \in L^q([0,T]; L^p(\mathbb{R}^d))$ by Theorem 18. The proof is complete. ■

6 Counterexamples to certain extensions

It would be interesting to try to generalize the results of this paper to random vectorfields $b(x,t,\omega)$, possibly adapted. However our approach faces a fundamental difficulty: it is very easy to exibith a counterexample which shows that in some cases the regularizing effect disappears. Consider in one dimension the case

$$b(x,t,\omega) = \sqrt{|x - W_t(\omega)|}$$

namely the stochastic differential equation

$$dX_t^x = b\left(X_t^x, t, .\right) dt + dW_t, \quad t \geq 0, \quad X_0^x = x$$

If (X_t^x) is a solution, then $Y_t^x = X_t^x - W_t$ solves

$$dY_t^x = \sqrt{|Y_t^x|} dt \quad t \geq 0, \quad Y_0^x = x$$

and viceversa. Hence the non-uniqueness for the latter equation transfer to the former. In terms of stochastic transport equation, an equation of the form

$$\partial_t u\left(t,x\right) + \left(b_0\left(x - W\left(t\right)\right) \cdot \nabla u\left(t,x\right)\right) dt + \nabla u\left(t,x\right) \circ dW\left(t\right) = 0$$

may have several pathologies if b_0 is only Hölder continuous.

Unfortunately the previous example is not so artificial: something similar happens in the nonlinear case. Let us argue only formally. Consider the example in $\mathbb{R}^2$

$$\partial_t \xi\left(t,x\right) + \left(u\left(t,x\right) \cdot \nabla \xi\left(t,x\right)\right) dt + \nabla \xi\left(t,x\right) \circ dW\left(t\right) = 0$$

where $\xi = \partial_2 u_1 - \partial_1 u_2$. This is the vorticity equation of a 2D ideal fluid described by a stochastic version of Euler equation. Following [16], this equation is (formally) equivalent to the stochastic ordinary equation depending on a parameter $a \in \mathbb{R}^2$

$$dX_t^a = \left[\int_{\mathbb{R}^2} K(X_t^a - X_t^{a'})\xi_0(X_t^{a'})da'\right] dt + dW_t$$

for a suitable kernel K, ξ_0 being the initial condition of the vorticity equation. This problem is equivalent to

$$dY_t^a = \left[\int_{\mathbb{R}^2} K(Y_t^a - Y_t^{a'})\xi_0(X_t^{a'})da'\right] dt$$

by the change of variable $Y_t^a = X_t^a - W_t$, and the equation for (Y_t^a) corresponds to the classical vorticity equation

$$\frac{\partial_t \xi'\left(t,x\right)}{\partial t} + \left(u'\left(t,x\right) \cdot \nabla \xi'\left(t,x\right)\right) dt = 0 \qquad \xi' = \partial_2 u_1' - \partial_1 u_2'$$

with initial condition ξ_0. This means that the stochastic vorticity equation is (at least formally) equivalent to the deterministic one. There is no advantage to introduce that kind of stochastic perturbation.

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